Classical Plate Model

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April 12, 2012

1 Introduction

If a structure has one of its dimensions much smaller than the other two, such as a flat or curved panel, we can simplify the analysis of such a structure using plate or shell models. In this chapter, we focus on flat panels which can be modeled using plate models for simplicity. Mathematically speaking, a plate can be considered as a degenerated shell. Figure 1 provides a simple sketches of a rectangular plate with constant thickness. In general it is not necessary for the plate to be rectangular nor the thickness must be constant. The only requirement is that the thickness must be small in comparison to the in-plane dimensions. For the rectangular plate in Figure 1, we require $h/a \ll 1$ and $h/b \ll 1$. Usually, we also have $a \approx b$. If it is a circular plate with radius r, we then require $h/r \ll 1$. The small aspect ratio is not a defined value but a qualitative assessment and an engineering judgement of the analyst. For us to develop the plate theory, we need to introduce two terminologies: *reference surface* and *transverse normal*. The reference surface of the plate is defined along the two larger dimensions, denoted using x_1, x_2 in Figure 1. There are infinite many choices of reference surface although the mid-plane is usually chosen as the reference surface. Note in the undeformed state, the reference surface of a plate structure is actually a plane. The transverse normal is formed by the material points along the thickness direction, denoted using x_3 in Figure 1. Without loss of generality, we locate the origin of x_3 on the reference surface in this note. In other words, $x_3 = 0$ denotes the reference surface. For every point in the reference surface denoted by (x_1, x_2), there erects a transverse normal. If the mid-plane is chosen as the reference surface, then x_3 ranges from $-h/2$ to $h/2$. If the thickness is not constant, we could have the thickness as a function of x_1, x_2. For the structure to be reasonable modeled as a plate, we also require that the thickness varies smoothly along the reference surface of the plate.

Although we could use the finite element method to routinely analyze complex structures, simple plate models are often used in the preliminary design stage because they can provide valuable insight into the behavior of the structures with much less effort. There

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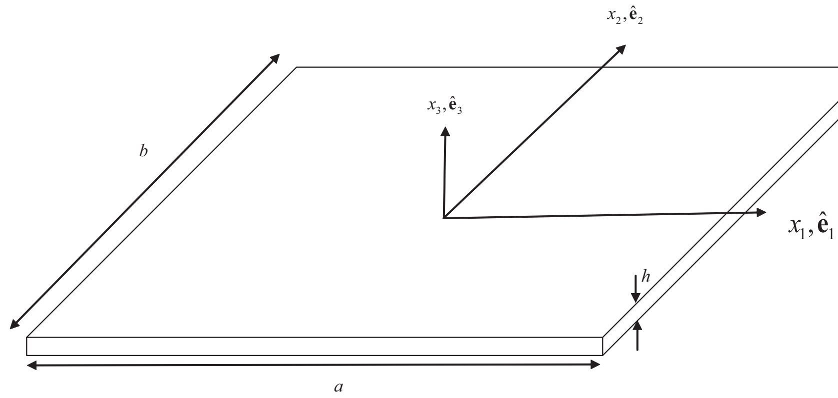

Figure 1: Sketch of a plate.

are different plate models with different accuracy. The simplest one is the so-called classical plate model, also called Kirchhoff plate theory, which usually can provide a reasonable prediction for thin plates. There are at least three ways to derive this plate model: Newtonian method based on free body diagrams, variational method as an application of the Kantorovich method, and variational asymptotic method. Both the Newtonian method and the variational method are based on various ad hoc assumptions including Kirchhoff kinematic assumptions and kinetic assumptions for the 3D stress field within the structure. For this reason, we also term both the Newtonian method and the variational method as ad hoc approaches. Most textbooks only present the Newtonian method as it is intuitive for understanding. However, it is tedious and error-prone for development of new models and analysis of real structures. On the contrary, the variational method is systematic and easy to handle real structures. Mainly for this reason, the variational method is commonly employed in the literature to derive new plate models. The variational asymptotic method is a recent addition to the plate literature and it has the merits of the variational method without using ad hoc assumptions. We will present the details of these three methods for constructing the classical plate model for isotropic, homogeneous plate (plate-like structure made of a single isotropic material) to appreciate the advantages and disadvantages of different methods.

Fundamentally speaking, a plate model, no matter how rudimentary or how sophisticated it is, is a two-dimensional (2D) model. It seeks to replace the governing equations of the original three-dimensional (3D) structure with a set of equations in terms of two fundamental variables, the two coordinates describing the plate reference surface. In other words, we need to replace the original 3D kinematics, kinetics, and energetics in terms of their 2D counterparts.

As plate models can be considered as an approximation to the 3D elasticity theory, it is appropriate for us to review the basics of the 3D elasticity theory. For simplicity, we restrict ourselves to material and geometric linear problems only. The theory of linear elasticity contains three parts including kinematics, kinetics and energetics. The kinematics deal with a continuous displacement field (u_i) (Here and throughout this chapter, Greek indices assume values 1 and 2 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated) and a continuous strain field (ε_{ij}) satisfying the following strain-displacement relations at any material point in the body:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

The kinetics deals with a continuous stress field (σ_{ij}) satisfying the following equilibrium equations at any material point in the body:

$$\sigma_{ji,j} + f_i = 0 \tag{2}$$

The energetics deals with the constitutive behavior of the material. For an isotropic elastic material, it deals with the following constitutive relations satisfied at any material point in the body:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{3}$$

It is commonly called the generalized Hooke's law for isotropic elastic materials. The 6×6 matrix is the compliance matrix with E as the Young's modulus and ν as the Possion's ratio. The constitutive relations in Eq. (3) can be simply inverted to obtain a 6×6 stiffness matrix.

The 15 equations in Eqs. (1), (2), (3) form the complete system to solve the 15 unknowns ($u_i, \varepsilon_{ij}, \sigma_{ij}$, note the symmetry of ε_{ij} and σ_{ij}). Clearly boundary is also part of the body, which implies that the above equations should also hold for points on the boundary. However, along the boundaries, we also know some information which can be considered as given to the plate structure in question. For example some boundary points are fixed. Hence along the boundary, we have some additional equations to satisfy. If the displacement of some boundary surfaces is prescribed to be u_i^*, then we require the displacement field to satisfy

$$u_i = u_i^* \tag{4}$$

on such boundary surfaces. If the traction of some boundary surfaces is prescribed to be t_i, then we require the stress field to satisfy

$$\sigma_{ij} n_i = t_j \tag{5}$$

2 Ad Hoc Approaches

In view of what we have learned about beams, plates can be understood analogously. Like a beam has the cross-section much smaller than the reference axis, a plate has the length of the transverse normal much smaller than the reference surface dimensions. Like beams are modeled using 1D models posted over the beam reference axis, plates are modeled using 2D models posted over the plate reference surface.

Using ad hoc approaches to develop beam models, we introduced some kinematic assumptions regarding the deformation of the cross-section. Likewise, to use ad hoc approaches to develop plate models, we will introduce some kinematic assumptions regarding the deformation of the transverse normal. Specifically, in deriving the classical plate model, we need to first introduce the so-called Kirchhoff assumptions which enables us to express the 3D displacements in terms of the 2D plate displacements, and the 3D strain field in terms of 2D plate strains. Then assumptions of the stress field are also used to relate the 3D stress field with the 3D strain field as what we did similarly in deriving the classical beam model. Although these assumptions are commonly used in our textbooks, they are not emphatically pointed as one set of many possible assumptions. Students might mistakenly think these are the assumptions must be made for plate theory or, even worse, they might think that these assumptions represent a universal truth for plate-like structures. The reality is that these assumptions are usually reasonably justified for isotropic homogeneous plates and become questionable for plate made of general anisotropic, heterogeneous materials such as composite laminates or sandwich panels. These assumptions are not absolutely needed if one uses the variational asymptotic method to construct the plate model, as we will show later.

2.1 Kinematics

As we have pointed out that the derivation of the classical plate model using the Newtonian method and the variational method starts from some kinematic assumptions which were originally made by Kirchhoff. We will first discuss his assumptions and their implications for kinematics.

2.1.1 The displacement field based on Kirchhoff assumptions

The Kirchhoff assumptions are

1. The transverse normal is infinitely rigid along its own direction.
2. The transverse normal of the plate remains straight during deformation.
3. The transverse normal remains normal to the reference surface of the plate during deformation.

Clearly these assumptions are completely analogous to the Euler-Bernoulli assumptions we used in deriving the classical beam model if we replace transverse normal with cross-section and plate reference surface with beam reference axis. Experimental observations show that these assumptions are reasonable for thin panels made of isotropic homogenous materials. When these conditions are not met, the classical plate model derived based on these assumptions may be inaccurate. Now, let us discuss the mathematical implication of the Kirchhoff assumptions.

As shown in Figure 1, we introduce a set of unit vectors $\hat{\boldsymbol{e}}_i$ with coordinates x_i to facilitate the derivation of our plate model. This set of axes is attached at a point of the plate structure, $\hat{\boldsymbol{e}}_3$ is along the transverse normal, and $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ define the plate

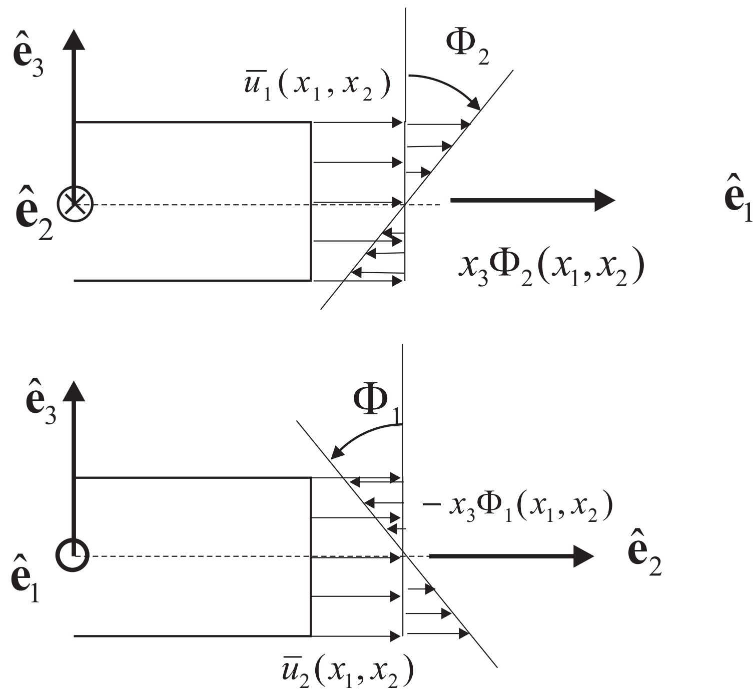

Figure 2: Decomposition of the in-plane displacement field.

reference surface. Let $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$, and $u_3(x_1, x_2, x_3)$ be the displacement of an arbitrary material point of the plate in the $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ directions, respectively.

The first Kirchhoff assumption states that the normal material line is infinitely rigid in its own direction, which implies every material point of the transverse normal moves rigidly along the transverse direction, which also implies that the transverse displacement of every material point in the plate with the same in-plane location (x_1, x_2) remains the same. In other words, the transverse displacement field of the plate structure can be described as a function of x_1, x_2 such that

$$u_3(x_1, x_2, x_3) = \bar{u}_3(x_1, x_2) \tag{6}$$

The second Kirchhoff assumption states that the transverse normal remains straight during deformation. This implies the in-plane displacement field of the plate is at most linear of coordinate x_3, which is

$$\begin{aligned} u_1(x_1, x_2, x_3) &= \bar{u}_1(x_1, x_2) + x_3\Phi_2(x_1, x_2) \\ u_2(x_1, x_2, x_3) &= \bar{u}_2(x_1, x_2) - x_3\Phi_1(x_1, x_2) \end{aligned} \tag{7}$$

Here $\bar{u}_\alpha$ can be considered as the in-plane displacements of material points on the reference surface ($x_3 = 0$). Although the center of rotation is not necessarily at the origin of x_3, rotation around any other point can still be expressed using Eq. (7) as any in-plane displacements introduced by the shifting of the rotation center can be incorporated into the unknown functions $\bar{u}_\alpha(x_1, x_2)$. The physical meaning of the in-plane displacement expressions in Eqs. (7) is explained by the sketch in Figure 2. Note the sign convention:

the rigid body translations of the transverse normal $\bar{u}_i(x_1, x_2)$ are positive in the direction of the axes $\hat{\boldsymbol{e}}_i$; the rigid body rotations of the transverse normal $\Phi_\alpha(x_1, x_2)$ are positive if they rotate about the axes $\hat{\boldsymbol{e}}_\alpha$, respectively. Figure 3 depicts these various sign conventions. The reason there is a negative sign in the last term of Eq. (7) is because a positive Φ_1 will create a negative in-plane displacement along x_2 direction for a positive x_3 (see Figure 2).

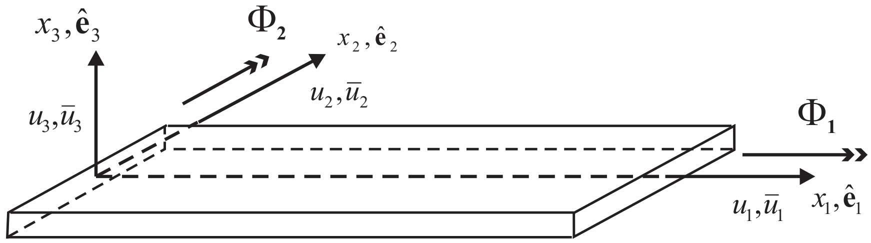

Figure 3: Sign convention for the displacements and rotations of a plate.

The third Kirchhoff assumption states that the transverse normal remains normal to the reference surface during deformation. This implies the equality of the slope of the reference surface and of the rotation of the transverse normal, as depicted in Figure 4

$$\Phi_1 = \bar{u}_{3,2}; \quad \Phi_2 = -\bar{u}_{3,1} \tag{8}$$

where comma denotes partial derivative with respect to in-plane coordinates, *i.e.*, $()_{,\alpha} = \frac{\partial()}{\partial x_\alpha}$. The minus sign in the second equation is a consequence of the sign convention on the displacements and rotations. Substituting Eqs. (8) into Eq. (7), we can eliminate the rotations of the transverse normal from the in-plane displacement field. The complete 3D displacement field for a plate-like structure implied by the Kirchhoff assumptions writes

$$\begin{aligned} u_1(x_1, x_2, x_3) &= \bar{u}_1(x_1, x_2) - x_3\bar{u}_{3,1} \\ u_2(x_1, x_2, x_3) &= \bar{u}_2(x_1, x_2) - x_3\bar{u}_{3,2} \\ u_3(x_1, x_2, x_3) &= \bar{u}_3(x_1, x_2) \end{aligned} \tag{9}$$

Here, we have to emphasize that the fact that in reality that the 3D displacements $u_i(x_1, x_2, x_3)$ are generally 3D unknown functions of x_1, x_2, x_3 as determined by physics. We have assumed a specific functional form for them in virtue of Kirchhoff assumptions so that u_α must be a linear combination of x_3 and some unknown 2D functions $\bar{u}_i$ and their derivatives which are functions of x_1, x_2 only. The Kirchhoff assumptions can be equivalently considered as constraining the structure in such a way that it must behave according to these assumptions, although we might not be able to apply such constraints physically. Because of these constraints, the overall system is stiffer than the original structure. In other words, for a structure under the same load, displacements u_i obtained using the classical plate model based on the Kirchhoff assumptions will be generally smaller than those obtained using a theory (for example 3D elasticity) without such assumptions. One or all of the three Kirchhoff assumptions can be removed or replaced by other assumptions. For example, one can remove the third Kirchhoff assumptions, which implies the transverse normal remains straight during deformation but it not necessarily remains as normal to

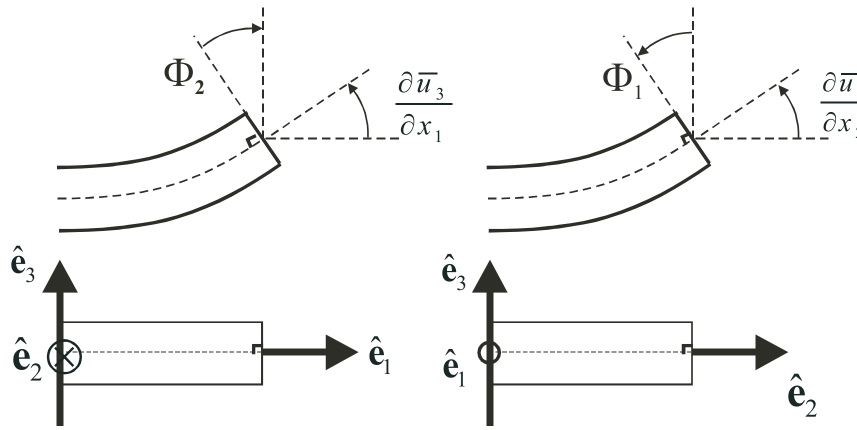

Figure 4: Plate reference surface slope and transverse normal rotation.

the reference surface. This is actually the starting point of the derivation of the Reissner-Mindlin plate model. As the Reissner-Mindlin plate model has one less assumption, it is expected that the displacements obtained by this model will be larger than those obtained using the classical plate model based on the Kirchhoff assumptions.

Clearly, the complete 3D displacement field of the plate can be expressed in terms of three two-dimensional displacements $\bar{u}_i(x_1, x_2)$. This important simplification resulting from the Kirchhoff assumptions allows the development of the classical plate model in terms of $\bar{u}_i$ which are unknowns functions of the in-plane coordinates x_1 and x_2 only, a 2D formulation. In other words, by using the Kirchhoff assumptions, we relate the 3D displacements, $u_i(x_1, x_2, x_3)$, in terms of 2D plate displacements, $\bar{u}_i(x_1, x_2)$.

2.1.2 The strain field

To deal with geometrical linear problem, we use the infinitesimal strain field defined in 3D linear elasticity as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{10}$$

Substituting the displacement field in Eqs. (9), we obtain the following 3D strain field as

$$\begin{aligned}
\varepsilon_{11}(x_1, x_2, x_3) &= \bar{u}_{1,1} - x_3\bar{u}_{3,11} \\
\varepsilon_{22}(x_1, x_2, x_3) &= \bar{u}_{2,2} - x_3\bar{u}_{3,22} \\
2\varepsilon_{12}(x_1, x_2, x_3) &= \bar{u}_{1,2} + \bar{u}_{2,1} - 2x_3\bar{u}_{3,12} \\
\varepsilon_{13}(x_1, x_2, x_3) &= \varepsilon_{23}(x_1, x_2, x_3) = \varepsilon_{33}(x_1, x_2, x_3) = 0
\end{aligned} \tag{11}$$

The vanishing of transverse normal strain ε_{33} is a direct consequence of the first Kirchhoff assumption as we assumed that the transverse normal is infinitely rigid which implies no strain exist in the transverse direction. The fact the in-plane strains $\varepsilon_{\alpha\beta}$ are linear functions of x_3 is a direct consequence of the second Kirchhoff assumption as we assumed that the transverse normal remains straight during deformation. The vanishing of transverse shear strains $\varepsilon_{\alpha 3}$ is a direct consequence of the third Kirchhoff assumption as we assumed that the transverse normal remains normal during deformation. That is, the 90 degree angle between transverse normal line and the reference surface remains as 90 degree, implying that the corresponding shear strain components are zero.

At this point it is convenient to introduce the following notation for the 2D plate strains

$$\epsilon_{\alpha\beta}(x_1,x_2)=\frac{1}{2}(\bar{u}_{\alpha,\beta}+\bar{u}_{\beta,\alpha}); \quad \kappa_{\alpha\beta}(x_1,x_2)=-\bar{u}_{3,\alpha\beta} \tag{12}$$

where $\epsilon_{\alpha\beta}$ are the in-plane plate strains, $\kappa_{\alpha\beta}$ the curvature of deformed reference surfaces. These strain measures $\epsilon_{\alpha\beta}, \kappa_{\alpha\beta}$ are usually collectively terms as the classical plate strain measures. Eq. (12) can be considered as the 2D plate strain-displacement relations.

Using the definition in Eqs. (12), we can express the 3D strain field in Eq. (11) as

$$\begin{aligned}\varepsilon_{\alpha\beta}(x_1,x_2,x_3)&=\epsilon_{\alpha\beta}+x_3\kappa_{\alpha\beta}\\ \varepsilon_{13}(x_1,x_2,x_3)&=\varepsilon_{23}(x_1,x_2,x_3)=\varepsilon_{33}(x_1,x_2,x_3)=0\end{aligned} \tag{13}$$

The original 3D strain field is expressed in terms of the classical plate strain measures, which are 2D functions of x_1, x_2 and we have now completed the expressions for 3D kinematics including the displacement field $u_i(x_1,x_2,x_3)$ and the strain field $\varepsilon_{ij}(x_1,x_2,x_3)$ in terms of 2D kinematics including plate displacement variables $\bar{u}_i(x_1,x_2)$ and the classical plate strain measures $\epsilon_{\alpha\beta}(x_1,x_2)$ and $\kappa_{\alpha\beta}(x_1,x_2)$.

2.2 Kinetics

Having known the strain field, we can obtain the stress field in the plate structure using the generalized 3D Hooke's law if the material is linear elastic. For example, for an isotropic material, we have

$$\begin{aligned}\sigma_{11}&=(\lambda+2G)\varepsilon_{11}+\lambda\varepsilon_{22}\\ \sigma_{22}&=(\lambda+2G)\varepsilon_{22}+\lambda\varepsilon_{11}\\ \sigma_{12}&=2G\varepsilon_{12}\\ \sigma_{33}&=\lambda(\varepsilon_{11}+\varepsilon_{22})\\ \sigma_{13}&=\sigma_{23}=0\end{aligned} \tag{14}$$

where $\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}$ and $G=\frac{E}{2(1+\nu)}$ is the shear modulus. Although this stress field naturally flow from the generalized Hooke's law using the strain field in Eq. (13), it does not agree with the experimental measurements very well. We have to introduce additional assumptions regarding the stress field to provide more accurate approximation of the reality. Because the thickness of the plate is much smaller comparing to the in-plane dimension of the plate, we can assume that transverse stresses $\sigma_{i3}\approx 0$ in comparison to $\sigma_{\alpha\beta}$. This assumption, particularly $\sigma_{33}\approx 0$, clearly conflicts with the stress field in Eq. (14) resulted from the strain field which is obtained from the displacement field based on the Kirchhoff assumptions. The reason is that the first assumption of Kirchhoff, transverse normal remains rigid in its own direction, clearly violates the reality. We all know that when the plate is deformed, the plate will deform in its thickness direction due to Poisson's effect. For this very reason, we overrule the previous assumptions for obtaining kinematics and introduce the following assumptions for the stress field:

$$\sigma_{i3} = 0 \tag{15}$$

Note $\sigma_{\alpha 3}$ already vanishes in Eq. (14) due to the third Kirchhoff assumption and isotropy of the material. If the material is general anisotropic, $\sigma_{\alpha 3}$ in Eq. (14) will not vanish although we still have the transverse shear strain $\varepsilon_{\alpha 3} = 0$ due to the third Kirchhoff assumption. The stress assumption in Eq. (15) is usually called plane stress assumption. Here, to comply with this assumption for isotropic material, we have to also implicitly assume the transverse normal strain ε_{33} is not zero. According to the generalized Hooke's law, we know

$$\sigma_{33} = (\lambda + 2G)\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22}) \tag{16}$$

In view of the assumption in Eq. (15), we have

$$\varepsilon_{33} = -\frac{\lambda}{\lambda + 2G}(\varepsilon_{11} + \varepsilon_{22}) = \frac{\nu}{\nu - 1}(\varepsilon_{11} + \varepsilon_{22}) \tag{17}$$

This implication directly contradicts with the strain field in Eq. (13) obtained using the Kirchhoff assumptions except when $\nu = 0$ which in general is not true.

Substituting the transverse normal strain in Eq. (17) along with those additional strain measures in Eq. (13) into the generalized Hooke's law, we end up with the following stress field:

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) \\
\sigma_{22} &= \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) \\
\sigma_{12} &= 2G\varepsilon_{12} \\
\sigma_{33} &= 0 \\
\sigma_{13} &= \sigma_{23} = 0
\end{aligned} \tag{18}$$

Clearly the above stress field is not the same as those in Eq. (14), which implies that the stress field in Eq. (18) conflicts with our starting Kirchhoff assumptions. This kind of contradictions is common in structural models derived based on ad hoc assumptions. Nevertheless, such inconsistencies are used in the derivation of the classical plate model and commonly taught in textbooks. These contractions can be partially justified by the fact that we need to rely on the Kirchhoff assumptions to obtain a simple expression of the 3D kinematics in terms of 2D kinematics and we also use the stress assumptions in Eq. (15) so that the results can better agree with reality. A sad fact is that such inconsistencies are seldom clearly pointed out and criticized. As a summary, to derive the classical plate model based on ad hoc assumptions, we have to first use the Kirchhoff assumptions to related 3D kinematics with 2D kinematics, and then use the stress assumption in Eq. (15) to obtain the 3D stress field. In other words, in our further derivations, we use the 3D strains as expressed in Eqs. (13) and the 3D stresses as expressed in Eq. (18), despite of the fact that they are obtained by using two sets of conflicting assumptions.

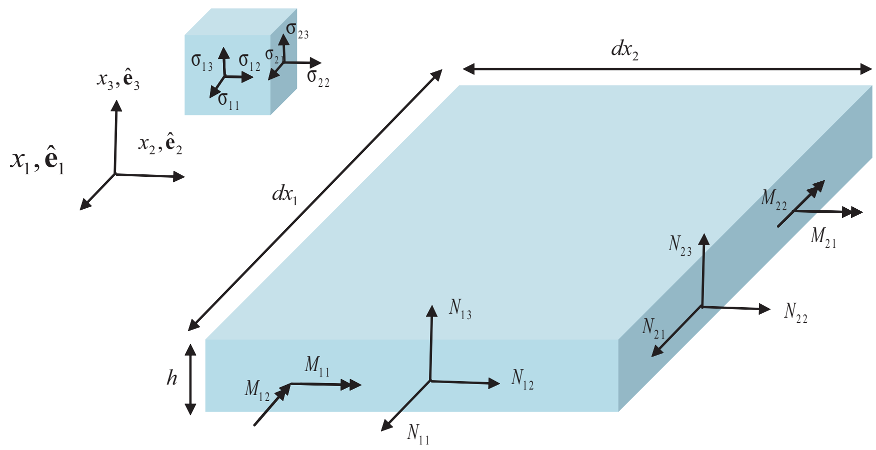

Figure 5: Sign convention for plate stress resultants

We also need to note that the transverse shear stresses in Eq. (18) vanish because the transverse shear strains vanish based on the Kirchhoff assumptions. When we assume the transverse normal remains normal to the reference surface during deformation, we effectively assume that the plate is infinitely rigid in transverse shear. Hence, the transverse shear stresses, although exist in general, cannot be obtained based on constitutive relations but must be determined from equilibrium considerations as will be shown later.

To complete the 2D plate model, we also need to introduce a set of 2D kinetic variables called plate stress resultants to relate with it 3D counterparts, the 3D stress field. The plate stress resultants are defined as follows:

$$\begin{aligned} \langle \sigma_{\alpha\beta} \rangle &= N_{\alpha\beta} \\ \langle x_3 \sigma_{\alpha\beta} \rangle &= M_{\alpha\beta} \end{aligned} \tag{19}$$

where the angle brackets denote integration over the thickness. Because $\sigma_{12} = \sigma_{21}$, we have $N_{12} = N_{21}$ and $M_{12} = M_{21}$. The sign convention is determined by the definition as depicted in Figure 5 for a differential plate segment. The transverse shear stress resultants $N_{\alpha 3}$ are defined similarly as the first equation in Eq. (19) such as

$$\langle \sigma_{1\alpha} \rangle = N_{\alpha 3} \tag{20}$$

although as we have pointed out that $\sigma_{1\alpha}$ in Eq. (18) vanish due to the third Kirchhoff assumption and the Hooke's law. However, such stress values are not zero as they are needed to balance the vertical load on the plate, which is the primary loading mechanism. For this we have to admit the fact that we cannot obtain the transverse shear stresses directly from the constitutive considerations. Nevertheless, we can obtain $N_{\alpha 3}$ from equilibrium considerations. From $N_{\alpha 3}$, we can approximately estimate $\sigma_{1\alpha}$. For example, we can assume the transverse shear stress is approximately uniform through the thickness, then we have $\sigma_{1\alpha} \approx N_{\alpha 3}/h$. Another way to estimate the transverse shear stresses are through the 3D equilibrium equations with the knowledge of in-plane stresses $\sigma_{\alpha\beta}$. As will be shown later, $N_{\alpha 3}$ are not kinetic variables in the 2D classical plate model and are only used for deriving the equilibrium equations using the Newtonian approach.

It is timely noted that to complete the kinetics part, we need to establish governing equations among the 2D plate kinetic variables $N_{\alpha\beta}, M_{\alpha\beta}$ which will be furnished by either Newtonian method or the variational method later.

2.3 Energetics

Substituting the 3D strain field in Eqs. (13) into the 3D stresses in Eq. (18), then into Eq. (19), we have

$$\begin{aligned}
N_{11} &= \left\langle \frac{E}{1-\nu^2} \right\rangle \epsilon_{11} + \left\langle \frac{E\nu}{1-\nu^2} \right\rangle \epsilon_{22} + \left\langle \frac{x_3 E}{1-\nu^2} \right\rangle \kappa_{11} + \left\langle \frac{x_3 E\nu}{1-\nu^2} \right\rangle \kappa_{22} \\
N_{22} &= \left\langle \frac{E\nu}{1-\nu^2} \right\rangle \epsilon_{11} + \left\langle \frac{E}{1-\nu^2} \right\rangle \epsilon_{22} + \left\langle \frac{x_3 E\nu}{1-\nu^2} \right\rangle \kappa_{11} + \left\langle \frac{x_3 E}{1-\nu^2} \right\rangle \kappa_{22} \\
N_{12} &= \left\langle \frac{E}{2(1+\nu)} \right\rangle 2\epsilon_{12} + \left\langle \frac{x_3 E}{2(1+\nu)} \right\rangle 2\kappa_{12} \\
M_{11} &= \left\langle \frac{x_3 E}{1-\nu^2} \right\rangle \epsilon_{11} + \left\langle \frac{x_3 E\nu}{1-\nu^2} \right\rangle \epsilon_{22} + \left\langle \frac{x_3^2 E}{1-\nu^2} \right\rangle \kappa_{11} + \left\langle \frac{x_3^2 E\nu}{1-\nu^2} \right\rangle \kappa_{22} \\
M_{22} &= \left\langle \frac{x_3 E\nu}{1-\nu^2} \right\rangle \epsilon_{11} + \left\langle \frac{x_3 E}{1-\nu^2} \right\rangle \epsilon_{22} + \left\langle \frac{x_3^2 E\nu}{1-\nu^2} \right\rangle \kappa_{11} + \left\langle \frac{x_3^2 E}{1-\nu^2} \right\rangle \kappa_{22} \\
M_{12} &= \left\langle \frac{x_3 E}{2(1+\nu)} \right\rangle 2\epsilon_{12} + \left\langle \frac{x_3^2 E}{2(1+\nu)} \right\rangle 2\kappa_{12}
\end{aligned} \tag{21}$$

The constitutive relations in Eqs. (21) can be rewritten in the following matrix form as

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ N_{22} \\ M_{11} \\ M_{12} \\ M_{22} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ \epsilon_{22} \\ \kappa_{11} \\ 2\kappa_{12} \\ \kappa_{22} \end{Bmatrix} \tag{22}$$

where A is the extension stiffness, B is the extension-bending coupling stiffness, and D is the bending stiffness. For a plate made of a single homogenous isotropic plate, we have

$$A = \frac{Eh}{1-\nu^2}\Delta \quad B = \frac{E\langle x_3 \rangle}{1-\nu^2}\Delta \quad D = \frac{E\langle x_3^2 \rangle}{1-\nu^2}\Delta \quad \text{with} \quad \Delta = \begin{bmatrix} 1 & 0 & \nu \\ 0 & \frac{1-\nu}{2} & 0 \\ \nu & 0 & 1 \end{bmatrix} \tag{23}$$

If the origin of x_3 is located at the mid-plane, *i.e.*, $-\frac{h}{2} \le x_3 \le \frac{h}{2}$, we have $\langle x_3 \rangle = 0$ and $\langle x_3^2 \rangle = \frac{h^3}{12}$. Clearly the extension-bending coupling stiffness matrix B vanishes under this choice of x_3.

Eq. (22) can be considered as the constitutive relations for the classical plate model, the 2D counterpart of the 3D generalized Hooke's law. The 6×6 symmetric matrix is commonly called classical plate stiffness matrix. Because of the assumptions we have used, the restriction that the plate is made of a single isotropic material, and the choice of mid-plane as the reference surface, the bending behavior is automatically decoupled from extension, implied by the fact that the extension-bending coupling stiffness matrix B vanishes. That is also the reason that why in traditional textbooks on the theory of plates, only the bending behavior is primarily taught. Many situations may result in nonzero B matrix. Then the extension and bending behavior should be studied together. For a composite plate structure, the stiffness matrix could be fully populated such that A, B, and D are fully populated 3×3 symmetric matrix, and the extension and bending behavior are fully coupled for general cases.

2.4 Equilibrium equations

In our classical plate problem, we are solving for the unknown plate displacements ($\bar{u}_i$), plate strains ($\epsilon_{\alpha\beta}, \kappa_{\alpha\beta}$), and stress resultants ($N_{\alpha\beta}, M_{\alpha\beta}$), a total of 15 unknowns, noting $\epsilon_{12} = \epsilon_{21}$, $\kappa_{12} = \kappa_{21}$, $N_{12} = N_{21}$, and $M_{12} = M_{21}$. Thus far, we have obtained six equations for the 2D strain-displacement relations in Eq. (12), and six equations for the 2D constitutive relations in Eq. (22), a total of 12 equations. We are lacking of three equations to form a complete system. These three equations can be derived using either Newtonian method or variational method.

2.4.1 Newtonian method

To use Newtonian method to derive the equilibrium equations of the classical plate model, we need to consider the equilibrium of a differential plate element using some free body diagrams, which is the focus of this section.

Consider a plate structure subjected to a general loading consists of distributed forces p_i and moments q_α in the reference surface and distributed forces P_i and moments Q_α along the boundary of the reference surface. The distributed surface forces $p_1(x_1, x_2)$, $p_2(x_1, x_2)$, and $p_3(x_1, x_2)$ act in the direction $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$, respectively. The same convention is used for the distributed boundary loads P_1, P_2, and P_3. The distributed surface moments $q_1(x_1, x_2)$ and $q_2(x_1, x_2)$ act about the axes $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$, respectively. The distributed moments along the boundary Q_1 and Q_2 act about the same axes. One or more sets of concentrated forces and moments could be applied at any in-plane locations. Note here, we consider the distributed loads in terms of distributed forces p_i and distributed moments q_α acting at the origin of x_3 and they are functions of x_1 and x_2 only. In other words the distribution is only along the reference surface axis and not distributed along the thickness. In reality, in the original 3D structure, within the framework of 3D elasticity, there are distributed body forces as functions of x_i and distributed surfaces tractions along the boundary surfaces. The 2D loads should relate with the 3D loads in such a way that they are statically equivalent: summation of forces and summation of moments in three directions of the 3D loads should be equal to those of the 2D loads. How to achieve it systematically will be given in the next section when we derive the classical plate model using the variational method. One may wonder why there is no moments (q_3 or Q_3) about $\hat{\boldsymbol{e}}_3$. The reason is that the classical plate model cannot sustain such loads which will become clear when we derive the loads using the variational method. The distributed forces have unit as force per unit area, N/m^2 or Pascals in the SI system. The distributed

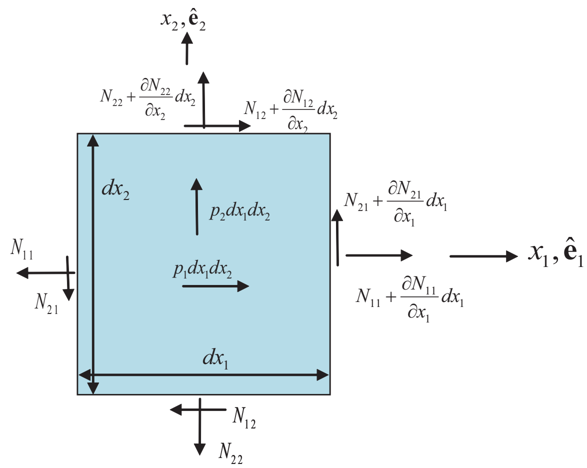

Figure 6: Free body diagram for in-plane forces.

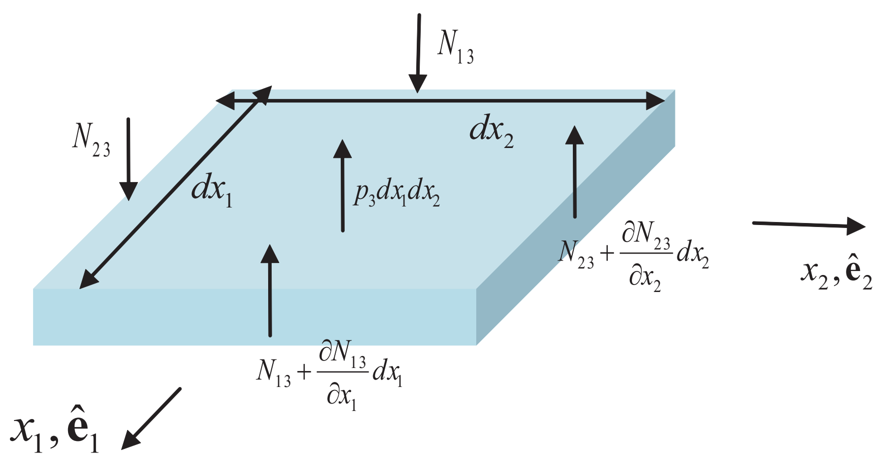

Figure 7: Free body diagram for the transverse shear forces.

moments have unit as moment per unit area, $N \cdot m/m^2$ in the SI system.

The equilibrium equations can be derived considering free body diagrams of a differential plate element. Let us focus on the force equilibrium along the in-plane directions of the plate first. Consider the differential plate element as depicted in Figure 6. Summing all the forces in the $\hat{\boldsymbol{e}}_1$ direction yields the following equation

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = 0 \tag{24}$$

Summing all the forces in the $\hat{\boldsymbol{e}}_2$ direction yields the following equation

$$\frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = 0 \tag{25}$$

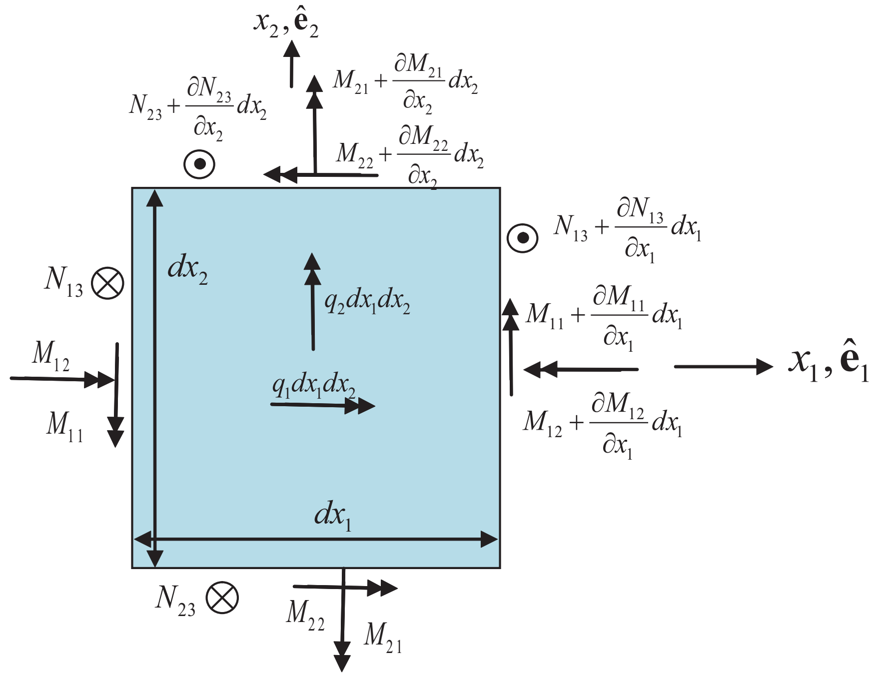

Figure 8: Free body diagram for moments and transverse shear forces.

Next, let us consider the force equilibrium along $\hat{\boldsymbol{e}}_3$ direction for a differential plate element. which is depicted in Figure 7. A summation of the forces along $\hat{\boldsymbol{e}}_3$ gives the transverse force equilibrium equation in this direction

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + p_3 = 0 \tag{26}$$

Equilibrium also implies the summation of moments along all the directions should vanish. As sketched in Figure 8, summing the moments about an axis parallel to $\hat{\boldsymbol{e}}_1$ yields

$$-\frac{\partial M_{12}}{\partial x_1} - \frac{\partial M_{22}}{\partial x_2} + q_1 + N_{23} = 0 \tag{27}$$

Summing the moments about an axis parallel to $\hat{\boldsymbol{e}}_2$ yields

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} + q_2 - N_{13} = 0 \tag{28}$$

The moment equilibrium about the $\hat{\boldsymbol{e}}_3$ direction can be obtained from a free body diagram similar to Figure 7, we will obtain the identity that $N_{12} = N_{21}$, but this equation bring no new information about equilibrium as this identity is already satisfied by the equality of in-plane shear stresses $\sigma_{12} = \sigma_{21}$.

The transverse shear stress resultants $N_{\alpha 3}$ can be eliminated from the equilibrium equations by taking a derivative of Eq. (27) with respect to x_2 and taking a derivative of Eq. (28) with respect to x_1, then using Eq. (26), to yield the bending moment equilibrium equations

$$(M_{11,1} + M_{21,2} + q_2)_{,1} + (M_{12,1} + M_{22,2} - q_1)_{,2} + p_3 = 0 \tag{29}$$

The three equations in Eqs. (24), (25), and (29) are the last three equations we need to complete the classical plate theory.

In summary, the classical plate theory is characterized by the following sets of equations

- Kinematics in terms of six strain-displacement equations in Eq. (12).
- Kinetics in terms of three equilibrium equations in Eqs. (24), (25), and (29).
- Energetics in terms of six constitutive equations in Eq. (21).

These equations plus some appropriate boundary conditions can be used to solve for the plate displacements $\bar{u}_i$, plate strains $\epsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, plate stress resultants $N_{\alpha\beta}$ and $M_{\alpha\beta}$, a total of 15 unknown 2D functions of the in-plane coordinates x_1 and x_2. After solved for these fields, we can recover the 3D displacement field using Eq. (9), the 3D strain field using Eq. (13), and the 3D stress field using Eq. (18).

If the extension-bending coupling stiffness matrix B vanishes, the Kirchhoff plate problem is decoupled into the following two simpler problems.

- *The in-plane problem*: this problem involves eight unknowns including $\bar{u}_\alpha$, $\epsilon_{\alpha\beta}$ and $N_{\alpha\beta}$. The corresponding eight governing equations are the first three strain-displacement equations in Eq. (12), the first three constitutive equations in Eq. (21), and the two equilibrium equations in Eqs. (24) and (25).
- *The bending problem*: this problem involves seven unknowns including $\bar{u}_3$, $\kappa_{\alpha\beta}$ and $M_{\alpha\beta}$. The corresponding seven governing equations are the last three strain-displacement equations in Eq. (12), the last three constitutive equations in Eq. (21), and the equilibrium equation in Eq. (29).

Next, let us show how we can eliminate the plate strains and stress resultants to develop a displacement formulation for the bending problem. Substituting the 2D strain-displacement relations in Eq. (12) into the 2D constitutive relations in Eq. (21), then into the moment equilibrium equation in Eq. (29), we obtain the following displacement formulation of the classical plate theory for the bending problem:

$$\bar{u}_{3,1111} + 2\bar{u}_{3,1122} + \bar{u}_{3,2222} = \frac{q_{2,1} - q_{1,2} + p_3}{\mathcal{D}} \tag{30}$$

with $\mathcal{D} = \frac{Eh^3}{12(1-\nu^2)}$, denoting the plate bending stiffness. The basic equation of Kirchhoff plate bending theory in Eq. (30) is a biharmonic partial differential equation for the transverse displacement, which can be written in a more compact manner as

$$\nabla^4 \bar{u}_3 = \frac{q_{2,1} - q_{1,2} + p_3}{\mathcal{D}} \tag{31}$$

What is lacking is a discussion of boundary conditions, which will be supplied later.

2.5 Variational method

The equilibrium equations of the classical plate model can be derived in a more systematic fashion using the variational method based on the Kantorovich method. From the view point of the Kantorovich method, our objective is to reduce the original 3D problem to a 2D problem so we need to approximate the original 3D fields in terms of 2D unknown functions of the in-plane coordinates x_α and some known functions of the transverse coordinates x_3. To this end, we consider the displacement field based on the Kirchhoff assumptions, Eq. (9), as approximate trial functions for the 3D displacement field, and the stress field in Eq. (18) as approximate trial functions for the 3D stress field. For the original 3D structure, the load can be applied either as distributed body force f_i, surface tractions τ_i on the top surface, β_i on the bottom surface, and t_i on the lateral surface. The principle of virtual work of the plate structure can be stated as

$$\frac{1}{2}\int_S \delta U_{2D} \mathrm{d}S = \delta W \tag{32}$$

with U_{2D} understood as the 2D strain energy density defined over the reference surface plane denoted using S. Clearly the 2D strain energy density is the integration of the 3D strain energy density over the thickness such that

$$U_{2D} = \frac{1}{2}\left\langle \sigma_{ij}\varepsilon_{ij}\right\rangle \tag{33}$$

where the angle bracket denotes the integration over the thickness. The virtual work δW due to applied loads can be expressed as

$$\delta W = \int_S \left(\langle f_i \delta u_i\rangle + \beta_i \delta u_i(x_1, x_2, -\frac{h}{2}) + \tau_i \delta u_i(x_1, x_2, \frac{h}{2}) \right) \mathrm{d}S + \int_\Omega \langle t_i \delta u_i\rangle \, \mathrm{d}\Omega \tag{34}$$

Here Ω denotes the boundary curve of the plate reference surface (that is, the intersection of the boundary lateral surfaces of the plate with the plate reference surface), and the last two terms within the first integrand denote the virtual work evaluated at the bottom surface ($x_3 = -\frac{h}{2}$) and the top surface ($x_3 = \frac{h}{2}$), respectively. Substituting the 3D displacement field expressed in Eq. (9) into Eq. (34), we have

$$\delta W = \int_S (p_i \delta \bar{u}_i + q_\alpha \delta \Phi_\alpha) \, \mathrm{d}S + \int_\Omega (P_i \delta \bar{u}_i + Q_\alpha \delta \Phi_\alpha) \, \mathrm{d}\Omega \tag{35}$$

with

$$\begin{aligned} p_i(x_1, x_2) &= \langle f_i \rangle + \beta_i + \tau_i \\ q_1(x_1, x_2) &= \frac{h}{2}(\beta_2 - \tau_2) - \langle x_3 f_2 \rangle \\ q_2(x_1, x_2) &= \frac{h}{2}(\tau_1 - \beta_1) + \langle x_3 f_1 \rangle \\ P_i &= \langle t_i \rangle \\ Q_1 &= -\langle x_3 t_2 \rangle \\ Q_2 &= \langle x_3 t_1 \rangle \end{aligned} \tag{36}$$

Note $\Phi_1 = \bar{u}_{3,2}$ and $\Phi_2 = -\bar{u}_{3,1}$ due to the third Kirchhoff assumption. Here we actually provided a systematic way to obtain the distributed forces $p_i(x_1, x_2)$ and moments $q_\alpha(x_1, x_3)$ along the reference surface, and the distributed forces P_i and moments Q_α along the boundary curve we used in the Newtonian method based on the body forces and surface tractions applied on the original 3D structure.

Substituting the 3D stress field expressed in Eq. (18) into Eq. (33), we have

$$U_{2D} = \frac{1}{2}\left\langle \sigma_{\alpha\beta}\varepsilon_{\alpha\beta}\right\rangle = \frac{1}{2}\left\langle \frac{E}{1-\nu^2}(\varepsilon_{11}^2 + 2\nu\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}^2) + G(2\varepsilon_{12})^2 \right\rangle \tag{37}$$

Substituting the 3D strain field expressed in Eqs. (13) into the above equation, we have

$$\begin{aligned} 2U_{2D} &= \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ \epsilon_{22} \\ \kappa_{11} \\ 2\kappa_{12} \\ \kappa_{22} \end{Bmatrix} \begin{bmatrix} \left\langle \frac{E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{\nu E}{1-\nu^2}\right\rangle & \left\langle \frac{x_3 E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{x_3 \nu E}{1-\nu^2}\right\rangle \\ 0 & \left\langle \frac{E}{2(1+\nu)}\right\rangle & 0 & 0 & \left\langle \frac{x_3 E}{2(1+\nu)}\right\rangle & 0 \\ \left\langle \frac{\nu E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{E}{1-\nu^2}\right\rangle & \left\langle \frac{x_3 \nu E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{x_3 E}{1-\nu^2}\right\rangle \\ \left\langle \frac{x_3 E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{x_3 \nu E}{1-\nu^2}\right\rangle & \left\langle \frac{x_3^2 E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{x_3^2 \nu E}{1-\nu^2}\right\rangle \\ 0 & \left\langle \frac{x_3 E}{2(1+\nu)}\right\rangle & 0 & 0 & \left\langle \frac{x_3^2 E}{2(1+\nu)}\right\rangle & 0 \\ \left\langle \frac{x_3 \nu E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{x_3 E}{1-\nu^2}\right\rangle & \left\langle \frac{x_3^2 \nu E}{1-\nu^2}\right\rangle & 0 & \left\langle \frac{x_3^2 E}{1-\nu^2}\right\rangle \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ \epsilon_{22} \\ \kappa_{11} \\ 2\kappa_{12} \\ \kappa_{22} \end{Bmatrix} \\ &= \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ \epsilon_{22} \\ \kappa_{11} \\ 2\kappa_{12} \\ \kappa_{22} \end{Bmatrix} \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ \epsilon_{22} \\ \kappa_{11} \\ 2\kappa_{12} \\ \kappa_{22} \end{Bmatrix} \end{aligned} \tag{38}$$

where A, B, D are the same as those given in Eq. (23).

Carrying out the partial derivatives of U_{2D} in Eq. (38) and in view of Eq. (21), we obtain

$$\begin{aligned} \frac{\partial U_{2D}}{\partial \epsilon_{11}} &= \frac{Eh}{1-\nu^2}(\epsilon_{11} + \nu\epsilon_{22}) = N_{11} \\ \frac{\partial U_{2D}}{\partial 2\epsilon_{12}} &= \frac{Eh}{2(1+\nu)} 2\epsilon_{12} = N_{12} \\ \frac{\partial U_{2D}}{\partial \epsilon_{22}} &= \frac{Eh}{1-\nu^2}(\nu\epsilon_{11} + \epsilon_{22}) = N_{22} \\ \frac{\partial U_{2D}}{\partial \kappa_{11}} &= \frac{Eh^3}{12(1-\nu^2)}(\kappa_{11} + \nu\kappa_{22}) = M_{11} \\ \frac{\partial U_{2D}}{\partial 2\kappa_{12}} &= \frac{Eh^3}{24(1+\nu)} 2\kappa_{12} = M_{12} \\ \frac{\partial U_{2D}}{\partial \kappa_{22}} &= \frac{Eh^3}{12(1-\nu^2)}(\nu\kappa_{11} + \kappa_{22}) = M_{22} \end{aligned} \tag{39}$$

Here, we restrict to the case that the plate is isotropic and homogeneous and the midplane is the reference plane. This gives another way to define the plate stress resultants as conjugates to the 2D plate strains in terms of the 2D strain energy density, *i.e.*, the stress resultants can be defined as the partial derivative of the 2D strain energy density with respect to the corresponding 2D plate strain measures and these equations can also be written in the same matrix form as Eq. (22). In other words the variational method provides another way to derive the same energetics as we have presented previously in Section 2.3.

Substituting Eqs. (35), into Eq. (32), we can rewrite the principal of virtual work energy in a 2D form as

$$\int_S \delta U_{2D} \mathrm{d}S = \int_S \left(p_i \delta \bar{u}_i + q_1 \delta \bar{u}_{3,2} - q_2 \delta \bar{u}_{3,1}\right) \mathrm{d}S + \int_\Omega \left(P_i \delta \bar{u}_i + Q_1 \delta \bar{u}_{3,2} - Q_2 \delta \bar{u}_{3,1}\right) \mathrm{d}\Omega \quad (40)$$

which implies the following

$$0 = \int_S \left(\delta U_{2D} - p_i \delta \bar{u}_i - q_1 \delta \bar{u}_{3,2} + q_2 \delta \bar{u}_{3,1}\right) \mathrm{d}S - \int_\Omega \left(P_i \delta \bar{u}_i + Q_1 \delta \bar{u}_{3,2} - Q_2 \delta \bar{u}_{3,1}\right) \mathrm{d}\Omega \quad (41)$$

The variation of 2D strain energy density U_{2D} can be evaluated based on Eq. (39) as

$$\begin{aligned} \delta U_{2D} &= N_{11} \delta \epsilon_{11} + N_{12} \delta(2\epsilon_{12}) + N_{22} \delta \epsilon_{22} + M_{11} \delta \kappa_{11} + M_{12} \delta(2\kappa_{12}) + M_{22} \delta \kappa_{22} \\ &= N_{11} \delta \bar{u}_{1,1} + N_{12} \delta(\bar{u}_{1,2} + \bar{u}_{2,1}) + N_{22} \delta \bar{u}_{2,2} - M_{11} \delta \bar{u}_{3,11} - 2M_{12} \delta \bar{u}_{3,12} - M_{22} \delta \bar{u}_{3,22} \end{aligned} \quad (42)$$

Carrying out integration by parts for the integral term in Eq. (41), we can obtain the corresponding equilibrium equations governing the plate. For the simplicity of presentation, we first collect the terms related with $\bar{u}_\alpha$ in Eq. (41) as

$$\begin{aligned} 0 = &\int_S \left[-(N_{11,1} + N_{12,2} + p_1) \delta \bar{u}_1 - (N_{12,1} + N_{22,2} + p_2) \delta \bar{u}_2\right] \mathrm{d}S \\ &+ \int_\Omega \left[(n_1 N_{11} + n_2 N_{12} - P_1) \delta \bar{u}_1 + (n_1 N_{12} + n_2 N_{22} - P_2) \delta \bar{u}_2\right] \mathrm{d}\Omega \end{aligned} \quad (43)$$

where n_1 and n_2 denote the components of the outward normal $\boldsymbol{n}$ of the boundary curve (see Figure 9). It is more natural to express the boundary conditions using the normal and tangent coordinates of the boundary curve. Denoting the displacement components along the boundary curve as $\bar{u}_n$ along the normal direction and $\bar{u}_s$ along the tangent direction, we have

$$\bar{u}_1 = n_1 \bar{u}_n - n_2 \bar{u}_s \qquad \bar{u}_2 = n_2 \bar{u}_n + n_1 \bar{u}_s \quad (44)$$

The boundary terms in Eq. (43) now become

$$\begin{aligned} &\int_\Omega \left[(n_1 N_{11} + n_2 N_{12} - P_1)(n_1 \delta \bar{u}_n - n_2 \delta \bar{u}_s) + (n_1 N_{12} + n_2 N_{22} - P_2)(n_2 \delta \bar{u}_n + n_1 \delta \bar{u}_s)\right] \mathrm{d}\Omega \\ &= \int_\Omega (N_{nn} - P_n) \delta \bar{u}_n + (N_{ns} - P_s) \delta \bar{u}_s \mathrm{d}\Omega \end{aligned} \quad (45)$$

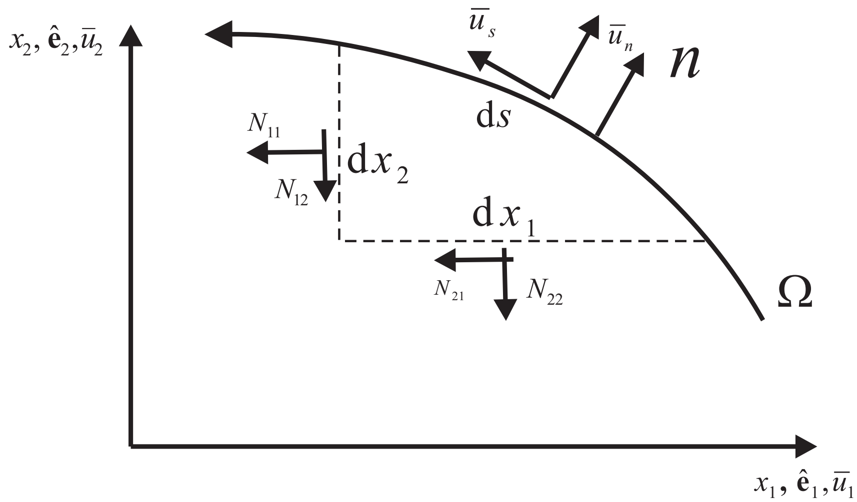

Figure 9: The local coordinate system along the boundary curve.

with

$$\begin{aligned} N_{nn} &= n_1^2 N_{11} + n_2^2 N_{22} + 2n_1 n_2 N_{12} \\ N_{ns} &= n_1 n_2 (N_{22} - N_{11}) + (n_1^2 - n_2^2) N_{12} \\ P_n &= P_1 n_1 + P_2 n_2 \\ P_s &= -P_1 n_2 + P_2 n_1 \end{aligned} \tag{46}$$

As $\bar{u}_1$ and δu_2 can vary independently, the corresponding Euler-Lagrange equations are

$$N_{11,1} + N_{12,2} + p_1 = 0 \qquad N_{12,1} + N_{22,2} + p_2 = 0 \tag{47}$$

which are the same as the equilibrium equations we obtained in Eqs. (24) and (25) using the Newtonian method. The boundary conditions can be deduced from the last line of Eq. (45). Following calculus of variations, if a displacement variable ($\bar{u}_n$ or $\bar{u}_s$) is prescribed, then it variation must be zero and the boundary term related with the variation of this displacement variable ($\delta\bar{u}_n$ or $\delta\bar{u}_s$) will vanish. If a displacement variable is free to vary, then to vanish the corresponding boundary term related with this displacement, the coefficients in front of the variation of that variable must be zero, that is we have

$$N_{nn} = P_n \qquad N_{ns} = P_s \tag{48}$$

Next, let us collect the terms related with $\delta\bar{u}_3$ in Eq. (41). We have

$$\begin{aligned} 0 = &- \int_S \left[(M_{11,1} + M_{12,2} + q_2)_{,1} + (M_{12,1} + M_{22,2} - q_1)_{,2} + p_3 \right] \delta\bar{u}_3 \mathrm{d}S \\ &- \int_\Omega \left[(n_1 M_{11} + n_2 M_{12} - Q_2)\delta\bar{u}_{3,1} + (n_1 M_{12} + n_2 M_{22} + Q_1)\delta\bar{u}_{3,2} \right] \mathrm{d}\Omega \\ &+ \int_\Omega \left[n_1 (M_{11,1} + M_{12,2} + q_2) + n_2 (M_{12,1} + M_{22,2} - q_1) - P_3 \right] \delta\bar{u}_3 \mathrm{d}\Omega \end{aligned} \tag{49}$$

The boundary terms in the second line of Eq. (49) can be expressed using the derivatives of $\bar{u}_3$ along with normal direction and tangent direction as

$$\begin{aligned}&\int_\Omega [(n_1M_{11}+n_2M_{12}-Q_2)(n_1\delta\bar{u}_{3,n}-n_2\delta\bar{u}_{3,s})(n_1M_{12}+n_2M_{22}+Q_1)(n_2\delta\bar{u}_{3,n}+n_1\delta\bar{u}_{3,s})]\,\mathrm{d}\Omega\\&\quad=\int_\Omega (M_{nn}-Q_n)\delta\bar{u}_{3,n}+(M_{ns}-Q_s)\delta\bar{u}_{3,s}\end{aligned} \tag{50}$$

with

$$\begin{aligned}M_{nn}&=n_1^2M_{11}+n_2^2M_{22}+2n_1n_2M_{12}\\M_{ns}&=n_1n_2(M_{22}-M_{11})+(n_1^2-n_2^2)M_{12}\\Q_n&=Q_2n_1-Q_1n_2\\Q_s&=-Q_2n_2-Q_1n_1\end{aligned} \tag{51}$$

If we also introduce the following notation

$$V_3=n_1(M_{11,1}+M_{12,2}+q_2)+n_2(M_{12,1}+M_{22,2}-q_1) \tag{52}$$

Then Eq. (49) can be simplified as

$$\begin{aligned}0=&-\int_S [(M_{11,1}+M_{12,2}+q_2)_{,1}+(M_{12,1}+M_{22,2}-q_1)_{,2}+p_3]\,\delta\bar{u}_3\mathrm{d}S\\&+\int_\Omega (V_3-P_3)\delta\bar{u}_3-(M_{nn}-Q_n)\delta\bar{u}_{3,n}-(M_{ns}-Q_s)\delta\bar{u}_{3,s}\mathrm{d}\Omega\end{aligned} \tag{53}$$

We also need to realized that $\delta\bar{u}_3$ and $\delta\bar{u}_{3,s}$ are not independent quantities along the boundary curve Ω. One more integration by parts is needed, so that we have

$$\begin{aligned}0=&-\int_S [(M_{11,1}+M_{12,2}-q_1)_{,1}+(M_{12,1}+M_{22,2}-q_2)_{,2}+p_3]\,\delta\bar{u}_3\mathrm{d}S\\&+\int_\Omega [V_3-P_3+(M_{ns}-Q_s)_{,s}]\delta\bar{u}_3-(M_{nn}-Q_n)\delta\bar{u}_{3,n}\mathrm{d}\Omega-[(M_{ns}-Q_s)\delta\bar{u}_3]_\Omega\end{aligned} \tag{54}$$

where the notation $[\cdot]_\Omega$ indicates the end points of the boundary curve. If it is a rectangular plate, the end points of the boundary curve will be defined by its four corners. If it is a circular plate, the boundary curve has not end points and the last term in Eq. (54) vanish.

Although both the Newtonian method and the variational method based on the same set of ad hoc assumptions necessary to obtain the displacement field in Eq. (9), the strain field in Eqs. (13), and the stress field in Eq. (18), there are some difference between these two methods.

- We does not have to introduce the transverse shear stress resultants for the derivation using the variational approach.
- The variational method can establish a rational connection between the applied loads in the original 3D structure and the final 2D plate model.

- Although lack of being intuitive, the variational approach is more systematic. As far as one is careful about the derivation, it is not easy to make a sign error like commonly happen in Newtonian approach particularly for deriving the boundary conditions.

- As the variational approach is based on the Kantrovich method, it is easy to extend this derivation for higher-order models by using a different set of assumptions for the 3D displacement field in terms of 2D unknown functions, while such extensions using the Newtonian approach are much more difficult.

- The concept of total vertical force and corner forces are a natural byproduct of the variational approach, while we have to introduce these two concepts in an ad hoc manner to overcome the problems associated with the boundary conditions to be applied along the free edge of a plate.

However, because both methods are based on a host of ad hoc assumptions, they feature the same set of contradictions as we discussed carefully in previous sections. In the next section, we will use the variational asymptotic method to construct the classical plate model without invoking any ad hoc assumptions thus avoiding the awkward self-contractions.

3 Variational Asymptotic Method

The whole purpose of the plate model is to approximate the original 3D model with a 2D model formulated in terms of unknown functions of the two in-plane coordinates describing the reference surface of the plate. Our motivation comes from the fact that the thickness is much smaller than the in-plane dimensions of the plate structure. This fact of smallness of the thickness compared to the plate in-plane dimensions can be exploited using the variational asymptotic method to derive the classical plate model. Let us denote h as the thickness of the plate and L as the characteristic dimension of the plate reference surface. Then we know that $\delta = h/L$ as a small parameter. Suppose the 3D displacements are $u_i(x_1, x_2, x_3)$, then the 3D strains as defined in linear elasticity are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{55}$$

To proceed using the variational asymptotic method, we need to have some very basic knowledge of order analysis. For a continuous differentiable function, $f(x)$ for $x \in [a, b]$. If we denote the order of $f(x)$ as $\bar{f}$, then $\frac{df}{dx}$ is of the order of $\frac{\bar{f}}{b-a}$, denoting as $\frac{df}{dx} \sim \frac{\bar{f}}{b-a}$. Then it is obvious that $u_{i,\alpha} \sim \bar{u}_i/L$ and $u_{i,3} \sim \bar{u}_i/h$, and $u_{i,\alpha} \ll u_{i,3}$ because $\delta = h/L \ll 1$.

The 3D strain field can be written explicitly as

$$\begin{aligned} \varepsilon_{11} &= u_{1,1} \\ 2\varepsilon_{12} &= u_{1,2} + u_{2,1} \\ 2\varepsilon_{13} &= u_{1,3} + u_{3,1} \\ \varepsilon_{22} &= u_{2,2} \\ 2\varepsilon_{23} &= u_{2,3} + u_{3,2} \\ \varepsilon_{33} &= u_{3,3} \end{aligned} \tag{56}$$

The total potential energy of the original 3D structure is given as follows

$$\Pi = \frac{1}{2} \int_S U_{2D} \mathrm{d}S - W \tag{57}$$

with twice of the 2D strain energy density expressed in the following form as

$$2U_{2D} = \left\langle 2G \left(\rho\varepsilon_{\alpha\alpha}^2 + \varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta} \right) + 4G\varepsilon_{\alpha 3}\varepsilon_{\alpha 3} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(\varepsilon_{33} + \rho\varepsilon_{\alpha\alpha} \right)^2 \right\rangle$$

with $\rho = \nu/(1-\nu)$. Note although this form is different from that in Eq. (33), they are identical to each other after some algebraic manipulations.

In view of Eq. (34), the work done by applied loads in the original 3D structure can be obtained as

$$W = \int_S \left(\langle f_i u_i \rangle + \beta_i u_i(x_1, x_2, -\frac{h}{2}) + \tau_i u_i(x_1, x_2, \frac{h}{2}) \right) \mathrm{d}S + \int_\Omega \langle t_i u_i \rangle \mathrm{d}\Omega \tag{58}$$

We have assumed that the 3D strain field is small as we are working within the framework of linearity elasticity, *i.e.*, $\hat{\epsilon} = O(\varepsilon_{ij}) \ll 1$ with $\hat{\epsilon}$ denoting the characteristic magnitude of the 3D strain field. From Eqs. (56), we can conclude that

$$u_i = O(L\hat{\epsilon}) \tag{59}$$

The 2D strain energy density will be of the order of $\bar{\mu}h\hat{\epsilon}^2$ with $\bar{\mu}$ denoting the order of the elastic constants. The condition of the boundedness of deformations for $h/L \to 0$ puts some constraints on the order of the external forces. It is clear that the work done must be of the same order as the strain energy, *i.e.*, $f_i u_i h \sim t_i u_i \sim \bar{\mu}h\hat{\epsilon}^2$. In view of Eq. (59), we have

$$f_i h \sim t_i \sim \bar{\mu}\frac{h}{L}\hat{\epsilon} \tag{60}$$

Substituting the strain field in Eq. (56) into the total potential energy of the original structure in Eq. (57) and dropping smaller terms, we obtain:

$$2\Pi = \left\langle G u_{1,3}^2 + G u_{2,3}^2 + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{3,3}^2 \right\rangle \tag{61}$$

Note these kept terms, in the order of $\bar{\mu}L^2\hat{\epsilon}^2$, are much larger than those neglected in the strain energy and in the work done. The behavior of the structure is governed by the principle of minimum total potential energy. The quadratic form in Eq. (61) will reach its absolute minimum zero if the following conditions can be satisfied:

$$u_{1,3} = u_{2,3} = u_{3,3} = 0 \tag{62}$$

which has the following solution

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1, x_2) \tag{63}$$
$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1, x_2) \tag{64}$$
$$u_3(x_1, x_2, x_3) = \bar{u}_3(x_1, x_2) \tag{65}$$

where $\bar{u}_i$ are arbitrary unknown 2D functions of in-plane coordinates x_1, x_2. Although we have found expressions for the 3D displacement field in terms of 2D functions of x_1, x_2, we are not sure whether we have included all the terms corresponding to the classical plate model yet. We need to continue our variational asymptotic procedure by perturbing the displacement field such that

$$\begin{aligned} u_1(x_1, x_2, x_3) &= \bar{u}_1(x_1, x_2) + v_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) &= \bar{u}_2(x_1, x_2) + v_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) &= \bar{u}_3(x_1, x_2) + v_3(x_1, x_2, x_3) \end{aligned} \tag{66}$$

with v_i asymptotically smaller than $\bar{u}_i$. Because $\bar{u}_i$ are three arbitrary functions, for definiteness of the expression in Eq. (66), we need to introduce three constraints for newly introduced 3D functions v_i. The choice of three constraints is directly related with how we define the three 2D functions $\bar{u}_i(x_1, x_2)$ in terms of the 3D displacement field $u_i(x_1, x_2, x_3)$. If we choose the constraints as

$$\langle v_i \rangle = 0 \tag{67}$$

It implies the following definitions of $\bar{u}_i(x_1, x_2)$ in terms of 3D displacements as

$$h\bar{u}_i(x_1, x_2) = \langle u_i(x_1, x_2, x_3) \rangle \tag{68}$$

That is, we define the 2D plate displacements $\bar{u}_i$ as the average of the corresponding 3D displacements u_i through the thickness. Substituting this displacement field in Eq. (66) into Eq. (56), we can obtain the following 3D strain field as

$$\begin{aligned} \varepsilon_{11} &= \epsilon_{11} + v_{1,1} \\ 2\varepsilon_{12} &= 2\epsilon_{12} + v_{1,2} + v_{2,1} \\ 2\varepsilon_{13} &= \bar{u}_{3,1} + v_{1,3} + v_{3,1} \\ \varepsilon_{22} &= \epsilon_{22} + v_{2,2} \\ 2\varepsilon_{23} &= \bar{u}_{3,2} + v_{2,3} + v_{3,2} \\ \varepsilon_{33} &= v_{3,3} \end{aligned} \tag{69}$$

Here we let $\epsilon_{\alpha\beta} = \frac{1}{2}(\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha})$ as we introduced previously for the definition of in-plane plate strains.

Substituting the displacement field in Eqs. (66) and the 3D strain field in Eqs. (69) into the total potential energy of the original 3D structures in Eq. (57) and dropping smaller terms, we have

$$\begin{aligned} 2\Pi = & \left\langle G(\bar{u}_{3,1} + v_{1,3})^2 + G(\bar{u}_{3,2} + v_{2,3})^2 + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(v_{3,3} + \rho\epsilon_{\alpha\alpha})^2 \right\rangle \\ & - \int_S p_i \bar{u}_i \, \mathrm{d}S + \int_\Omega P_i \bar{u}_i \mathrm{d}\Omega \end{aligned} \tag{70}$$

The load related terms p_i and P_i are defined the same as Eq. (36). The v_i related terms in Eq. (70) will reach the absolute minimum value zero if the following conditions are satisfied:

$$\bar{u}_{3,1} + v_{1,3} = 0 \tag{71}$$

$$\bar{u}_{3,2} + v_{2,3} = 0 \tag{72}$$

$$v_{3,3} + \rho\epsilon_{\alpha\alpha} = 0 \tag{73}$$

which has the following solution

$$v_\alpha = -x_3 \bar{u}_{3,\alpha} \qquad v_3 = -x_\alpha \rho \epsilon_{\alpha\alpha} \tag{74}$$

with the unknown functions of x_1 and x_2 can be absorbed into $\bar{u}_i(x_1, x_2)$. If we choose the origin of x_3 located at the mid-plane, the constraints in Eq. (67) are also satisfied. Otherwise, some functions of x_1, x_2 (constants as far as x_3 is concerned) should be introduced to satisfy the constraints in Eq. (67).

Substituting the solutions for v_i in Eqs. (74) into Eq. (66), we can express the 3D displacement field as

$$\begin{aligned} u_1 &= \bar{u}_1(x_1, x_2) - x_3 \bar{u}_{3,1} \\ u_2 &= \bar{u}_2(x_1, x_2) - x_3 \bar{u}_{3,2} \\ u_3 &= \bar{u}_3(x_1, x_2) - x_3 \rho \epsilon_{\alpha\alpha} \end{aligned} \tag{75}$$

Now, we know that the asymptotical expansion of the 3D displacement field will be spanned by $\bar{u}_i$ as no new degrees of freedom will appear according to the variational asymptotic method. However, we are still not sure whether we have included all the orders needed for the classical plate model. For this purpose, we perturb the displacement field one more time such that

$$\begin{aligned} u_1 &= \bar{u}_1(x_1, x_2) - x_3 \bar{u}_{3,1} + w_1(x_1, x_2, x_3) \\ u_2 &= \bar{u}_2(x_1, x_2) - x_3 \bar{u}_{3,2} + w_2(x_1, x_2, x_3) \\ u_3 &= \bar{u}_3(x_1, x_2) - x_3 \rho \epsilon_{\alpha\alpha} + w_3(x_1, x_2, x_3) \end{aligned} \tag{76}$$

with the constraints on v_i passed onto w_i following the same reasoning we have used for obtaining Eq. (67). That is we have

$$\langle w_i \rangle = 0 \tag{77}$$

The 3D strain field corresponding to the displacement field in Eq. (76) is

$$
\begin{aligned}
\varepsilon_{11} &= \epsilon_{11} + x_3\kappa_{11} + w_{1,1} \\
2\varepsilon_{12} &= 2\epsilon_{12} + 2x_3\kappa_{12} + w_{1,2} + w_{2,1} \\
2\varepsilon_{13} &= -x_3\rho\epsilon_{\alpha\alpha,1} + w_{1,3} + w_{3,1} \\
\varepsilon_{22} &= \epsilon_{22} + x_3\kappa_{22} + w_{2,2} \\
2\varepsilon_{23} &= -x_3\rho\epsilon_{\alpha\alpha,2} + w_{2,3} + w_{3,2} \\
\varepsilon_{33} &= -\rho\epsilon_{\alpha\alpha} + w_{3,3}
\end{aligned}
\tag{78}
$$

Here we let $\kappa_{\alpha\beta} = -\bar{u}_{3,\alpha\beta}$ as we defined previously. Clearly from these equations, we can estimate that $\epsilon_{\alpha\beta} \sim h\kappa_{\alpha\beta} \sim \hat{\epsilon}$.

Substituting the displacement field in Eqs. (76) and the 3D strain field in Eqs. (78) into the total potential energy of the original 3D structures in Eq. (57) and dropping smaller terms, we have

$$
\begin{aligned}
2\Pi = &\left\langle G w_{1,3}^2 + G w_{2,3}^2 + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(w_{3,3} + x_3\rho\kappa_{\alpha\alpha})^2 \right\rangle \\
&- \int_S p_i\bar{u}_i + q_1\bar{u}_{3,2} - q_2\bar{u}_{3,1}\,\mathrm{d}S + \int_\Omega P_i\bar{u}_i + Q_1\bar{u}_{3,2} - Q_2\bar{u}_{3,1}\mathrm{d}\Omega
\end{aligned}
\tag{79}
$$

with q_α, Q_α defined the same as those in Eqs. (36). The minimization of this functional in Eq. (79) will be reached by the following conditions:

$$
\begin{aligned}
w_1 &= w_2 = 0 \\
w_{3,3} &+ x_3\rho\kappa_{\alpha\alpha} = 0
\end{aligned}
$$

which can be solved along with the constraints in Eq. (77), yielding

$$
\begin{aligned}
w_1 &= w_2 = 0 \\
w_3 &= -\frac{1}{2}\rho\kappa_{\alpha\alpha}(x_3^2 - \frac{h^2}{12})
\end{aligned}
\tag{80}
$$

Now we have obtained for all the contributions to the classical plate model and it can be easily verified that any further perturbation will not add any major terms to this plate model as far as the total potential energy of the structure is concerned.

Substituting Eq. (80) into Eq. (76), we obtain the complete 3D displacement field of the classical plate model according to the variational asymptotic method as

$$
\begin{aligned}
u_1 &= \bar{u}_1(x_1, x_2) - x_3\bar{u}_{3,1} \\
u_2 &= \bar{u}_2(x_1, x_2) - x_3\bar{u}_{3,2} \\
u_3 &= \bar{u}_3(x_1, x_2) - \underline{\rho\left(x_3\epsilon_{\alpha\alpha} + \frac{1}{2}(x_3^2 - \frac{h^2}{12})\kappa_{\alpha\alpha}\right)}
\end{aligned}
\tag{81}
$$

Comparing to the displacement field based on the Kirchhoff assumptions in Eqs. (9), the variational asymptotic method obtained additional terms which are underlined in

Eq. (81). In other words, for plates made of a single isotropic material, the first Kirchhoff assumption is not valid, *i.e.*, the transverse normal can deform in its own direction. As far as the classical plate model for an isotropic homogeneous plate is concerned, the other two Kirchhoff assumptions are still valid.

Substituting the solutions for w_i in Eqs. (80) into Eqs. (78) and dropping the terms smaller than the order of $\hat{\epsilon}$, the complete 3D strain field of the classical plate model according to the variational asymptotic method is

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} \\ 2\varepsilon_{13} &= 2\varepsilon_{23} = 0 \\ \varepsilon_{33} &= -\rho(\epsilon_{\alpha\alpha} + x_3\kappa_{\alpha\alpha}) \end{aligned} \tag{82}$$

Comparing to the strain field obtained based on the Kirchhoff assumptions in Eq. (13), ε_{33} is different.

The complete stress field using the Hooke's law will be

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) \\ \sigma_{22} &= \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) \\ \sigma_{12} &= 2G\varepsilon_{12} \\ \sigma_{33} &= 0 \\ \sigma_{13} &= \sigma_{23} = 0 \end{aligned} \tag{83}$$

which is the same as those we assumed previously in Eq. (18) in the ad hoc approaches, although none of the assumptions has been used in obtaining this.

Substituting the solutions for w_i into Eq. (79), we will obtain the potential energy of the classical plate model and carry out the variation will result in the same variational statement as that in Eq. (40), which implies we will have the same 2D constitutive relations as those in Eq. (22), the same 2D governing different equations as those in Eqs. (24), (25), and (29), and the same boundary conditions as those derived using the variational methods. In other words, the plate behavior for an isotropic homogeneous elastic plate using the classical plate model will be the same no matter whether the equations are derived using the ad hoc approaches, the Newtonian method or the variational method, or the variational asymptotic method. The differences of the variational asymptotic method is that the 3D displacement field and 3D strain field will be different and also the theory derived using the variational asymptotic method is self consistent.

3.1 A shortcut for the variational asymptotic derivation

We have used three perturbations to derive the classical plate model. A shortcut is possible for us to derive the same model using one perturbation, which is what we adopted in the formulation of VAPAS, a general-purpose code for modeling composite plates.

To construct 2D classical plate model, the 3D displacement field must be expressed in terms of the three function $\bar{u}_i(x_1, x_2)$. Let us introduce the following change of variables

$$
\begin{aligned}
u_1 &= \underline{\bar{u}_1(x_1, x_2) - x_3 \bar{u}_{3,1}} + w_1(x_1, x_2, x_3) \\
u_2 &= \underline{\bar{u}_2(x_1, x_2) - x_3 \bar{u}_{3,2}} + w_2(x_1, x_2, x_3) \\
u_3 &= \underline{\bar{u}_3(x_1, x_2)} + w_3(x_1, x_2, x_3)
\end{aligned} \tag{84}
$$

The underline terms can be understood as the displacements introduced by the deformation of the plate reference surface in terms of $\bar{u}_i(x_1, x_2)$ if one assumes that the transverse normal is not deformable. The reality that the transverse normal is deformable will be captured by w_i which are called generalized warping functions as the transverse normal can deform both in-plane and out-of-plane which are asymptotically smaller than those underlined terms. Although w_i are not the same as those used in Eqs. (76), the constraints in Eq. (77) can be used if we define $\bar{u}_i$ according to the following definitions:

$$
\begin{aligned}
h\bar{u}_3(x_1, x_2) &= \langle u_3 \rangle \\
h\bar{u}_\alpha(x_1, x_2) &= \langle u_\alpha(x_1, x_2, x_3) \rangle + \langle x_3 \rangle \bar{u}_{3,\alpha}
\end{aligned} \tag{85}
$$

The 3D strain field corresponding to the displacement field in Eq. (84) is

$$
\begin{aligned}
\varepsilon_{11} &= \epsilon_{11} + x_3 \kappa_{11} + w_{1,1} \\
2\varepsilon_{12} &= 2\epsilon_{12} + 2x_3 \kappa_{12} + w_{1,2} + w_{2,1} \\
2\varepsilon_{13} &= w_{1,3} + w_{3,1} \\
\varepsilon_{22} &= \epsilon_{22} + x_3 \kappa_{22} + w_{2,2} \\
2\varepsilon_{23} &= w_{2,3} + w_{3,2} \\
\varepsilon_{33} &= w_{3,3}
\end{aligned} \tag{86}
$$

Substituting the displacement field in Eqs. (84) and the 3D strain field in Eqs. (86) into the total potential of the original 3D structures in Eq. (57) and dropping smaller terms, we have

$$
\begin{aligned}
2\Pi = &\left\langle G w_{1,3}^2 + G w_{2,3}^2 + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(w_{3,3} + \rho(\epsilon_{\alpha\alpha} + x_3 \kappa_{\alpha\alpha})^2 \right) \right\rangle \\
&- \int_S p_i \bar{u}_i + q_1 \bar{u}_{3,2} - q_2 \bar{u}_{3,1} \, \mathrm{d}S + \int_\Omega P_i \bar{u}_i + Q_1 \bar{u}_{3,2} - Q_2 \bar{u}_{3,1} \mathrm{d}\Omega
\end{aligned} \tag{87}
$$

The warping functions that minimize the above energy functional are given by

$$
w_1 = w_2 = 0 \qquad w_3 = -\rho \left(x_3 \epsilon_{\alpha\alpha} + \frac{1}{2}(x_3^2 - \frac{h^2}{12}) \kappa_{\alpha\alpha} \right) \tag{88}
$$

Substituting the solutions for w_i into Eq. (84), we obtain the same displacement as Eq. (81). Substituting the solutions for w_i into Eq. (86), we obtain the same strain field as Eq. (82). Using the 3D Hooke's law, we will obtain the same stress field as in Eq. (83). In other words, we obtained the same solution for relating the original 3D elasticity to the classical plate model as we derived previously using three perturbations in the previous section in a much quicker way.

4 Problems