Assume that matter moves with velocity $\underset{\sim}{v} + \underset{\sim}{V}^m$.

Assume body moves with velocity $\underset{\sim}{v}$.

The balance laws are valid for all sub-parts of the body.

density corresponds to density of matter in body.

kinetic energy & internal energy, linear & angular momentum are attributes of matter.

Balance of Mass:-

$$\boxed{\frac{d}{dt}\int_{B(t)} \rho \, dv = -\int_{\partial B(t)} \rho \underset{\sim}{V}^m \cdot \underset{\sim}{n} \, da}$$

$$\Rightarrow \boxed{\dot{\rho} + \rho \, \mathrm{div}\, \underset{\sim}{v} = - \mathrm{div}\left(\rho \underset{\sim}{V}^m\right)}$$

Balance of Linear Momentum:

$$\boxed{\frac{d}{dt}\int_{B(t)} \rho\left(\underset{\sim}{v} + \underset{\sim}{V}^m\right) dv = \int_{\partial B(t)} -\rho\left(\underset{\sim}{v} + \underset{\sim}{V}^m\right) \otimes \underset{\sim}{V}^m \cdot \underset{\sim}{n} \, da + \int_{\partial B(t)} \left(I + \underset{\sim}{J}\right) \underset{\sim}{n} \, da}$$

$$\Rightarrow \boxed{\left(\dot{\rho} + \rho \, \mathrm{div}\, \underset{\sim}{v}\right)\left(\underset{\sim}{v} + \underset{\sim}{V}^m\right) + \rho \, \dot{\overline{\left(\underset{\sim}{v} + \underset{\sim}{V}^m\right)}} = \mathrm{div}\left(I + \underset{\sim}{J}\right) - \mathrm{div}\left(\left(\underset{\sim}{v} + \underset{\sim}{V}^m\right) \otimes \rho \underset{\sim}{V}^m\right)}$$

Balance of Angular Momentum:

$$\boxed{\frac{d}{dt}\int_{B(t)} \mathbf{x} \times \rho(\mathbf{v} + \mathbf{V}^m)\,dv = \int_{\partial B(t)} \mathbf{x} \times (\underline{I}+\underline{J})\mathbf{n}\,da - \int_{\partial B(t)} \mathbf{x} \times \rho(\mathbf{v}+\mathbf{V}^m)\otimes\mathbf{V}^m\cdot\mathbf{n}\,da}$$

$$\Rightarrow (\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v})\left[\mathbf{x}\times(\mathbf{v}+\mathbf{V}^m)\right] + \rho\,\mathbf{v}\times(\mathbf{v}+\mathbf{V}^m) + \rho\,\mathbf{x}\times\overline{(\mathbf{v}+\mathbf{V}^m)}^{\,\cdot}$$

$$= -\underline{X}(\underline{I}+\underline{J}) + \mathbf{x}\times\mathrm{div}(\underline{I}+\underline{J})$$

$$-\Big[\rho\mathbf{V}^m\times(\mathbf{v}+\mathbf{V}^m) + \mathbf{x}\times\left[\nabla(\mathbf{v}+\mathbf{V}^m)\rho\mathbf{V}^m\right] + \mathbf{x}\times(\mathbf{v}+\mathbf{V}^m)\,\mathrm{div}(\rho\mathbf{V}^m)\Big]$$

$$\{\underline{X}(A\underline{B})\}_i = e_{ijk}A_{jr}B_{rk}$$

$$e_{irk}\left(x_r(T_{lk}+J_{lk})\right)_{,k} = e_{ikl}(T_{lk}+J_{lk}) + e_{irl}x_r(T_{lk}+J_{lk})_{,k}$$

$$\left[e_{irl}x_r(v_l+V^m_l)V^m_k\right]_{,k} = e_{ikl}(v_l+V^m_l)V^m_k + e_{irl}x_r(v_l+V^m_l)_{,k}V^m_k + e_{irl}x_r(v_l+V^m_l)V^m_{k,k}$$

from balance of mass

$$\Rightarrow -\mathrm{div}(\rho\mathbf{V}^m)\cancel{\left[\mathbf{x}\times(\mathbf{v}+\mathbf{V}^m)\right]}\ (1) + \cancel{\rho\,\mathbf{v}\times(\mathbf{v}+\mathbf{V}^m)}\ (3) + \mathbf{x}\times\mathrm{div}(\rho\mathbf{V}^m)(\mathbf{v}+\mathbf{V}^m) + \cancel{\mathbf{x}\times\mathrm{div}(\underline{I}+\underline{J})}\ (2) - \mathbf{x}\times\mathrm{div}\left[(\mathbf{v}+\mathbf{V}^m)\times\rho\mathbf{V}^m\right]$$

$$= -\underline{X}(\underline{I}+\underline{J}) + \cancel{\mathbf{x}\times\mathrm{div}(\underline{I}+\underline{J})}\ (2) - \cancel{\rho\mathbf{V}^m\times(\mathbf{v}+\mathbf{V}^m)}\ (3) - \mathbf{x}\times\left[\mathrm{grad}(\mathbf{v}+\mathbf{V}^m)\rho\mathbf{V}^m\right] - \cancel{\mathbf{x}\times(\mathbf{v}+\mathbf{V}^m)\,\mathrm{div}(\rho\mathbf{V}^m)}\ (1)$$

$$\Rightarrow \mathbf{x}\times\mathrm{div}\left[\left[\mathbf{v}+\mathbf{V}^m\right]\otimes(\rho\mathbf{V}^m)\right] = -\underline{X}(\underline{I}+\underline{J})$$

$$-\mathbf{x}\times\mathrm{div}\left[(\mathbf{v}+\mathbf{V}^m)\times\rho\mathbf{V}^m\right]$$

$$\Rightarrow \quad \underline{\underline{X}}(\underline{\underline{T}}+\underline{\underline{J}}) = \underset{\sim}{0}$$

$$\Rightarrow \quad \boxed{(\underline{\underline{T}}+\underline{\underline{J}})_{skw} = \underline{\underline{0}}}$$

Balance of Energy:

$$\boxed{\begin{aligned}\frac{d}{dt}\int_{\mathcal{B}(t)} \rho\left[\frac{1}{2}(\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot(\underset{\sim}{v}+\underset{\sim}{V}^m)+\varepsilon\right] dv &= \int_{\partial\mathcal{B}(t)} \left\{\frac{1}{2}\rho(\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot(\underset{\sim}{v}+\underset{\sim}{V}^m)\right\}(-\underset{\sim}{V}^m\cdot\underset{\sim}{n})\, da \\ &+ \int_{\partial\mathcal{B}(t)} -\rho\varepsilon\, \underset{\sim}{V}^m\cdot\underset{\sim}{n}\, da \\ &+ \int_{\partial\mathcal{B}(t)} (\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot(\underline{\underline{T}}+\underline{\underline{J}})\underset{\sim}{n}\, da\end{aligned}}$$

$$\begin{aligned}LHS &= (\dot\rho+\rho\,\mathrm{div}\,\underset{\sim}{v})\left[\frac{1}{2}|\underset{\sim}{v}+\underset{\sim}{V}^m|^2+\varepsilon\right] + \rho(\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\overline{(\underset{\sim}{v}+\underset{\sim}{V}^m)}^{\cdot} + \rho\dot\varepsilon \\ &= -\mathrm{div}(\rho\underset{\sim}{V}^m)\left[\frac{1}{2}|\underset{\sim}{v}+\underset{\sim}{V}^m|^2+\varepsilon\right] + (\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\mathrm{div}(\underline{\underline{T}}+\underline{\underline{J}}) \\ &\quad - (\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\mathrm{div}\{(\underset{\sim}{v}+\underset{\sim}{V}^m)\otimes\rho\underset{\sim}{V}^m\} \\ &\quad + (\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\mathrm{div}(\rho\underset{\sim}{V}^m)(\underset{\sim}{v}+\underset{\sim}{V}^m) \\ &\quad + \rho\dot\varepsilon \\ &= +\mathrm{div}(\rho\underset{\sim}{V}^m)\left[\frac{1}{2}|\underset{\sim}{v}+\underset{\sim}{V}^m|^2\right] + \cancel{(\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\mathrm{div}(\underline{\underline{T}}+\underline{\underline{J}})} \, ① \\ &\quad ② \cancel{-\mathrm{div}(\rho\underset{\sim}{V}^m)\varepsilon} - (\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\mathrm{div}\{(\underset{\sim}{v}+\underset{\sim}{V}^m)\otimes\rho\underset{\sim}{V}^m\}. \\ &\quad + \rho\dot\varepsilon\end{aligned}$$

$$\underline{\underline{R.H.S.}} = -\operatorname{div}\left\{\tfrac{1}{2}\rho\,|\underset{\sim}{v}+\underset{\sim}{V}^m|^2\,\underset{\sim}{V}^m\right\}$$

$$-\operatorname{div}\left\{\rho\,\mathcal{E}\,\underset{\sim}{V}^m\right\}$$

$$+\operatorname{div}(\underline{I}+\underline{J})\cdot(\underset{\sim}{v}+\underset{\sim}{V}^m) + (\underline{I}+\underline{J}):\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m).$$

$$= -\operatorname{div}(\rho\underset{\sim}{V}^m)\left[\tfrac{1}{2}|\underset{\sim}{v}+\underset{\sim}{V}^m|^2 + \cancel{\mathcal{E}}\right] + \operatorname{grad}\left(\tfrac{1}{2}|\underset{\sim}{v}+\underset{\sim}{V}^m|^2\right)\cdot\rho\underset{\sim}{V}^m$$

$$+\operatorname{grad}(\mathcal{E})\cdot\rho\underset{\sim}{V}^m$$

$$+\cancel{\operatorname{div}(\underline{I}+\underline{J})\cdot(\underset{\sim}{v}+\underset{\sim}{V}^m)} + (\underline{I}+\underline{J}):\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m).$$

(1) (2)

$$\Rightarrow \operatorname{div}(\rho\underset{\sim}{V}^m)|\underset{\sim}{v}+\underset{\sim}{V}^m|^2 - \operatorname{grad}\left(\tfrac{1}{2}|\underset{\sim}{v}+\underset{\sim}{V}^m|^2\right)\cdot\rho\underset{\sim}{V}^m$$

$$-\operatorname{grad}\mathcal{E}\cdot(\rho\underset{\sim}{V}^m) - (\underline{I}+\underline{J}):\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m)$$

$$-(\underset{\sim}{v}+\underset{\sim}{V}^m)\cdot\operatorname{div}\left\{(\underset{\sim}{v}+\underset{\sim}{V}^m)\otimes\rho\underset{\sim}{V}^m\right\} + \rho\mathring{\mathcal{E}} = 0$$

$$\boxed{a_i(a_i c_j)_{,j} = a_i a_{i,j} c_j + a_i a_i c_{j,j}.}$$

$$\Rightarrow \cancel{\operatorname{div}(\rho\underset{\sim}{V}^m)|\underset{\sim}{v}+\underset{\sim}{V}^m|^2} - \operatorname{grad}\left(|\underset{\sim}{v}+\underset{\sim}{V}^m|^2\right)\cdot(\rho\underset{\sim}{V}^m)$$

$$-\operatorname{grad}\mathcal{E}\cdot(\rho\underset{\sim}{V}^m) - (\underline{I}+\underline{J}):\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m)$$

$$-\cancel{|\underset{\sim}{v}+\underset{\sim}{V}_m|^2\operatorname{div}(\rho\underset{\sim}{V}^m)} + \rho\mathring{\mathcal{E}} = 0$$

(3) (3)

$$\Rightarrow \rho\mathring{\mathcal{E}} = (\underline{I}+\underline{J}):\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m)$$

$$+\operatorname{grad}\left(\mathcal{E}+|\underset{\sim}{v}+\underset{\sim}{V}^m|^2\right)\cdot\rho\underset{\sim}{V}^m.$$

Now angular momentum $\Rightarrow (\underline{I}+\underline{J})_{skew} = \underline{0}$.

$$\Rightarrow \rho\mathring{\mathcal{E}} = \left(\underline{I}_{sym}:\left[\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m)\right]_{sym}\right.$$

$$+\underline{J}_{sym}:\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m) + 2\left[(\underset{\sim}{v}+\underset{\sim}{V}^m)\otimes\rho\underset{\sim}{V}^m\right]:\operatorname{grad}(\underset{\sim}{v}+\underset{\sim}{V}^m)$$

$$+\operatorname{grad}\mathcal{E}\cdot\rho\underset{\sim}{V}^m.$$

$$\Rightarrow \rho\dot{\varepsilon} = \underline{T} : [\mathrm{grad}(\underset{\sim}{v} + \underset{\sim}{V}^m)] + \rho\underset{\sim}{V}^m \cdot \mathrm{grad}\,\varepsilon$$

$$+ \left[\underline{J} + 2(\underset{\sim}{v} + \underset{\sim}{V}^m) \otimes \rho\underset{\sim}{V}^m\right] : \mathrm{grad}(\underset{\sim}{v} + \underset{\sim}{V}^m)$$

The last two re-writings do not seem to help.

$\boxed{\text{So, let us assume } \underline{T} \text{ *and* } \underline{J} \text{ are individually symmetric (angular momentum balance is taken care of)}}$

&

$$\rho\dot{\varepsilon} = \underline{T} : (\underline{D} + \mathrm{grad}\,\underset{\sim}{V}^m) + \rho\underset{\sim}{V}^m \cdot \mathrm{grad}\,\varepsilon$$

$$+ \underline{J} : \mathrm{grad}(\underset{\sim}{v} + \underset{\sim}{V}^m) + \rho\underset{\sim}{V}^m \cdot \mathrm{grad}(|\underset{\sim}{v} + \underset{\sim}{V}^m|^2)$$

\# from a quick check I did it does not look like $\mathrm{grad}(|\underset{\sim}{v} + \underset{\sim}{V}^m|^2)$ can be objective and neither the unit vector in its direction. Therefore, it looks like for Balance of energy to be objective one cannot have $\underline{J} = \underline{0}$.

\# So to make balance of energy objective, require the constraint

$$\boxed{\underline{J} : \left[\mathrm{grad}(\underset{\sim}{v} + \underset{\sim}{V}^m)\right]_{\mathrm{sym}} = -\rho\underset{\sim}{V}^m \cdot \mathrm{grad}(|\underset{\sim}{v} + \underset{\sim}{V}^m|^2)}$$

Remark 1 -: How is this analysis different from doing conventional continuum mechanics on control volumes moving with arbitrary velocities?

— In the case mentioned above, it is possible to write the balance laws in at least one way where the arbitrary velocity of the control volume does not figure in the balance laws.

Not so, in the case with matter moving w.r.t. the body.

Remark 2 -: In many descriptions of fracture a 'cohesive law' is required prescribe how fracture patterns evolve. This cohesive law is a 'traction-separation' specification. Is it possible to interpret the appearance of $\underline{J}$ (a traction) as such a cohesive law predicated from invariance requirements? Note that $\underset{\sim}{V}^m$ will end up depending on stress gradients &.', $\underline{J}$ too, thus having a component from material stress field $\underline{I}$.

Remark 3 -: It seems to make vague physical sense to me that movement of matter relative to the body (resulting in fracture) should have an implication on contact traction on surfaces.