Vartional phase-field modeling of brittle and cohesive fracture

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1. Griffith's linear elastic fracture mechanics

Without being historically exact one can say that fracture mechanics has started with the work of Alan Arnold Griffith (1893–1963). As Griffith's work is the basic of variational phase-field fracture model developed 80 years later, let's examine what he did.

In 1921, Griffith conducted experiments on fracture of glass fibers [1]. He found two things: (1) the fracture strength of glass is significantly smaller than the theoretical value (coming from breaking the atomic bonds) and (2) small glass fibers are stronger than larger fibers. He concluded that small naturally occurring defects existing in the fibers make them weak. Precisely these defects amplify the stress field in front of their tips and thus rendering the fracture stress much smaller compared with the theoretical strength. Actually Griffith was aware of the work of Charles Inglis (1875–1952) conducted seven years ago about stress concentration due to an elliptical hole [2].

With defects now in his mind, he made specimens with artificial surface cracks (to overcome natural defects) of varying sizes and quantified the relationship between the remote tensile stress σ and the crack size or length a. What he found is an inverse relation between the strength and the crack length. In symbols, he found that $\sigma\sqrt{a} = C$ where C is a constant. To find this constant, he carried out an energy-based analysis that basically led to the born of what is now known as fracture mechanics.

Griffith computed the energy of the system which consists of the stored elastic strain energy and the surface energy *i.e.*, energy due to the creation of the new crack surfaces. He considered a unit thickness infinite plate with a surface crack of length a subjected to a remote tensile stress σ normal to the crack. The energy of this system is given by

$$U = U_0 - \pi a^2 \frac{\sigma^2}{2E_0} + 2a\gamma_s \tag{1.1}$$

where he used Inglis' solution to obtain $-\pi a^2\sigma^2/(2E_0)$ the elastic strain energy released due to the crack's presence; U_0 is the elastic strain energy of the plate without crack. What is interesting here is the last term $2a\gamma_s$ the surface energy associated with a crack of length a where γ_s is the energy required to creare a unit surface area.

The first derivative of U with respect to the crack length a is

$$\frac{\partial U}{\partial a} = -2\pi a \frac{\sigma^2}{2E_0} + 2\gamma_s \tag{1.2}$$

And the vanishing derivative condition gives us

$$\frac{\partial U}{\partial a} = 0 \Longrightarrow \pi a \frac{\sigma^2}{2E_0} = \gamma_s \Longrightarrow \sigma = \sqrt{\frac{2E_0\gamma_s}{\pi a}} \tag{1.3}$$

which is the well-known Griffith's equation relating the remote stress to the crack length.

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The work of Griffith which is applicable only to brittle materials (*e.g.* glasses) was ignored for almost 20 years. It was not until the modifications made by Orowan and particularly George Rankin Irwin (1907–1998) that, a new field has emerged: Linear Elastic Fracture Mechanics (LEFM). Irwin introduced the concept of energy release rate $\mathcal{G}$ which is the negative of the derivative of the elastic strain energy with respect to the crack length and he and Orowan replaced γ_s by G_f — the critical energy release rate or fracture energy — to take into account other dissipative processes such as plastic deformation, i.e.,

$$\mathcal{G} = -\frac{\partial}{\partial a}\left(U_0 - \pi a^2 \frac{\sigma^2}{2E_0}\right) = 2\pi a \frac{\sigma^2}{2E_0}, \qquad G_\mathrm{f} = \gamma_s \tag{1.4}$$

Another significant contribution to fracture mechanics was made by James Rice (1940 –) in 1968, the famous J-integral [3] which is equal to the fracture energy release rate $\mathcal{G}$.

By recalling the irreversibility of the crack propagation, $\dot{a} \geq 0$, the Griffith crack propagation criterion is given then:

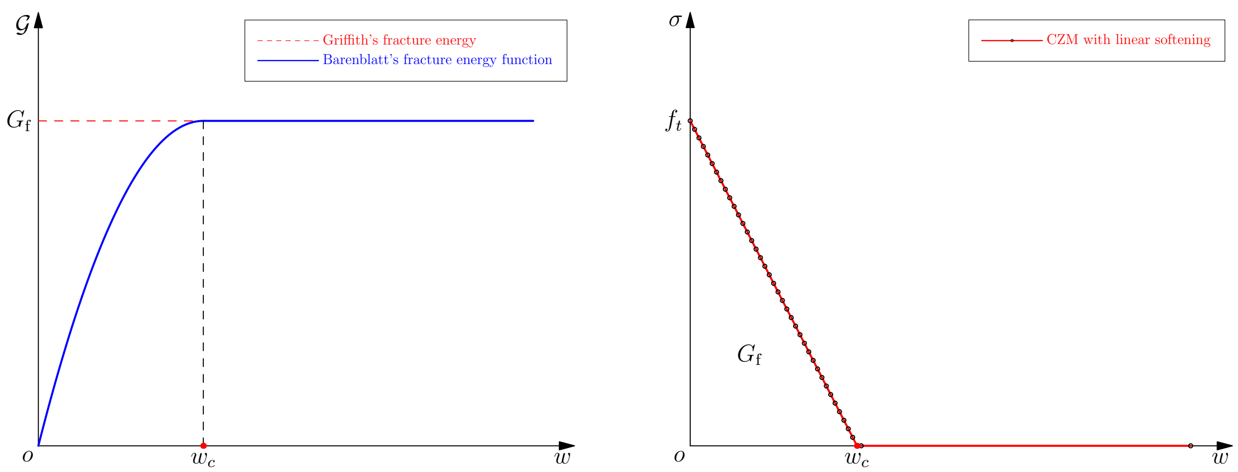
$$\mathcal{G} - G_\mathrm{f} \leq 0, \quad \dot{a} \geq 0, \quad \dot{a}\big(\mathcal{G} - G_\mathrm{f}\big) \equiv 0 \tag{1.5}$$

That is, for quasi-static loading case, the crack propagates (i.e., $\dot{a} > 0$) if $\mathcal{G} = G_\mathrm{f}$ and otherwise remains stationary $\dot{a} = 0$ for $\mathcal{G} < G_\mathrm{f}$.

Fracture mechanics has been a great success as it provides the engineers a continuum mechanics tool to quantitatively predict the structural integrity of large structures using data such as fracture toughness (a concept introduced by Irwin which is related to the fracture energy) which can be experimentally measured using laboratory scale specimens. Furthermore it helps material scientists to improve existing materials and design new ones by looking at their fracture toughness.

2. Barenblatt's cohesive zone model

In 1959, Grigory Isaakovich Barentblatt (1927–2018) [4] proposed the celebrating cohesive zone model (CZM). Barentblatt's CZM solved two major issues of classical fracture mechanics: crack nucleation and stress singularity at the crack tip. Its idea is to lump what is going on in the fracture process zone (FPZ) ahead of a traction-free crack tip on a surface, and approximate this zone by a traction-separation law that relates the cohesive traction transfered across the crack surfaces σ and the crack opening w. In this case, the fracture energy release rate $\mathcal{G}(w)$ is no longer a constant but a non-convex function of the crack opening w as shown in Figure 1.



(a) Fracture toughness *versus* fracture energy function

(b) Linear traction – separation law

Figure 1: Barenblatt's CZM [4] approximation of Griffith's [1] brittle fracture

Thus, instead of instant dissipation upon the creation of a unit fracture surface in Griffith's theory, in Barenblatt's CZM, the energy is released gradually:

$$\int_0^\infty \sigma(w)\,\mathrm{d}w = G_\mathrm{f} \tag{2.1}$$

The traction corresponding to zero crack opening is the tensile strength f_t of the material. It is then able to define a material characteristic length

$$l_\mathrm{ch} = \frac{E_0 G_\mathrm{f}}{f_t^2} \tag{2.2}$$

which is called Irwin's internal length [5], with E_0 being Young's modulus of the material. It characterizes (is proportional to) the size of the FPZ. Hence, it also measure the brittleness of the material: the smaller (compared to the structural size) it is, the more brittle the material behaves. Note that the CZM applies not only to cohesive cracks, but also to brittle one with a small Irwin's length.

It was A. Hillerborg et al. [6] who assumed that the cohesive crack may develop anywhere, even if no *a priori* existing macrocrack is actually present and named this extension as the fictitious crack model. They also implemented the CZM in a finite element framework to model fracture of concrete beams [6]. Another notable work is of Xu and Needleman [7] for dynamic fracture. CZM is usually implemented using the so-called interface elements. Recent works of Paulino's group and Papoulia's demonstrate that CZM and *advanced* interface elements are indeed a powerful tool for fracture simulations [8, 9].

3. The variational approach to fracture

In 1998 *i.e.*, 78 years after Griffith's work Francfort and Marigo reformulated Griffith's energetic theory in a variational framework and coined it the *variational approach to brittle fracture* [10]. This approach generalizes LEFM by allowing crack nucleation and arbitrary crack propagation within a single framework.

In this energetic approach, crack propagation results from the competition between the bulk energy away from the crack and the surface energy on the crack, denoted by Γ. From this point of view, the total energy functional $\mathscr{E}$ in a quasi-static loading regime reads

$$\mathscr{E}(\boldsymbol{u}, \Gamma) := \int_{\Omega\setminus\Gamma} \psi_0(\boldsymbol{\epsilon}(\boldsymbol{u}), \Gamma)\mathrm{d}V + \int_\Gamma G_\mathrm{f}\,\mathrm{d}A - \mathscr{P} \tag{3.1}$$

for the external potential energy $\mathscr{P}$. The initial (elastic) strain energy density function $\psi_0(\boldsymbol{\epsilon}(u), \Gamma)$ is expressed as usual in terms of the standard linearized strain tensor $\boldsymbol{\epsilon}(\boldsymbol{u})$ and also depends on the crack set Γ.

Motivated by the above facts, Francfort and Margio [10] recast brittle fracture as an energy minimization problem. That is, the pair $(\boldsymbol{u}(t), \Gamma(t))$ is a global minimizer of the potential energy functional $\mathscr{E}$, i.e.,

$$(\boldsymbol{u}(t), \Gamma(t)) = \mathrm{Arg}\big\{\min \mathscr{E}(\boldsymbol{u}, \Gamma)\big\} \tag{3.2}$$

among all $\Gamma \supseteq \Gamma(t)$ ($\supseteq \Gamma_0$ if a pre-defined crack Γ_0 exists) and all $\boldsymbol{u} = \boldsymbol{u}^*(t)$ on $\partial\Omega_u$. Note that for the case in which the crack path Γ may be not smooth "enough", an infimum energy rather than a minimal one should be sought. In such a variational approach to brittle fracture, cracks should propagate along the path of least energy. In particular, for an *a priori* existing crack constrained to propagate along a pre-defined crack path, Griffith's criterion is exactly retrieved [11]. However, the most significant merit is that with this variational approach to fracture it is possible to deal with crack nucleation in an initially perfectly sound solid and to determine intrinsically crack paths in a variationally consistent manner, bypassing the underlying assumptions of LEFM.

4. The variational phase-field model for fracture

Though there exist other schemes, e.g., the eigenerrosion method [12, 13], the adaptive finite elements, to approximate the variational approach to brittle fracture, the variational phase-field model for fracture, set forth in [14], might be the most versatile. This numerically more amenable counterpart of the Francfort–Marigo's variational approach to brittle fracture is motivated by the Ambrosio–Tortorelli elliptic regularization [15] of the Mumford–Shah functional [16] in image segmentation problems. The idea is to introduce a continuous scalar field — the crack phase-field or damage field $d(\boldsymbol{x})$ — that takes either a value of zero for intact material, or a value of unity for completely broken material or a value between 0 and 1 for partially broken material. This field helps to approximate the Griffith surface energy–a surface integral–as a volume integral over the computational domain Ω. So, in this regularized model, for quasi-static fracture of solids under the infinitesimal strain regime, the displacement field $\boldsymbol{u}$ and damage field d are minimizers of the following total energy functional of the solid

$$\mathscr{E}(\boldsymbol{u}, d) = \int_{\Omega} \left[\omega(d)\psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) + \psi_0^-(\boldsymbol{\epsilon}(\boldsymbol{u}))\right] \mathrm{d}V + \int_{\Omega} \frac{G_\mathrm{f}}{c_\alpha} \left[\frac{1}{b}\alpha(d) + b\left|\nabla d\right|^2\right] \mathrm{d}V - \mathscr{P}(\boldsymbol{u}) \tag{4.1}$$

where the first integral is the stored strain energy influenced now by the crack phase-field, the second one denotes the fracture energy à la Griffith, and $\mathscr{P}(\boldsymbol{u})$ denotes the external potential energy as before. The phase-field length scale b is a regularization parameter that controls the crack band *i.e.*, where $d(\boldsymbol{x})$ is non-zero; $\omega(d)$ is the energetic degradation function and $\alpha(d)$ represents the crack geometric function, with $c_\alpha = 4\int_0^1 \sqrt{\alpha(\beta)}\,\mathrm{d}\beta$ being the scaling constant. The positive and negative parts of the elastic strain energy density $\psi_0(\boldsymbol{\epsilon}(\boldsymbol{u}))$ are denoted by $\psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u}))$ and $\psi_0^-(\boldsymbol{\epsilon}(\boldsymbol{u}))$, respectively. This positive/negative decomposition plays an important role in capturing the tension/compression asymmetry of fracture and in removing spurious compressive fracture – fracture does not occur in domains under compression; see [17] for the discussion of several positive/negative decomposition schemes of the elastic strain energy $\psi_0(\boldsymbol{\epsilon}(\boldsymbol{u}))$.

Where does the name *phase field* come from? In the similar phase-field models for phase transformation based upon the Ginzburg–Landau equation [18] in physics and materials science, for multiphase fluid based upon the Cahn–Hillard [19] or Allan–Cahn [20] equation in fluid mechanics, the so-called order parameter is also introduced to discriminate the distinct states or phases. Here, the crack phase-field $d(\boldsymbol{x})$ bears the same role in discriminating two phases of the solid: the intact phase and the completely broken phase, Kuhn and R. Müller called this model a phase-field fracture model (PFM) [21] in 2010, ten years after the work of Bourdin, Francfort and Marigo! To highlight its variational argument, we believe a better term is variational phase-field fracture model. In the same year, the later mechanician, Christian Miehe, published two papers [22, 23] presenting a more intuitive approach alternative to the formal and mathematically demanding formulation of Francfort–Marigo–Bourdin paper. Indeed Miehe was successful in making PFMs much more attractive in the engineering community: we have seen a surge in publications on PFM since 2010. Another key player is probably Thomas Hughes who proposed the fourth-order PFM [24] and presented many keynote lectures on applications of PFM to dynamic fracture and ductile fracture at various Complas (Computational Plasticity) conferences.

Looking at Eq. (4.1) one can see that there are many possibilities in choosing the degradation function $\omega(d)$ and the crack geometric function $\alpha(d)$. We refer to [17] for an extensive discussion. The three most common (second order) PFMs are listed in Table 1. The `AT2` model, developed in [14, 22] is probably the most widely used PFM in engineering even though it lacks an elastic domain *i.e.*, damage becomes non-zero immediately when the load (no matter how small it is) is applied. The `AT1` model of [25], possessing an elastic domain, is getting more attention. While both the `AT1` and `AT2` models apply only to brittle fracture, the `PF-CZM` (phase field cohesive zone model) of [26] is the first PFM that applies to both brittle and cohesive fracture.

Herein we need to stress the most important difference between `AT1/2` and `PF-CZM`. That is, the latter converges to the CZM while all the others to the Griffith LEFM, in the Gamma-convergence [27], to be discussed shorly.

model	$\alpha(d)$	$\omega(d)$	fracture type	length-scale	support	Parameters
AT2	d^2	$(1-d)^2$	brittle	$b = \dfrac{27}{256} l_{\mathrm{ch}}$	∞	$E_0, \nu_0, G_{\mathrm{f}}, b$
AT1	d	$(1-d)^2$	brittle	$b = \dfrac{3}{8} l_{\mathrm{ch}}$	$4b$	$E_0, \nu_0, G_{\mathrm{f}}, b$
PF-CZM	$2d - d^2$	$\dfrac{(1-d)^p}{(1-d)^p + a_1 d \cdot P(d)}$	brittle/cohesive	$b = \alpha_b l_{\mathrm{ch}}$	πb	$E_0, \nu_0, f_t, G_{\mathrm{f}}$+ TSL

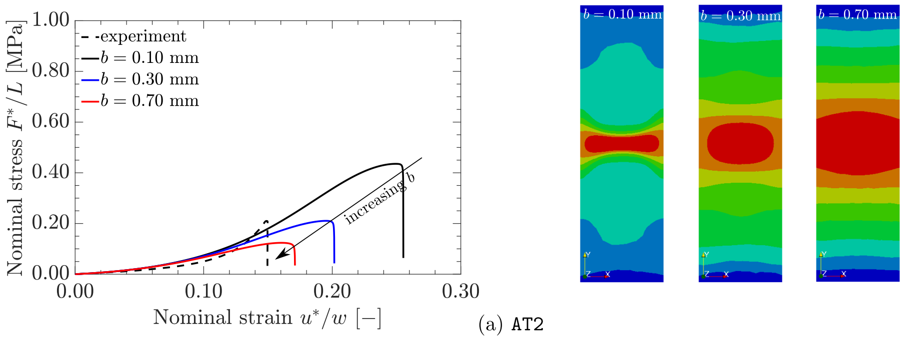
Table 1: Common PFMs for brittle and cohesive fracture. The AT2 model has an infinite support of the damage field whereas the AT1 and PF-CZM both have a finite support. TSL is short for traction-separation law, see *e.g.* [28]. Notice that the models have been presented in the chronological order. Note that $P(d) = 1 + a_2 d + a_3 d^2$, with $a_1 := \frac{4 l_{\mathrm{ch}}}{\pi b}$ encoding the strength of the material. Theoretical the coefficient $\alpha_b \leq \frac{8}{3\pi}$ is an arbitrary parameter so long it guarantees the convexity of the energetic degradation function $\omega(d)$.

5. Role of the length scale and crack nucleation

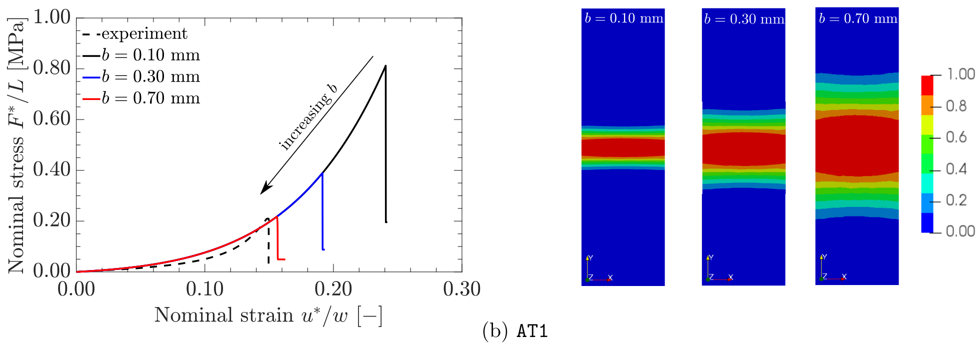
Using the tools from free discontinuity problems, it can be shown that when the phase-field length scale b approaches zero in the vanishing limit, the PFM solution converges to the solution of the original problem. This is known as Gamma-convergence [27]. Therefore, in the early days of PFMs the length scale was considered as a merely numerical parameter. However during verification attempts against experiment data, it was soon realized that for the AT1/2 models the length scale must be considered as a material parameter relating to Irwin's internal length l_{ch} so that the failure strength f_t can be matched in the uniaxial traction state [25, 29]. Due to this new interpretation of the length scale, Marigo re-named it as a gradient damage model [25] since it belong to the gradient damage model developed by Frémond and Nedjar in 1996 [30]. Still, as recently pointed out in [31] such models cannot capture crack nucleation in intact solids with no pre-existing defects. To demonstrate this issue, we model the uniaxial tensile test of a biological tissue, and we used different values for the length scale. To match the experiment a large length scale have to be used, but this value results in an erroneously wide crack band (Figure 2).

For the PF-CZM, on the other hand, the phase-field length scale is merely a numerical parameter (and it can also be treated as a physical one related to possibly the microscopic inhomogeneity of the material) so long it is small enough compared to the structural size (usually $1/100 \sim 1/50$). As seen from Figure 2c, all length scales yield the same load-displacement responses and the peak is similar to the experiment. Therefore, we can use a small length scale to guarantee the Gamma-convergence. See [32] for details regarding the formulation for large deformation anisotropic fracture of hyperelastic solids.

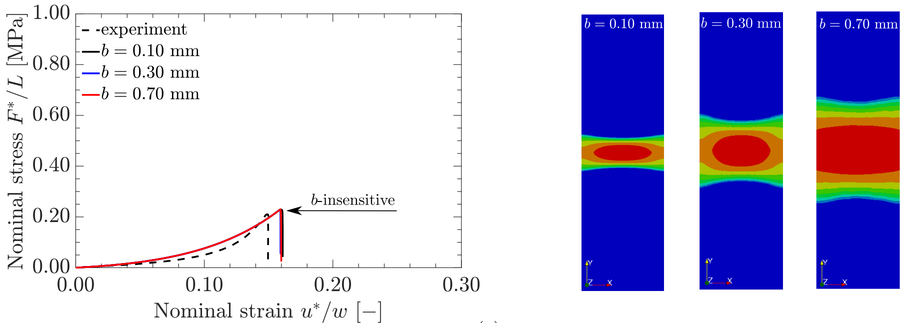
We present another example reported by Bourdin himself [33] so that our discussion would be more convincing. This example is the popular L-shape panel test of concrete fracture. The specimen size is about 500 mm. Using a small length scale ($b = 3.125$ mm), Bourdin got a crack path quite matching the experiment, but the peak load is highly overestimated. To match the failure strength of concrete, a much larger phase-field length scale $b = 126.32$ mm has to be used. With this length scale, a better load–displacement curve is obtained, but the crack path is completely polluted by the too diffuse regularization (Figure 3a). Comparatively, this is not an issue for the PF-CZM at all – both the crack path and the load – displacement curve can be well captured, both insensitive to the phase-field length scale as shown in Figure 3b. This is because the PF-CZM incorporates the stress-based crack nucleation criterion, the energy-based crack propagation criterion and the variational principle based crack path chooser into one standalone framework.



(a) AT2

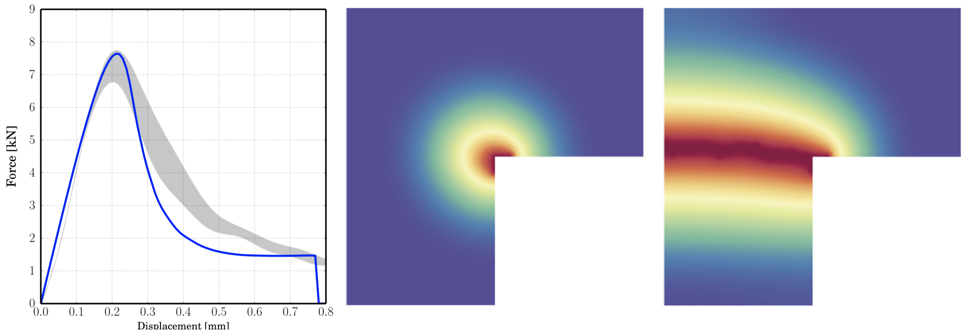


(b) AT1

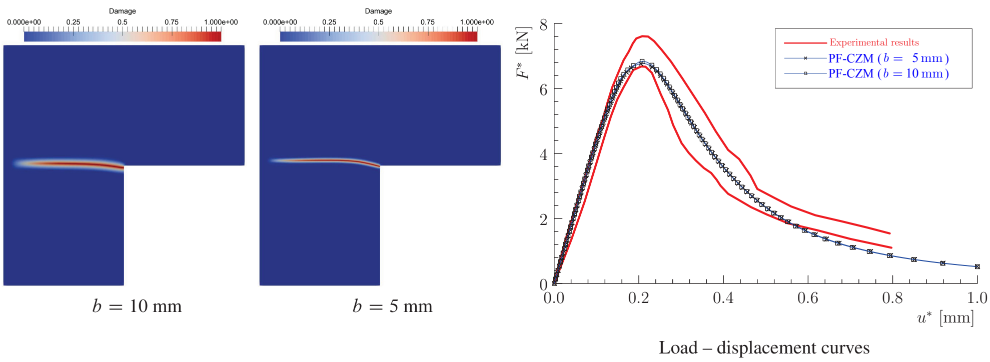


(c) PF-CZM

Figure 2: Uniaxial tensile test of biological tissues: study of load-deformation and crack pattern for homogeneous specimen (without singularity) using different PFMs. Results are given using three different length scales $b = \{0.10 \text{ mm}, 0.30 \text{ mm}, 0.70 \text{ mm}\}$ with element size $h = b/3$.



(a) Another demonstration of the fact that AT1/2 cannot capture crack nucleation properly, taken from [33]



(b) The demonstration that PF-CZM can capture crack nucleation properly regardless the phase-field length scale, taken from [17]

Figure 3: Phase-field modeling of the L-shape panel: Crack patterns and load – displacement curves predicted by various models

6. Numerical implementation of variational phase-field fracture models

Compared with discrete fracture numerical methods such as XFEM [34], the implementation of a PFM is quite simple. We do not need to introduce the cumbersome crack tracking algorithm nor any *ad hoc* criteria for determining the crack orientation/branching. Only standard continuum finite elements with both displacement and damage degrees of freedom (dofs) are needed in the numerical implementation of a PFM. Compared to those classical continuum damage models in which particular strategies (e.g., the mixed stabilized finite elements recently advocated by Cervera and coworkers [35]) have to be employed, the solution of PFMs do not depend on the underlying spatial discretization. That is, just as elastostatic problems, almost all the existing numerical methods, e.g., irreducible and mixed FEs, meshless or meshfree methods, particle methods (MPM, RKPM) and even the more recent peridynamics, can be employed. There is no stability issue nor mesh bias dependence [36]. Moreover, for the `PF-CZM` the solution is also insensitive to the phase-field length scale. In other words, users need only to focus on the physical problems of interest rather than worry about those numerical issues/parameters.

Upon finite element discretization, the solution of a PFM boils down to the solution of two equations with constraints $d \in [0,1],\ \dot{d} \geq 0$:

$$\mathbf{r}^u := \mathbf{f}^{\text{ext}} - \int_\Omega \mathbf{B}^{\text{T}} \boldsymbol{\sigma} \, \mathrm{d}V - \mathbf{M}\ddot{\mathbf{a}} = \mathbf{0} \tag{6.1a}$$

$$\mathbf{r}^d := -\int_{\mathcal{B}} \left[\bar{\mathbf{N}}^{\text{T}} \Big(\omega' \bar{Y} + \frac{1}{c_\alpha b} \alpha' G_{\text{f}} \Big) + \frac{2b}{c_\alpha} G_{\text{f}} \bar{\mathbf{B}}^{\text{T}} \nabla d \right] \mathrm{d}V \leq \mathbf{0} \tag{6.1b}$$

$$d \in [0,1], \quad \dot{d} \geq 0 \tag{6.1c}$$

where $\bar{Y}$ is the effective crack driving force. Note that standard parallelization technique such as domain decomposition method can be readily reused. Moreover, for the `AT1` and `PF-CZM` in which the regularized crack band is of finite support, the second equation (6.1b) can be considered only within an *a priori* selected sub-domain.

A crucial step in solving the above discretized governing equations is to deal with the irreversibily and boundedness conditions (6.1c). One universal option is using a bound constrained optimization solver, e.g., the "reduced-space active set Newton method" [37] as in [38, 39, 40]. Other options include the (augmented) Lagrangian [41] or penalty methods [42]. However, these particular treatments are not easily implemented in common software platforms. In the literature, the implicit method of [22] has been most widely adopted for its simplicity. That is, the effective crack driving force $\bar{Y}$ in Eq. (6.1b) is replaced by its maximum value $\mathcal{H}$ that is ever reached

$$\mathcal{H} = \max_{n \in [0,T]} \big(\bar{Y}_0, \bar{Y}_n \big), \tag{6.2}$$

where the initial threshold $\bar{Y}_0 = 0$ for the `AT2` model with no linear elastic stage and $\bar{Y}_0 = \frac{1}{2} f_t^2 / E_0$ for the `AT1`/`PF-CZM` with a finite critical stress. With this redefinition, the damage evolution law Eq. (6.1b) becomes an equality.

As other coupled multi-physics problems, the solver used to solve these equations is more involved as the monolithic solver using the standard Newton-Raphson method usually does not work, since the energy functional $\mathscr{E}$ is non-convex with respect to $(\boldsymbol{u}, d)$. Therefore, the most common solver is the alternate minimization or staggered (AM/staggered) solver in which one first fixes the damage dof and solve for the displacement dof, and then solves for the damage dof using the updated displacement dof. Though the one-pass AM/staggered solver has been widely adopted, it would result in delay of damage evolution, inaccurate post-peak regimes and spurious damage widening when the crack arrives at the external boundary. These issues are largely alleviated in the iterative multi-pass AM/staggered solver.

Staggered solvers are easy to implement and stable but their convergence is extremely slow. Among many monolithic solvers proposed, the BFGS solver — a popular solver in nonlinear optimization problems — has proven to be promising: the BFGS-based monolithic solver with the most crude initial guess of the system matrix (BFGS-SM0) is about 4 to 8 times faster than the AM/staggered solver [43], and with the optimized system modification (BFGS-SM1) the computational efficiency can be further boosted [44]; see Figure 4 for the comparison. We have also demonstrated that this gain in computational efficiency also applies to thermo-mechanical fracture problems [45].

We note again that PFMs can be implemented in any numerical discretization method and the `PF-CZM` is insensitive to those numerical issues/parameters. The only requirement is that the phase-field length is sufficiently resolved by the spatial discretization.

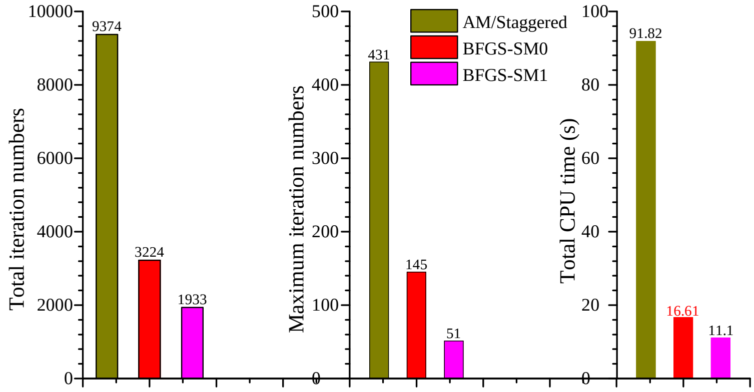


Figure 4: Performance of the staggered solver and the BFGS solver.

7. Applications

Variational phase-field fracture models have been applied to many problems. And in [17] a quite comprehensive review has been provided, so we herein present some of our recent works covering static and dynamic fracture of both brittle and quasi-brittle materials.

7.1. Quasi-static brittle fracture

Figure 5 presents a `PF-CZM` simulation of the peeling test of a biological tissue. It is taken from our work in [32]. See [46, 47] for other works on finite deformation fracture of rubber-like materials using phase-field models.

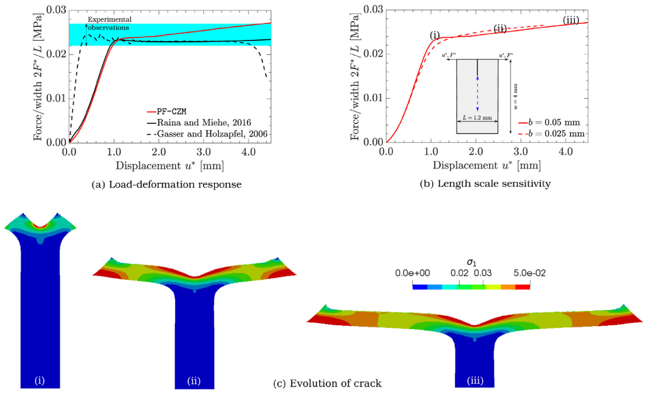


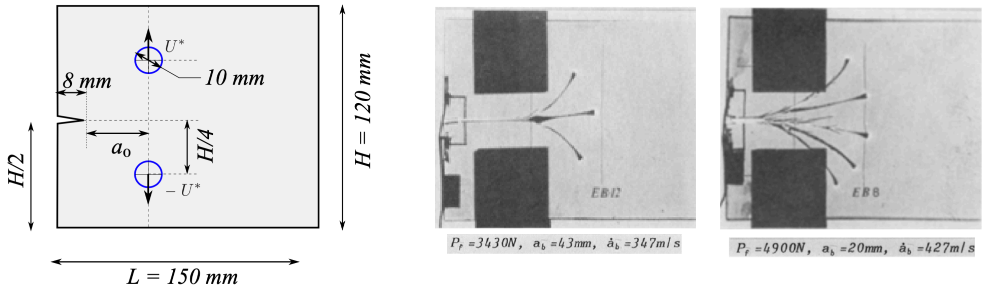
Figure 5: Peeling test of a biological tissue simulated with the `PFCZM`.

7.2. Dynamic brittle fracture

PFMs have been applied to dynamic fracture as reported in [48, 49, 50, 51, 52, 53, 54]. The idea is simple: the rigorously derived and well studied variational approach to quasi-static fracture is maintained and inertial effects are incorporated. Borden et al. used the Hamilton's principle of least action to derive the governing equations [51].

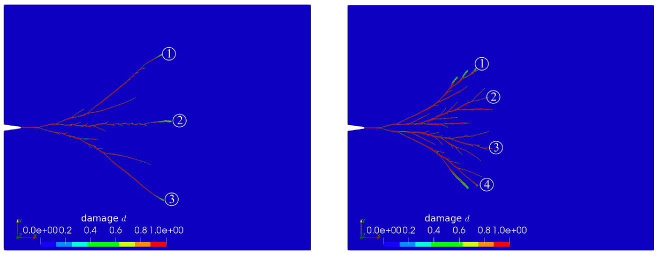
From the results reported in the literature, and particularly from our work in [56], we can say that:

- Phase-field fracture simulation results are very encouraging. Indeed, PFMs can capture several dynamic fracture phenomena: crack branching, crack arrest, fragmentation and multiple branching (Figure 6);
- Phase-field simulations results are similar to predictions of peridynamics and discontinuous Galerkin extrinsic cohesive elements [56];



(a) Geometry and loading

(b) Experimental observations



(a) $P_f = 3000$ N, $a_b = 10$ mm, $\dot{a}_b = 600$ m/s

(b) $P_f = 4500$ N, $a_b = 8$ mm, $\dot{a}_b = 610$ m/s

Figure 6: Multiple branching experiment of Arakawa and Takahashi [55] and our PFM simulation [56].

- The `AT1` model and `PF-CZM` are spatial and temporal convergent for dynamic brittle fracture problems;
- Only a few quantitative assessment of dynamic brittle fracture has been done. More work are needed.

7.3. Quasi-static cohesive fracture

Only when it comes to 3D fracture problems with non-planar crack surfaces that the strength of PFMs is obvious. One can capture complex 3D crack paths involving merging, branching, twisting with a relatively simple implementation (quite the same implementation as a 2D implementation, precisely) and the entire fracture process is done over a fixed finite element mesh. To illustrate this, we show in Figure 7 and 8 some simulations of mixed-mode I/III fracture taken from our recent work [57].

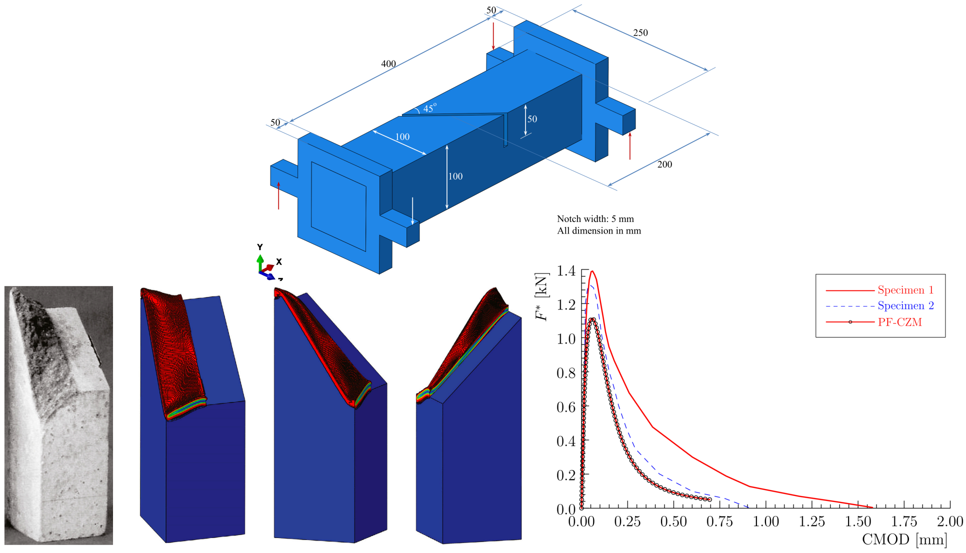


Figure 7: Mixed mode I + III fracture of a prismatic skew notched concrete beam under torsion.

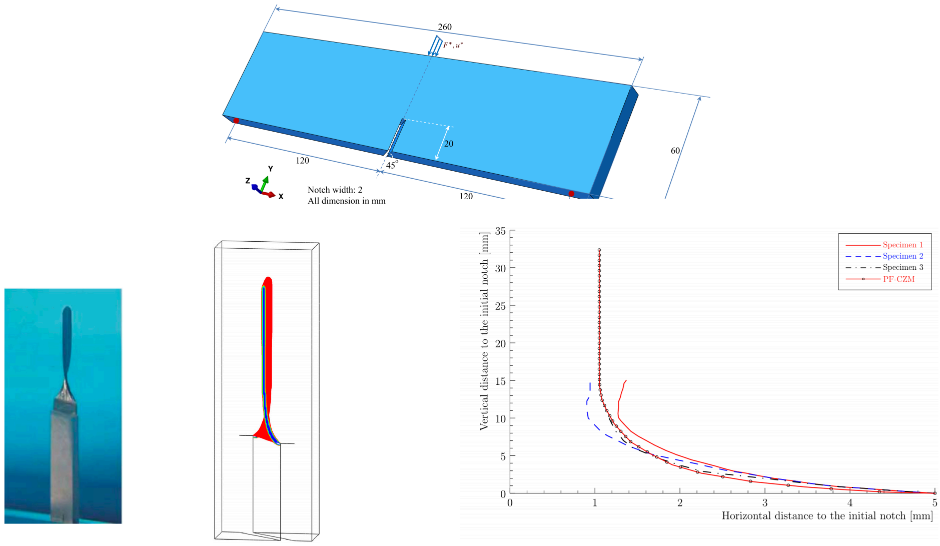


Figure 8: Mixed mode I + III failure of a skewly notched PMMA beam under three-point bending.

8. Conclusions

It has been a century since Griffith's seminal work on brittle fracture. Let's us summarize this adventure. Griffith set up an energy principle for brittle crack propagation. Irwin developed an equivalent approach with the concept of stress intensity factor and established the framework of linear elastic fracture mechanics. Barenblatt proposed the well-known cohesive zone model and greatly broadened the scope of fracture mechanics to nonlinear territories. Rice proposed the J-integral and proved its relation to the fracture energy release rate.

About eighty years after Griffith, with the help of modern variational calculus, Francfort and Marigo were able to cast Griffith's energetic principle into a variational framework. This variational approach to fracture generalizes Griffith's energetic principle by allowing crack nucleation and non-predefined crack paths without resorting to external criteria. Then comes Bourdin's phase-field approximation of the variational approach. For simple problem, Gammacomvergence says that when length scale is going to zero, Bourdin's approximate model converges to the original problem with sharp cracks. Our recent work on the phase-field cohesive zone model then sheds light on the promising prospect in unifying fracture mechanics and continuum damage mechanics into a single framework.

Variational phase-field fracture models have been applied to more and more problems. We can cite multi-physics fracture (hydraulic fracture, hydrogen-assisted cracking, thermo-elastic fracture *etc.*), ductile fracture, fatigue, anisotropic fracture *e.g.* fracture of fiber reinforced composites, biological tissues. However, to be a practical tool one needs to lower down the computational cost of these models.

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