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A nonlinear generalization of the Koiter–Sanders–Budiansky bending strain measure

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Abstract

A physically motivated kinematical property to be demanded of a bending strain measure, beyond the ones that are conventionally agreed upon, is proposed. Guided by the criterion, a bending strain measure is proposed for a first-order, nonlinear, elastic shell theory that has the property of vanishing in rigid and pure-stretch deformations of the shell. The measure is motivated by the Koiter–Sanders–Budiansky (KSB) bending measure of the first-order, linear shell theory, and is shown to linearize to the KSB measure about the undeformed shell. On retaining the standard measure for membrane straining, work-conjugate stress and stress-couple resultants to the membrane and bending strains are derived, as is the expression for the internal virtual work in terms of the proposed strains, stress and stress-couple resultants. The bending strain is calculated for some simple deformations and compared with other generalizations of the KSB measure of the linear theory. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Shell; Bending strain; Polar decomposition; Koiter–Sanders–Budiansky

1. Introduction

It is perhaps fair to say that unlike the direct functional relationship that exists between the deformed length of a curve, the curve itself and the Right Cauchy-Green deformation tensor ($\mathbf{f}^T \cdot \mathbf{f}$; $\mathbf{f}$ being the deformation gradient), there does not exist as precise a connection between the conventional measures of bending used in shell theories and the physical notion of bending itself. The difficulty seems to lie in formulating a precise statement of the physical notion of bending. The truth of the above statement is borne out in the fact that if a cylindrical or spherical surface were to be subjected to a uniform radial expansion, a deformation that is intuitively attributed to a 'stretching' of the surface and not to

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'bending', the classical bending strain measure characterized by the change in the second fundamental form is affected by the mid-surface stretching inherent in such a deformation mode and does not vanish in such deformations. However, in the context of a first-order linear theory of shells, the bending measure proposed by Koiter (1960), Sanders (1959) and Budiansky and Sanders (1963) possesses the property of vanishing in deformations characterized by a uniform normal deflection of the reference shell (Niordson, 1985) up to the accuracy inherent in a theory employing linear kinematic measures, apart from having other desirable properties, as pointed out in Budiansky and Sanders (1963). As will be shown in this paper, a shell deformation characterized by a uniform normal deflection can be shown to correspond to a non-bending deformation in a fairly precise way.

In this paper, an effort is made to formalize the physical notion of bending of a surface so that kinematical necessary conditions to be demanded of a bending strain measure can be laid down. The intention is not to prescribe conditions that make the choice of the bending strain measure unique — indeed, guided by the situation regarding strain and deformation measures in the three dimensional theory, there is no good reason to do so, since a given *physically reasonable* constitutive equation in terms of one strain measure can always be transformed to produce a form appropriate for another strain measure simply by substitution of kinematical entities. However, even in the three dimensional theory, there are some necessary conditions, based on the physical notion the strain or deformation measure represents, that are satisfied by all such measures, e.g. vanishing or reducing to its reference evaluation (which obviously does not depend on the deformation) in rigid deformations. Satisfying such necessary conditions is important from the point of view of constitutive assumptions, especially in theories where exact mid-surface kinematics is used so that properly invariant constitutive assumptions need not be considered as linearization of some nonlinear statement, and the theory is complete in and of itself. Such conditions can also make clear the essential coupling that exists between the existence of bending moments and purely membrane deformations of a curved shell. For instance, a radial expansion of a cylindrical shell cannot be associated in any reasonable way with a bending deformation of the shell. However, it is impossible to deny that such a deformation necessarily results in the occurrence of bending moments. To state such observations precisely, it is absolutely essential to have a clear kinematic definition of a bending deformation of a shell. One of the goals of this work is to produce such a set of requirements for a bending strain measure for a first-order, nonlinear shell theory.

Based on these criteria, two bending strain measures for a first-order, nonlinear, elastic shell theory are proposed. The measures possess the property of vanishing in rigid and pure-stretch deformations of arbitrary magnitude of the shell midsurface, thus generalizing the KSB measure. The result is based on an examination of the polar decomposition of the deformation gradient, when viewed as a two-point tensor field between two non-trivial manifolds, as would be the case for a curved shell. One of these measures is discussed in detail since it has many similarities with the KSB measure of the linear theory. It is shown that the proposed bending strain measure, when linearized about the reference geometry, is identical to the KSB measure of the linear theory. The work conjugate stress and stress-couple resultants, corresponding to the membrane strain characterized by the change in the first fundamental form and the proposed bending strain measure are also derived, as is an expression for the internal virtual work solely in terms of these measures. The stress and stress-couple resultants, when evaluated at the reference geometry, reproduce the corresponding KSB measures of the linear theory. Constitutive equations for the general modified stress and stress-couple resultants in terms of the derived membrane and bending strain measure, along with the statement of internal virtual work can form the basic ingredients for a numerical implementation of the theory via the finite element method. Finally, some elementary deformations are considered to illustrate the predictions of the proposed bending strain measure. Koiter (1966) and Budiansky (1968) have presented bending strain measures for a first-order, finite deformation elastic shell theory, which linearize to the KSB measure. These measures are compared with the measure proposed in this paper in the context of a biaxial stretching of a cylinder.

2. Geometry, notation and terminology

A shell mid-surface is thought of as a 2D surface in ambient 3D space (the qualification 'mid-surface' will not be used in all instances — it is hoped that the meaning will be clear from the context). Both the reference and deformed shell are parametrized by the same coordinate system $\{\xi^\alpha\}, \alpha = 1, 2$ (convected coordinates). Points on the reference geometry are denoted generically by $\mathbf{X}$ and on the deformed geometry by $\mathbf{x}$. The reference unit normal is denoted by $\mathbf{N}$ and the unit normal on the deformed geometry by $\mathbf{n}$. A subscript comma refers to partial differentiation, e.g. $\partial(\,)/\partial\xi^\alpha = (\,)_{,\alpha}$. Summation over repeated indices will be assumed. The metric tensors in the reference and deformed geometry will be denoted by $\mathbf{G}$ and $\mathbf{g}$, while the curvature tensors on the two geometries will be denoted by $\mathbf{B}$ and $\mathbf{b}$, respectively. Tensor indices will be raised or lowered with the use of $G^{\alpha\beta}$ and $G_{\alpha\beta}$, where these are the contravariant and covariant components of the tensor $\mathbf{G}$ and mutual inverses of each other. The convected coordinate basis vectors in the reference geometry will be referred to by the symbols $\{\mathbf{E}_\alpha\}$ and those in the deformed geometry by $\{\mathbf{e}_\alpha\}$, $\alpha = 1, 2$. A suitable number of dots placed between two tensors represent the operation of contraction, while the symbol $\otimes$ will represent a tensor product. Given a tensor $\mathbf{A}$ of any order, its derivative $\partial\mathbf{A}/\partial\mathbf{x}$ refers to the tensor product $(\partial\mathbf{A}/\partial\xi^\alpha)\otimes\mathbf{e}^\alpha$, with a similar interpretation for derivatives on the reference geometry.

If a property is said to hold 'locally' about some point, then the property holds in some neighborhood of the point. In referring to tangent spaces at a reference point and the deformed point, no distinction will be made if one can be obtained by a constant translation of the other, i.e. the distinction arising from the position of the base point will be ignored. A deformation will be considered as a smooth, orientation-preserving, one-to-one map.

3. Bending strain measure

In setting out to propose a bending strain measure, it is useful to collect some physical notions that would seem appropriate for a bending strain measure to embody. *First and foremost, being a strain measure, it should be a tensor that vanishes in rigid deformations. Secondly, it should be based on a 'proper' tensorial comparison of the deformed and undeformed curvature fields,* $\mathbf{b} = \partial\mathbf{n}/\partial\mathbf{x}$ *and* $\mathbf{B} = \partial\mathbf{N}/\partial\mathbf{X}$. *And thirdly, a vanishing bending strain at a point should be associated with any deformation that leaves the orientation of the unit normal field locally unaltered around that point.* The first two criteria are physically intuitive requirements that are conventionally adopted in shell theory. The third criterion is intended to incorporate the intuitive notion that bending at a point is associated with a change, under deformation, of the orientation of tangent planes in a neighborhood of that point. As is shown in the following, a sufficient condition for the third criterion to be satisfied is that the deformation of the shell be a pure stretch locally.

With the above criteria in mind, the classical bending strain measure

$$\mathbf{K} = K_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta = \left(\mathbf{x}_{,\alpha}\cdot\mathbf{n}_{,\beta} - \mathbf{X}_{,\alpha}\cdot\mathbf{N}_{,\beta}\right)\mathbf{E}^\alpha\otimes\mathbf{E}^\beta \tag{1}$$

is considered. Let $\mathbf{f} = \partial\mathbf{x}/\partial\mathbf{X}$ be the deformation gradient. Then, the components $K_{\alpha\beta}$ can also be expressed as

$$K_{\alpha\beta} = \mathbf{E}_\alpha\cdot\mathbf{f}^{\mathrm{T}}\cdot\mathbf{n}_{,\beta} - \mathbf{E}_\alpha\cdot\mathbf{N}_{,\beta}, \tag{2}$$

from which it is apparent that the tensor $\mathbf{K}$ vanishes in rigid deformations. Defining $\mathbf{b}^* := \mathbf{f}^{\mathrm{T}}\cdot(\partial\mathbf{n}/\partial\mathbf{x})\cdot\mathbf{f}$, it is also clear that

$$\mathbf{K} = \mathbf{b}^* - \mathbf{B}. \tag{3}$$

In trying to accommodate the third criterion, it is noted that the components $K_{\alpha\beta}$ can also be expressed as,

$$K_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{r}^{\mathrm{T}} \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{N}_{,\beta}, \tag{4}$$

where $\mathbf{f} = \mathbf{r} \cdot \mathbf{U}$ is the right polar decomposition of the deformation gradient, and if it were possible to define a bending strain measure on the $\{\mathbf{E}^\alpha\}$ basis with components,

$$K'_{\alpha\beta} := \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{r}^{\mathrm{T}} \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{n}_{,\beta}, \tag{5}$$

then it is clear that at points where the deformation is a pure stretch, the matrix $K'_{\alpha\beta}$ would vanish. Moreover, the rotation tensor $\mathbf{r}$ is the identity on the tangent space at such points for pure stretch deformations which implies that the unit normal does not change orientation, a result that follows from the definition of the domain and range of $\mathbf{U}(\mathbf{X})$, which happen to be the reference tangent space at $\mathbf{X}$ (Marsden and Hughes, 1994). Consequently, a covariant tensor on the reference geometry with components $K'_{\alpha\beta}$ on the $\{\mathbf{E}^\alpha\}$ basis would satisfy the third criterion laid out above *if* it could be shown that the set of deformations that leave the orientation of tangent planes unaltered locally is identical to the set of deformations whose deformation gradient field around the point in question is a pure stretch field. However, before this issue is discussed, a more elementary problem is to be confronted. $\mathbf{U}(\mathbf{X})$ is a tensor on the tangent space at $\mathbf{X}$ onto itself and $\mathbf{n}_{,\beta}$ is a vector belonging to the tangent space of $\mathbf{x}(\mathbf{X})$ in the deformed geometry, so that, at first glance, $\mathbf{U} \cdot \mathbf{n}_{,\beta}$ does not make sense as a reasonable tensor operation. But this turns out to be an insignificant obstacle since, for reasons mentioned above,

$$\mathbf{n}(\mathbf{x}(\mathbf{X})) = \mathbf{N}(\mathbf{X}) \tag{6}$$

for all $\mathbf{X}$ in any neighborhood in which the deformation is a pure stretch, which implies that $\mathbf{n}_{,\alpha} = \mathbf{N}_{,\alpha}$ in that neighborhood. Hence, redefining a bending strain matrix as

$$\tilde{K}_{\alpha\beta} := \mathbf{E}_\alpha \cdot \mathbf{f}^{\mathrm{T}} \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta} = \mathbf{x}_{,\alpha} \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta} \tag{7}$$

reveals that the tensor

$$\tilde{\mathbf{K}} := \mathbf{b}^* - \mathbf{U} \cdot \frac{\partial \mathbf{N}}{\partial \mathbf{X}} = \tilde{K}_{\alpha\beta} \mathbf{E}^\alpha \otimes \mathbf{E}^\beta \tag{8}$$

embodies all the three criteria that were defined at the outset to be desirable in a bending strain measure if locally pure stretch deformations are the only ones that leave the orientation of tangent planes unaltered locally under deformation.

The question now arises as to whether pure stretch deformations are the only ones that leave the orientation of the tangent plane unaltered. That this is not the case can be seen by considering deformations whose rotation tensor corresponds to a ‘drill’ rotation, i.e. a rotation tensor whose finite rotation vector is oriented along $\mathbf{N}$ (reflections on the tangent space being ruled out due to the restriction on deformations to satisfy the constraint of orientation preservation). Consequently, the bending strain measure $\tilde{\mathbf{K}}$ does not satisfy the third criterion for the class of all deformations. Moreover, the rotation tensor $\mathbf{r}$ cannot be uniquely factored into a ‘drill’ factor followed by another rotation, and hence incorporating its effect in $\tilde{\mathbf{K}}$, as was done with the stretch tensor $\mathbf{U}$, does not seem to be straightforward.

If the definition of a bending strain measure laid down earlier is adhered to strictly, then $\tilde{\mathbf{K}}$ does not qualify as a measure of bending in a first-order nonlinear shell theory, as would be the case for all other existing proposals known to the author. However, it is noted here that the set of deformations that leave the orientation of tangent planes unaltered locally can be divided into two classes — deformations that have a pure stretch deformation gradient locally, and those that have their local rotation tensor field consisting of either 'drill' rotations or the identity tensor. It has been shown that $\tilde{\mathbf{K}}$ vanishes in one of these subclasses (pure stretch) and it will be shown that the other existing proposals fail to do so in this subclass (Section 6.2). For the rest of the deliberations in this paper, the logical inconsistency between the third defining criterion for a bending measure and reference to $\tilde{\mathbf{K}}$ as a bending measure will simply be accepted for the lack of a better alternative.

Before proceeding further, it is also easy to see, based on the line of argument pursued in developing $\tilde{\mathbf{K}}$, that the quantity

$$\mathbf{E}_\alpha \cdot \mathbf{r}^{\mathrm{T}} \cdot \mathbf{n}_{,\beta} \mathbf{E}^\alpha \otimes \mathbf{E}^\beta \tag{9}$$

can serve as a measure of bending deformation, reducing to its reference value in the case of rigid and pure stretch deformations. An associated strain measure would be

$$(\mathbf{E}_\alpha \cdot \mathbf{r}^{\mathrm{T}} \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{N}_{,\beta})\mathbf{E}^\alpha \otimes \mathbf{E}^\beta,$$

which satisfies the requirements of a bending strain measure in a restricted sense, as outlined above for $\tilde{\mathbf{K}}$. It is also easily verified that all the above deformation and strain measures are invariant under superposed rigid body deformations. In the following, we only pursue the strain measure $\tilde{\mathbf{K}}$ in order to make connections with existing work in the literature.

With a view toward providing a bending strain measure for a first-order theory restricted by the Kirchhoff hypothesis, it is reasonable to demand a symmetric measure to effect a reduction in the number of stress-couple resultant components that enter the theory (Budiansky and Sanders, 1963; Sanders, 1963). With this motivation, a symmetrized measure

$$\check{\mathbf{K}} := \mathbf{b}^* - \frac{1}{2}\left(\mathbf{U} \cdot \frac{\partial \mathbf{N}}{\partial \mathbf{X}} + \frac{\partial \mathbf{N}}{\partial \mathbf{X}} \cdot \mathbf{U}\right) \equiv \check{K}_{\alpha\beta} \mathbf{E}^\alpha \otimes \mathbf{E}^\beta \tag{10}$$

is introduced, where

$$\check{K}_{\alpha\beta} = \mathbf{x}_{,\alpha} \cdot \mathbf{n}_{,\beta} - \frac{1}{2}(\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta} + \mathbf{N}_{,\alpha} \cdot \mathbf{U} \cdot \mathbf{E}_\beta). \tag{11}$$

If the deformation is a pure stretch locally then $\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta}$ is symmetric in α and β since it is identical to $\mathbf{x}_{,\alpha} \cdot \mathbf{n}_{,\beta}$ (which is symmetric, as can be demonstrated through a simple calculation using the identity $\mathbf{x}_{,\alpha} \cdot \mathbf{n} = 0$), and hence $\check{K}_{\alpha\beta}$ vanishes in this case.

4. Linearization of $\check{\mathbf{K}}$

A connection between the linearization of the measure $\check{\mathbf{K}}$ and the KSB measure of the linear theory, $\hat{\mathbf{K}}$, is established in this section. Henceforth the symbol δ shall represent a variation of its argument. In performing the linearization, the tensor will be considered as expressed in terms of components on the $\{\mathbf{E}^\alpha\}$ basis which is not deformation dependent. Consequently, it shall suffice to consider the linearization of only the components of the tensor. The normal $\mathbf{n}$ and its admissible variation, $\delta\boldsymbol{\theta}$, will be considered independent of $\mathbf{x}$ and $\delta\mathbf{x}$ in performing the linearization with the understanding that in the final result $\mathbf{n}$ is a function of the position mapping $\mathbf{x}$ and $\delta\boldsymbol{\theta}$ is a function of the mappings $\mathbf{x}$ and $\delta\mathbf{x}$ for a first-order shell theory.

The linearization of $\check{K}_{\alpha\beta}$, $\Lambda\check{K}_{\alpha\beta}$, at $\{\mathbf{X}, \mathbf{N}\}$ in the direction of an admissible variation $\{\delta\mathbf{x}, \delta\boldsymbol{\theta}\}$ is given by

$$\Lambda\check{K}_{\alpha\beta}(\mathbf{X}, \mathbf{N}; \delta\mathbf{x}, \delta\boldsymbol{\theta}) = \check{K}_{\alpha\beta}(\mathbf{X}, \mathbf{N}) + \delta\check{K}_{\alpha\beta}(\mathbf{X}, \mathbf{N}; \delta\mathbf{x}, \delta\boldsymbol{\theta}). \tag{12}$$

Now, $\check{K}_{\alpha\beta}(\mathbf{X}, \mathbf{N})=0$. Let $\mathbf{C}:=\mathbf{f}^{\mathrm{T}} \cdot \mathbf{f}$ be the Right Cauchy Green deformation tensor.
Then

$$\delta\check{K}_{\alpha\beta}(\mathbf{X}, \mathbf{N}; \delta\mathbf{x}, \delta\boldsymbol{\theta}) = \delta\mathbf{x}_{,\alpha} \cdot \mathbf{N}_{,\beta} + \mathbf{X}_{,\alpha} \cdot [\delta\boldsymbol{\theta} \times \mathbf{N}]_{,\beta} - \frac{1}{2}\left[\mathbf{E}_\alpha \cdot \left(\frac{\partial\mathbf{U}}{\partial\mathbf{C}}(\mathbf{G}):\delta\mathbf{C}(\mathbf{X}, \delta\mathbf{x})\right) \cdot \mathbf{N}_{,\beta} + \mathbf{N}_{,\alpha} \cdot \left(\frac{\partial\mathbf{U}}{\partial\mathbf{C}}(\mathbf{G}):\delta\mathbf{C}(\mathbf{X}, \delta\mathbf{x})\right) \cdot \mathbf{E}_\beta\right]. \tag{13}$$

From Hoger and Carlson (1984),

$$\frac{\partial\mathbf{U}}{\partial\mathbf{C}}(\mathbf{C}'):\delta\mathbf{C} = A_1^{C'}\mathbf{C}' \cdot \delta\mathbf{C} \cdot \mathbf{C}' - A_2^{C'}\{\mathbf{C}' \cdot \delta\mathbf{C} + \delta\mathbf{C} \cdot \mathbf{C}'\} + A_3^{C'}\delta\mathbf{C} \tag{14}$$

where

$$A_1^{C'} := \frac{\left(I_{C'} + 2\sqrt{II_{C'}}\right)^{-\frac{3}{2}}}{2\sqrt{II_{C'}}}; \quad A_2^{C'} := -A_1^{C'}\left(I_{C'} + \sqrt{II_{C'}}\right);$$
$$A_3^{C'} := A_1^{C'}\left(I_{C'}^2 + 3I_{C'}\sqrt{II_{C'}} + 3II_{C'}\right); \tag{15}$$

$$I_{C'} := \mathrm{tr}(\mathbf{C}' \cdot \mathbf{G}^{\mathrm{T}}) = C'_{\alpha\beta}G^{\alpha\beta}; \quad II_{C'} := \det(\mathbf{C}') = \frac{(\mathbf{x}_{,1} \times \mathbf{x}_{,2}) \cdot (\mathbf{x}_{,1} \times \mathbf{x}_{,2})}{(\mathbf{X}_{,1} \times \mathbf{X}_{,2}) \times (\mathbf{X}_{,1} \times \mathbf{X}_{,2})}. \tag{16}$$

Consequently,

$$\frac{\partial\mathbf{U}}{\partial\mathbf{C}}(\mathbf{G}):\delta\mathbf{C} = \frac{1}{2}\delta\mathbf{C}. \tag{17}$$

Now $\mathbf{C} = \mathbf{x}_{,\alpha} \cdot \mathbf{x}_{,\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$ implies $\delta\mathbf{C}(\mathbf{X}, \delta\mathbf{x}) = (\delta\mathbf{x}_{,\alpha} \cdot \mathbf{X}_{,\beta} + \mathbf{X}_{,\alpha} \cdot \delta\mathbf{x}_{,\beta})\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$. Therefore, on defining $B_\alpha^\gamma := \mathbf{N}_{,\alpha} \cdot \mathbf{E}^\gamma$, $\mathbf{E}_\alpha \cdot (\frac{\partial\mathbf{U}}{\partial\mathbf{C}}(\mathbf{G}):\delta\mathbf{C}) \cdot \mathbf{N}_{,\beta} = \frac{1}{2}(\delta\mathbf{x}_{,\alpha} \cdot \mathbf{X}_{,\delta} + \mathbf{X}_{,\alpha} \cdot \delta\mathbf{x}_{,\delta})B_\beta^\delta$. Hence,

$$\Lambda\check{K}_{\alpha\beta}(\mathbf{X}, \mathbf{N}; \delta\mathbf{x}, \delta\boldsymbol{\theta}) = \delta\mathbf{x}_{,\alpha} \cdot \mathbf{N}_{,\beta} + \mathbf{X}_{,\alpha} \cdot [\delta\boldsymbol{\theta} \times \mathbf{N}]_{,\beta} - \frac{1}{2}\left\{\frac{1}{2}(\delta\mathbf{x}_{,\alpha} \cdot \mathbf{X}_{,\delta} + \mathbf{X}_{,\alpha} \cdot \delta\mathbf{x}_{,\delta})B_\beta^\delta + \frac{1}{2}(\delta\mathbf{x}_{,\delta} \cdot \mathbf{X}_{,\beta} + \mathbf{X}_{,\delta} \cdot \delta\mathbf{x}_{,\beta})B_\alpha^\delta\right\} \tag{18}$$

Defining the symmetric part of the displacement gradient tensor as $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{d}_{,\alpha} \cdot \mathbf{E}_\beta + \mathbf{E}_\alpha \cdot \mathbf{d}_{,\beta})\mathbf{E}^\alpha \otimes \mathbf{E}^\beta =: \varepsilon_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$, where $\mathbf{d}$ is the displacement field, and $\boldsymbol{\vartheta}$ the linearized rotation vector corresponding to the orthogonal transformation that orients the normal to the shell, it is found that

$$\Lambda\check{K}_{\alpha\beta}(\mathbf{X},\mathbf{N};\mathbf{d},\boldsymbol{\vartheta}) = \mathbf{d}_{,\alpha}\cdot\mathbf{N}_{,\beta} + \mathbf{E}_\alpha\cdot(\boldsymbol{\vartheta}\times\mathbf{N})_{,\beta} - \frac{1}{2}\left(\varepsilon_{\alpha\delta}B^\delta_\beta + \varepsilon_{\alpha\beta}B^\delta_\alpha\right)$$
$$\equiv \check{K}_{\alpha\beta}\ \text{(KSB bending strain components)}, \tag{19}$$

thus establishing the fact that $\check{\mathbf{K}}$ can indeed be legitimately considered a nonlinear generalization of the KSB bending strain measure.

5. Work conjugate stress measures

In terms of the stress resultant tensor, **s**, and stress-couple resultant tensor, **m**, on the deformed geometry, tensors of membrane stress and bending moment,

$$\mathbf{N}_0 := j\mathbf{f}^{-1}\cdot\mathbf{s}\cdot\mathbf{f}^{-\mathrm{T}} \quad\text{and}\quad \mathbf{M}_0 := j\mathbf{f}^{-1}\cdot\mathbf{m}\cdot\mathbf{f}^{-\mathrm{T}}, \tag{20}$$

can be defined on the reference geometry. In the above, $j = \sqrt{II_C}$ defined in Eq. (16) above. If two new tensors of membrane and bending stress are now defined through the relations

$$\mathbf{N} := \mathbf{N}_0 - j\mathbf{f}^{-1}\cdot\mathbf{b}\cdot\mathbf{m}^{\mathrm{T}}\cdot\mathbf{f}^{-\mathrm{T}} \quad\text{and}\quad \mathbf{M} := \frac{1}{2}\left(\mathbf{M}_0 + \mathbf{M}_0^{\mathrm{T}}\right) \tag{21}$$

(Sanders, 1963; Fox, 1990) then it is well known that when the deformation of the shell is restricted by the Kirchhoff hypothesis, the three-dimensional statement of internal virtual work can be expressed as (Sanders, 1963; Budiansky, 1968),

$$\int_{A_0}\left(\mathbf{N}:\frac{1}{2}\delta\mathbf{C} + \mathbf{M}:\delta\mathbf{b}^*\right)\mathrm{d}A_0, \tag{22}$$

where A_0 is the two dimensional region occupied by the reference shell mid-surface.

Of course, an alternative way of expressing **C** is

$$\mathbf{C} \equiv \mathbf{f}^{\mathrm{T}}\cdot\mathbf{f} = \mathbf{f}^{\mathrm{T}}\cdot\left(\mathbf{x}_{,\alpha}\cdot\mathbf{x}_{,\beta}\mathbf{e}^\alpha\otimes\mathbf{e}^\beta\right)\cdot\mathbf{f} =: \mathbf{f}^{\mathrm{T}}\cdot\left(g_{\alpha\beta}\mathbf{e}^\alpha\otimes\mathbf{e}^\beta\right)\cdot\mathbf{f} =: \mathbf{g}^*. \tag{23}$$

It is clear from Eq. (22) that **N** and **M** are work-conjugate measures to the membrane strain measure $\mathbf{E} := \frac{1}{2}(\mathbf{C}-\mathbf{G})$ and the bending strain measure **K** defined in Eq. (3).

The work-conjugate stress pair to the strains $\{\mathbf{E},\check{\mathbf{K}}\}$ is now sought. In other words, tensors $\dot{\mathbf{N}}$ and $\dot{\mathbf{M}}$ need to be determined so that the relationship

$$\dot{\mathbf{N}}:\frac{1}{2}\delta\mathbf{C} + \dot{\mathbf{M}}:\delta\check{\mathbf{K}} = \mathbf{N}:\frac{1}{2}\delta\mathbf{C} + \mathbf{M}:\delta\mathbf{b}^* \tag{24}$$

is satisfied. In components with respect to the $\{\mathbf{E}_\alpha\}$ and $\{\mathbf{E}^\alpha\}$ bases, Eq. (24) reduces to

$$\dot{N}^{\alpha\beta}\frac{1}{2}\delta C_{\alpha\beta} + \dot{M}^{\alpha\beta}\delta\check{K}_{\alpha\beta} = N^{\alpha\beta}\frac{1}{2}\delta C_{\alpha\beta} + M^{\alpha\beta}\delta b^*_{\alpha\beta}. \tag{25}$$

From the definition of $\check{K}_{\alpha\beta}$ (11),

$$M^{\alpha\beta}\delta b^*_{\alpha\beta} = M^{\alpha\beta}\delta\check{K}_{\alpha\beta} + M^{\varepsilon\eta}\frac{1}{2}\left(\mathbf{E}_\varepsilon\cdot\delta\mathbf{U}\cdot\mathbf{N}_{,\eta} + \mathbf{E}_\eta\cdot\delta\mathbf{U}\cdot\mathbf{N}_{,\varepsilon}\right). \tag{26}$$

On performing somewhat tedious algebraic manipulations using Eqs. (14)–(16), it can be shown that

$$M^{\varepsilon\eta}\frac{1}{2}(\mathbf{E}_\varepsilon\cdot\delta\mathbf{U}\cdot\mathbf{N}_{,\eta}+\mathbf{E}_\eta\cdot\delta\mathbf{U}\cdot\mathbf{N}_{,\varepsilon}) = M^{\varepsilon\eta}\Pi^{\alpha\beta}_{\varepsilon\eta}\frac{1}{2}\delta C_{\alpha\beta}$$

where

$$\Pi^{\alpha\beta}_{\varepsilon\eta} = \frac{A^C_1}{2}\left(C^{\ \alpha}_{\varepsilon}C^{\ \beta}_{\nu}B^{\nu}_{\eta} + C^{\ \beta}_{\varepsilon}C^{\ \alpha}_{\nu}B^{\nu}_{\eta} + C^{\ \alpha}_{\eta}C^{\ \beta}_{\nu}B^{\nu}_{\varepsilon} + C^{\ \beta}_{\eta}C^{\ \alpha}_{\nu}B^{\nu}_{\varepsilon}\right) + \frac{A^C_3}{2}\left(G^{\ \alpha}_{\varepsilon}B^{\beta}_{\eta} + G^{\ \beta}_{\varepsilon}B^{\beta}_{\eta} + G^{\ \alpha}_{\eta}B^{\beta}_{\varepsilon} + G^{\ \beta}_{\eta}B^{\alpha}_{\varepsilon}\right)$$
$$- \frac{A^C_2}{2}\left(C^{\ \alpha}_{\varepsilon}B^{\beta}_{\eta} + C^{\ \beta}_{\varepsilon}B^{\alpha}_{\eta} + C^{\ \alpha}_{\eta}B^{\beta}_{\varepsilon} + C^{\ \beta}_{\eta}B^{\alpha}_{\varepsilon} + G^{\ \alpha}_{\varepsilon}C^{\beta\kappa}B_{\eta\kappa} + G^{\ \alpha}_{\eta}C^{\beta\kappa}B_{\varepsilon\kappa} + G^{\ \beta}_{\varepsilon}C^{\alpha\kappa}B_{\eta\kappa}\right.$$
$$\left. + G^{\ \beta}_{\eta}C^{\alpha\kappa}B_{\varepsilon\kappa}\right).$$

Consequently, from Eqs. (25) and (26) it can be seen that the pair

$$\dot{\mathbf{N}} := \left(N^{\alpha\beta} + M^{\varepsilon\eta}\Pi^{\alpha\beta}_{\varepsilon\eta}\right)\mathbf{E}_\alpha\otimes\mathbf{E}_\beta; \quad \dot{\mathbf{M}} := \mathbf{M} \tag{27}$$

is work-conjugate to the strain pair $\{\mathbf{E}, \check{\mathbf{K}}\}$, and $\dot{\mathbf{N}}$ and $\dot{\mathbf{M}}$ are both symmetric. As expected, if the reference configuration is considered to be the deformed geometry then $\dot{\mathbf{N}}$ is given by the expression

$$\left\{N^{\alpha\beta} + \frac{1}{2}\left(M^{\alpha\eta}B^{\beta}_{\eta} + M^{\beta\eta}B^{\alpha}_{\eta}\right)\right\}\mathbf{E}_\alpha\otimes\mathbf{E}_\beta,$$

which is exactly the expression for the KSB membrane stress in the linear theory. It is also clear at this point that the exact expression for the internal virtual work (22) can be written as

$$\int_{A_0}\left(\dot{\mathbf{N}}:\delta\mathbf{E} + \dot{\mathbf{M}}:\delta\check{\mathbf{K}}\right)\,\mathrm{d}A_0.$$

It should be noted here that even though the expression for $\dot{\mathbf{N}}$ is quite cumbersome, in actual applications it is replaced by a constitutive equation in terms of $\mathbf{E}$ and $\check{\mathbf{K}}$ so that it does not have to be dealt with directly. In the context of numerical solution procedures based on implicit finite element methods, the variation of $\check{K}_{\alpha\beta}$ would be required and is given by

$$\delta\check{K}_{\alpha\beta} = \delta b^*_{\alpha\beta} - \frac{1}{2}\Pi^{\varepsilon\eta}_{\alpha\beta}\delta C_{\varepsilon\eta} = \left\{\delta\mathbf{x}_{,\alpha}\cdot\mathbf{n}_{,\beta} + \mathbf{x}_{,\alpha}\cdot(\delta\boldsymbol{\theta}\times\mathbf{n})_{,\beta}\right\} - \frac{1}{2}\Pi^{\varepsilon\eta}_{\alpha\beta}\delta C_{\varepsilon\eta},$$

where $\delta\mathbf{x}$ and $\delta\boldsymbol{\theta}$ are admissible variations as defined in Section 4.

6. Examples

6.1. Uniform normal deflection

In this example, a deformation characterized by a uniform normal deflection of the reference shell is considered. It is shown that such a deformation indeed corresponds to a situation where the orientation of the tangent planes around any point are not altered, as is perhaps intuitively obvious. Hence, it would seem reasonable to demand that a bending strain measure vanish for such a deformation, and it is shown that the measure $\check{\mathbf{K}}$ does so vanish for most reasonable values of the magnitude of the deflection. In Niordson (1985, pp. 128–129), it is shown that the KSB measure also vanishes in such deformations, up to the accuracy of the linear theory. However, Niordson (1985, p. 129) states "it is difficult to find a sound physical reason for preferring a measure having this particular property...". It is hoped that the preceding sections of this paper and this example furnish a convincing physical reason for such a preference.

A deformation of the form

$$\mathbf{x}(\mathbf{X}) = \mathbf{X} + f\mathbf{N}(\mathbf{X}) \tag{28}$$

is considered, where f does not vary with $\mathbf{X}$. Since $\mathbf{N}_{,\alpha} \cdot \mathbf{N} = 0$, the deformation gradient $\mathbf{f}(\mathbf{X})$ given by

$$\mathbf{f}(\mathbf{X}) = \mathbf{x}_{,\alpha} \otimes \mathbf{E}^{\alpha} \tag{29}$$

maps the reference tangent space at $\mathbf{X}$ onto itself. Hence, $\mathbf{f}$ may be written in the form

$$\mathbf{f} = f_{\alpha\beta}\mathbf{E}^{\alpha} \otimes \mathbf{E}^{\beta} =: \mathbf{F}$$

where

$$F_{\alpha\beta} \equiv f_{\alpha\beta} = \mathbf{X}_{,\alpha} \cdot \mathbf{f} \cdot \mathbf{X}_{,\beta} = \mathbf{X}_{,\alpha} \cdot \mathbf{x}_{,\beta} = \mathbf{X}_{,\alpha} \cdot \mathbf{X}_{,\beta} + f\mathbf{X}_{,\alpha} \cdot \mathbf{N}_{,\beta} = \mathbf{X}_{,\alpha} \cdot \mathbf{X}_{,\beta} - f\mathbf{N} \cdot \mathbf{X}_{,\alpha\beta} \tag{30}$$

and hence $\mathbf{F}$ is symmetric. The matrix $F_{\alpha\beta}$ is given by

$$[F_{\alpha\beta}] = \begin{bmatrix} \mathbf{X}_{,1} \cdot \mathbf{X}_{,1} + f\mathbf{N}_{,1} \cdot \mathbf{X}_{,1} & \mathbf{X}_{,1} \cdot \mathbf{X}_{,2} + f\mathbf{N}_{,2} \cdot \mathbf{X}_{,1} \\ \mathbf{X}_{,2} \cdot \mathbf{X}_{,1} + f\mathbf{N}_{,1} \cdot \mathbf{X}_{,2} & \mathbf{X}_{,2} \cdot \mathbf{X}_{,2} + f\mathbf{N}_{,2} \cdot \mathbf{X}_{,2} \end{bmatrix}. \tag{31}$$

Because the orientation of the tangent planes to the shell is not altered by deformation, ideally the measure $\check{\mathbf{K}}$ should vanish without qualification. However, this is not the case as shown below.

Introducing local lines of curvature orthogonal coordinates, i.e. the coordinates $\{\xi^{\alpha}\}$ are chosen so that $(\partial\mathbf{N}/\partial\mathbf{X}) \cdot \mathbf{X}_{,\alpha} = \lambda_{\alpha}\mathbf{X}_{,\alpha}$ (no sum on α), where λ_{α} are the eigenvalues of the symmetric tensor $\partial\mathbf{N}/\partial\mathbf{X}$ and $\mathbf{X}_{,\alpha} \cdot \mathbf{X}_{,\beta} = 0$ if $\alpha \neq \beta$, the matrix $F_{\alpha\beta}$ can be brought to the form

$$[F_{\alpha\beta}] = \begin{bmatrix} \mathbf{X}_{,1} \cdot \mathbf{X}_{,1}(1 + f\lambda_1) & 0 \\ 0 & \mathbf{X}_{,2} \cdot \mathbf{X}_{,2}(1 + f\lambda_2) \end{bmatrix}, \tag{32}$$

and it is clear that only under certain circumstances is the symmetric tensor $\mathbf{F}$ positive-definite and hence a pure stretch (by the uniqueness of the polar decomposition). Some such important cases are:

1. If the deflection is small in the sense of line $|f\lambda_{\alpha}| < 1$, $\alpha = 1, 2$, i.e. the magnitude of the product of the deflection and the principal curvature of the reference shell is small. This is always the case within the assumptions of the linear theory.
2. If the reference shell is convex and $f > 0$ ('outward' deflection), or the reference shell is concave and $f < 0$ ('inward deflection').
3. If λ_1 and λ_2 do not have the same sign but $f\lambda_{\alpha} > -1$, $\alpha = 1, 2$.

In all of these cases the bending tensor $\check{\mathbf{K}}$ vanishes.

It is also pointed out here that in the cases when the deformation gradient is not a pure stretch it necessarily differs from one by a 'drill' rotation (reflections not being considered even though kinematically possible, a special case being a 'saddle-shaped' reference geometry with arbitrarily large values of f) since the overall deformation gradient is such that the orientation of the tangent planes is unaltered.

Another point to be noted is that the magnitude of f cannot be unrestricted for arbitrary reference geometries once physical deformations restricted by balance laws of linear and angular momentum and realistic constitutive equations are considered, since it is easily observed that unrestricted deformations of this class, for arbitrary reference geometries, can lead to self intersection of the shell.

6.2. Biaxial stretching of a cylinder

A quadrant of a cylindrical surface of radius R and length L is subjected to a stretch in the axial direction in the amount α and a uniform radial displacement of Δr (the geometry for this example is adapted from Fox (1990), where the case of uniform radial expansion is considered). A rectangular Cartesian basis $\{\mathbf{c}_i\}$, $i = 1, 3$ is used for ambient space. The axis of the cylinder is parallel to $\mathbf{c}_1$. The cylindrical surface is parametrized by the coordinates $\{\xi^\alpha\}$, $\alpha = 1, 2$, where $\xi^1 \in [0, L]$ and $\xi^2 \in [0, R]$. The reference and deformed position maps are given by

$$\mathbf{X} = \xi^1\mathbf{c}_1 + \xi^2\mathbf{c}_2 + \sqrt{R^2 - (\xi^2)^2}\,\mathbf{c}_3$$

$$\mathbf{x} = \alpha\xi^1\mathbf{c}_1 + \left(1 + \frac{\Delta r}{R}\right)\xi^2\mathbf{c}_2 + \left(1 + \frac{\Delta r}{R}\right)\sqrt{R^2 - (\xi^2)^2}\,\mathbf{c}_3; \quad \alpha > 0, \Delta r > 0. \tag{33}$$

Now,

$$\mathbf{X}_{,1} = \mathbf{c}_1; \qquad \mathbf{X}_{,2} = \mathbf{c}_2 - \frac{\xi^2}{\sqrt{R^2 - (\xi^2)^2}}\mathbf{c}_3,$$

$$\mathbf{x}_{,1} = \alpha\mathbf{X}_{,1}; \quad \mathbf{x}_{,2} = \left(1 + \frac{\Delta r}{R}\right)\mathbf{X}_{,2}, \tag{34}$$

which proves that the deformation gradient $\mathbf{F} = \mathbf{x}_{,\alpha} \otimes \mathbf{E}^\alpha$ maps the reference tangent space into itself at every point of the shell and hence the orientation of tangent planes remains unaltered under deformation, thus constituting a non-bending deformation. Hence, the deformation gradient can be written in the form

$$\mathbf{F} = F_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$$

where

$$F_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{F} \cdot \mathbf{E}_\beta = \mathbf{X}_{,\alpha} \cdot \mathbf{x}_{,\beta},$$

$$[F_{\alpha\beta}] = \begin{bmatrix} \alpha(\mathbf{X}_{,1} \cdot \mathbf{X}_1) & 0 \\ 0 & \left(1 + \frac{\Delta r}{R}\right)(\mathbf{X}_{,2} \cdot \mathbf{X}_2) \end{bmatrix}. \tag{35}$$

The deformation gradient is symmetric, positive-definite and, therefore, a pure stretch. Consequently, $\check{\mathbf{K}} = \check{K}_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$ vanishes in this deformation.

Next, the measure $\mathbf{K} = K_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$ is considered. From the above analysis, it is clear that $\mathbf{n} = \mathbf{N}$, which can also be directly confirmed from the expression $\mathbf{x}_{,1} \times \mathbf{x}_{,2} = \alpha(1 + \frac{\Delta r}{R})\mathbf{X}_{,1} \times \mathbf{X}_{,2}$. Therefore,

$$\mathbf{n}_{,1} = \mathbf{N}_{,1} = \mathbf{0}; \qquad \mathbf{n}_{,2} = \mathbf{N}_{,2} = \frac{1}{R}\mathbf{c}_2 - \frac{\xi^2}{R\sqrt{R^2 - (\xi^2)^2}}\mathbf{c}_3. \tag{36}$$

Now, $b^*_{\alpha\beta} = \mathbf{x}_{,\alpha} \cdot \mathbf{N}_{,\beta}$, $B_{\alpha\beta} = \mathbf{X}_{,\alpha} \cdot \mathbf{N}_{,\beta}$ and $K_{\alpha\beta} = b^*_{\alpha\beta} - B_{\alpha\beta}$ imply

$$[b^*_{\alpha\beta}] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{R + \Delta r}{R^2 - (\xi^2)^2} \end{bmatrix}; \quad [B_{\alpha\beta}] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{R}{R^2 - (\xi^2)^2} \end{bmatrix}; \quad [K_{\alpha\beta}] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{\Delta r}{R^2 - (\xi^2)^2} \end{bmatrix}. \tag{37}$$

It is noted that $K_{\alpha\beta}$ differ from 0 by a first order term in the normal displacement.

The nonlinear bending strain measure proposed by Koiter (1966) is considered next. The measure is given by the following expression;

$$\mathbf{K}'' = K''_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$$

where

$$K''_{\alpha\beta} = K_{\alpha\beta} - \frac{1}{2}\left(E_{\gamma\beta}B^\gamma_\alpha + E_{\gamma\alpha}B^\gamma_\beta\right). \tag{38}$$

Now, $G^{\alpha\beta} = (\mathbf{X}_{,\alpha} \cdot \mathbf{X}_{,\beta})^{-1}$; $E_{\alpha\beta} = \frac{1}{2}(C_{\alpha\beta} - G_{\alpha\beta})$; $B^\gamma_\alpha = B_{\alpha\mu}G^{\mu\gamma}$ imply

$$[G^{\mu\gamma}] = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{R^2 - (\xi^2)^2}{R^2} \end{bmatrix}; \quad [E_{\alpha\beta}] = \frac{1}{2}\begin{bmatrix} \alpha^2 - 1 & 0 \\ 0 & \dfrac{2\Delta r R + \Delta r^2}{R^2 - (\xi^2)^2} \end{bmatrix}; \quad [B^\gamma_\alpha] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{R} \end{bmatrix}$$
$$\left[K''_{\alpha\beta}\right] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{-\Delta r^2}{2R\left[R^2 - (\xi^2)^2\right]} \end{bmatrix}. \tag{39}$$

The components of Koiter's measure differ from 0 by second order terms in the normal displacement for this deformation, which means that in the context of a linear theory, it vanishes for such a deformation. In fact, it can be shown that Koiter's measure is also a valid generalization of the KSB measure of the linear theory, in the sense that its linearization is the KSB measure. However, in the context of exact kinematics, it does not vanish in non-bending deformations.

Finally, a nonlinear bending strain measure proposed by Budiansky (1968) is considered. The measure is given by the following expression;

$$\mathbf{K}''' = K'''_{\alpha\beta}\mathbf{E}^\alpha \otimes \mathbf{E}^\beta$$

where

$$K'''_{\alpha\beta} = jb^*_{\alpha\beta} - B_{\alpha\beta} - \frac{1}{2}\left(E_{\gamma\beta}B^\gamma_\alpha + E_{\gamma\alpha}B^\gamma_\beta\right) - E^\gamma_\gamma B_{\alpha\beta}; \; C \equiv \det(C_{\alpha\beta}); \quad G \equiv \det(G_{\alpha\beta}); \quad j \equiv \sqrt{\frac{C}{G}} \tag{40}$$

Now,

$$j=\alpha\left(1+\frac{\Delta r}{R}\right);\quad E^{\gamma}_{\gamma}=\frac{1}{2}\left\{(\alpha^2-1)+\frac{2\Delta r}{R}+\left(\frac{\Delta r}{R}\right)^2\right\}. \tag{41}$$

Consequently,

$$\left[K'''_{\alpha\beta}\right]=\begin{bmatrix}0 & 0\\ 0 & R\left(\alpha-\frac{\alpha^2}{2}-\frac{1}{2}\right)+\left(2\Delta r+\frac{\Delta r^2}{R}\right)(\alpha-1)\end{bmatrix}, \tag{42}$$

and it is interesting to note that for $\alpha = 1$ (radial expansion) $K'''_{\alpha\beta}$ vanishes. Defining $u^1 := \alpha\xi^1 - \xi^1$,

$$(\alpha-1)=\frac{u^1}{\xi^1};\qquad \left(\alpha-\frac{\alpha^2}{2}-\frac{1}{2}\right)=-\frac{(u^1)^2}{(\xi^1)^2}. \tag{43}$$

Consequently, the components of Budiansky's measure differ from zero by second order terms in the displacement. As shown by Budiansky (1968), the measure also linearizes to the KSB measure of the linear theory and, in that sense, is a valid generalization of the KSB measure.

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