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Local integration of 2-D fractional telegraph equation via local radial point interpolant approximation

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Abstract. In this article, a general type of two-dimensional time-fractional telegraph equation explained by the Caputo derivative sense for $(1 < \alpha \leq 2)$ is considered and analyzed by a method based on the Galerkin weak form and local radial point interpolant (LRPI) approximation subject to given appropriate initial and Dirichlet boundary conditions. In the proposed method, so-called meshless local radial point interpolation (MLRPI) method, a meshless Galerkin weak form is applied to the interior nodes while the meshless collocation method is used for the nodes on the boundary, so the Dirichlet boundary condition is imposed directly. The point interpolation method is proposed to construct shape functions using the radial basis functions. In the MLRPI method, it does not require any background integration cells so that all integrations are carried out locally over small quadrature domains of regular shapes, such as circles or squares. Two numerical examples are presented and satisfactory agreements are achieved.

1 Introduction

The fractional derivative and fractional differential equation have been applied to describe some phenomena in physics and engineering, *e.g.* colored noise, dielectric polarization, boundary layer effects in ducts, electromagnetic waves, electrode-electrolyte polarization, allometric scaling laws in biology and ecology, fractional kinetics, quantitative finance, quantum evolution of complex systems, power-law phenomena in fluid and complex networks, viscoelastic mechanics, etc. [1–6]. Moreover, the fractional telegraph equation (typical fractional diffusion-wave equation) has been applied into signal analysis for transmission, the modeling of the reaction diffusion, propagation of electrical signals and the random walk of suspension flows, etc. [7, 8].

The classical telegraph equation is the result of the variational link between the voltage wave and the current wave on the well-proportioned transmission line (therefore it is called also the transmission line equation). However, the classical telegraph equation could not well describe the abnormal diffusion phenomena during the finite long transmits progress, where the voltage wave or the current wave possibly exist [9, 10]. So, it is necessary to study the fractional telegraph equation, including the time and (or) space fractional derivatives. The present paper considers the following general class of time-fractional telegraph equation of order $(1 < \alpha \leq 2)$:

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} + \gamma_1 \frac{\partial^{\alpha-1} u(\mathbf{x},t)}{\partial t^{\alpha-1}} + \gamma_2 u(\mathbf{x},t) = \gamma_3 \Delta u + f(\mathbf{x},t), \qquad \mathbf{x} \in \Omega,\ t \in [0,T], \tag{1}$$

subject to the compatible initial conditions

$$u(\mathbf{x},0) = \varphi(\mathbf{x}), \qquad \frac{\partial u}{\partial t}(\mathbf{x},0) = \psi(\mathbf{x}), \qquad \mathbf{x} \in \Omega \tag{2}$$

and the boundary conditions

$$u(\mathbf{x},t) = g_0(y,t) \quad \text{for} \quad x=-1, \qquad u(\mathbf{x},t) = g_1(y,t) \quad \text{for} \quad x=1, \qquad \mathbf{x} \in \Omega,\ t \in [0,T], \tag{3}$$

$$u(\mathbf{x},t) = h_0(x,t) \quad \text{for} \quad y=-1, \qquad u(\mathbf{x},t) = h_1(x,t) \quad \text{for} \quad y=1, \qquad \mathbf{x} \in \Omega,\ t \in [0,T], \tag{4}$$

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where $\mathbf{x} = (x, y)$ is spatial variable, $\Omega = [-1, 1]^2 = \{(x, y) : -1 \le x, y \le 1\}$ and, γ_1, γ_2 and γ_3 are constants. Also, $f(\mathbf{x}, t)$ is the source function with sufficient smoothness and, $\varphi(\mathbf{x})$, $\Psi(\mathbf{x})$, $g_0(y, t)$, $g_1(y, t)$, $h_0(x, t)$ and $h_1(x, t)$ are given continuous functions. Moreover, in eq. (1), the time-fractional derivatives are in the sense of Caputo which is defined by

$$D_t^\alpha F(t) = \begin{cases} \dfrac{1}{\Gamma(k-\alpha)} \displaystyle\int_0^t (t-\xi)^{k-\alpha-1} F^{(k)}(\xi)\, \mathrm{d}\xi, & k-1 < \alpha < k, \quad t > 0, \\ F^{(k)}(t) & \alpha = k. \end{cases} \tag{5}$$

The main shortcoming of mesh-based methods such as the finite element method (FEM), the finite volume method (FVM) and the boundary element method (BEM) is that these numerical methods rely on meshes or elements. In the two last decades, in order to overcome the mentioned difficulties, some techniques, so-called meshless methods, have been proposed [11] which use a set of nodes scattered within the domain of the problem as well as sets of nodes on the boundaries of the domain to represent (but not to discretize) the domain of the problem. There are some classes of meshless methods: Meshless methods based on weak forms such as the element free Galerkin (EFG) method [12–14], meshless methods based on collocation techniques (strong forms) such as the meshless collocation method based on radial basis functions (RBFs) [11,15–19] and meshless methods based on the combination of weak forms and collocation technique.

In the literature, various type of meshless weak form methods have been reported, *e.g.*, the diffuse element method (DEM) [20], the smooth particle hydrodynamic (SPH) [21, 22], the reproducing kernel particle method (RKPM) [23], the boundary node method (BNM) [24], the partition of unity finite element method (PUFEM) [25], the finite sphere method (FSM) [26], the boundary point interpolation method (BPIM) [27] and the boundary radial point interpolation method (BRPIM) [28]. The weak forms are used to derive a set of algebraic equations through a numerical integration process using a set of quadrature domain that may be constructed globally or locally in the domain of the problem. In the global weak form methods, global background cells are needed for numerical integration in computing the algebraic equations.

To avoid the use of global background cells, Atluri proposed a local weak form and presented the meshless local Petrov-Galerkin (MLPG) method [29–39]. Liu applied the concept of MLPG and developed a meshless local radial point interpolation (MLRPI) method [14, 40–46]. When a local weak form is used for a field node, the numerical integrations are carried out over a local quadrature domain defined for the node, which also is the local domain. The local domain usually has a regular and simple shape for an internal node (such as sphere, square, etc.), and the integration is done numerically within the local domain. Hence the domain and boundary integrals in the weak form methods can easily be evaluated over the regularly shaped sub-domains and their boundaries.

In this paper, we focus on the numerical solution of the eqs. (1)–(4) using the meshless local radial point interpolation (MLRPI) method. Two illustrative examples are given and satisfactory results are achieved.

2 The time discretization approximation

According to eq. (5), $\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}$ can be written as follows:

$$\frac{\partial^\alpha u(\mathbf{x}, t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(2-\alpha)} \displaystyle\int_0^t \frac{\partial^2 u(\mathbf{x}, \xi)}{\partial \xi^2} (t-\xi)^{1-\alpha}\, \mathrm{d}\xi, & 1 < \alpha < 2, \\ \dfrac{\partial^2 u(\mathbf{x}, t)}{\partial t^2}, & \alpha = 2, \end{cases} \tag{6}$$

and, regarding $(0 < \alpha - 1 < 1)$, we have

$$\frac{\partial^{\alpha-1} u(\mathbf{x}, t)}{\partial t^{\alpha-1}} = \begin{cases} \dfrac{1}{\Gamma(2-\alpha)} \displaystyle\int_0^t \frac{\partial u(\mathbf{x}, \xi)}{\partial \xi} (t-\xi)^{1-\alpha}\, \mathrm{d}\xi, & 1 < \alpha < 2, \\ \dfrac{\partial u(\mathbf{x}, t)}{\partial t}, & \alpha = 2. \end{cases} \tag{7}$$

In order to discretize the problem for $1 < \alpha < 2$ in the time direction, we substitute $t^{(n+1)}$ into eqs. (6) and (7), then the integrals can be partitioned as

$$\begin{aligned} \frac{\partial^\alpha u\left(\mathbf{x}, t^{(n+1)}\right)}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t^{(n+1)}} \frac{\partial^2 u(\mathbf{x}, \xi)}{\partial \xi^2} \left(t^{(n+1)} - \xi\right)^{1-\alpha} \mathrm{d}\xi \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \int_{t^{(k)}}^{t^{(k+1)}} \frac{\partial^2 u(\mathbf{x}, \xi)}{\partial \xi^2} \left(t^{(n+1)} - \xi\right)^{1-\alpha} \mathrm{d}\xi \end{aligned} \tag{8}$$

and

$$\frac{\partial^{\alpha-1}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha-1}} = \frac{1}{\Gamma(2-\alpha)}\int_0^{t^{(n+1)}}\frac{\partial u(\mathbf{x},\xi)}{\partial\xi}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\,\mathrm{d}\xi$$
$$= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\int_{t^{(k)}}^{t^{(k+1)}}\frac{\partial u(\mathbf{x},\xi)}{\partial\xi}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\,\mathrm{d}\xi, \tag{9}$$

where $t^{(0)}=0$, $t^{(n+1)}=t^{(n)}+\triangle t$, $n=0,1,2,\ldots,M$. Also, n can be increased to the time length with $\triangle t$ as the time step which $M\triangle t=T$. Approximations of the first- and second-order derivatives due to the forward finite difference formulae are defined as

$$\frac{\partial^2 u(\mathbf{x},\sigma)}{\partial t^2} = \frac{u\left(\mathbf{x},t^{(n+1)}\right)-2u\left(\mathbf{x},t^{(n)}\right)+u\left(\mathbf{x},t^{(n-1)}\right)}{\triangle t^2}+o(\triangle t^2), \tag{10}$$

$$\frac{\partial u(\mathbf{x},\sigma)}{\partial t} = \frac{u\left(\mathbf{x},t^{(n+1)}\right)-u\left(\mathbf{x},t^{(n)}\right)}{\triangle t}+o(\triangle t), \tag{11}$$

where $\sigma\in[t^{(n)},t^{(n+1)}]$. Replacement of eqs. (10) and (11) into eqs. (8) and (9), respectively, gives

$$\frac{\partial^{\alpha}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha}} = \frac{1}{\Gamma(2-\alpha)}\int_0^{t^{(n+1)}}\frac{\partial^2 u(\mathbf{x},\xi)}{\partial\xi^2}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\,\mathrm{d}\xi$$
$$= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\frac{u^{(k+1)}-2u^{(k)}+u^{(k-1)}}{\triangle t^2}\int_{t^{(k)}}^{t^{(k+1)}}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\,\mathrm{d}\xi, \tag{12}$$

and

$$\frac{\partial^{\alpha-1}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha-1}} = \frac{1}{\Gamma(2-\alpha)}\int_0^{t^{(n+1)}}\frac{\partial u(\mathbf{x},\xi)}{\partial\xi}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\,\mathrm{d}\xi$$
$$= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\frac{u^{(k+1)}-u^{(k)}}{\triangle t}\int_{t^{(k)}}^{t^{(k+1)}}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\,\mathrm{d}\xi, \tag{13}$$

where $u^{(k)}=u(\mathbf{x},t^{(k)})$, $k=0,1,2,\ldots,M$. By considering $t^{n+1}-\xi=r$, the integral is easily obtained as

$$\int_{t^{(k)}}^{t^{(k+1)}}\left(t^{(n+1)}-\xi\right)^{1-\alpha}\mathrm{d}\xi = \frac{-1}{(2-\alpha)}r^{2-\alpha}\Big|_{t^{(n-k+1)}}^{t^{(n-k)}} = \frac{1}{(2-\alpha)}\triangle t^{2-\alpha}\left[(n-k+1)^{2-\alpha}-(n-k)^{2-\alpha}\right]. \tag{14}$$

Rearrangement of eqs. (12) and (13) by assumption $b_k=(k+1)^{2-\alpha}-(k)^{2-\alpha}$ leads to

$$\frac{\partial^{\alpha}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha}} = \frac{\triangle t^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=0}^{n}b_k\left[u^{(n-k+1)}-2u^{(n-k)}+u^{(n-k-1)}\right]$$
$$= a_0\left\{u^{(n+1)}-2u^{(n)}+u^{(n-1)}+\sum_{k=1}^{n}b_k\left[u^{(n-k+1)}-2u^{(n-k)}+u^{(n-k-1)}\right]\right\}, \tag{15}$$

and

$$\frac{\partial^{\alpha-1}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha-1}} = \frac{\triangle t^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{k=0}^{n}b_k\left[u^{(n-k+1)}-u^{(n-k)}\right]$$
$$= a_0'\left\{u^{(n+1)}-u^{(n)}+\sum_{k=1}^{n}b_k\left[u^{(n-k+1)}-u^{(n-k)}\right]\right\}, \tag{16}$$

where $a_0=\frac{\triangle t^{-\alpha}}{\Gamma(3-\alpha)}$, $a_0'=\frac{\triangle t^{1-\alpha}}{\Gamma(3-\alpha)}$ and $n=0,1,2,\ldots,M$. We note that eq. (1) at $t=t^{(n+1)}$ due to the θ-weighted finite difference formulation is as follows:

$$\frac{\partial^{\alpha}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha}}+\gamma_1\frac{\partial^{\alpha-1}u\left(\mathbf{x},t^{(n+1)}\right)}{\partial t^{\alpha-1}}+\gamma_2 u^{(n+1)} = \gamma_3\left[\theta\Delta u^{(n+1)}+(1-\theta)\Delta u^{(n)}\right]+f^{(n+1)}, \tag{17}$$

where, $0 < \theta < 1$ is a constant, $\Delta u^{(n)} = \Delta u(\mathbf{x}, t^{(n)})$ and $f^{(n)} = f(\mathbf{x}, t^{(n)})$. We set $\theta = \frac{1}{2}$ for simplicity, and substitute eqs. (15) and (16) into eq. (17), then we obtain

$$\begin{aligned} & a_0 \left\{ u^{(n+1)} - 2u^{(n)} + u^{(n-1)} + \sum_{k=1}^{n} b_k \left[u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)} \right] \right\} \\ & + \gamma_1 a_0' \left\{ u^{(n+1)} - u^{(n)} + \sum_{k=1}^{n} b_k \left[u^{(n-k+1)} - u^{(n-k)} \right] \right\} + \gamma_2 u^{(n+1)} = \\ & \frac{1}{2}\gamma_3 \left[\Delta u^{(n+1)} + \Delta u^{(n)} \right] + f^{(n+1)}, \end{aligned} \tag{18}$$

or, equivalently,

$$\begin{aligned} \frac{1}{2}\gamma_3 \Delta u^{(n+1)} - (\gamma_2 + a_0 + \gamma_1 a_0')\, u^{(n+1)} = & -\frac{1}{2}\gamma_3 \Delta u^{(n)} - (2a_0 + \gamma_1 a_0')\, u^{(n)} + a_0 u^{(n-1)} \\ & + \sum_{k=1}^{n} a_0 b_k \left[u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)} \right] \\ & + \sum_{k=1}^{n} \gamma_1 a_0' b_k \left[u^{(n-k+1)} - u^{(n-k)} \right] - f^{(n+1)}. \end{aligned} \tag{19}$$

3 The modified radial point interpolation scheme

In the conventional point interpolation method (PIM) there is a main difficulty that inverse of the polynomial moment matrix (it will be defined later) does not often exist. This condition could always be possible depending on the locations of the nodes in the support domain and the terms of monomials used in the basis. If an inappropriate polynomial basis is chosen for a given set of nodes, it may yield in a badly conditioned or even singular moment matrix [11]. In order to avoid the singularity problem in the polynomial point interpolation method (PIM), the radial basis function (RBF) is used to develop the radial point interpolation method (RPIM) for meshless weak form techniques [41,47,48]. The combination of RPIM and polynomial PIM is described as following: Consider a continuous function $u(\mathbf{x})$ defined in a domain Ω, which is represented by a set of field nodes. The $u(\mathbf{x})$ at a point of interest $\mathbf{x}$ is approximated in the form of

$$u(\mathbf{x}) = \sum_{i=1}^{n} R_i(\mathbf{x}) a_i + \sum_{j=1}^{m} p_j(\mathbf{x}) b_j = \mathbf{R}^T(\mathbf{x})\mathbf{a} + \mathbf{P}^T(\mathbf{x})\mathbf{b}, \tag{20}$$

where $R_i(\mathbf{x})$ is a radial basis function (RBF), n is the number of RBFs, $p_j(\mathbf{x})$ is monomial in the space coordinate $x^T = [x, y]$, and m is the number of polynomial basis functions. The $p_j(\mathbf{x})$ in eq. (20) is built using Pascal's triangle and a complete basis is usually preferred. the linear basis functions are given by

$$\mathbf{P}^T(\mathbf{x}) = \{1, x, y\}, \qquad m = 3, \tag{21}$$

and the quadratic basis functions are

$$\mathbf{P}^T(\mathbf{x}) = \left\{1, x, y, x^2, y^2, xy\right\}, \qquad m = 6. \tag{22}$$

When $m = 0$, only RBFs are used. Otherwise, the RBF is augmented with m polynomial basis functions. Coefficients a_i and b_j are unknown which should be determined. There are a number of types of RBFs, and the characteristics of RBFs have been widely investigated [15,49,50]. In the current work, we have chosen the thin plate spline (TPS) as radial basis functions in eq. (20). This RBF is defined as follows:

$$R(\mathbf{x}) = r^{2m} \ln(r), \qquad m = 1, 2, 3, \dots . \tag{23}$$

Since $R(\mathbf{x})$ in eq. (23) belongs to C^{2m-1} (all continuous function to the order $2m-1$), so higher-order thin plate splines must be used for higher-order partial differential operators. For the second-order partial differential equation (1), $m = 2$ is used for thin plate splines (*i.e.* second-order thin plate splines). In the radial basis function $R_i(\mathbf{x})$, the variable is only the distance between the point of interest x and a node at $\mathrm{x}_i = (x_i, y_i)$, *i.e.* $r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ for 2-D. In order to determine a_i and b_j in eq. (20), a support domain is formed for the point of interest at $\mathbf{x}$, and n field nodes are included in the support domain (see fig. 1). Coefficients a_i and b_j in eq. (20) can be determined by enforcing

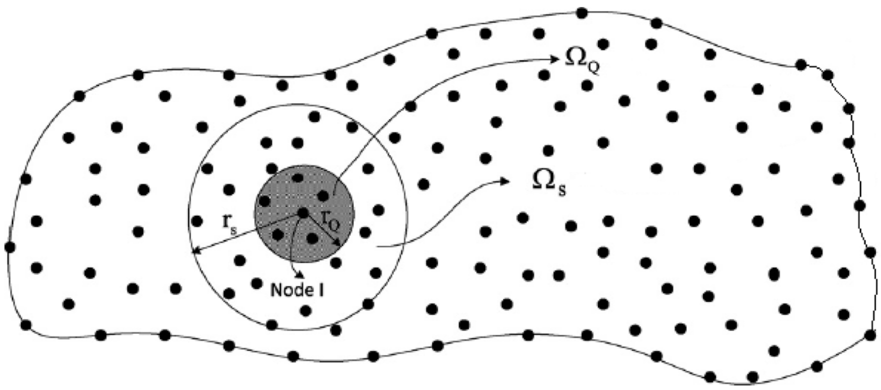

Fig. 1. Node I is an interior node. Ω_s and Ω_Q are local support and local quadrature domains, respectively.

eq. (20) to be satisfied at these n nodes surrounding the point of interest $\mathbf{x}$. This leads to the system of n linear equations, one for each node. The matrix form of these equations can be expressed as

$$\mathbf{U}_s = \mathbf{R}_n\mathbf{a} + \mathbf{P}_m\mathbf{b}, \tag{24}$$

where the vector of function values $\mathbf{U}_s$ is

$$\mathbf{U}_s = \{u_1\ u_2\ u_3\ \dots\ u_n\}^T, \tag{25}$$

the RBFs moment matrix is

$$\mathbf{R}_n = \begin{pmatrix} R_1(r_1) & R_2(r_1) & \dots & R_n(r_1) \\ R_1(r_2) & R_2(r_2) & \dots & R_n(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_1(r_n) & R_2(r_n) & \dots & R_n(r_n) \end{pmatrix}_{n\times n}, \tag{26}$$

and the polynomial moment matrix is

$$\mathbf{P}_m = \begin{pmatrix} 1 & x_1 & y_1 & \dots & p_m(\mathbf{x}_1) \\ 1 & x_2 & y_2 & \dots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & y_n & \dots & p_m(\mathbf{x}_n) \end{pmatrix}_{n\times m}. \tag{27}$$

Also, the vector of unknown coefficients for RBFs is

$$\mathbf{a}^T = \{a_1\ a_2\ \dots\ a_n\} \tag{28}$$

and the vector of unknown coefficients for polynomial is

$$\mathbf{b}^T = \{b_1\ b_2\ \dots\ b_m\}. \tag{29}$$

We notify that in eq. (26), r_k in $R_i(r_k)$ is defined as

$$r_k = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}. \tag{30}$$

We mention that there are $m + n$ variables in eq. (24). The additional m equations can be added using the following m constraint conditions:

$$\sum_{i=1}^{n} p_j(\mathbf{x}_i)a_i = \mathbf{P}_m^T\mathbf{a} = 0, \qquad j = 1, 2, \dots, m. \tag{31}$$

Combining eqs. (24) and (31) yields the following system of equations in the matrix form:

$$\tilde{\mathbf{U}}_s = \begin{pmatrix} \mathbf{U}_s \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_n & \mathbf{P}_m \\ \mathbf{P}_m^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \mathbf{G}\tilde{\mathbf{a}}, \tag{32}$$

where

$$\tilde{\mathbf{U}}_s = \{u_1\ u_2\ \dots\ u_n\ 0\ 0\ \dots\ 0\}, \qquad \tilde{\mathbf{a}}^T = \{a_1\ a_2\ \dots\ a_n\ b_1\ b_2\ \dots\ b_m\}. \tag{33}$$

Because the matrix $\mathbf{R}_n$ is symmetric, the matrix G will also be symmetric. Solving eq. (32), we obtain

$$\tilde{\mathbf{a}} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = G^{-1}\tilde{\mathbf{U}}_s. \tag{34}$$

Equation (20) can be rewritten as

$$u(\mathbf{x}) = \mathbf{R}^T(\mathbf{x})\mathbf{a} + \mathbf{P}^T(\mathbf{x})\mathbf{b} = \left\{\mathbf{R}^T(\mathbf{x})\mathbf{P}^T(\mathbf{x})\right\} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}. \tag{35}$$

Now, using eq. (34), we obtain

$$u(\mathbf{x}) = \left\{\mathbf{R}^T(\mathbf{x})\ \mathbf{P}^T(\mathbf{x})\right\} G^{-1}\tilde{\mathbf{U}}_s = \tilde{\mathbf{\Phi}}^T(\mathbf{x})\tilde{\mathbf{U}}_s, \tag{36}$$

where $\tilde{\mathbf{\Phi}}^T(\mathbf{x})$ can be rewritten as

$$\tilde{\mathbf{\Phi}}^T(\mathbf{x}) = \{\mathbf{R}^T(\mathbf{x})\ \mathbf{P}^T(\mathbf{x})\}G^{-1} = \{\phi_1(\mathbf{x})\ \phi_2(\mathbf{x})\ \dots\ \phi_n(\mathbf{x})\ \phi_{n+1}(\mathbf{x})\ \dots\ \phi_{n+m}(\mathbf{x})\}. \tag{37}$$

The first n functions of the above vector function are called the RPIM shape functions corresponding to the nodal displacements and we show by the vector $\mathbf{\Phi}^T(\mathbf{x})$ so that it is

$$\mathbf{\Phi}^T(\mathbf{x}) = \{\phi_1(\mathbf{x})\ \phi_2(\mathbf{x})\ \dots\ \phi_n(\mathbf{x})\}, \tag{38}$$

then eq. (36) is converted to the following one:

$$u(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x})\mathbf{U}_s = \sum_{i=1}^{n} \phi_i(\mathbf{x})u_i. \tag{39}$$

The derivatives of $u(\mathbf{x})$ are easily obtained as

$$\frac{\partial u(\mathbf{x})}{\partial x} = \sum_{i=1}^{n} \frac{\partial \phi_i(\mathbf{x})}{\partial x}u_i, \qquad \frac{\partial u(\mathbf{x})}{\partial y} = \sum_{i=1}^{n} \frac{\partial \phi_i(\mathbf{x})}{\partial y}u_i. \tag{40}$$

Note that $\mathbf{R}_n^{-1}$ usually exists for arbitrary scattered nodes and therefore the augmented matrix G is theoretically non-singular [51, 52]. In addition, the order of polynomial used in eq. (20) is relatively low. We add that the RPIM shape functions have the Kronecker delta function property, that is

$$\phi_i(\mathbf{x}_j) = \begin{cases} 1, & i = j, \quad j = 1, 2, \dots, n, \\ 0, & i \neq j, \quad i, j = 1, 2, \dots, n. \end{cases} \tag{41}$$

This is because the RPIM shape functions are created to pass through nodal values.

4 The meshless local weak form formulation

Instead of giving the global weak form, the meshless local Galerkin weak form method constructs the weak form over local quadrature cell such as Ω_q, which is a small region taken for each node in the global domain Ω (see fig. 1). The local quadrature cells overlap each other and cover the whole global domain Ω. The local quadrature cells could be of any geometric shape and size. In this paper they are taken to be of circular shape. The local weak form of eq. (19) for $\mathbf{x}_i = (x_i, y_i) \in \Omega_q^i$ can be written as

$$\begin{aligned}
&\int_{\Omega_q^i} \left[\frac{1}{2}\gamma_3 \Delta u^{(n+1)} - (\gamma_2 + a_0 + \gamma_1 a_0')u^{(n+1)}\right] v(\mathbf{x})\mathrm{d}\Omega = \\
&\int_{\Omega_q^i} \left[-\frac{1}{2}\gamma_3 \Delta u^{(n)} - (2a_0 + \gamma_1 a_0')u^{(n)} + a_0 u^{(n-1)}\right] v(\mathbf{x})\mathrm{d}\Omega \\
&+ \int_{\Omega_q^i} \left(\sum_{k=1}^{n} a_0 b_k [u^{(n-k+1)} - 2u^{(n-k)} + u^{(n-k-1)}]\right) v(\mathbf{x})\,\mathrm{d}\Omega \\
&+ \int_{\Omega_q^i} \left(\sum_{k=1}^{n} \gamma_1 a_0' b_k [u^{(n-k+1)} - u^{(n-k)}]\right) v(\mathbf{x})\,\mathrm{d}\Omega - \int_{\Omega_q^i} f^{(n+1)} v(\mathbf{x})\,\mathrm{d}\Omega,
\end{aligned} \tag{42}$$

where Ω_q^i is the local quadrature domain associated with the point i, *i.e.*, it is a circle centered at $\mathbf{x}_i$ of radius r_q and, $v(\mathbf{x})$ is the Heaviside step function [53,54],

$$v(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_q, \\ 0, & \mathbf{x} \notin \Omega_q, \end{cases} \tag{43}$$

as the test function in each local quadrature domain. Using the divergence theorem, eq. (42) yields the following expression:

$$\begin{aligned}
&-(\gamma_2 + a_0 + \gamma_1 a_0') \int_{\Omega_q^i} u^{(n+1)} v(\mathbf{x})\, \mathrm{d}\Omega - \frac{1}{2}\gamma_3 \int_{\Omega_q^i} \nabla u^{(n+1)} \nabla v\, \mathrm{d}\Omega + \frac{1}{2}\gamma_3 \int_{\partial\Omega_q^i} v \frac{\partial u^{(n+1)}}{\partial n}\, \mathrm{d}\Gamma = \\
&\frac{1}{2}\gamma_3 \int_{\Omega_q^i} \nabla u^{(n)} \nabla v\, \mathrm{d}\Omega - \frac{1}{2}\gamma_3 \int_{\partial\Omega_q^i} v \frac{\partial u^{(n)}}{\partial n}\, \mathrm{d}\Gamma - (2a_0 + \gamma_1 a_0') \int_{\Omega_q^i} u^{(n)} v(\mathbf{x})\, \mathrm{d}\Omega + a_0 \int_{\Omega_q^i} u^{(n-1)} v(\mathbf{x})\, \mathrm{d}\Omega \\
&+ \sum_{k=1}^{n} a_0 b_k \left[\int_{\Omega_q^i} u^{(n-k+1)} v(\mathbf{x})\, \mathrm{d}\Omega - 2\int_{\Omega_q^i} u^{(n-k)} v(\mathbf{x}) \mathrm{d}\Omega + \int_{\Omega_q^i} u^{(n-k-1)} v(\mathbf{x}) \mathrm{d}\Omega \right] \\
&+ \sum_{k=1}^{n} \gamma_1 a_0' b_k \left[\int_{\Omega_q^i} u^{(n-k+1)} v(\mathbf{x})\, \mathrm{d}\Omega - \int_{\Omega_q^i} u^{(n-k)} v(\mathbf{x})\, \mathrm{d}\Omega \right] \\
&- \int_{\Omega_q^i} f^{(n+1)} v(\mathbf{x})\, \mathrm{d}\Omega,
\end{aligned} \tag{44}$$

where $\partial\Omega_q^i$ is the boundary of Ω_q^i, $n = (n_1, n_2)$ is the outward unit normal to the boundary $\partial\Omega_q^i$, and

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_1 + \frac{\partial u}{\partial y} n_2$$

is the normal derivative, *i.e.*, the derivative in the outward normal direction to the boundary $\partial\Omega_q^i$. Because the derivative of the Heaviside step function $v(\mathbf{x})$ is equal to zero, then the local weak form equation (44) is changed into the following simple integral equation:

$$\begin{aligned}
&-(\gamma_2 + a_0 + \gamma_1 a_0') \int_{\Omega_q^i} u^{(n+1)}\, \mathrm{d}\Omega + \frac{1}{2}\gamma_3 \int_{\partial\Omega_q^i} \frac{\partial u^{(n+1)}}{\partial n}\, \mathrm{d}\Gamma = \\
&-\frac{1}{2}\gamma_3 \int_{\partial\Omega_q^i} \frac{\partial u^{(n)}}{\partial n}\, \mathrm{d}\Gamma + a_0 \int_{\Omega_q^i} u^{(n-1)}\, \mathrm{d}\Omega - (2a_0 + \gamma_1 a_0') \int_{\Omega_q^i} u^{(n)}\, \mathrm{d}\Omega \\
&+ \sum_{k=1}^{n} a_0 b_k \left[\int_{\Omega_q^i} u^{(n-k+1)}\, \mathrm{d}\Omega - 2\int_{\Omega_q^i} u^{(n-k)}\, \mathrm{d}\Omega + \int_{\Omega_q^i} u^{(n-k-1)}\, \mathrm{d}\Omega \right] \\
&+ \sum_{k=1}^{n} \gamma_1 a_0' b_k \left[\int_{\Omega_q^i} u^{(n-k+1)}\, \mathrm{d}\Omega - \int_{\Omega_q^i} u^{(n-k)}\, \mathrm{d}\Omega \right] - \int_{\Omega_q^i} f^{(n+1)}\, \mathrm{d}\Omega.
\end{aligned} \tag{45}$$

Applying the moving least squares (MLS) approximation for the unknown functions, the local integral equation (45) is transformed into a system of algebraic equations with used unknown quantities, as described in the next section.

5 Discretization and numerical implementation of MLRPI method

In this section, we consider eq. (45) to see how to obtain discrete equations. Consider N regularly located points on the boundary and domain of the problem ($\Omega = [-1,1]^2 = \{(x,y) : -1 \le x, y \le 1\}$) so that the distance between to consecutive nodes in each direction is constant and equal to h. Assuming that $u(\mathbf{x}_i, k\triangle t)$ for all $k = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, N$ are known, our aim is to compute $u(\mathbf{x}_i, (n+1)\triangle t)$, $i = 1, 2, \ldots, N$. So, we have N unknowns and to compute these unknowns we need N equations. As it will be described, corresponding to each node we obtain one equation. For nodes which are located in the interior of the domain, *i.e.*, for $\mathbf{x}_i \in$ interior Ω, to obtain the discrete equations from the locally weak forms (45), substituting approximation formula (39) into local integral equations (45) yields:

$$\begin{aligned}
&-(\gamma_2+a_0+\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n+1)}+\frac{1}{2}\gamma_3\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\,\mathrm{d}\Gamma\right)u_j^{(n+1)}=\\
&-\frac{1}{2}\gamma_3\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\,\mathrm{d}\Gamma\right)u_j^{(n)}-(2a_0+\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n)}+a_0\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n-1)}\\
&+\sum_{k=1}^{n}a_0 b_k\left[\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n-k+1)}-2\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n-k)}+\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n-k-1)}\right]\\
&+\sum_{k=1}^{n}\gamma_1 a_0' b_k\left[\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n-k+1)}-\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(n-k)}\right]-\int_{\Omega_q^i}f^{(n+1)}\,\mathrm{d}\Omega. \qquad (46)
\end{aligned}$$

We had supposed $b_k=(k+1)^{2-\alpha}-(k)^{2-\alpha}$, $k=1,2,\ldots,n$ in sect. 3, in addition, we assume $b_{-1}=0$ and $b_0=1$. By these assumptions eq. (46) is converted to the following equation

$$\begin{aligned}
&\left[-(\gamma_2+a_0+\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)+\frac{1}{2}\gamma_3\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\,\mathrm{d}\Gamma\right)\right]u_j^{(n+1)}=\\
&\sum_{s=1}^{n}\left\{[\gamma_1 a_0'(b_{n-s+1}-b_{n-s})+a_0(b_{n-s-1}-2b_{n-s}+b_{n-s+1})]\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(s)}\right\}\\
&+[a_0(b_{n-1}-2b_n)-\gamma_1 a_0' b_n]\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(0)}+a_0 b_n\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(-1)}\\
&-\frac{1}{2}\gamma_3\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\,\mathrm{d}\Gamma\right)u_j^{(n)}-\int_{\Omega_q^i}f^{(n+1)}\,\mathrm{d}\Omega. \qquad (47)
\end{aligned}$$

According to the initial conditions that were introduced in eq. (2), we apply the following assumptions:

$$u_j^{(0)}=\varphi(\mathbf{x}_j),\qquad u_j^{(-1)}=u_j^{(1)}-2\triangle t\psi(\mathbf{x}_j), \qquad (48)$$

where the second relation is the result of central finite difference formula, then we conclude the following

$$\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(0)}=\int_{\Omega_q^i}\varphi(\mathbf{x})\,\mathrm{d}\Omega, \qquad (49)$$

$$\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(-1)}=\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(1)}-2\triangle t\int_{\Omega_q^i}\psi(\mathbf{x})\,\mathrm{d}\Omega. \qquad (50)$$

Therefore, applying eqs. (49) and (50) into (47) yields

$$\begin{aligned}
&\left[-(\gamma_2+a_0+\gamma_1 a_0')\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)+\frac{1}{2}\gamma_3\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\,\mathrm{d}\Gamma\right)\right]u_j^{(n+1)}=\\
&\sum_{s=1}^{n}\left\{[\gamma_1 a_0'(b_{n-s+1}-b_{n-s})+a_0(b_{n-s-1}-2b_{n-s}+b_{n-s+1})]\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(s)}\right\}\\
&+a_0 b_n\sum_{j=1}^{N}\left(\int_{\Omega_q^i}\phi_j\,\mathrm{d}\Omega\right)u_j^{(1)}-\frac{1}{2}\gamma_3\sum_{j=1}^{N}\left(\int_{\partial\Omega_q^i}\frac{\partial\phi_j}{\partial n}\,\mathrm{d}\Gamma\right)u_j^{(n)}-\int_{\Omega_q^i}f^{(n+1)}\,\mathrm{d}\Omega\\
&+[a_0(b_{n-1}-2b_n)-\gamma_1 a_0' b_n]\int_{\Omega_q^i}\varphi(\mathbf{x})\,\mathrm{d}\Omega-2a_0 b_n\triangle t\int_{\Omega_q^i}\psi(\mathbf{x})\,\mathrm{d}\Omega. \qquad (51)
\end{aligned}$$

For nodes which are located on the boundary, we have the following cases:

(b1) For nodes on the boundary $-1 \leq y \leq 1$ and $x = -1$, using eq. (3) we have

$$u^{n+1}(\mathbf{x}_i) = g_0(\mathbf{x}_i, (n+1)\triangle t). \tag{52}$$

(b2) For nodes on the boundary $-1 \leq y \leq 1$ and $x = 1$, using eq. (3) we have

$$u^{n+1}(\mathbf{x}_i) = g_1(\mathbf{x}_i, (n+1)\triangle t). \tag{53}$$

(b3) For nodes on the boundary $-1 \leq x \leq 1$ and $y = -1$, using eq. (4) we have

$$u^{n+1}(\mathbf{x}_i) = h_0(\mathbf{x}_i, (n+1)\triangle t). \tag{54}$$

(b4) For nodes on the boundary $-1 \leq x \leq 1$ and $y = 1$, using eq. (4) we have

$$u^{n+1}(\mathbf{x}_i) = h_1(\mathbf{x}_i, (n+1)\triangle t). \tag{55}$$

The matrix forms of eqs. (51)–(55) for all N nodal points in domain and on boundary of the problem are given below:

$$\begin{aligned}
&\left[-(\gamma_2 + a_0 + \gamma_1 a_0') \sum_{j=1}^{N} A_{ij} + \frac{1}{2}\gamma_3 \sum_{j=1}^{N} B_{ij}\right] u_j^{(n+1)} = \\
&\sum_{s=1}^{n} \left\{ [\gamma_1 a_0'(b_{n-s+1} - b_{n-s}) + a_0(b_{n-s-1} - 2b_{n-s} + b_{n-s+1})] \sum_{j=1}^{N} A_{ij} u_j^{(s)} \right\} + a_0 b_n \sum_{j=1}^{N} A_{ij} u_j^{(1)} \\
&- \frac{1}{2}\gamma_3 \sum_{j=1}^{N} B_{ij} u_j^{(n)} - F_i^{(n+1)} + [a_0(b_{n-1} - 2b_n) - \gamma_1 a_0' b_n]\, \Phi_i - 2a_0 b_n \triangle t \Psi_i,
\end{aligned} \tag{56}$$

where

$$A_{ij} = \int_{\Omega_q^i} \phi_j \, \mathrm{d}\Omega, \qquad B_{ij} = \int_{\partial\Omega_q^i} \frac{\partial \phi_j}{\partial n} \, \mathrm{d}\Gamma, \qquad F_i^{(n+1)} = \int_{\Omega_q^i} F(\mathbf{x}, (n+1)\triangle t) \mathrm{d}\Omega, \tag{57}$$

$$\Phi_i = \int_{\Omega_q^i} \varphi(\mathbf{x}) \, \mathrm{d}\Omega, \qquad \Psi_i = \int_{\Omega_q^i} \psi(\mathbf{x}) \, \mathrm{d}\Omega. \tag{58}$$

By considering the following notations,

$$\begin{aligned}
\mathbf{A}_{ij} &= -(\gamma_2 + a_0 + \gamma_1 a_0') \sum_{j=1}^{N} A_{ij} + \frac{1}{2}\gamma_3 \sum_{j=1}^{N} B_{ij}, \\
\alpha_{n,s} &= \gamma_1 a_0'(b_{n-s+1} - b_{n-s}) + a_0(b_{n-s-1} - 2b_{n-s} + b_{n-s+1}), \\
\beta_n &= a_0 b_n, \\
\lambda &= -\frac{1}{2}\gamma_3, \\
\delta_n &= a_0(b_{n-1} - 2b_n) - \gamma_1 a_0' b_n, \\
\mu_n &= -2a_0 b_n \triangle t, \\
\mathbf{F}^{n+1} &= \left[F_1^{(n+1)}, F_2^{(n+1)}, \ldots, F_N^{(n+1)}\right]^T, \\
\Phi &= [\Phi_1, \Phi_2, \ldots, \Phi_N]^T, \\
\Psi &= [\Psi_1, \Psi_2, \ldots, \Psi_N]^T, \\
U^{n+1} &= \left[u_1^{(n+1)}, u_2^{(n+1)}, \ldots, u_N^{(n+1)}\right]^T,
\end{aligned}$$

eq. (56) changes to the following matrix form:

$$\mathbf{A}U^{(n+1)} = \lambda BU^{(n)} + \sum_{s=1}^{n} \left\{ \alpha_{n,s} AU^{(s)} \right\} + \beta_n AU^{(1)} + \delta_n \Phi + \mu_n \Psi - \mathbf{F}^{n+1}. \tag{59}$$

Furthermore, to satisfy eqs. (52)–(55), for all nodes belonging to the boundary, *i.e.* $\mathbf{x}_i \in \partial\Omega$, we set, for each step,

$$\Phi_i = \Psi_i = 0, \qquad \forall j : A_{ij} = B_{ij} = 0, \qquad \mathbf{A}_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \tag{60}$$

and, for the vector $\mathbf{F}^{(n+1)}$, we assign:

1) If $\mathbf{x}_i$ belongs to the boundary (b1) described in eqs. (52)–(55) then $\mathbf{F}_i^{(n+1)} = -g_0(\mathbf{x}_i, (n+1)\triangle t)$.
2) If $\mathbf{x}_i$ belongs to the boundary (b2) described in eqs. (52)–(55) then $\mathbf{F}_i^{(n+1)} = -g_1(\mathbf{x}_i, (n+1)\triangle t)$.
3) If $\mathbf{x}_i$ belongs to the boundary (b3) described in eqs. (52)–(55) then $\mathbf{F}_i^{(n+1)} = -h_0(\mathbf{x}_i, (n+1)\triangle t)$.
4) If $\mathbf{x}_i$ belongs to the boundary (b4) described in eqs. (52)–(55) then $\mathbf{F}_i^{(n+1)} = -h_1(\mathbf{x}_i, (n+1)\triangle t)$.

We notice here that, when $n = 0$, we use directly (46) and then, for $n > 0$, it is straightforward to use eq. (59).

6 Stability and convergency of the method

In this section, we give discussion on the stability and convergency of the method. Let us introduce $H^1(\Omega)$, $H_0^1(\Omega)$ and $H^m(\Omega)$ space which are helpful in our analysis. Let $\Omega \subseteq R^2$ be convex set [55]

$$\begin{aligned} H^1(\Omega) &:= \left\{ \upsilon \in L^2(\Omega), \quad D^{\mathbb{a}} \in L^2(\Omega), \quad |\mathbb{a}| < 1 \right\}, \\ H_0^1(\Omega) &:= \left\{ \upsilon \in H^1(\Omega), \quad \upsilon|_{\partial\Omega} = 0 \right\}, \\ H^m(\Omega) &:= \left\{ \upsilon \in L^2(\Omega), \quad D^{\mathbb{a}} \in L^2(\Omega), \quad \text{for all,} \quad |\mathbb{a}| < m \right\}, \end{aligned}$$

where $\Omega \subset R^d$, $d = 2, 3$. The inner products of $L^2(\Omega)$ and $H^1(\Omega)$ are defined, respectively, by

$$(u, \upsilon) = \int_\Omega u\upsilon \, \mathrm{d}\Omega, \qquad (u, \upsilon)_1 = (u, \upsilon) + (\nabla u, \nabla \upsilon), \tag{61}$$

and the norms in L^2 and H^1 are

$$\|\upsilon\| = (\upsilon, \upsilon)^{1/2}, \qquad |\upsilon|_1 = (\nabla\upsilon, \nabla\upsilon)^{1/2}. \tag{62}$$

Also, suppose $\mathbb{k} > 0$, then we define the weighted H^1 norm as

$$\|\upsilon\|_{\omega,1} = \left[\|\upsilon\|^2 + \mathbb{k}|\upsilon|_1^2 \right]^{1/2}. \tag{63}$$

For a positive N, let $\tau = \triangle t = T/N$, $t_n = n\tau (0 \leq n \leq N)$. The time domain $[0, T]$ is covered by $\{t_n | 0 \leq n \leq N\}$. Given the grid function $\mathbb{w} = \{\mathbb{w}^n | 0 \leq n \leq N\}$, from [56] and [57] we denote

$$\mathbb{w}^{n-1/2} = \frac{1}{2}(\mathbb{w}^n + \mathbb{w}^{n-1}), \qquad \partial_t \mathbb{w}^{n-1/2} = \frac{\mathbb{w}^n - \mathbb{w}^{n-1}}{\tau}. \tag{64}$$

Lemma 1 (see [58]). *Suppose* $1 < \alpha < 2$, $\mathbb{g} \in \mathcal{C}^2[0, T]$. *It holds that*

$$\left| \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\mathbb{g}'(\xi)}{(t_n - \xi)^{\alpha-1}} \mathrm{d}\xi - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[a_0 \mathbb{g}(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \mathbb{g}(t_k) - a_{n-1}\mathbb{g}(0) \right] \right| \leq$$
$$\frac{1}{\Gamma(3-\alpha)} \left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1 + 2^{1-\alpha}) \right] \max_{0 \leq t \leq t_n} |\mathbb{g}''(t)| \tau^{3-\alpha}, \tag{65}$$

where

$$a_k = (k+1)^{2-\alpha} - k^{2-\alpha}. \tag{66}$$

Lemma 2 (see [58]). *Suppose* $0 < \alpha - 1 < 1$, $\mathbb{g} \in \mathcal{C}^2[0,T]$. *It holds that*

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{\mathbb{g}'(\xi)}{(t_n-\xi)^\alpha}\,\mathrm{d}\xi - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0\mathbb{g}(t_n) - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathbb{g}(t_k) - a_{n-1}\mathbb{g}(0)\right]\right| \leq \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\right]\max_{0\leq t\leq t_n}|\mathbb{g}''(t)|\tau^{2-\alpha}, \tag{67}$$

where

$$b_k = (k+1)^{1-\alpha} - k^{1-\alpha}. \tag{68}$$

Let $\mathcal{V}(\mathbf{x},t) = u_t(\mathbf{x},t)$, eq. (1) is reduced to

$$\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial\mathcal{V}(\mathbf{x},\xi)}{\partial\xi}(t-\xi)^{1-\alpha}\,\mathrm{d}\xi + \gamma_1\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(\mathbf{x},\xi)}{\partial\xi}(t-\xi)^{-\alpha}\,\mathrm{d}\xi + \gamma_2 u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + f(\mathbf{x},t). \tag{69}$$

Let us assume the following notations:

$$\mathcal{V}(\mathbf{x},t_n) = \mathcal{V}^n, \qquad u(\mathbf{x},t_n) = u^n, \qquad f(\mathbf{x},t_n) = f^n, \qquad \mathbf{x}\in\Omega, \qquad 1\leq n\leq N.$$

Applying Lemmas 1 and 2, we have

$$\begin{aligned}&\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\mathcal{V}^n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathcal{V}^k - a_{n-1}U_t^0\right]\\&+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0u^n - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^k - b_{n-1}U^0\right] + \gamma_2u^n =\\&\Delta u^n + f^n + (\mathbb{R}_t^\alpha)^n + (\mathbb{r}_t^\alpha)^n, \qquad 1\leq n\leq N,\end{aligned}$$

where

$$|(\mathbb{R}_t^\alpha)^n| \leq \frac{1}{\Gamma(3-\alpha)}\left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1-2^{1-\alpha})\right]\max_{0\leq t\leq t_n}|u''(t)|\tau^{3-\alpha} \tag{70}$$

and

$$|(\mathbb{r}_t^\alpha)^n| \leq \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1-2^{-\alpha})\right]\max_{0\leq t\leq t_n}|u''(t)|\tau^{2-\alpha}. \tag{71}$$

Then, it holds, for $2\leq n\leq N$, that

$$\begin{aligned}&\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\mathcal{V}^{n-1} - \sum_{k=1}^{n-2}(a_{n-k-2}-a_{n-k-1})\mathcal{V}^k - a_{n-2}U_t^0\right]\\&+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0u^{n-1} - \sum_{k=1}^{n-2}(b_{n-k-2}-b_{n-k-1})u^k - b_{n-2}U^0\right] + \gamma_2u^{n-1} =\\&\Delta u^{n-1} + f^{n-1} + (\mathbb{R}_t^\alpha)^{n-1} + (\mathbb{r}_t^\alpha)^{n-1}, \qquad 2\leq n\leq N.\end{aligned}$$

On the other hand, it easily holds

$$\begin{aligned}&-\sum_{k=1}^{n-2}(a_{n-k-2}-a_{n-k-1})\mathcal{V}^k + a_{n-2}U_t^0 - \sum_{k=1}^{n-2}(b_{n-k-2}-b_{n-k-1})u^k + b_{n-2}U^0 =\\&-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathcal{V}^{k-1} + a_{n-1}U_t^0 - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^{k-1} + b_{n-1}U^0.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\mathcal{V}^{n-1} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathcal{V}^{k-1} - a_{n-1}U_t^0\right]\\&+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0u^{n-1} - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^{k-1} - b_{n-1}U^0\right] + \gamma_2u^{n-1} =\\&\Delta u^{n-1} + f^{n-1} + (\mathbb{R}_t^\alpha)^{n-1} + (\mathbb{r}_t^\alpha)^{n-1}, \qquad 2\leq n\leq N.\end{aligned}$$

Using the notations

$$\mathfrak{w}^n = \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\mathcal{V}^n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathcal{V}^k - a_{n-1}U_t^0\right] + \gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0u^n - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^k - b_{n-1}U^0\right] + (\mathbb{R}_t^\alpha)^n + (\mathbb{r}_t^\alpha)^n,$$

and $\mathfrak{w}^{n-1/2} = \frac{1}{2}(\mathfrak{w}^n + \mathfrak{w}^{n-1})$, we have the following equality:

$$\begin{aligned}\mathfrak{w}^{n-1/2} = \frac{1}{2}(\mathfrak{w}^n + \mathfrak{w}^{n-1}) = \\ &\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\mathcal{V}^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathcal{V}^{k-1/2} - a_{n-1}U_t^0\right]\\ &+ \gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0u^{n-1/2} - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^{k-1/2} - b_{n-1}U^0\right]\\ &+ (\mathbb{R}_t^\alpha)^{n-1/2} + (\mathbb{r}_t^\alpha)^{n-1/2}, \quad 1 \le n \le N. \end{aligned} \tag{72}$$

According to Lemmas 1 and 2

$$|\mathscr{R}_3^{n-1/2}| = |(\mathbb{R}_t^\alpha)^{n-1/2} + (\mathbb{r}_t^\alpha)^{n-1/2}| \le \mathcal{C}_3\tau^{3-\alpha} + \mathcal{C}_3\tau^{2-\alpha} \le \mathcal{C}_3\tau^{2-\alpha}. \tag{73}$$

Using the Taylor expansion and ref. [59], it follows that

$$\mathcal{V}^{n-1/2} = \partial_t u^{n-1/2} + (\mathscr{R}_1^1)^{n-1/2}, \tag{74}$$

$$u^{n-1/2} = \frac{u^n + u^{n-1}}{2} + (\mathscr{R}_1^2)^{n-1/2}, \tag{75}$$

$$\mathfrak{w}^{n-1/2} + \gamma_2 u^{n-1/2} = \Delta u^{n-1/2} + f^{n-1/2} + \mathscr{R}_2^{n-1/2}, \tag{76}$$

and, also, there is a constant $\mathcal{C}_1$ such that

$$|(\mathscr{R}_1^1)^{n-1/2}| \le \mathcal{C}_1\tau^2, \tag{77}$$

$$|(\mathscr{R}_1^2)^{n-1/2}| \le \mathcal{C}_1\tau^2. \tag{78}$$

Then, from refs. [60–63], we have

$$|\mathscr{R}_2^{n-1/2}| \le \mathcal{C}_2\tau^{3-\alpha} + \mathcal{C}_2\tau \le \mathcal{C}_2\tau. \tag{79}$$

Thus, from (77), (78) and (79), we can obtain

$$\begin{aligned}&\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\partial_t u^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\partial_t u^{k-1/2} - a_{n-1}U_t^0\right]\\ &+ \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0(\mathscr{R}_1^1)^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\mathscr{R}_1^1)^{k-1/2}\right]\\ &+ \gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0u^{n-1/2} - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^{k-1/2} - b_{n-1}U^0\right]\\ &+ \gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0(\mathscr{R}_1^2)^{n-1/2} - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\mathscr{R}_1^2)^{k-1/2}\right]\\ &+ \mathscr{R}_3^{n-1/2}, \qquad 1 \le n \le N,\end{aligned}$$

therefore, we derive

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\partial_t u^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\partial_t u^{k-1/2}-a_{n-1}U_t^0\right]$$
$$+\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0(\mathscr{R}_1^1)^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\mathscr{R}_1^1)^{k-1/2}\right]$$
$$+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0 u^{n-1/2}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})u^{k-1/2}-b_{n-1}U^0\right]$$
$$+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0(\mathscr{R}_1^2)^{n-1/2}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\mathscr{R}_1^2)^{k-1/2}\right]+\gamma_2 u^{n-1/2}=$$
$$\Delta u^{n-1/2}+f^{n-1/2}+\mathfrak{R}^{n-1/2},\qquad 1\le n\le N, \tag{80}$$

where

$$\mathfrak{R}^{n-1/2}=\left\{\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0(\mathscr{R}_1^1)^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\mathscr{R}_1^1)^{k-1/2}\right]\right.$$
$$+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0(\mathscr{R}_1^1)^{n-1/2}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\mathscr{R}_1^1)^{k-1/2}\right]$$
$$\left.+(\mathscr{R}_3)^{k-1/2}\right\}+(\mathscr{R}_2)^{k-1/2}, \tag{81}$$

then, we conclude,

$$|\mathfrak{R}^{n-1/2}|=\left[\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[\mathcal{C}_1\tau^2-(1-a_{n-k})\mathcal{C}_1\tau^2\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[\mathcal{C}_1\tau^2-(1-b_{n-k})\mathcal{C}_1\tau^2\right]\right]+\mathcal{C}_3\tau^{2-\alpha}+(\mathcal{C}_2\tau). \tag{82}$$

Finally, it holds

$$|\mathfrak{R}^{n-1/2}|\le\mathcal{C}\tau^{2-\alpha}. \tag{83}$$

Omitting the small term $\mathfrak{R}^{n-1/2}$ and substituting the function u^n with its numerical approximation $\mathcal{U}^n$ in (80), we obtain the following difference scheme:

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0\partial_t\mathcal{U}^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\partial_t\mathcal{U}^{k-1/2}-a_{n-1}U_t^0\right]$$
$$+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[b_0\mathcal{U}^{n-1/2}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\mathcal{U}^{k-1/2}-b_{n-1}U^0\right]+\gamma_2\mathcal{U}^{n-1/2}=$$
$$\Delta\mathcal{U}^{n-1/2}+f^{n-1/2},\qquad 1\le n\le N. \tag{84}$$

Also, the weak formulation of eq. (80) for $u\in H_0^1(\Omega)$ and $\nu\in H_0^1(\Omega)$ is

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0(\partial_t u^{n-1/2},\nu)-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\partial_t u^{k-1/2},\nu)-a_{n-1}(U_t^0,\nu)\right]$$
$$+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0(u^{n-1/2},\nu)-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(u^{k-1/2},\nu)-a_{n-1}(U^0,\nu)\right]+\gamma_2(u^{n-1/2},\nu)=$$
$$\Delta(u^{n-1/2},\nu)+(f^{n-1/2},\nu)+(\mathfrak{R}^{n-1/2},\nu),\qquad 1\le n\le N. \tag{85}$$

6.1 Stability

Theorem 1. *The scheme* (84) *is unconditionally stable in the sense that for all* $\tau > 0$, *it holds*

$$\|\mathcal{U}^n\|_{\omega,1} \leq \|\mathcal{U}^0\|_{\omega,1} + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)}\|U_t^0\|^2 + \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)}\|U^0\|^2 + 2\kappa t_n \max_{0\leq n\leq N}\|f^{n-1/2}\|. \tag{86}$$

Proof. Choosing $\nu = \partial_t \mathcal{U}^{n-1/2}$ in (85) and noticing $a_0 = b_0 = 1$, $\mu_1 = \Gamma(3-\alpha)\tau^{\alpha-1}$ and $\mu_2 = \Gamma(2-\alpha)\tau^{\alpha}$, we have following equation for $1 \leq n \leq N$:

$$\begin{aligned}
&\frac{1}{\mu_1}\|\partial_t\mathcal{U}^{n-1/2}\|^2 + \left(\nabla\mathcal{U}^{n-1/2}, \nabla\partial_t\mathcal{U}^{n-1/2}\right) + \frac{\gamma_1}{\mu_2}\left(\mathcal{U}^{n-1/2}, \partial_t\mathcal{U}^{n-1/2}\right) + \gamma_2\left(\mathcal{U}^{n-1/2}, \partial_t\mathcal{U}^{n-1/2}\right) = \\
&\frac{1}{\mu_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left(\partial\mathcal{U}_t^{k-1/2}, \partial_t\mathcal{U}^{n-1/2}\right) - a_{n-1}\left(U_t^0, \partial_t\mathcal{U}^{n-1/2}\right)\right] \\
&+\frac{\gamma_1}{\mu_2}\left[\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\left(\mathcal{U}^{k-1/2}, \partial_t\mathcal{U}^{n-1/2}\right) - b_{n-1}\left(U^0, \partial_t\mathcal{U}^{n-1/2}\right)\right] \\
&+\left(f^{n-1/2}, \partial_t\mathcal{U}^{n-1/2}\right), \qquad 1\leq n\leq N.
\end{aligned} \tag{87}$$

Since

$$(\nabla\mathcal{U}^{n-1/2}, \nabla\partial_t\mathcal{U}^{n-1/2}) = \left(\frac{\mathcal{U}^n - \mathcal{U}^{n-1}}{2}, \frac{\mathcal{U}^n + \mathcal{U}^{n-1}}{\tau}\right) = \frac{1}{2\tau}(\|\nabla\mathcal{U}^n\|^2 - \|\nabla\mathcal{U}^{n-1}\|^2) \tag{88}$$

and a_{k-1}, b_{k-1}, $(a_{n-k-1} - a_{n-k})$ and $(b_{n-k-1} - b_{n-k})$ are all positive, eq. (87) yields

$$\begin{aligned}
&\frac{1}{\mu_1}\|\partial_t\mathcal{U}^{n-1/2}\|^2 + \frac{1}{2\tau}(\|\nabla\mathcal{U}^n\|^2 - \|\nabla\mathcal{U}^{n-1}\|^2) \\
&+\frac{\gamma_1}{\mu_2}|(\mathcal{U}^{n-1/2}, \partial_t\mathcal{U}^{n-1/2})| + \gamma_2|(\mathcal{U}^{n-1/2}, \partial_t\mathcal{U}^{n-1/2})| = \\
&\frac{1}{\mu_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})|(\partial_t\mathcal{U}^{k-1/2}, \partial_t\mathcal{U}^{n-1/2})| - a_{n-1}|(U_t^0, \partial_t\mathcal{U}^{n-1/2})|\right] \\
&+\frac{\gamma_1}{\mu_2}\left[\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})|(\mathcal{U}^{k-1/2}, \partial_t\mathcal{U}^{n-1/2})| - b_{n-1}|(U^0, \partial_t\mathcal{U}^{n-1/2})|\right] \\
&+|(f^{n-1/2}, \partial_t\mathcal{U}^{n-1/2})|, \qquad 1\leq n\leq N.
\end{aligned} \tag{89}$$

Assuming $\gamma_1 = 1$, then we rewrite the last equation as

$$\begin{aligned}
&\frac{\tau}{\mu_1}\|\partial_t\mathcal{U}^{n-1/2}\|^2 + (\|\nabla\mathcal{U}^n\|^2 - \|\nabla\mathcal{U}^{n-1}\|^2) \\
&+\frac{\gamma_1\tau}{\mu_2}(\|\mathcal{U}^{n-1/2}\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2) + 2\gamma_2\tau(\|\mathcal{U}^{n-1/2}\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2) \leq \\
&\frac{\tau}{\mu_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\|\partial_{t\mathcal{U}}^{k-1/2}\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2) - a_{n-1}(\|U_t^0\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2)\right] \\
&+\frac{\gamma_2\tau}{\mu_2}\left[\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\|\mathcal{U}^{k-1/2}\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2) - b_{n-1}(\|U^0\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2)\right] \\
&+2\tau(\|f^{n-1/2}\|^2 + \|\partial_t\mathcal{U}^{n-1/2}\|^2), \qquad 1\leq n\leq N.
\end{aligned} \tag{90}$$

Considering $\Bbbk > 0$ is the number depending on the time step and $\gamma_1 = 1$, we obtain

$$\begin{aligned}
&\frac{\Bbbk\tau}{\mu_1}\|\partial_t \mathcal{U}^{n-1/2}\|^2 + \Bbbk\|\nabla \mathcal{U}^n\|^2 - \Bbbk\|\nabla \mathcal{U}^{n-1}\|^2 + \frac{\Bbbk\gamma_1\tau}{\mu_2}\|\mathcal{U}^{n-1/2}\|^2 + \|\mathcal{U}^n\|^2 - \|\mathcal{U}^{n-1}\|^2 = \\
&\frac{\Bbbk\tau}{\mu_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\|\partial_t \mathcal{U}^{k-1/2}\|^2 - a_{n-1}\|U_t^0\|^2\right] \\
&+ \frac{\Bbbk\gamma_2\tau}{\mu_2}\left[\sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\|\mathcal{U}^{k-1/2}\|^2 - b_{n-1}\|U^0\|^2\right] \\
&+ 2\Bbbk\tau\|f^{n-1/2}\|^2, \qquad 1 \le n \le N,
\end{aligned} \tag{91}$$

Let us put

$$\mathbb{E}^0 = \|\mathcal{U}0\|^2 + \Bbbk\|\nabla \mathcal{U}^0\|^2 \tag{92}$$

and

$$\mathbb{E}^n = \|\mathcal{U}^n\|^2 + \Bbbk\|\nabla \mathcal{U}^n\|^2 + \frac{\Bbbk\tau}{\mu_1}\sum_{k=1}^{n} a_{n-k}\|\partial_t \mathcal{U}^{k-1/2}\|^2 + \frac{\Bbbk\gamma_2\tau}{\mu_2}\sum_{k=1}^{n} b_{n-k}\|\mathcal{U}^{k-1/2}\|^2. \tag{93}$$

Consequently, we have the following inequality:

$$\begin{aligned}
&\mathbb{E}^n \le \mathbb{E}^{n-1} + \frac{\Bbbk\tau a_{n-1}}{\mu_1}\|U_t^0\|^2 + \frac{\Bbbk\gamma_1\tau b_{n-1}}{\mu_2}\|U^0\|^2 + 2\Bbbk\tau\|f^{n-1/2}\|^2 \le \\
&\mathbb{E}^0 + \frac{\Bbbk\tau}{\mu_1}\sum_{k=1}^{n} a_{k-1}\|U_t^0\|^2 + \frac{\gamma_1\Bbbk\tau}{\mu_2}\sum_{k=1}^{n} b_{k-1}\|U^0\|^2 + 2\Bbbk\tau n\max_{0\le k\le N}\|f^{n-1/2}\|^2.
\end{aligned}$$

It is easy to verify that $\sum_{k=1}^{n} a_{k-1} = n^{2-\alpha}$ and $\sum_{k=1}^{n} b_{k-1} = n^{1-\alpha}$. Then, we have

$$\begin{aligned}
\frac{\tau}{\mu_1}\sum_{k=1}^{n} a_{k-1}\|U_t^0\|^2 &= \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)}\|U_t^0\|^2 \\
\frac{\tau}{\mu_2}\sum_{k=1}^{n} b_{k-1}\|U^0\|^2 &= \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)}\|U^0\|^2,
\end{aligned}$$

and the proof is complete.

6.2 Convergency

Theorem 2. *Assume $u \in H_0^1$ be the exact solution of* (80) *and $\mathcal{U} \in H_0^1$ be approximate solution of* (84)*, then we have the following inequality*

$$\|\mathscr{E}^n\|_{\omega,1}^2 \le \|\mathscr{E}^0\|_{\omega,1}^2 + 2\Bbbk t_n \mathcal{C}^2\tau^{4-2\alpha}, \tag{94}$$

in which $\mathscr{E}^n = u^n - \mathcal{U}^n$.

Proof. Subtracting (80) from weak formulation of (84) and choosing $\nu = \mathscr{E}^{n-1/2}$ we obtain

$$\begin{aligned}
&\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0(\partial_t\mathscr{E}^{n-1/2}, \nu) - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(\partial_t\mathscr{E}^{k-1/2}, \nu)\right] \\
&+ \gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0(\mathscr{E}^{n-1/2}, \nu) - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})(\mathscr{E}^{k-1/2}, \nu)\right] + \gamma_2(\mathscr{E}^{n-1/2}, \nu) = \\
&\Delta(\mathscr{E}^{n-1/2}, \nu) + (\mathfrak{R}^{n-1/2}, \nu), \qquad 1 \le n \le N.
\end{aligned} \tag{95}$$

Taking $\mathbb{\nu} = \mathscr{E}^{n-1/2}$, we find

$$
\begin{aligned}
&\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0(\partial_t\mathscr{E}^{n-1/2},\mathscr{E}^{n-1/2}) - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\partial_t\mathscr{E}^{k-1/2},\mathscr{E}^{n-1/2})\right]\\
&+\gamma_1\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0(\mathscr{E}^{n-1/2},\mathscr{E}^{n-1/2}) - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\mathscr{E}^{k-1/2},\mathscr{E}^{n-1/2})\right]\\
&+\gamma_2(\mathscr{E}^{n-1/2},\mathscr{E}^{n-1/2}) =\\
&\Delta(\mathscr{E}^{n-1/2},\mathscr{E}^{n-1/2}) + (\mathfrak{R}^{n-1/2},\mathscr{E}^{n-1/2}), \qquad 1\le n\le N.
\end{aligned}
\tag{96}
$$

If we set $\gamma_1 = 1$, then we have

$$
\begin{aligned}
&\frac{\tau}{\mathbb{\mu}_1}\|\partial_t\mathscr{E}^{n-1/2}\|^2 + (\|\nabla\mathscr{E}^n\|^2 - \|\nabla\mathscr{E}^{n-1}\|^2) + \frac{\gamma_1\tau}{\mathbb{\mu}_2}(\|\mathscr{E}^{n-1/2}\|^2) + \gamma_2\tau(\|\mathscr{E}^{n-1/2}\|^2) \le\\
&\frac{\tau}{\mathbb{\mu}_1}\left[\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\|\partial_t\mathscr{E}^{n-1/2}\|^2)\right] + \frac{\gamma_2\tau}{\mathbb{\mu}_2}\left[\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\|\mathscr{E}^{n-1/2}\|^2)\right]\\
&+2\tau\|\mathfrak{R}^{n-1/2}\|^2, \qquad 1\le n\le N,
\end{aligned}
\tag{97}
$$

i.e.,

$$
\begin{aligned}
&\gamma_2\tau\|\mathscr{E}^{n-1/2}\|^2 + \|\nabla\mathscr{E}^n\|^2 + \frac{\tau}{\mathbb{\mu}_1}\left[\sum_{k=1}^{n}a_{n-k}\|\partial_t\mathscr{E}^{n-1}\|^2\right] + \frac{\gamma_2\tau}{\mathbb{\mu}_2}\left[\sum_{k=1}^{n}b_{n-k}\|\mathscr{E}^{n-1/2}\|^2\right] \le\\
&\|\nabla\mathscr{E}^{n-1}\|^2 + \frac{\tau}{\mathbb{\mu}_1}\left[\sum_{k=1}^{n-1}a_{n-k-1}\|\partial_t\mathscr{E}^{n-1/2}\|^2)\right] + \frac{\gamma_2\tau}{\mathbb{\mu}_2}\left[\sum_{k=1}^{n-1}b_{n-k-1}\|\mathscr{E}^{n-1/2}\|^2\right]\\
&+2\tau\|\mathfrak{R}^{n-1/2}\|^2, \qquad 1\le n\le N.
\end{aligned}
$$

Employing a similar procedure as in Theorem 1, we can derive

$$
\begin{aligned}
\mathcal{E}^n = \|\mathscr{E}^n\|^2 + \mathbb{k}\|\nabla\mathscr{E}^n\|^2 + \frac{\mathbb{k}\tau}{\mathbb{\mu}_1}\left[\sum_{k=1}^{n}a_{n-k}\|\partial_t\mathscr{E}^{n-1}\|^2\right]\\
+\frac{\mathbb{k}\gamma_2\tau}{\mathbb{\mu}_2}\left[\sum_{k=1}^{n}b_{n-k}\|\mathscr{E}^{n-1/2}\|^2\right]
\end{aligned}
\tag{98}
$$

to obtain

$$
\mathcal{E}^n \le \mathcal{E}^0 + 2\tau\mathbb{k}\sum_{k=1}^{n}\|\mathfrak{R}^{n-1/2}\|^2. \tag{99}
$$

Finally, applying the inequality (83) for $1\le n\le N$ completes the proof.

7 Numerical experiments

In this section, we show the results obtained for two examples using the meshless method described above. In both examples, the domain integrals are evaluated with 16 points Gaussian quadrature rule while the boundary integrals are evaluated with 7 points Gaussian quadrature rule. To show the behaviour of the solution and the efficiency of the proposed method, the following root mean square (RMS) error is applied to make comparison

$$
RMS = \sqrt{\frac{\sum_{i=1}^{N}\left(U_{\text{exact}}(\mathbf{x}_i) - U_{\text{approx}}(\mathbf{x}_i)\right)^2}{N}},
$$

where $U_{\text{exact}}(\mathbf{x}_i)$ and $U_{\text{approx}}(\mathbf{x}_i)$ are achieved by exact and approximate solution on points $\mathbf{x}_i$ and N is number of nodal points. In both problems the regular node distribution is used. Also, in order to implement the meshless local

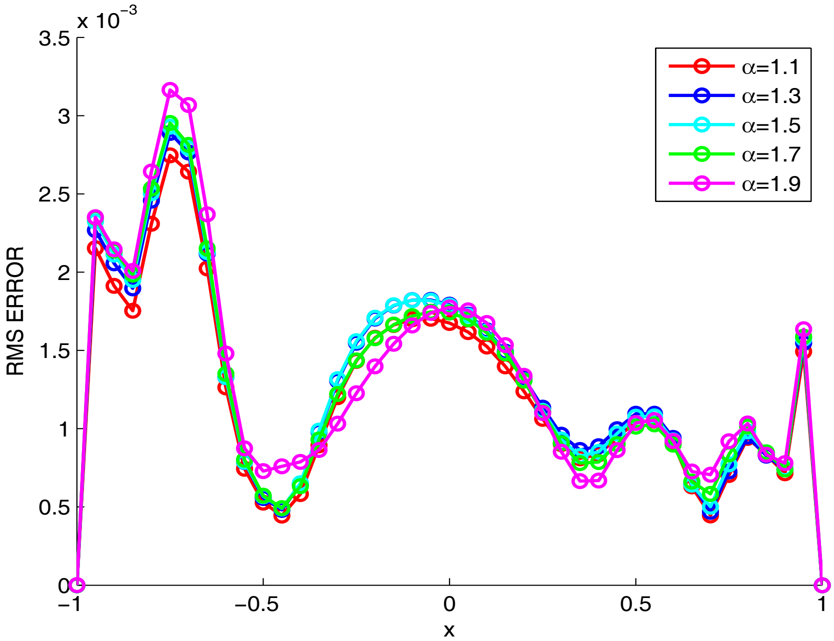

Fig. 2. Diagram of the RMS error of $u(x, 0)$ at $t = 0.5$ with $\triangle t = 0.01$ and $N = 1681(h = 0.05)$ for Example 1.

weak form, the radius of the local quadrature domain $r_q = 0.7h$ is selected, where h is the distance between the nodes in the x- or y-direction. The size of r_q is such that the union of these sub-domains must cover the whole global domain. The radius of the support domain to moving least squares approximation is $r_s = 4r_q$. This size is significant enough to have a sufficient number of nodes (n) and gives an appropriate approximation. Also, the quadratic basis function (22) is used, *i.e.* $m = 6$ is taken.

Example 1. We set $\gamma_1 = \gamma_2 = \gamma_3 = 1$, the exact solution of problems (1)–(4) is taken as

$$u(x, y, t) = \exp(t)\sin(10x^2 + 10y^2),$$

where $\varphi(x, y)$, $\psi(x, y)$, $g_0(y, t)$, $g_1(y, t)$, $h_0(x, t)$ and $h_1(x, t)$ are defined accordingly and also $f(x, y, t)$ is given by

$$\begin{aligned} f(x, y, t) = &-\frac{2e^t \Gamma(2-\alpha, t)\sin\left(10\left(x^2+y^2\right)\right)}{\Gamma(2-\alpha)} \\ &+ 400e^t x^2 \sin\left(10\left(x^2+y^2\right)\right) \\ &+ 3e^t \sin\left(10\left(x^2+y^2\right)\right) \\ &+ 400e^t y^2 \sin\left(10\left(x^2+y^2\right)\right) \\ &- 40e^t \cos\left(10\left(x^2+y^2\right)\right), \end{aligned} \tag{100}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}\exp(-t)\,\mathrm{d}t, \tag{101}$$

$$\Gamma(z, a) = \int_z^\infty t^{a-1}\exp(-t)\,\mathrm{d}t. \tag{102}$$

Figure 2 presents the RMS error of numerical MLRPI solutions $u(x, 0, 0.5)$ for different values of α. As it is seen in this figure, the error is almost the same for all α's. The RMS error *versus* N (the number of total nodal points) is plotted for different values of time steps ($\triangle t$) at $t = 1$ for $\alpha = 1.5$ in fig. 3. As it is seen in this figure, the error decreases by decreasing $\triangle t$ and increasing N that means the convergence occurs with respect to both N and $\triangle t$.

Example 2. In this example, we take $\gamma_1 = \gamma_3 = 1$, and $\gamma_2 = 10$. We assume that

$$u(x, y, t) = \exp(-t)\sin(\pi(x + y))$$

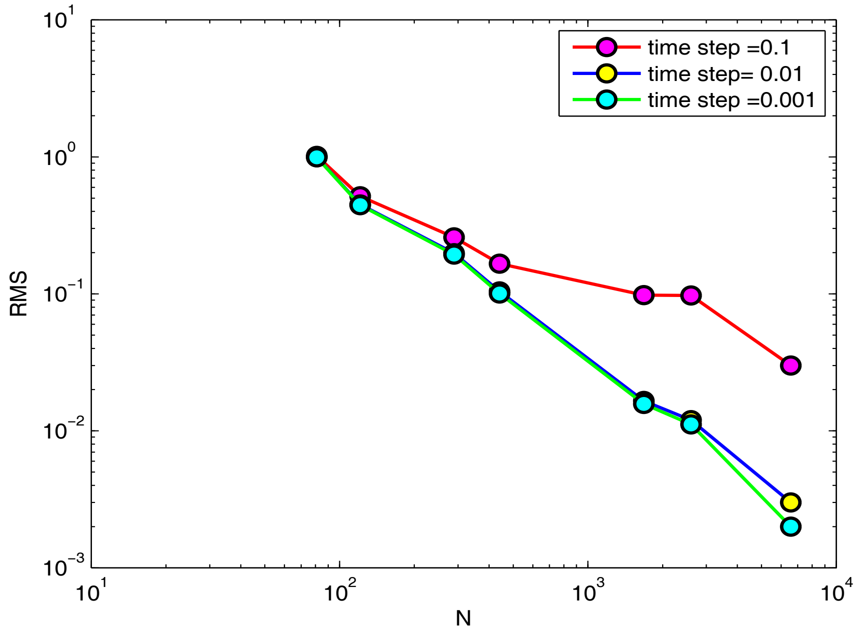

Fig. 3. Diagram of the RMS error at $t = 1$ *versus* N with $\alpha = 1.5$ for Example 1.

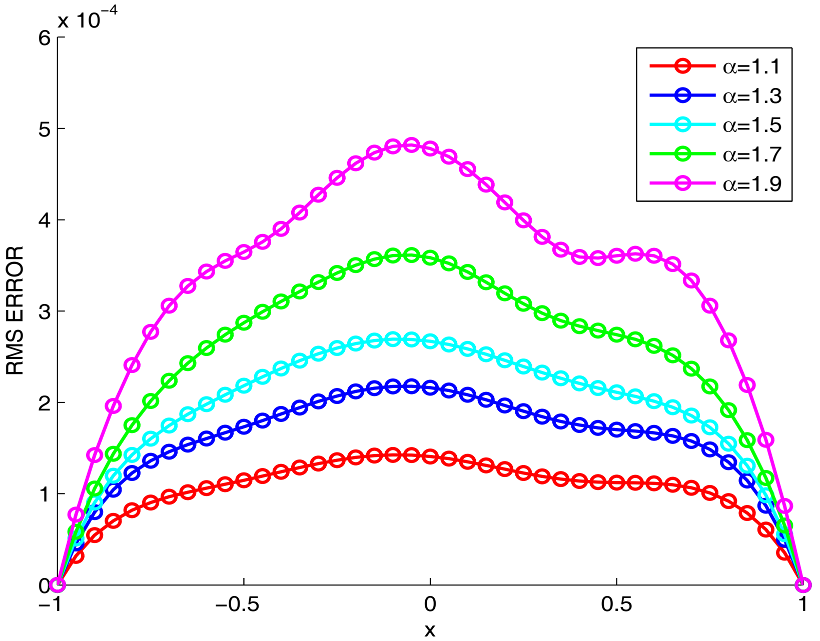

Fig. 4. Diagram of the RMS error of $u(x, 0)$ at $t = 0.5$ with $\triangle t = 0.01$ and $N = 1681(h = 0.05)$ for Example 2.

is the exact solution of the problems (1)–(4), where $\varphi(x, y)$, $\psi(x, y)$, $g_0(y, t)$, $g_1(y, t)$, $h_0(x, t)$ and $h_1(x, t)$ are defined accordingly and also $f(x, y, t)$ is given by

$$f(x, y, t) = 2\left(5 + \pi^2\right) e^{-t} \sin(\pi(x + y)). \tag{103}$$

Figure 4 shows the RMS error of the numerical MLRPI solutions $u(x, 0, 0.5)$ for different values of α. As it is seen in this figure, the error becomes larger by increasing α. In fig. 5, the RMS error *versus* N (the number of total nodal points) is plotted for different values of the time step ($\triangle t$) at $t = 1$ for $\alpha = 1.5$. As it is seen in this figure, the error decreases by decreasing $\triangle t$ and increasing N that means again the convergence holds with respect to both N and $\triangle t$. The descending behavior of the RMS error for the long-time integration by increasing time in fig. 6 shows the stability of MLRPI scheme for Example 2.

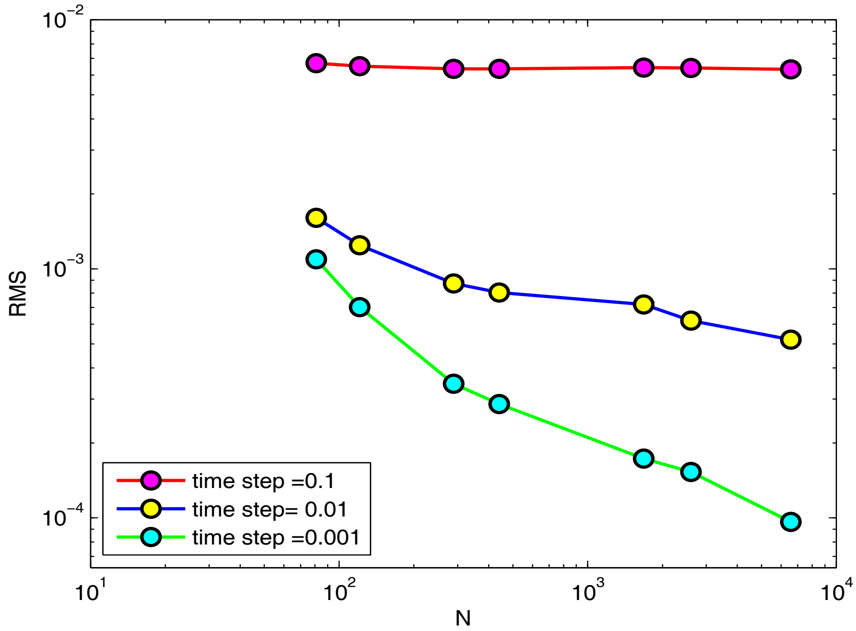

Fig. 5. Diagram of the RMS error at $t = 1$ *versus* N with $\alpha = 1.5$ for Example 2.

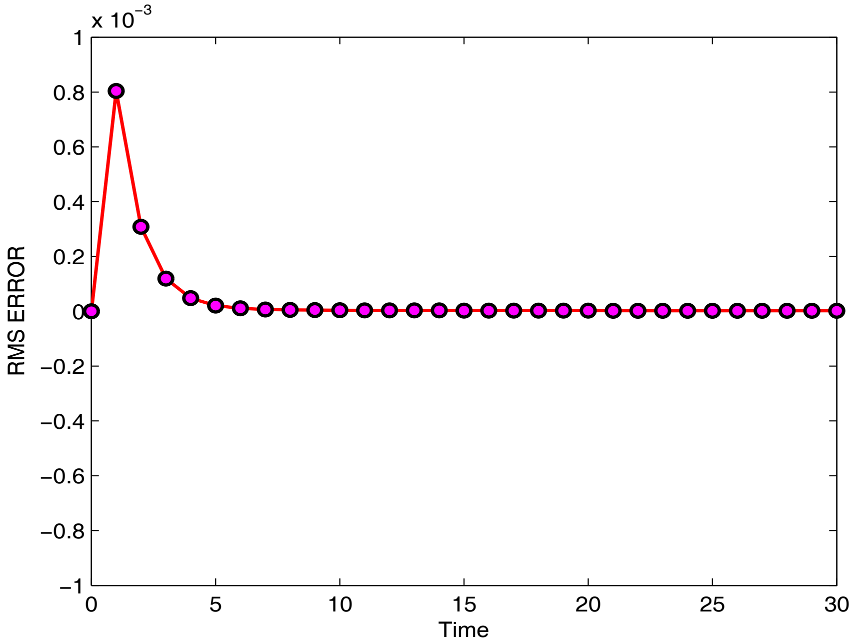

Fig. 6. The behavior of RMS for the long-time integration with $\triangle t = 0.01$ and $(N = 441)h = 0.1$ when $\alpha = 1.5$ for Example 2.

8 Conclusions

In this paper, the meshless local radial point interpolation method has been applied to solve a general class of two-dimensional time-fractional telegraph equation $(1 < \alpha \leq 2)$. The present method provides a local quadrature domain and a local support domain for each node so that the integration and the interpolation are done on these domains. In this method, the shape functions have been constructed by the point interpolation augmented to radial basis functions. A time discretization has been done in Caputo sense using finite difference techniques. The Heaviside step function was used as the test function in the local weak form method in MLRPI. In the current work, to demonstrate the accuracy and usefulness of this method, two numerical examples have been presented. As demonstrated by the computational results, it is not difficult to implement the proposed method for similar problems.

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