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Wear of an Elastic Block

PIERO VILLAGGIO
Università di Pisa, Dipartimento di Ingegneria Strutturale, Via Diotisalvi, 2; 56126 Pisa, Italy

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Abstract. The classical theory of wearing, proposed by Reye 140 years ago and universally accepted for predicting the life of a component of a machine, can be rendered more precise by considering the effective stresses causing the smoothing of two surfaces mutually sliding in the presence of friction.

Sommario. La teoria classica dell'usura meccanica, proposta da Reye 140 anni fa ed accettata universalmente per prevedere la vita di un componente di una macchina, può essere posta in forma più precisa considerando gli sforzi effettivi che causano il logorio di due superfici che scorrono una rispetto all'altra in presenza d'attrito.

Key words: Wear, Elasticity.

1. Introduction

In 1860 the mathematician and geometer Reye proposed a simple and elegant theory for explaining the consumption of a solid body when it slides with friction on a rough surface [8]. Reye's model became very popular in Europe (in Italy was promulgated by Panetti [7]), and it is still taught in university courses of applied mechanics. But, strangely enough, this theory has been totally ignored in English and American literature.

The onset and the extent of wearing can be described by the following example. Let us consider a long parallelepiped of length $2h$, height ℓ, and unit thickness lying on a rigid plane and pushed against this plane by a uniform pressure p (Figure 1). Let us assume that the material constituting the parallelepiped is elastic, homogeneous, and isotropic, with Young's modulus E and Poisson's ratio ν. If the block is at rest it will be kept in equilibrium by a uniform pressure $-p$ exerted by supporting plane. Let us now assume that we shift the block horizontally by applying a uniform tangential stress q acting from left to right (Figure 1), on the upper face. The sliding will begin as soon as q reaches the value fp, where f is the friction factor, and, in this case, an opposite tangential stress will be exerted by the supporting

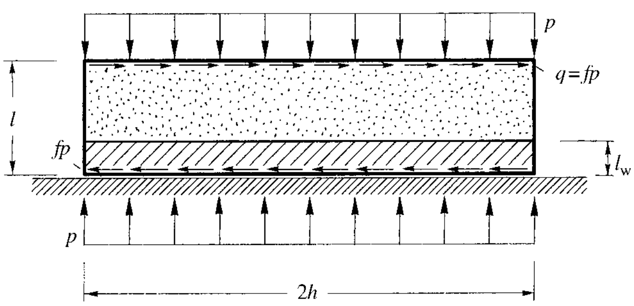

Figure 1. Block under uniform pressure.

base along the lower face of the block. According Reye's theory, this tangential stress will produce the deterioration of a thin layer of material just adjacent to the base. This layer, called the *worn region*, has constant width ℓ_w proportional to the tangential stress and, consequently, to the pressure p. Then we can write the relation

$$\ell_w = kp = \text{constant}, \tag{1a}$$

where k is a coefficient depending on the nature of the two surfaces at contact.

However, this result, despite of its wide acceptance, is exposed to two serious objections. The first is that uniform states of normal compressive stress p and tangential stress fp are not the correct elastic solutions for the block under consideration since they yield non-zero tangential tractions on the lateral faces, which are instead free. If the block is sufficiently slender ($\ell/2h < 1/2$ according to Girkmann's [3, Ziff. 42] analysis) this partial violation of the boundary conditions is not very influential. But, if the ratio $\ell/2h$ is close to one, then Reye's approximation is unacceptable. The second objection is that, in presence of friction, the lower face of the parallelepiped block is simultaneously loaded by a normal pressure p and a tangential stress fp. Then, if wearing is caused by the work done by tangential stresses acting along the base, it is necessary to consider that here the maximum shear stress is not fp but

$$|\tau|_{\max} = \frac{1}{2} p\sqrt{1 + 4f^2}, \tag{2}$$

and hence the effective height of the worn region is not given by (1) but by a formula of the type

$$\ell_w = k\frac{p}{2}\sqrt{1 + 4f^2}. \tag{1b}$$

We incidentally observe that a criterion like (2) was proposed by Tabor [10] in his theory of plastic wear.

Now the question arises of freeing Reye's theory from its incongruities without compromising its simplicity. Our purpose is to see how much the worn region changes when we consider exact elastic solutions for the block and the effective largest shear stresses on the base. Elastic solutions for mixed boundary conditions are very few, and hence we limit ourselves to examine two simple cases for which the corresponding solutions are explicit, though obtained by a semi-inverse method.

2. The Rectangular Block

We consider a rectangular sheet of length $2h$ and height ℓ like that shown in Figure 2. Its thickness is unit. The lower edge of the rectangle is placed on a rigid rough plane, the upper edge is simultaneously loaded by a vertical compressive force P and by a horizontal force fP acting (from left to right) on the same point O_1 at distance e from the axis of symmetry, the lateral edges are unloaded. Upon the combined action of the two forces the rectangle would tend to slide towards right, but its movement is contrasted by the shear stresses due to friction issuing from the support. In order to determine these stresses we must solve a problem of plane elasticity. On assuming that the thickness of the block is small with respect to the other dimensions, the stress state may be regarded as a 'generalised plane state' (cf. [6, Art. 146]). With respect to a system of Cartesian x, y-axes placed as shown in Figure 2

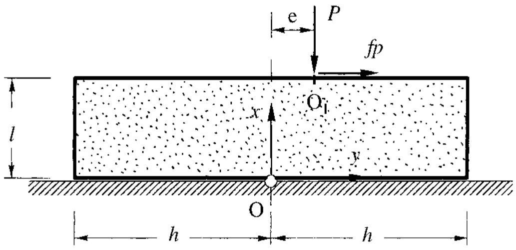

Figure 2. Block under a concentrated load.

the three significant stress components σ_x, σ_y, τ_{xy} are determined by a scalar stress function $F(x, y)$ through the formulae

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{3}$$

The displacement-components u, v in the x, y directions are also expressed in terms of $F(x, y)$ by the formulae (cf. [11, II, p.55])

$$Eu = \int \frac{\partial^2 F}{\partial y^2}\,\mathrm{d}x - \nu \frac{\partial F}{\partial x} + A - Cy, \tag{4a}$$

$$Ev = \int \frac{\partial^2 F}{\partial x^2}\,\mathrm{d}y - \nu \frac{\partial F}{\partial y} + B + Cx, \tag{4b}$$

where A, B, C are constant, E is Young's modulus and ν is Poisson's ratio.

A suitable explicit expression for $F(x, y)$, obtained by superposing some classical solutions in plane elasticity recorded in all texbooks of technical mechanics (cf. for instance, [9, §10]), is

$$F(x, y) = -\frac{1}{2}\frac{P}{2h}y^2 - \frac{1}{4h^3}P\mathrm{e}y^3 + \frac{fP}{4h^3}(\ell - x)(3h^2 - y^2)y. \tag{5}$$

By using (3), (4a), (4b) we can determine the stresses and the displacements. If we put $A = B = C = D$ it is easy to see that the boundary conditions

$$\sigma_y|_{y=\pm h} = \tau_{xy}|_{y=\pm h} = 0, \qquad v|_{x=0} = 0, \tag{6}$$

on three sides of the rectangle are pointwise satisfied. The boundary conditions on the side $x = \ell$ are instead satisfied only in the mean

$$\int_{-h}^{+h} \sigma_x|_{x=\ell}\,\mathrm{d}y = -P, \qquad \int_{-h}^{+h} y\sigma_x|_{x=\ell}\,\mathrm{d}y = -P\mathrm{e}, \qquad \int_{-h}^{+h} \tau_{xy}|_{x=\ell}\,\mathrm{d}y = fP. \tag{7}$$

In particular the stresses on the base are

$$\begin{aligned}
\sigma_y|_{x=0} &= 0, \qquad \sigma_x|_{x=0} = -\frac{P}{2h} - \frac{3}{2h^3}P\mathrm{e}y - \frac{3fP}{2h^3}\ell y, \\
\tau_{xy}|_{x=0} &= \frac{3fP}{4h^3}(h^2 - y^2).
\end{aligned} \tag{8}$$

The maximum shear stress due to the combination of these components is given by

$$|\tau|_{\max}|_{x=0} = \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2} = \frac{|P|}{4h}\sqrt{\left(1 + \frac{3ey}{h^2} + \frac{3f\ell y}{h^2}\right)^2 + \left(\frac{3f(h^2 - y^2)}{h^2}\right)^2}. \tag{9}$$

Then, if we accept Reye's hypothesis in the form (1b), relation (9), multiplied by k, yields the profile of the worn region.

The following figures show the behaviour of $\ell_w(y)$ for some particular values of the parameters. More precisely, we have put $P = 1$, $k = 1/4$, $f = 1/2$, $h = 1$, and have chosen the values $\ell = 1/2$, 1, e $= 0, 1/6$ (Figure 3). These values have been chosen to suitably amplify the shapes of the profiles. The maximum eccentricity e has been taken 1/6 and not more in order to ensure that the normal stress σ_x underneath the bottom is everywhere compressive.

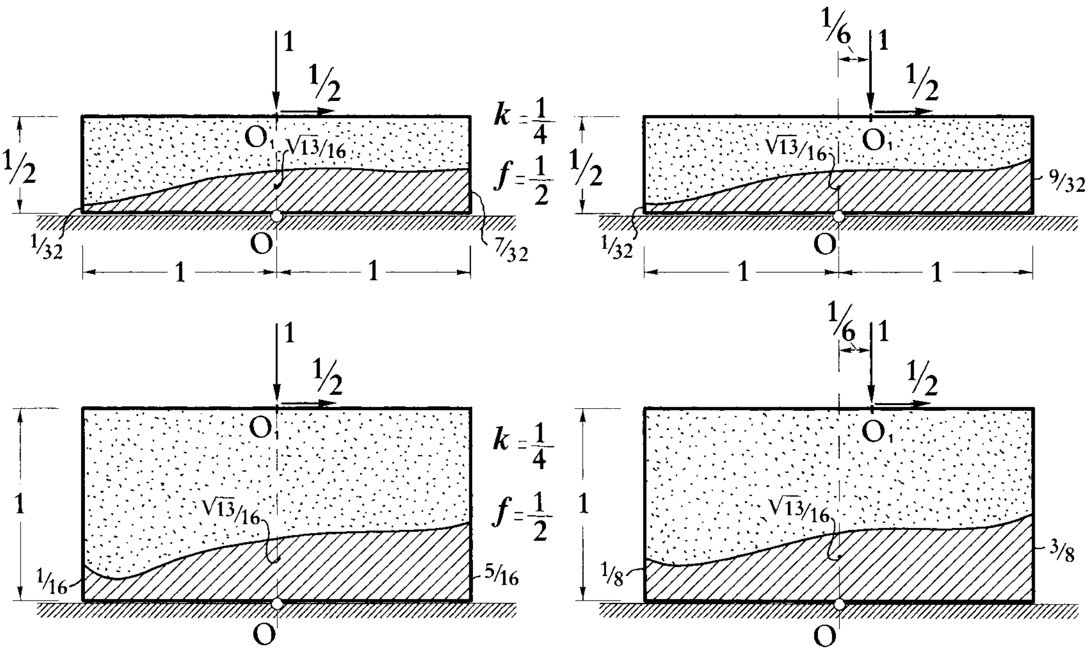

Figure 3. Worn region.

A look to the graphs of Figure 3 shows that the upper boundary of the worn region is never symmetric with respect the x-axis, even if the eccentricity e is zero. This is a consequence of the horizontal force fP which always accompanies P. Another, surprising, property of the solution is that the profile of the worn region is strongly influenced by the ratio h/ℓ. The profile is relatively flat for $h/\ell = 1/2$, but sensibly inclined for $h/\ell = 1$.

3. The Annular Block

The same argument can be applied in the treatement of an annular block pressed against a rigid drum like that sketched in Figure 4.

In this case it is convenient to represent the geometric quantities in a system of r, φ-polar coordinates with origin O placed at the center of the drum. In these coordinates, let r_0 be the radius of the drum, ℓ the height of the annulus, $2\omega(-\omega \leqslant \varphi \leqslant \omega)$ its angular opening. The

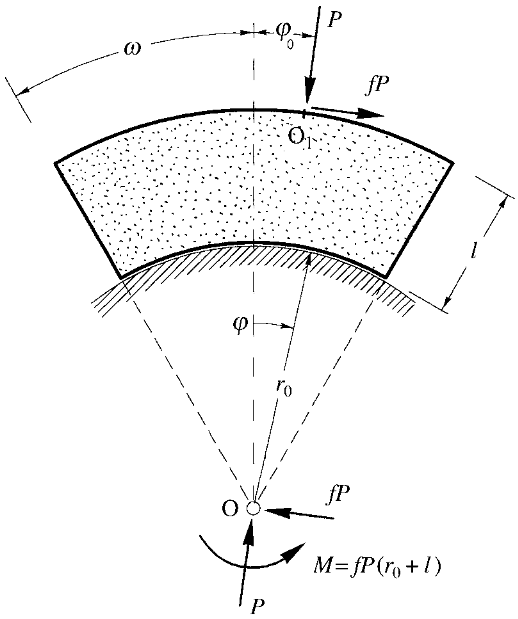

Figure 4. Annular block.

thickness of the annulus is taken to be one. The annulus is loaded at the point O_1 by a radial compressive force P and by a tangential force fP, where f is the friction factor, oriented as indicated in the figure. The polar coordinates of O_1 are $(r_0 + \ell)$, φ_0 respectively. We denote the stress components in the system by σ_r, σ_φ, $\tau_{r\varphi}$, the radial component of displacement by u and the tangential component by v.

In order to find the solution satisfying all the boundary conditions we apply a semi–inverse method considering an infinite plane wedge with vertex 0 and opening 2ω loaded at 0 by two forces P and fP directly opposed to those acting on the upper edge of the annulus (see Figure 4) and a couple $M = fP(r_0 + \ell)$. Then we exploit the elastic solution for the wedge by applying a method due to Galerkin [2] and superposing Worch's formulae ([11, II, p. 38–42]). The stress function becomes

$$\begin{aligned} F \;=\; & -\frac{N}{2\omega + \sin 2\omega} r\varphi \sin\varphi + \frac{T}{2\omega - \sin 2\omega} r\varphi \cos\varphi - \\ & -\frac{fP(r_0 + \ell)}{\sin 2\omega - 2\omega \cos 2\omega} (2\varphi \cos 2\omega - \sin 2\varphi), \end{aligned} \tag{10}$$

where we have put

$$N = P\cos\varphi_0 + fP\sin\varphi_0, \qquad T = P\sin\varphi_0 - fP\cos\varphi_0. \tag{11}$$

Note that N and T are respectively the vertical and horizontal components of the resultant of the two forces P and fP applied at 0.

On deriving the stress components from F through the formulae

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}, \qquad \sigma_\varphi = \frac{\partial^2 F}{\partial r^2}, \qquad \tau_{r\varphi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial r}\right), \tag{12}$$

we can verify that the boundary conditions of zero tractions along the lateral edges $\varphi = \pm\omega$ are pointwise satisfied, while those along upper edge $r = r_0 + \ell$ are satisfied only in the mean,

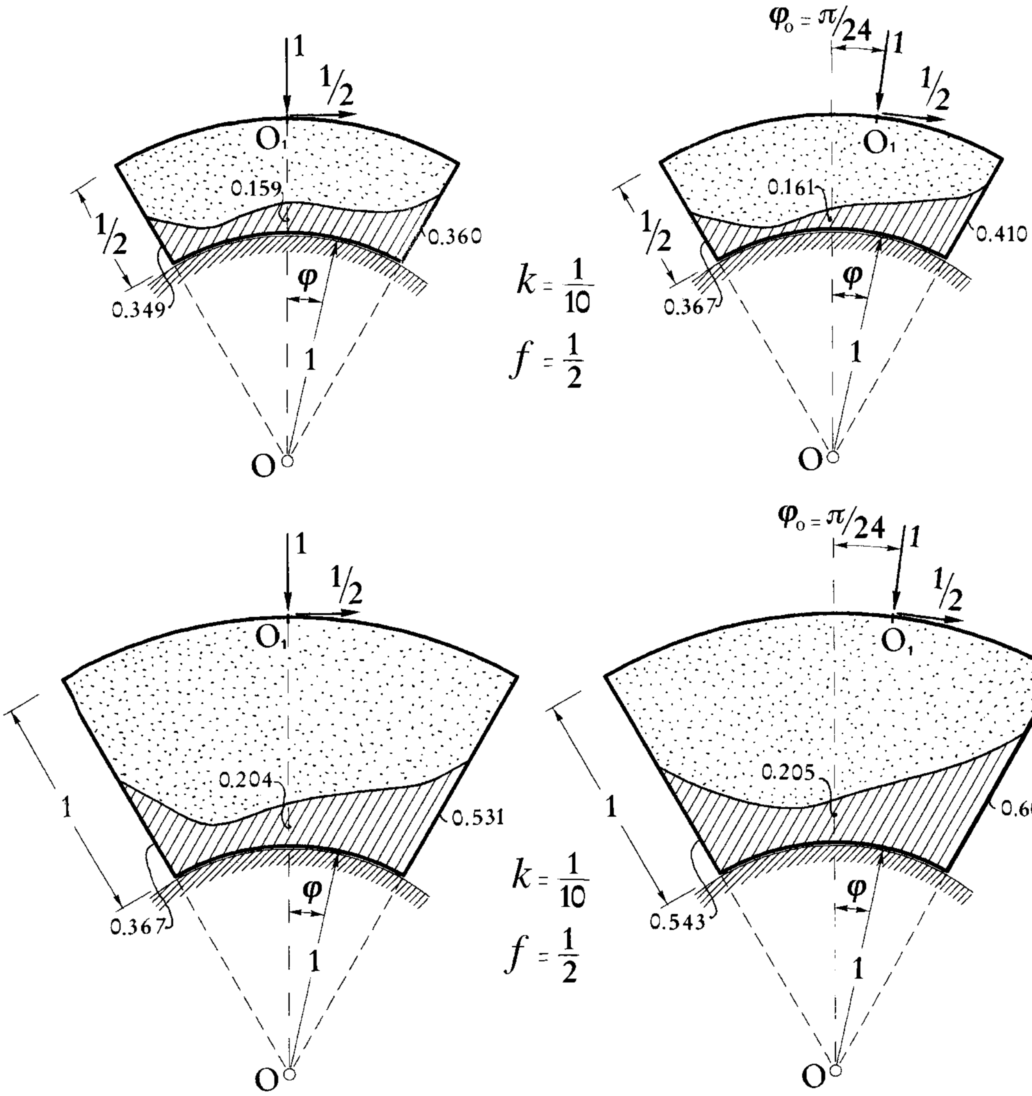

Figure 5. Worn region.

in the sense that the tractions here are statically equivalent to two forces P and fP applied at O_1 (Figure 4). From (12) we also derive the tractions along the lower edge $r = r_0$. The result is

$$\begin{aligned}\sigma_r|_{r=r_0} = &- \frac{2P}{[(2\omega)^2 - sin^2 2\omega]}\frac{1}{r_0} \times \\ &\times\ [2\omega(\cos(\varphi-\varphi_0) - f\sin(\varphi-\varphi_0)) - \\ &-\ \sin 2\varphi(\cos(\varphi+\varphi_0) + f\cos(\varphi+\varphi_0))] - \\ &-\ \frac{2fP(r_0+\ell)}{\sin 2\omega - 2\omega\cos 2\omega}\frac{\sin 2\varphi}{r_0^2},\end{aligned} \tag{13}$$

$$\tau_{r\varphi}|_{r=r_0} = \frac{fP(r_0+\ell)}{\sin 2\omega - 2\omega\cos 2\omega}\frac{1}{r_0^2}(\cos 2\varphi - \cos 2\omega). \tag{14}$$

Consequently the maximum shear stress can be written as

$$|\tau|_{\max}|_{r=r_0} = \frac{1}{2}\sqrt{\sigma_r^2|_{r=r_0} + 4\tau_{r\varphi}^2|_{r=r_0}}, \tag{15}$$

and this expression, multiplied by k, define the worn region.

We now imply illustrate formula (15) in a particular case by fixing $P = 1$, $k = 1/10$, $f = 1/2$, $r_0 = 1$, $\omega = \pi/8$, and the values $\ell = 1/2$, 1; $\varphi_0 = 0$, $\pi/24$. Here again φ_0 must be sufficiently small in order to prevent the loss of contact of the block with the drum. The graphs of the upper boundary $\ell_{\mathrm{w}}(\varphi)$ (in polar coordinates) of the worn region are collected in the Figure 5. Also in this case the region is never symmetric with respect the vertical axis, even when the angular eccentricity φ_0 is zero. The region is only symmetric when $\varphi_0 = 0$ and there is no friction, and, in this case, we recover a result recorded in the textbook by Ferrari and Romiti [1, p. 287].

From Figures 3 and 5 we can derive the conclusion that consideration of the true tangential stresses along the surface of sliding sensibly alter the profile of the worn region in a rectangular or in an annular block. The profile is never rectilinear nor sinusoidal, but undulated with a maximum at the end point of the interval of contact situated in the direction of motion, and two lobes symmetrically placed with respect to the axis of geometric symmetry. This result surprisingly agrees with the experiments described by Kragelskii [5, p. 23] and the computations of Goryacheva [4, p. 188].

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