Elastic Sheets: A Real Cliff-hanger!

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1 Introduction

The bending of thin elastic sheets is an old problem, dating back several centuries, to the Bernoullis and Euler, among other mechanicians of note from the 17th century. A thorough account of the history of the problem can be found in Levian. Though the problem is well understood, evaluation of the problem in light of modern applications of shell elasticity can sometimes yield interesting results. Zhiyan's presentation on how iso-strain leads to buckling in thin sheets (e.g. leaves, etc.) is a nice example of how rich the buckling behavior in thin sheets can be. Mahadevan's unpublished work on the development of a boundary layer for the curvature in a draping sheet, as will be discussed herein, provides a compelling theory, and suggests a method for the computation of the elastica curve in the cantilevered sheet problem. The problem becomes very interesting when one considers it as an optimization problem: leaves are essentially cantilevered beams, and plants wish to maximize their exposure to sunlight. What shape is the optimal shape for them to select, in order for the exposed area to be maximized? Do we observe such a geometry in nature?

The answer to this question arises from an extension of my study of elastica. The model problem I analyze is a first step in completing the story of how plants optimize their exposed surface area to obtain a competitive advantage in the jungle. The geometric non-linearity of the problem makes its study particularly interesting.

2 Problem Formulation

The formulation of the elastica problem can be found online in notes prepared by Prof. Suo. The equilibrium equations are solved for an incremental element in the sheet. The balance of moments and forces in the sheet gives rise to three coupled equations:

$$\frac{\partial n_1}{\partial s} = 0$$
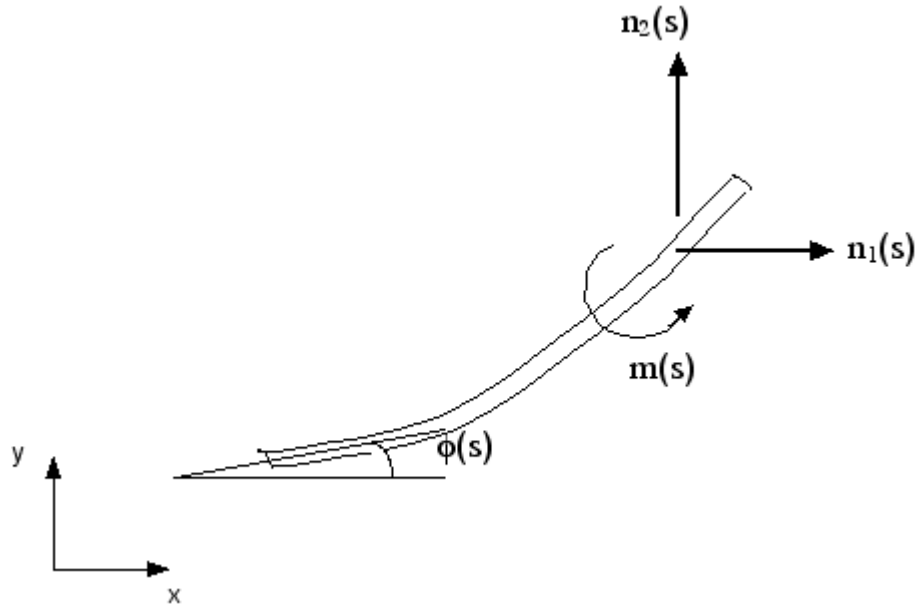
$$\frac{\partial n_2}{\partial s} = \rho g$$

Figure 1: The free-body diagram for the forces and moments an elastic sheet with thickness, t, linear density, ρ, and flexural stiffness, $B = EI$.

$$\frac{\partial m}{\partial s} = t(n_1 sin(\phi) - n_2 cos(\phi))$$

.

Our material law is from the Bernoulli-Euler beam theory:

$$m(s) = B\frac{\partial \phi}{\partial s}$$

.

Geometry relates the angle, ϕ, to the coordinate directions x and y:

$$\frac{\partial x}{\partial s} = cos(\phi)$$

$$\frac{\partial y}{\partial s} = sin(\phi)$$

.

These equations will can be solved with a shooting method; this is an alternative numerical method to the FEM solution, and could be used to verify the FEM output.

3 Computational Modeling and Numerical Results

I attempted to solve the problem with a variety of numerical methods after performing experiments.

3.1 ABAQUS

I naively thought the problem would be simple to solve in ABAQUS using a shell element with non-linear geometry (nlgeom) selected in the solver options. I set up the problem with quadratic interpolation functions, and used varying numbers of elements. I used values of density and Young's modulus for paper. Each time I attempted to run the solution, ABAQUS would respond by saying that the solution couldn't be reached due to deformations becoming too large for a single step. I proceeded to try COMSOL at this point, after asking Yuhang for help.

3.2 COMSOL Solver Set-Up

I built a three-dimensional thin model of a strip of paper in a similar manner to the model I constructed in ABAQUS, and used identical values for the material parameters. The sheet was made with length = 19 cm, width = 1 cm, and thickness = 0.01 cm. I selected non-linear elements, and solved the equations with the non-linear single-step solver option in COMSOL. The solution ran to completion, and offered output that stretched significantly, and is equivalent to the linear beam theory solution for similar geometry. For comparison, I include both plots in line.

3.3 Interpretation of Numerical Results

I encountered much difficulty in trying to simulate large deformations of thin elastic sheets with FEM. Though I don't consider myself to be a proficient FEM user, I attempted to solve the problem many ways, and have come to the conclusion that either a semi-analytical approach or the solutions from Elastica theory are the most appropriate to solve this particular problem. The experimental results are also interesting. I think my result highlights the importance of verification of solutions obtained with computation.

4 Experimental Verification

Experiments on the drape of two elastic materials were carried out. Care was taken to ensure that the sheets don't have an intrinsic curvature. The materials tested are paper and latex. In order to compare the results, the elastic-gravity length, $\lambda_g = (\frac{B}{\rho g t})^{\frac{1}{3}}$ is calculated, and all lengths are normalized by the elastic-gravity length for each material.
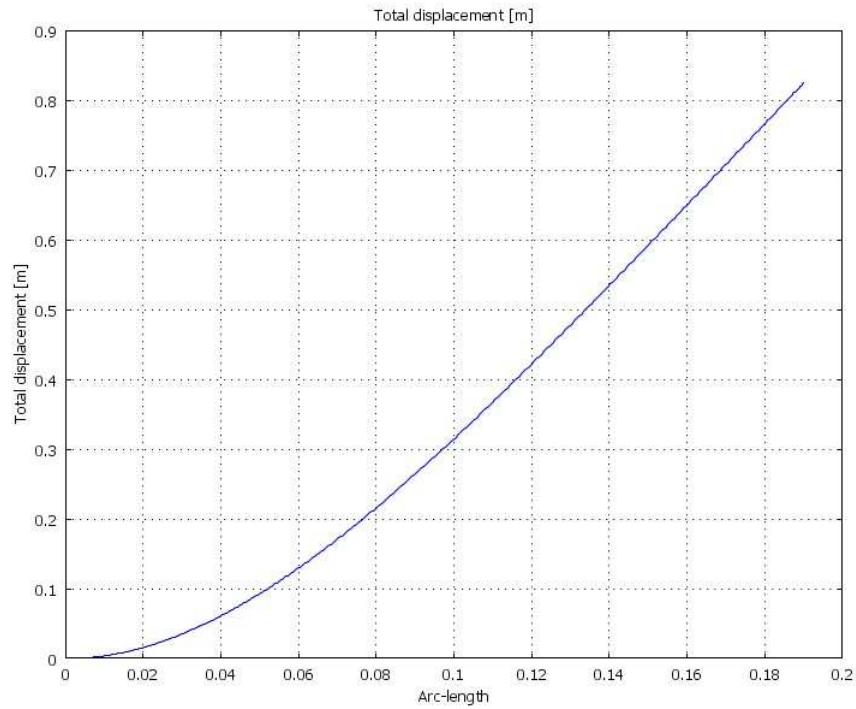
Figure 2: The output from COMSOL, showing the beam's deflection as a function of distance along the beam. Note that the horizontal extension as large as 19 cm is not experimentally attainable: the drape of the sheet begins to dominate before the sheet will extend this far over the edge horizontally; therefore I interpret the solution as unphysical. It should be noted that the sheet extends a great deal, and its arc-length is not conserved. Perhaps if I had penalized the stretching of the sheet correctly, the solution would have been accurate.
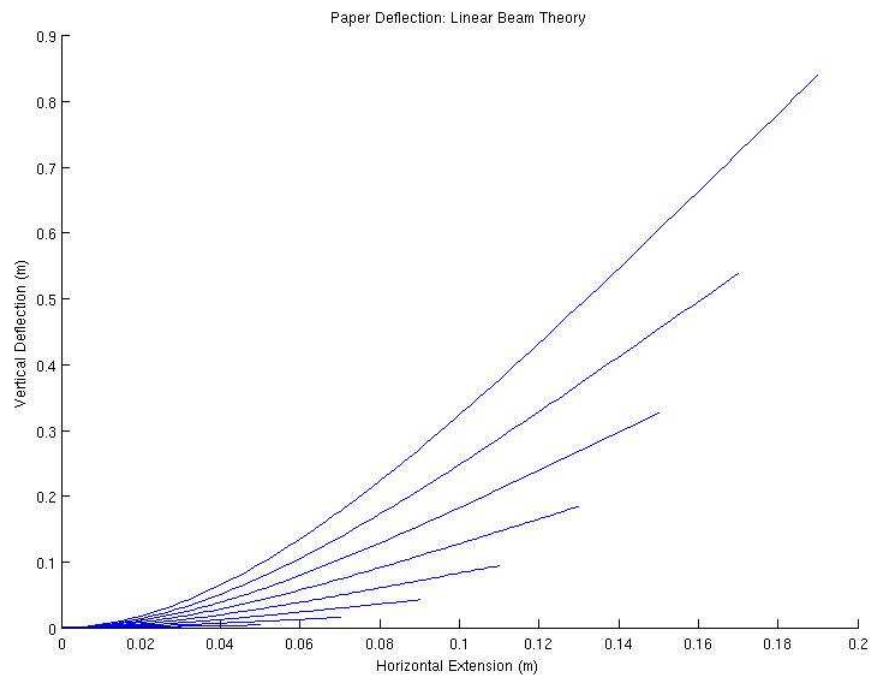
Figure 3: The calculated deflection using linear beam theory for a variety of lengths of beams, including the largest, at 19 cm. Compare this displacement curve with the COMSOL curve, and one sees immediately that they are nearly identical.
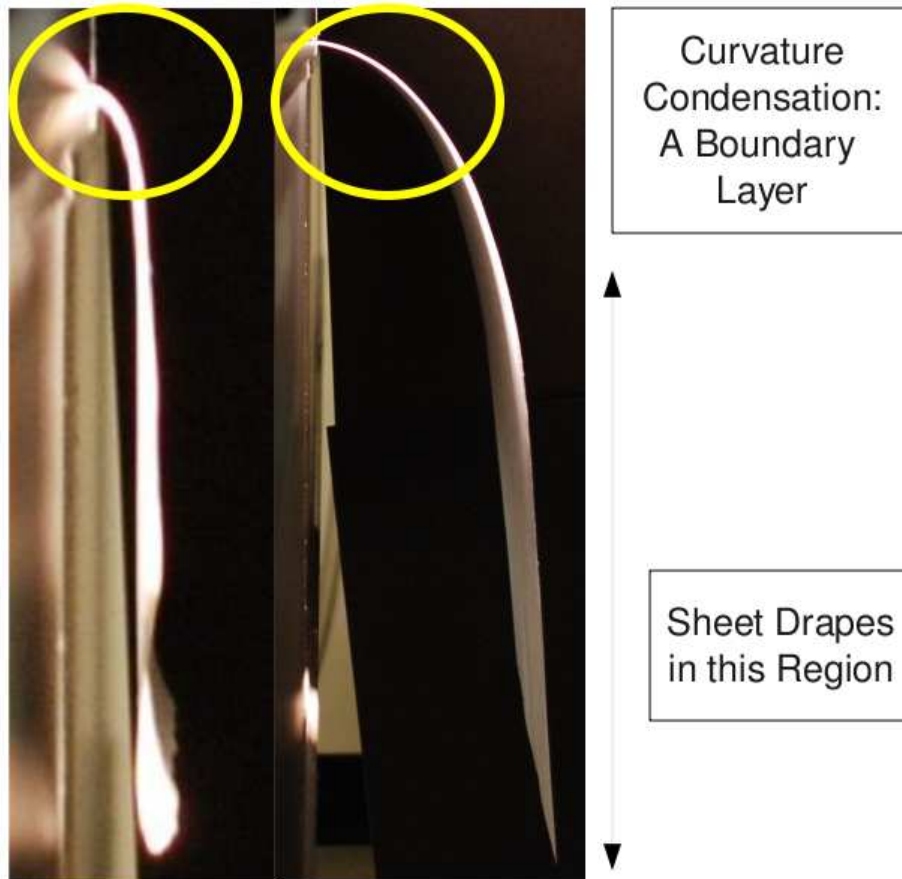
Figure 4: Here, we see two draping sheets of different materials. The arc-length allowed to drape is vastly different (on the left is latex, arc-length = 11 cm, whereas on the right is paper, arc-length = 25 cm), but the shapes exhibit qualitative similarity in that they can be broken up into two regimes: a regime with condensation of curvature (which I call a boundary layer), and a regime with vertical drape.
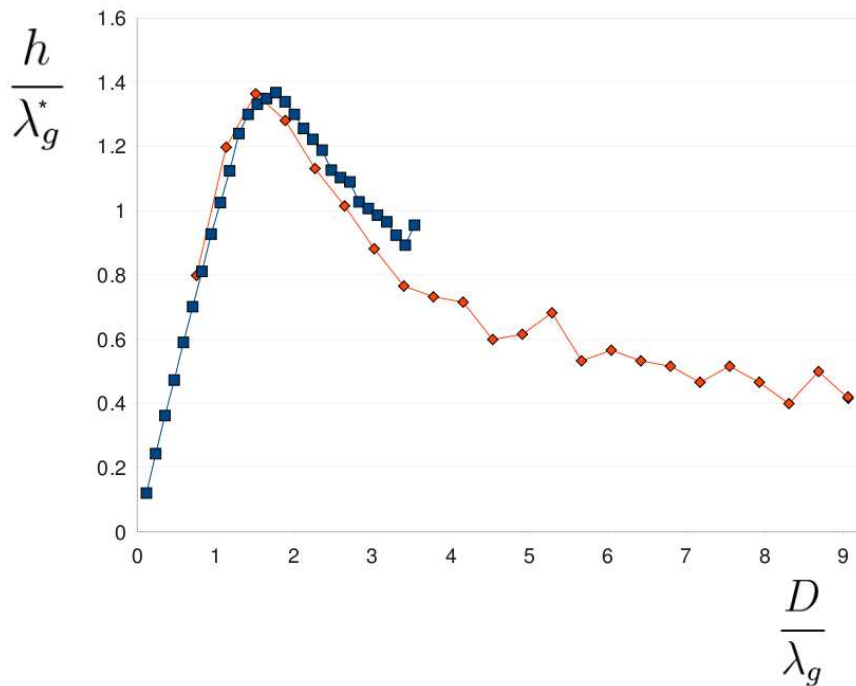
Figure 5: The curve of horizontally projected extension, h, as a function of arc-length, D. Note the reversal of h as D continues to increase.

4.1 Procedure

The sheets were allowed to extend over the edge with a fixed and measured arc-length. The sheet is cantilevered by a heavy, flat weight at the corner of a steel table. Photos are taken from the side-on view. The length is incremented, the cantilevering weight replaced, and another photo is taken. From the photos, the horizontally projected extension data is measured.

4.2 Results

The results show a non-linear dependence of horizontally projected extension, h, on the arc-length, D: In fact, the horizontally projected extension is two-valued for a given arc-length. We see this behavior in the graph of h vs. D normalized by λ_g. A fitting constant $F = 2.66$ is needed for the data from the paper to collapse onto the seemingly universal curve in Fig. 5.

4.3 Interpretation of Results

The experimental results show some surprising behavior of the horizontally projected extension, which becomes intuitive after one thinks about the effect of the

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added weight of the residual draping material. The result suggests a boundary layer where curvature is localized for large arc-lengths, and encourages semi-analytical development of a theory to describe the curve.

5   (Semi)-Analytical Solution

At an arc-length large relative to the elastic-gravity length, we expect more and more of the sheet to remain unbent, and to approach vertical in the limit of infinite sheet length. Experimentally, we see that the latex is able to acheive verticality in draping almost immediately, with an arc-length of $\simeq 11$ cm. In order to describe this system, we look to boundary layer theory. We anticipate that in the length of the draping sheet near the cantilever, the behavior is exactly that of a cantilevered beam, with a load yet to be determined. In the draping regime (the arc-length far from the cantilever), we expect the sheet to act as a point load on the end of a cantilvered beam of length ϵ, where ϵ is the thickness of the boundary layer. We anticipate that ϵ is related inversely to the arc-length, D, and directly to its elastic-gravity length. Since ϵ has dimensions of length, a natural scaling arises: $\epsilon = \frac{\lambda^{\frac{3}{2}}}{D^{\frac{1}{2}}}$. This boundary layer analysis was constructed in a private communication with Mahadevan.

We proceed to use this ϵ to predict behavior of our h vs. D curve. We anticipate that the behavior of the sheet after it has begun to move back toward the wall at which it is cantilevered will go as a $h \propto D^{-\frac{1}{2}}$ power law. Indeed, we observe in Fig. 6 something quite close, and expect that if we tested longer sheets, that our approximation would grow closer and closer to this behavior.

6   Conclusion

Though the FEM calculation didn't predict the non-linear behavior I anticipated based on the experimental results, I'm sure it is only due to the stretching of the sheet, which is non-physical. The experimental verification suggests a natural interpretation of the localization of curvature as a boundary layer in the limit as the sheet is allowed more and more drape. The semi-analytical boundary layer agrees fairly well with the limited set of experimental data taken.

References

[1] Raph Levien. The elastica: a mathematical history. Technical Report UCB/EECS-2008-103, EECS Department, University of California, Berkeley, Aug 2008.

[2] L Mahadevan. Personal communication. 2008.
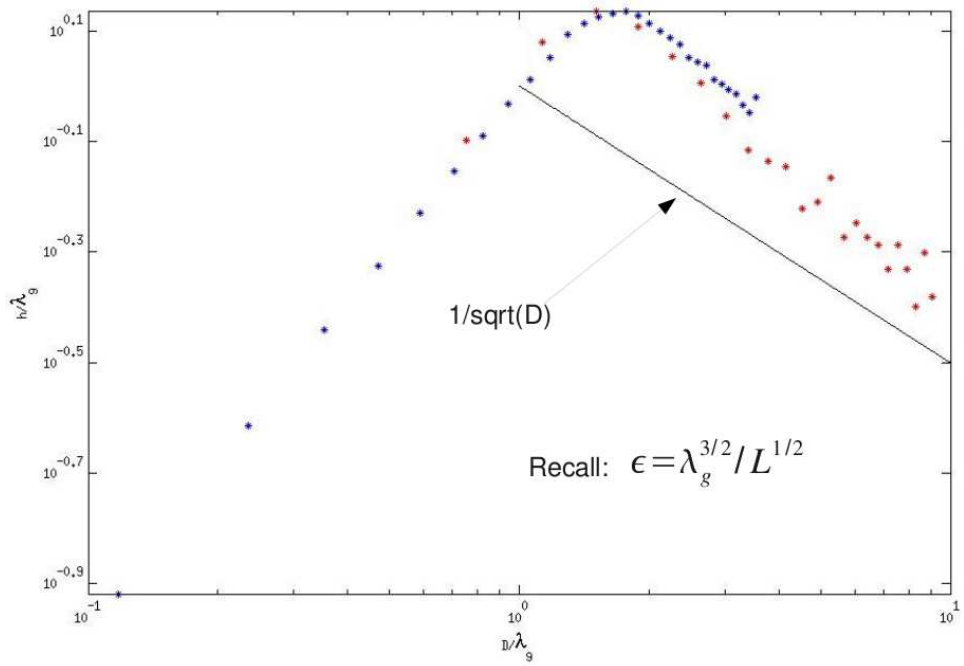
[3] Zhigang Suo. String and elastica, 2008.

Figure 6: We observe the behavior of the sheet in bending after the turning point from our boundary layer theory, wherein we predict $h \propto D^{-\frac{1}{2}}$ behavior. $D^{-\frac{1}{2}}$ is plotted in the black line for reference.