Plasticity implies the Volterra formulation: an example

Amit Acharya*

Abstract

A demonstration through an example is given of how the Volterra dislocation formulation in linear elasticity can be viewed as a (formal) limit of a problem in plasticity theory. Interestingly, from this point of view the Volterra dislocation formulation with discontinuous displacement, and non-integrable energy appears as a large-length scale limit of a smoother microscopic problem. This is in contrast to other formulations using SBV functions as well as the theory of Structured Deformations where the microscopic problem is viewed as discontinuous and singular, and the smoother plasticity formulation appears as a homogenized large length-scale limit.

With reference to Fig. 1, let the domain Ω be the unit disk, considered as the cross-section of a right circular cylinder on the $x-y$ plane, O the origin, and S the trace of a horizontal surface on the $x-y$ plane defined by $S = \{(x,y) : x \geq 0, y = 0\}$. The Volterra dislocation problem for a single straight dislocation may be defined as the following statement. Solve for a displacement field $u : \Omega \backslash S \to \mathbb{R}^3$ for which the limits

$$\begin{aligned} &u^+(x) := u(x,y), y \to 0^+, x > 0 \\ &u^-(x) := u(x,y), y \to 0^-, x > 0 \\ &(gradu)^+(x) := gradu(x,y), y \to 0^+, x > 0 \\ &(gradu)^-(x) := gradu(x,y), y \to 0^-, x > 0 \end{aligned}$$

*Dept. of Civil & Environmental Engineering, and Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh, PA 15213, email: acharyaamit@cmu.edu.

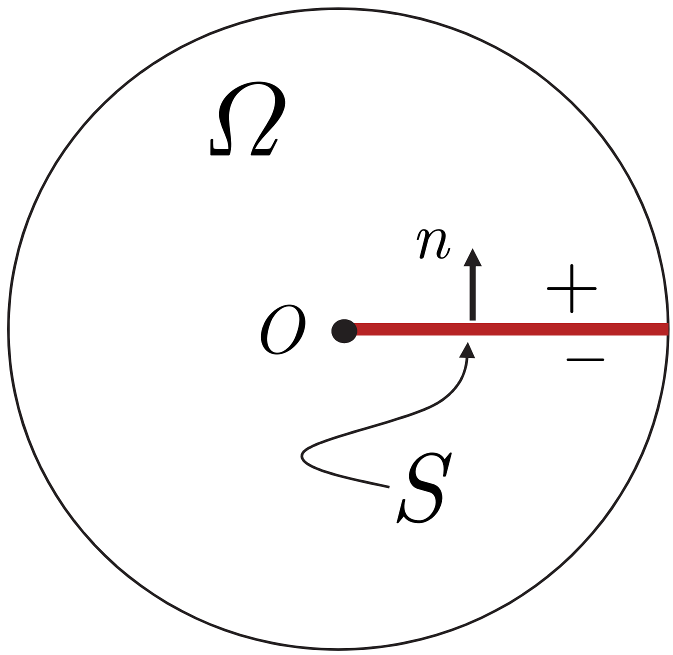

Figure 1: Schematic of setting for Volterra dislocation.

exist, and which satisfies

$$\begin{aligned} &div\,(C\,grad u) = 0 \quad \text{on} \quad \Omega \backslash S \\ &u^+ - u^- =: [\![u]\!] = b \\ &(C grad u^+ - C grad u^-)n =: [\![C grad u]\!]n = 0 \quad \text{for} \quad x \neq 0 \\ &(C grad u)\nu = 0 \quad \text{on} \ \partial\Omega, \end{aligned} \tag{1}$$

where $b \in \mathbb{R}^3$ is a given *constant* vector and ν is the unit normal field on $\partial\Omega$. It is now an easy observation that any such solution must have $|grad u(x, y)| \to \infty$ as $(x, y) \to (0, 0)$, since the line integral of the displacement gradient along any circular loop of arbitrarily small radius starting from the 'top' of S and ending at the 'bottom' of S must recover the finite value b. Moreover, the divergence is like $\frac{1}{r}$ where r is the distance of a point from the origin. This shows that the stress blows up like $\frac{1}{r}$ as well implying that the linear elastic energy density is not integrable for bodies of finite extent.

With reference to Fig. 2, the plasticity formulation for dislocations replaces the above statement with the following: Solve for $u^d : \Omega \to \mathbb{R}^3$ that

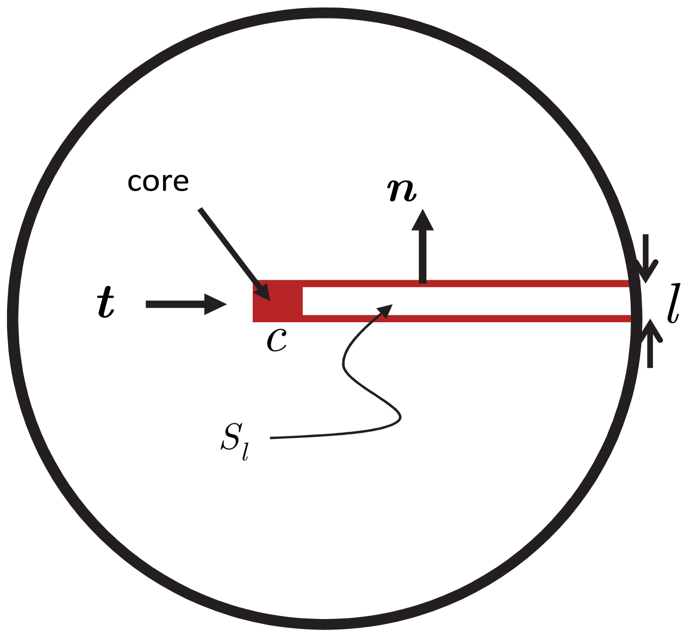

Figure 2: Schematic of setting for dislocation in plasticity theory.

satisfies

$$\begin{aligned} &div\,(CU^e) = 0 \quad \text{on} \quad \Omega \\ &(CU^e)\nu = 0 \quad \text{on} \;\; \partial\Omega \end{aligned} \tag{2}$$

where

$$\begin{aligned} &U^e := gradu^d - U^p \\ &U^p = \begin{cases} \;\; g(x)\,\frac{1}{l}\,(b \otimes n) \;\; \text{in} \;\; S_l \\ \;\; 0 \;\; \text{in} \;\; \Omega \backslash S_l \end{cases} \\ &S_l = \left\{ (x,y) : x \geq 0, y \in \left[-\frac{l}{2}, \frac{l}{2} \right] \right\}, \end{aligned} \tag{3}$$

and $n \in \mathbb{R}^2$ is the unit normal to the 'layer', $g(x) = 1$ for $x > c > 0$, $g(x) = 0$ for $x < 0$ and g is monotone increasing in $(0, c)$. Since the tangential component of each row of U^p is zero while its normal component has a non-zero derivative in the core region, *curl* $U^p =: -\alpha$ is non-vanishing only in the core (note that the jumps in U^p in the normal direction across the layer are not sensed by the *curl*). Moreover,

$$\int_A curlU^p\, e_z da = \int_{\partial A} U^p dx = b$$

for any area patch that encircles the core completely, i.e. the points of intersection of whose closed bounding curve (∂A) with the layer S_l have x-coordinates greater than c. In the above, e_z is the unit normal in the direction out of the plane.

In order to compare the total displacement and stress solutions of the Volterra formulation and the plasticity formulation, consider an orthogonal, Stokes-Helmholtz-like decomposition of the field U^p:

$$\begin{aligned} &U^p = gradz - \chi \\ &curl\chi = \alpha = -curlU^p \\ &div\chi = 0 \\ &\chi\nu = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4}$$

This has the important implication that *for all* values of $l \geq 0$ and $c > 0$, χ is a smooth field on Ω (including the origin). If $c = 0$ then χ would be a smooth field on the punctured domain $\Omega \backslash O$.

Using this smoothness of χ and performing a line integral of both sides of (4_1) along a curve

$$p : [0,1] \ni y \mapsto xt + \left(-\frac{l}{2} + yl\right) n, \quad x > c,$$

it can be deduced that

$$z(x,0^+)_{x>c} - z(x,0^-)_{x>c} =: [\![z]\!]_{x>c} = \int_p U^p dx = b \;\text{ as }\; l \to 0. \qquad (5)$$

We next note that (2) implies

$$\begin{aligned}&div\left(Cgradu^1\right) = -div\left(C\chi\right)\\&\text{where } \; u^1 := u^d - z\\&\text{and } \; \left(Cgradu^1\right)\nu = -\left(C\chi\right)\nu \;\text{ on }\; \partial\Omega,\end{aligned}$$

and for $c > 0$ $gradu^1$ is a smooth field on Ω *for all* values of $l \geq 0$. Again by integrating both sides of the statement $gradu^d - gradz = gradu^1$ along the curve p and taking the limit $l \to 0$ we obtain

$$u^d(x,0^+) - u^d(x,0^-) =: [\![u^d]\!]_{x>c} = [\![z]\!]_{x>c} = b.$$

Additionally, because of the smoothness of χ and $gradu^1$ in Ω, tractions in the plasticity formulation are always continuous on any internal surface of Ω, in particular

$$CU^e(x,0^+)n - CU^e(x,0^-)n =: [\![CU^e]\!]n = 0, \quad \text{for } \; c > 0.$$

even for points in the core.

Noting that as $l \to 0$, $S_l \to S$ and that $U^p = 0$ in $\Omega \backslash S_l$, we have, for $c > 0$,

$$\begin{aligned}&div\left(C\,gradu^d\right) = 0 \;\text{ on }\; \Omega\backslash S\\&[\![u^d]\!]_{x>c} = b\\&\left[Cgradu^d(x,0^+) - Cgradu^d(x,0^-)\right] n =: [\![Cgradu^d]\!]n = 0\\&(Cgradu^d)\nu = 0 \;\text{ on }\; \partial\Omega,\end{aligned}$$

and it is in this sense that the plasticity solution 'solves' the classical Volterra dislocation problem (1).

It should be noted that for $c > 0$ the plasticity problem has integrable linear elastic energy.

It can be shown that even for $l > 0$, small (compared to the radius of the body), the plasticity solution is a very good approximation of the Volterra solution in $\Omega \backslash S_l$, and comparison of finite element approximations [1] with the exact Volterra solution away from the core shows that even within S_l the correspondence is excellent.

References

[1] Zhang, X., Acharya, A., Walkington, N. J., & Bielak, J. (2015). A single theory for some quasi-static, supersonic, atomic, and tectonic scale applications of dislocations. Journal of the Mechanics and Physics of Solids, 84, 145-195.