3.6 AN ALTERNATIVE COROTATIONAL FORMULATION USING ENGINEERING STRAIN

In all of the previous developments, the coordinate axes x, z (and y) have remained fixed in direction even if, as in Sections 3.3.5 and 3.3.6, we have updated the coordinates. We will now apply a 'corotational' formulation and will show that it gives the same results as those previously obtained in Section 3.4. The procedure adopts a set of corotational axes (x_l, z_l -- Figure 3.7) which rotate with the element. In these circumstances, the engineering strain is given by

$$\varepsilon = \frac{1}{2\alpha_o}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}^{T}\begin{bmatrix} u_1 \\ u_2 \\ w_1 \\ w_2 \end{bmatrix} = \mathbf{b}_l^T \mathbf{p}_l = \frac{1}{2\alpha_o}\mathbf{c}_l^T \mathbf{p}_l. \tag{3.123}$$

In the above equation and throughout this section a subscript E for engineering will be implied but omitted on all strain and stress measures. Equation (3.123) is obvious but it could be derived by relating the shape-function approaches of the previous sections to the local coordinate system. Following from (3.123), the principle of virtual work gives

$$\mathbf{q}_{il} = \int \frac{\sigma}{2\alpha_o}\mathbf{c}_l \, \mathrm{d}V_o = A_o \sigma \mathbf{c}_l. \tag{3.124}$$

We can now apply standard transformation procedures [C2.2], to give

$$\mathbf{q}_i = \mathbf{T}^{\mathrm{T}}\mathbf{q}_{il} = A_o\sigma\mathbf{T}^{\mathrm{T}}\mathbf{c}_l \tag{3.125}$$

where the transformation matrix, $\mathbf{T}$, relates the local displacements, $\mathbf{p}_l$ to the 'global' cartesian displacements, $\mathbf{p}$, so that

$$\mathbf{p}_l = \mathbf{T}\mathbf{p} = \begin{bmatrix} c & 0 & s & 0 \\ 0 & c & 0 & s \\ -s & 0 & c & 0 \\ 0 & -s & 0 & c \end{bmatrix}\mathbf{p} = \frac{1}{2\alpha_n}\begin{bmatrix} x'_{21} & 0 & z'_{21} & 0 \\ 0 & x'_{21} & 0 & z'_{21} \\ -z'_{21} & 0 & x'_{21} & 0 \\ 0 & -z'_{21} & 0 & x'_{21} \end{bmatrix}\mathbf{p}. \tag{3.126}$$

The terms c and s in (3.126) are $\cos\theta$ and $\sin\theta$ respectively, where θ is illustrated in Figure 3.7. If $\mathbf{T}^{\mathrm{T}}$ is multiplied by $\mathbf{c}_l$ from (3.123), it can be shown that

$$\mathbf{T}^{\mathrm{T}}\mathbf{c}_l = \frac{1}{2\alpha_n}\mathbf{c}(\mathbf{x}') \tag{3.127}$$

where $\mathbf{c}(\mathbf{x}')$ is given in (3.92). Hence substitution into (3.125) gives

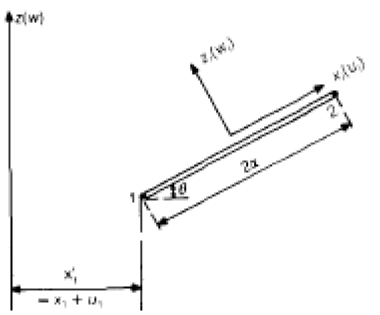
$$\mathbf{q}_i = \frac{A_o\sigma}{2\alpha_n}\mathbf{c}(\mathbf{x}') \tag{3.128}$$

which coincides with (3.107), which was obtained with the aid of 'fixed coordinates'.

We could now proceed to differentiate (3.128) to obtain the tangent stiffness matrix given by the components (3.111), (3.113) and (3.114). However, we will instead adopt the spirit of the corotational approach and firstly differentiate (3.124) to obtain a 'local tangent stiffness matrix'. From (3.123) and (3.124) this gives

$$\mathbf{K}_{tl} = \frac{\partial\mathbf{q}_{il}}{\partial\mathbf{p}_l} = EA_o\mathbf{c}_l\frac{\partial\varepsilon_l}{\partial\mathbf{p}_l} = \frac{EA_o}{2\alpha_o}\mathbf{c}_l\mathbf{c}_l^{\mathrm{T}}. \tag{3.129}$$

In order to relate this local stiffness matrix to the fixed cartesian coordinate system,



(3.125) can be differentiated to give

$$\delta \mathbf{q}_i = \mathbf{T}^T \frac{\partial \mathbf{q}_{il}}{\partial \mathbf{p}_l}\delta\mathbf{p}_l + \delta\mathbf{T}^T\mathbf{q}_{il} = \mathbf{T}^T\mathbf{K}_{tl}\mathbf{T}\delta\mathbf{p} + \delta\mathbf{T}^T\mathbf{q}_{il} = \mathbf{K}_t\delta\mathbf{p} \tag{3.130}$$

where use has been made of (3.126). Substitution from (3.127) into the first of the two stiffness terms in (3.130) gives

$$\mathbf{K}_{tl} = \frac{EA}{8\alpha_n^2\alpha_o}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T \tag{3.131}$$

which coincides with (3.111).

In order to deal with the second stiffness term in (3.130), the $\mathbf{T}$ matrix in (3.126) can be differentiated so that

$$\delta\mathbf{T}^T = \begin{bmatrix} -s & 0 & -c & 0 \\ 0 & -s & 0 & -c \\ c & 0 & -s & 0 \\ 0 & c & 0 & -s \end{bmatrix}\delta\theta. \tag{3.132}$$

From Figure 3.8, a unit vector normal to the rotating element is given by

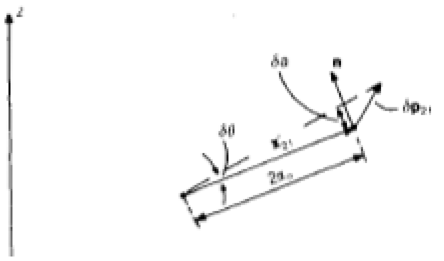
$$\mathbf{n} = \frac{1}{2\alpha_n}\begin{pmatrix} -z'_{21} \\ x'_{21} \end{pmatrix} \tag{3.133}$$

which is orthogonal to the truss vector, $\mathbf{x}'_{21}$. The infinitesimal relative displacement vector (Figure 3.8) can be expressed as

$$\delta\mathbf{p}_{21} = \begin{pmatrix} \delta u_{21} \\ \delta w_{21} \end{pmatrix}. \tag{3.134}$$

Resolving this vector in the direction $\mathbf{n}$ gives a scalar length:

$$\delta a = \mathbf{n}^T\delta\mathbf{p}_{21} = \mathbf{n}^T\begin{pmatrix} \delta u_{21} \\ \delta w_{21} \end{pmatrix} = \frac{1}{2\alpha_n}\begin{pmatrix} -z'_{21} \\ x'_{21} \end{pmatrix}^T\begin{pmatrix} \delta u_{21} \\ \delta w_{21} \end{pmatrix}. \tag{3.135}$$



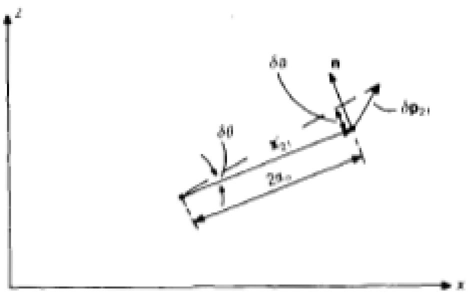


Figure 3.8 Small movement from new configuration.

Consequently, the angle $\delta\theta$ (Figure 3.8) is given by

$$\delta\theta = \frac{\delta a}{2\alpha_n} = \frac{1}{4\alpha_n^2}\begin{pmatrix} -z'_{21} \\ x'_{21} \end{pmatrix}^T \begin{pmatrix} \partial u_{21} \\ \partial w_{21} \end{pmatrix} = \frac{1}{4\alpha_n^2}\begin{pmatrix} z'_{21} \\ -z'_{21} \\ x'_{21} \\ -x'_{21} \end{pmatrix}^T \Delta\mathbf{p} = \frac{1}{4\alpha_n^2}\mathbf{z}^T\delta\mathbf{p}. \quad (3.136)$$

Hence, using (3.124) and (3.132), the second stiffness term in (3.130) is given by

$$\delta\mathbf{q}_i = \frac{A_o\sigma}{8\alpha_n^3}\begin{bmatrix} -z'_{21} & 0 & -x'_{21} & 0 \\ 0 & -z'_{21} & 0 & -x'_{21} \\ x'_{21} & 0 & -z'_{21} & 0 \\ 0 & x'_{21} & 0 & -z'_{21} \end{bmatrix}\mathbf{c}_f\mathbf{z}^T\delta\mathbf{p} \quad (3.137)$$

with $\mathbf{c}_f$ from (3.123). Alternatively

$$\delta\mathbf{q}_i = \frac{A_o\sigma}{8\alpha_n^3}\mathbf{z}\mathbf{z}^T\delta\mathbf{p} = \frac{A\sigma}{8\alpha_n^3}\begin{bmatrix} z'^2_{21} & & \text{symmetric} & \\ -z'^2_{21} & z'^2_{21} & & \\ -x'_{21}z'_{21} & z'_{21}x'_{21} & x'^2_{21} & \\ z'_{21}x'_{21} & -z'_{21}x'_{21} & -x'^2_{21} & x'^2_{21} \end{bmatrix}\delta\mathbf{p} = \mathbf{K}_{t\sigma}\delta\mathbf{p}. \quad (3.138)$$

It can easily be shown that the matrix $\mathbf{K}_{t\sigma}$ in (3.138) coincides with the sum of $\mathbf{K}_{t\sigma 1}$ and $\mathbf{K}_{t\sigma 2}$ from (3.113) and (3.114). Hence, identical solutions are produced by the two formulations using (a) a fixed cartesian system and (b) a rotating (corotational) coordinate system. A similar correspondence can be shown for the log-strain formulation.