

# THE METRIC-RESTRICTED INVERSE DESIGN PROBLEM

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ABSTRACT. We study a class of design problems in solid mechanics, leading to a variation on the classical question of equi-dimensional embeddability of Riemannian manifolds. In this general new context, we derive a necessary and sufficient existence condition, given through a system of total differential equations, and discuss its integrability. In the classical context, the same approach yields conditions of immersibility of a given metric in terms of the Riemann curvature tensor. In the present situation, the equations do not close in a straightforward manner, and successive differentiation of the compatibility conditions leads to a new algebraic description of integrability. We also recast the problem in a variational setting and analyze the infimum of the appropriate incompatibility energy, resembling the non-Euclidean elasticity. We then derive a  $\Gamma$ -convergence result for dimension reduction from 3d to 2d in the Kirchhoff energy scaling regime.

## 1. THE METRIC-RESTRICTED INVERSE DESIGN PROBLEM

Assume  $\mathcal{T}$  is a manifold of material types, differentiated by their structure, density, swelling-shrinkage rates, and other qualities. We assume that any material type  $T \in \mathcal{T}$  is naturally endowed with a prestrain  $\bar{g}(T)$ , where  $\bar{g} : \mathcal{T} \rightarrow \mathbb{R}_{\text{sym, pos}}^{2 \times 2}$  is a given smooth mapping, taking values in the symmetric positive definite tensors.

Suppose we need to manufacture a 2-dimensional membrane  $S \subset \mathbb{R}^3$ , where at any given point  $p \in S$  a material of type  $T(p) \in \mathcal{T}$  must be used for a given  $T : S \rightarrow \mathcal{T}$ . The question is how to print a thin film  $U \subset \mathbb{R}^2$  in a manner that the activation of the prestrain in the film would result in a deformation leading eventually to the desired surface shape  $S$ . In view of [16, 9, 19], the activation  $u : U \rightarrow \mathbb{R}^3$  must be an isometric immersion of the Riemannian manifold  $(U, \mathcal{G})$  into  $\mathbb{R}^3$ , where  $\mathcal{G}$  is the prestrain in the flat (referential) thin film. Our design problem requires hence that we find *an unknown reference configuration*  $U \subset \mathbb{R}^2$ , *an unknown material distribution*  $\mathfrak{T} : U \rightarrow \mathcal{T}$  and *an unknown deformation*  $u : U \rightarrow \mathbb{R}^3$  such that

- (i)  $S = u(U)$ ,
- (ii) For any  $x \in U$ , the point  $u(x)$  carries a material of type  $T(u(x))$ , i.e.  $T(u(x)) = \mathfrak{T}(x)$ ,
- (iii)  $\nabla u(x)^T \nabla u(x) = \mathcal{G}(x) := \bar{g}(\mathfrak{T}(x))$ .

If the membrane  $S \subset \mathbb{R}^3$  is a smooth surface, then letting  $g := \bar{g} \circ T : S \rightarrow \mathbb{R}_{\text{sym, pos}}^{2 \times 2}$ , the conditions (i)-(iii) simplify to finding a domain  $U \subset \mathbb{R}^2$  and a bijection  $u : U \rightarrow S$ , such that:

$$(1.1) \quad (\nabla u)^T (\nabla u)(x) = g(u(x)) \quad \forall x \in U.$$

The smoothness of  $g$  is determined by the regularity of  $\mathcal{T}$  and of the mappings  $\bar{g}$  and  $T$ .

The above described problem is natural as a design question in various areas of solid mechanics, even though the involved tensors are not intrinsic geometric objects. For example, it includes the subproblems and extensions to higher dimensions:

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*Date:* January 28, 2016.

- (a) Given the deformed configuration of an elastic 2d membrane and the rectangular Cartesian components of the Right Cauchy-Green tensor field of a deformation, mapping a flat undeformed reference of the membrane to it, find the flat reference configuration and the deformation of the membrane<sup>1</sup>.
- (b) Given the deformed configuration of a 3-dimensional body and the rectangular Cartesian components of the Right Cauchy-Green tensor field of the deformation, mapping a reference configuration to it, find the reference configuration and the deformation.
- (c) Suppose the current configuration of a 3d, plastically deformed body is given, and on it is specified the rectangular Cartesian components of a plastic distortion  $F_p$ . Find a reference configuration and a deformation  $\zeta$ , mapping this reference to the given current configuration, such that the latter is stress-free. Assume that the stress response of the material is such that the stress vanishes if and only if  $(\nabla\zeta(F_p)^{-1})^T(\nabla\zeta(F_p)^{-1}) = \text{Id}_3$ .

The inverse design problem that we study in this paper, can be further rephrased as follows. Let  $y : \Omega \rightarrow \mathbb{R}^3$ , be a smooth parametrization of  $S = y(\Omega)$ . Find a change of variable  $\xi : \Omega \rightarrow U$  so that the pull back of the Euclidean metric on  $S$  through  $y$  is realized by the following formula:

$$(1.2) \quad (\nabla y)^T \nabla y = (\nabla \xi)^T (g \circ y) \nabla \xi \quad \text{in } \Omega.$$

Clearly, once a solution  $\xi$  of (1.2) is found, the material type distribution  $\mathfrak{T}$ , which is needed for the construction of the printed film  $U$ , can be calculated by:

$$\mathfrak{T} := T \circ y \circ \xi^{-1} : U = \xi(\Omega) \rightarrow \mathcal{T},$$

since  $u = y \circ \xi^{-1}$  satisfies (1.1) and consequently the properties (i)-(iii) hold. Any  $\xi$  satisfying (1.2) is an isometry between the Riemannian manifolds  $(\Omega, \tilde{G})$  and  $(U, G \circ \xi^{-1})$ , with metrics:

$$(1.3) \quad \tilde{G} = (\nabla y)^T \nabla y \quad \text{and} \quad G = g \circ y \quad \text{on } \Omega.$$

For convenience of the reader, we gather some of our notational symbols in Figure 1.1.

The same problem can be set up for a three dimensional shape  $S \subset \mathbb{R}^3$ . In that case, the pre-strain mapping  $\bar{g}$  must take values in  $\mathbb{R}_{sym, pos}^{3 \times 3}$  and the printed prestrained reference configuration is modeled by  $U \subset \mathbb{R}^3$ . The equation to be solved is still given by (1.2), now posed in  $\Omega \subset \mathbb{R}^3$ . Equivalently, a solution  $u : U \rightarrow S$  to (1.1) is obtained as in (iii) above for  $\mathfrak{T} := T \circ u$  and it is the absolute minimizer of the prestrain elastic energy (see e.g. [19]):

$$(1.4) \quad E(u, U, \mathcal{G}) := \int_U \text{dist}^2((\nabla u)\mathcal{G}^{-1/2}, SO(3)) \, dx = 0.$$

Here,  $\text{dist}(F, SO(3))$  stands for the distance of a matrix  $F$  from the compact set  $SO(3)$ , with respect to the Hilbert-Schmidt norm. What distinguishes our problems from the classical isometric immersion problem in differential geometry, where one looks for an isometric mapping between two given manifolds  $(\Omega, \tilde{G})$  and  $(U, \mathcal{G})$ , is that the target manifold  $U = \xi(\Omega)$  and its Riemannian metric  $\mathcal{G} = G \circ \xi^{-1}$  are only given a-posteriori, after the solution is found. Note that only when  $G$  is constant, the target metric becomes a-priori well defined and can be extended over the whole of  $\mathbb{R}^n$ , as it is independent of  $\xi$ , and then the problem reduces to the classical case (see Example 5.4 and a few other similar cases in Examples 5.5 and 5.6).

<sup>1</sup>We thank Kaushik Bhattacharya for bringing this problem to our attention.

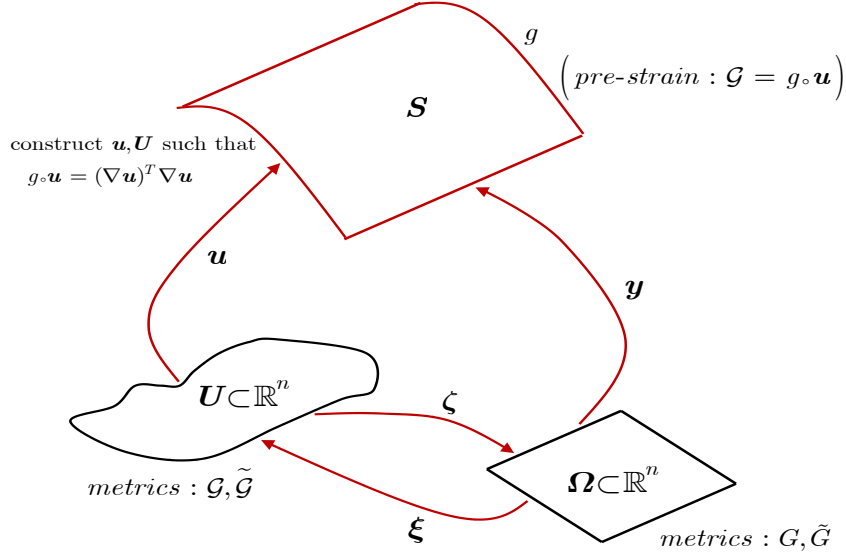


FIGURE 1.1. Geometry of the problem.

**Remark 1.1.** Note that the minimizing solution in the equidimensional problem (1.4) is unique up to rigid motions. Hence, If the configuration  $(U, \mathcal{G})$  is printed, the elastic body is bound to take the required shape  $S$  as requested in the design problem. Note that in case the required shape  $S$  is two dimensional, uniqueness fails due to one more degree of freedom as the deformation  $u$  of  $U \subset \mathbb{R}^2$  takes values in a higher dimensional space  $\mathbb{R}^3$ . In this case, other restrictions on  $u$  have to be imposed, to solve the original design problem. One particular remedy is to consider the membrane as a thin three dimensional body; we implement this approach in Section 3. Note that solving (1.1) directly in two dimensions implies that a compatible reference configuration can be found, as formulated in (a) above.

Another approach in the two dimensional case would be to use a parametrization  $y : \Omega \rightarrow S$  which is bending energy minimizing among all other mappings  $\tilde{y} : \Omega \rightarrow S$  that induce the same metric  $(\nabla y)^T \nabla y$ . This would necessitate a study of the bending energy effects, which is beyond the scope of the present paper. For the sake of comparison, we mention a parallel problem, studied in [15], where the authors calculate the prestrain to be printed in the thin film by minimizing a full three dimensional energy of deformation, consisting of both stretching and bending. The nature of the problem in [15] is different than ours, in as much as the material constraints are not present.

The paper is organized as follows. In Sections 2 and 3, we study the above mentioned variational formulation of (1.2) and analyze the infimum value of the appropriate incompatibility energy, resembling the non-Euclidean elasticity [19]. We derive a  $\Gamma$ -convergence result for the dimension reduction from 3d to 2d, in the Kirchhoff-like energy scaling regime, corresponding to the square of thickness of the thin film. In Sections 4-9, we formulate (1.2) as an algebraically constrained system of total differential equations, in which the second derivatives of  $\xi$  are expressed in terms of its first derivatives and the Christoffel symbols of the involved metrics. The idea is then to investigate the integrability conditions of this system. When this method is applied in the context of the standard Riemannian isometric immersion problem, the parameters involving  $\xi$

can be removed from the conditions and the intrinsic conditions of immersibility will be given in terms of the Riemann curvature tensors. In our situation, the equations do not close in a straightforward manner, and successive differentiation of the compatibility conditions leads to a more sophisticated algebraic description of solvability. This approach has been adapted in [1] for deriving compatibility conditions for the Left Cauchy-Green tensor.

**Acknowledgments.** This project is based upon work supported by, among others, the National Science Foundation. M.L. was partially supported by the NSF grants DMS-0846996 and DMS-1406730. M.R.P. was partially supported by the NSF grant DMS-1210258. A part of this work was completed while the second and the third authors visited the Forschungsinstitut für Mathematik at ETH (Zurich, Switzerland). The institute's hospitality is gratefully acknowledged.

## 2. A VARIATIONAL REFORMULATION OF THE PROBLEM (1.3)

In this section, we recast the problem (1.3) in a variational setting, similar to that of non-Euclidean elasticity [19]. Using the same arguments as in [11, 19], we will analyze the properties of the infimum value of the appropriate incompatibility energy, over the natural class of deformations of  $W^{1,2}$  regularity.

We will first discuss the problem (1.2) in the general  $n$ -dimensional setting and only later restrict to the case  $n = 2$  (or  $n = 3$ ). Hence, we assume that  $\Omega$  is an open, bounded, simply connected and smooth subset of  $\mathbb{R}^n$ . We look for a bilipschitz map  $\xi : \Omega \rightarrow U := \xi(\Omega)$ , satisfying (1.3) and which is orientation preserving:

$$(2.1) \quad \det \nabla \xi > 0 \quad \text{in } \Omega.$$

We begin by rewriting (1.3) as:

$$(2.2) \quad \tilde{G} = (G^{1/2} \nabla \xi)^T (G^{1/2} \nabla \xi).$$

Note that, in view of the polar decomposition theorem of matrices, a vector field  $\xi : \Omega \rightarrow \mathbb{R}^n$  is a solution to (2.2), augmented by the constraint (2.1), both valid a.e. in  $\Omega$ , if and only if:

$$(2.3) \quad \forall a.e. x \in \Omega \quad \exists R = R(x) \in SO(n) \quad G^{1/2} \nabla \xi = R \tilde{G}^{1/2},$$

where  $G^{1/2}(x)$  denotes the unique symmetric positive definite square root of  $G(x) \in \mathbb{R}_{sym, pos}^{n \times n}$ , while  $SO(n)$  stands for the set of special orthogonal matrices. Define:

$$(2.4) \quad E(\xi) = \int_{\Omega} \text{dist}^2(G^{1/2}(\nabla \xi) \tilde{G}^{-1/2}, SO(n)) \, dx \quad \forall \xi \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n),$$

where  $\text{dist}(F, SO(n))$  is the calculated distance of a matrix  $F$  from the compact set  $SO(n)$ , with respect to the Hilbert-Schmidt norm of matrices. It immediately follows that  $E(\xi) = 0$  if and only if  $\xi$  is a solution to (2.2) and hence to (1.3), together with (2.1). Also, note that  $E(\xi) < \infty$  if and only if  $\xi \in W^{1,2}(\Omega, \mathbb{R}^n)$ , as can be easily deduced from the inequality:

$$(2.5) \quad \forall F \in \mathbb{R}^{n \times n} \quad |F|^2 \leq C |G^{1/2} F \tilde{G}^{-1/2}|^2,$$

valid with a constant  $C > 0$  independent of  $x$  and  $F$ .

Finally, observe that, due to the uniform positive definiteness of the matrix field  $G$ :

$$(2.6) \quad |F|^2 = \sum_{i=1}^n |F e_i|^2 \leq C \sum_{i=1}^n \langle F e_i, G F e_i \rangle \leq C \text{trace}(F^T G F).$$

**Proposition 2.1.** (i) Assume that the metrics  $G, \tilde{G}$  are  $\mathcal{C}(\bar{\Omega}, \mathbb{R}^{n \times n})$  regular. Let  $\xi \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$  satisfy (1.3) for a.e.  $x \in \Omega$ . Then  $\xi \in W^{1,\infty}(\Omega, \mathbb{R}^n)$  must be Lipschitz continuous.

(ii) Assume additionally that for some  $k \geq 0$  and  $0 < \mu < 1$ ,  $G, \tilde{G} \in \mathcal{C}^{k,\mu}(\Omega, \mathbb{R}^{n \times n})$ . If (1.3), (2.1) hold a.e. in  $\Omega$  (so that  $E(\xi) = 0$ ), then  $\xi \in \mathcal{C}^{k+1,\mu}(\Omega, \mathbb{R}^n)$ .

*Proof.* The first assertion clearly follows from the boundedness of  $\tilde{G}$  and positive definiteness of  $G$ , through (2.6). To prove (ii), recall that for a matrix  $F \in \mathbb{R}^{n \times n}$ , the matrix of cofactors of  $F$  is  $\text{cof } F$ , with  $(\text{cof } F)_{ij} = (-1)^{i+j} \det \hat{F}_{ij}$ , where  $\hat{F}_{ij} \in \mathbb{R}^{(n-1) \times (n-1)}$  is obtained from  $F$  by deleting its  $i$ th row and  $j$ th column. Then, (1.3) implies that:

$$\det \nabla \xi = \left( \frac{\det \tilde{G}}{\det G} \right)^{1/2} =: a \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^+) \quad \text{and} \quad \text{cof} \nabla \xi = aG(\nabla \xi) \tilde{G}^{-1}.$$

Since  $\text{div}(\text{cof } \nabla \xi) = 0$  for  $\xi \in W^{1,\infty}$  (where the divergence of the cofactor matrix is always taken row-wise), we obtain that  $\xi$  satisfies the following linear system of differential equations, in the weak sense:

$$\text{div}(aG(\nabla \xi) \tilde{G}^{-1}) = 0.$$

Writing in coordinates  $\xi = (\xi^1 \dots \xi^n)$ , and using the Einstein summation convention, the above system reads:

$$\forall i = 1 \dots n \quad \partial_\alpha (aG_{ij} \tilde{G}^{\beta\alpha} \partial_\beta \xi^j) = 0.$$

The regularity result is now an immediate consequence of [13, Theorem 3.3] in view of the ellipticity of the coefficient matrix  $A_{ij}^{\alpha\beta} = aG_{ij} \tilde{G}^{\alpha\beta}$ .  $\blacksquare$

We now prove two further auxiliary results.

**Lemma 2.2.** *There exist constants  $C, M > 0$ , depending only on  $\|G\|_{L^\infty}$  and  $\|\tilde{G}\|_{L^\infty}$ , such that for every  $\xi \in W^{1,2}(\Omega, \mathbb{R}^n)$  there exists  $\bar{\xi} \in W^{1,2}(\Omega, \mathbb{R}^n)$  with the properties:*

$$\|\nabla \bar{\xi}\|_{L^\infty} \leq M, \quad \|\nabla \xi - \nabla \bar{\xi}\|_{L^2(\Omega)}^2 \leq CE(\xi) \quad \text{and} \quad E(\bar{\xi}) \leq CE(\xi).$$

*Proof.* Use the approximation result of Proposition A.1. in [11] to obtain the truncation  $\bar{\xi} = \xi^\lambda$ , for  $\lambda > 0$  having the property that if a matrix  $F \in \mathbb{R}^{n \times n}$  satisfies  $|F| \geq \lambda$  then:

$$|F|^2 \leq C \text{dist}^2(G^{1/2} F \tilde{G}^{-1/2}(x), SO(n)) \quad \forall x \in \Omega.$$

Then  $\|\nabla \xi^\lambda\|_{L^\infty} \leq C\lambda := M$  and further, since  $\nabla \xi = \nabla \xi^\lambda$  a.e. in the set  $\{|\nabla \xi| \leq \lambda\}$ :

$$\|\nabla \xi - \nabla \xi^\lambda\|_{L^2(\Omega)}^2 = \int_{\{|\nabla \xi| > \lambda\}} |\nabla \xi|^2 \leq c \int_{\{|\nabla \xi| > \lambda\}} \text{dist}^2(G^{1/2} \nabla \xi \tilde{G}^{-1/2}, SO(n)) \, dx \leq CE(\xi).$$

The last inequality of the lemma follows from the above by the triangle inequality.  $\blacksquare$

**Lemma 2.3.** *Let  $\xi \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ . Then there exists a unique weak solution  $\phi : \Omega \rightarrow \mathbb{R}^n$  to:*

$$(2.7) \quad \begin{cases} \text{div}(aG(\nabla \phi) \tilde{G}^{-1}) = 0 & \text{in } \Omega, \\ \phi = \xi & \text{on } \partial\Omega. \end{cases}$$

*Moreover, there is constant  $C > 0$ , depending only on  $G$  and  $\tilde{G}$ , and (in a nondecreasing manner) on  $\|\nabla \xi\|_{L^\infty}$ , such that:*

$$\|\nabla(\xi - \phi)\|_{L^2(\Omega)}^2 \leq CE(\xi).$$

*Proof.* Consider the functional:

$$I(\varphi) := \int_{\Omega} \langle G(\nabla\varphi)\tilde{G}^{-1}(x) : \nabla\varphi(x) \rangle dx = \int_{\Omega} |a^{1/2}G^{1/2}(\nabla\varphi)\tilde{G}^{-1/2}|^2 dx \quad \forall \varphi \in W^{1,2}(\Omega, \mathbb{R}^n).$$

The formula (2.5), in which we have implicitly used the coercivity of  $G$  and  $\tilde{G}$ , implies that:

$$\|\nabla\varphi\|_{L^2(\Omega)}^2 \leq CI(\varphi).$$

Therefore, in view of the strict convexity of  $I$ , the direct method of calculus of variations implies that  $I$  admits a unique critical point  $\phi$  in the set:

$$\{\varphi \in W^{1,2}(\Omega, \mathbb{R}^n); \varphi = \xi \text{ on } \partial\Omega\}.$$

By the symmetry of  $G$  and  $\tilde{G}$ , (2.7) is precisely the Euler-Lagrange equation of  $I$ , and hence it is satisfied, in the weak sense, by  $\phi$ .

Further, for the correction  $\psi = \xi - \phi \in W_0^{1,2}(\Omega, \mathbb{R}^n)$  it follows that:

$$\begin{aligned} \forall \eta \in W_0^{1,2}(\Omega, \mathbb{R}^n) \quad \int_{\Omega} aG(\nabla\psi)\tilde{G}^{-1} : \nabla\eta dx &= \int_{\Omega} aG(\nabla\xi)\tilde{G}^{-1} : \nabla\eta - \int_{\Omega} aG(\nabla\phi)\tilde{G}^{-1} : \nabla\eta \\ &= \int_{\Omega} aG(\nabla\xi)\tilde{G}^{-1} : \nabla\eta \\ &= \int_{\Omega} aG(\nabla\xi)\tilde{G}^{-1} : \nabla\eta - \int_{\Omega} \text{cof}\nabla\xi : \nabla\eta. \end{aligned}$$

Indeed, the last term above equals to 0, since the row-wise divergence of the cofactor matrix of  $\nabla\xi$  is 0, in view of  $\xi$  being Lipschitz continuous. Use now  $\eta = \psi$  to obtain:

$$\begin{aligned} \|\nabla\psi\|_{L^2(\Omega)}^2 &\leq CI(\psi) = C \int_{\Omega} (aG(\nabla\xi)\tilde{G}^{-1} - \text{cof}\nabla\xi) : \nabla\psi dx \\ &\leq C\|\nabla\psi\|_{L^2(\Omega)} \left( \int_{\Omega} |aG(\nabla\xi)\tilde{G}^{-1} - \text{cof}\nabla\xi|^2 \right)^{1/2} \\ &\leq C\|\nabla\psi\|_{L^2(\Omega)} E(\xi)^{1/2}. \end{aligned}$$

The last inequality above follows from:

$$\forall |F| \leq M \quad \forall x \in \Omega \quad |aGF\tilde{G}^{-1}(x) - \text{cof} F|^2 \leq C_M \text{dist}^2(G^{1/2}F\tilde{G}^{-1/2}, SO(n)),$$

because when  $G^{1/2}F\tilde{G}^{-1/2} \in SO(n)$  then the difference in the left hand side above equals 0.  $\blacksquare$

**Theorem 2.4.** *Assume that the metrics  $G, \tilde{G} \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^{n \times n})$  are Lipschitz continuous. Define:*

$$(2.8) \quad \kappa(G, \tilde{G}) = \inf_{\xi \in W^{1,2}(\Omega, \mathbb{R}^n)} E(\xi).$$

*Then,  $\kappa(G, \tilde{G}) = 0$  if and only if there exists a minimizer  $\xi \in W^{1,2}(\Omega, \mathbb{R}^n)$  with  $E(\xi) = 0$ . In particular, in view of Proposition 2.1, this is equivalent to  $\xi$  being a solution to (1.3) (2.1), and  $\xi$  is smooth if  $G$  and  $\tilde{G}$  are smooth.*

*Proof.* Assume, by contradiction, that for some sequence of deformations  $\xi_k \in W^{1,2}(\Omega, \mathbb{R}^n)$ , there holds  $\lim_{k \rightarrow \infty} E(\xi_k) = 0$ . By Lemma 2.2, replacing  $\xi_k$  by  $\tilde{\xi}_k$ , we may without loss of generality request that  $\|\nabla\xi_k\|_{L^\infty} \leq M$ .

The uniform boundedness of  $\nabla \xi_k$  implies, via the Poincaré inequality, and after a modification by a constant and passing to a subsequence, if necessary:

$$(2.9) \quad \lim_{k \rightarrow \infty} \xi_k = \xi \quad \text{weakly in } W^{1,2}(\Omega).$$

Consider the decomposition  $\xi_k = \phi_k + \psi_k$ , where  $\phi_k$  solves (2.7) with the boundary data  $\phi_k = \xi_k$  on  $\partial\Omega$ . By the Poincaré inequality, Lemma 2.3 implies for the sequence  $\psi_k \in W_0^{1,2}(\Omega)$ :

$$\lim_{k \rightarrow \infty} \psi_k = 0 \quad \text{strongly in } W^{1,2}(\Omega).$$

In view of the convergence in (2.9), the sequence  $\phi_k$  must be uniformly bounded in  $W^{1,2}(\Omega)$ , and hence by [23, Theorem 4.11, estimate (4.18)]:

$$\forall \Omega' \subset \subset \Omega \quad \exists C_{\Omega'} \quad \forall k \quad \|\phi_k\|_{W^{2,2}(\Omega')} \leq C_{\Omega'} \|\phi_k\|_{W^{1,2}(\Omega)} \leq C.$$

Consequently,  $\phi_k$  converge to  $\phi$  strongly in  $W_{loc}^{1,2}(\Omega)$ . Recalling that  $E(\xi_k)$  converge to 0, we finally conclude that:

$$E(\xi) = 0.$$

This proves the claimed result. ■

### 3. A DIMENSION REDUCTION RESULT FOR THE ENERGIES (2.4)

The variational problem induced by (2.4) is difficult due to the lack of convexity. One way of reducing the complexity of this problem is to assume that the target shapes are thin bodies, described by a “thin limit” residual theory, which is potentially easier to analyze. We concentrate on the case  $n = 3$  and the energy functional (2.4) relative to a family of thin films  $\Omega^h = \omega \times (-\frac{h}{2}, \frac{h}{2})$  with the midplate  $\omega \subset \mathbb{R}^2$  given by an open, smooth and bounded set. This set-up corresponds to a scenario where a target 3-dimensional thin shell is to be manufactured, rather than a 2-dimensional surface as in Figure 1.1. As we shall see below, an approximate realization of the ideal thin shell is delivered by solving the variational problem in the limit of the vanishing thickness  $h$ .

Assume further that the given smooth metrics  $G, \tilde{G} : \Omega^h \rightarrow \mathbb{R}^{3 \times 3}$  are thickness-independent:

$$G, \tilde{G}(x', x_3) = G, \tilde{G}(x') \quad \forall x = (x', x_3) \in \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right),$$

and denote:

$$(3.1) \quad E^h(\xi^h) = \frac{1}{h} \int_{\Omega^h} W(G^{1/2}(\nabla \xi^h) \tilde{G}^{-1/2}) \, dx \quad \forall \xi^h \in W^{1,2}(\Omega^h, \mathbb{R}^3).$$

The energy density  $W : \mathbb{R}^{3 \times 3} \rightarrow \bar{\mathbb{R}}_+$  is assumed to be  $\mathcal{C}^2$  regular close to  $SO(3)$ , and to satisfy the conditions of normalisation, frame invariance and bound from below:

$$\begin{aligned} \exists c > 0 \quad \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3) \quad W(R) = 0, \quad W(RF) = W(F), \\ W(F) \geq c \, \text{dist}^2(F, SO(3)). \end{aligned}$$

Following the approach of [11], which has been further developed in [12, 19, 3, 18] (see also [18] for an extensive review of the literature), we obtain the following  $\Gamma$ -convergence results, describing in a rigorous manner the asymptotic behavior of the approximate minimizers of the energy (3.1).

**Theorem 3.1.** *For a given sequence of deformations  $\xi^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$  satisfying:*

$$(3.2) \quad \exists C > 0 \quad \forall h \quad E^h(\xi^h) \leq Ch^2,$$

*there exists a sequence of vectors  $c^h \in \mathbb{R}^3$ , such that the following properties hold for the normalised deformations  $y^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$ :*

$$y^h(x', x_3) = \xi^h(x', hx_3) - c^h.$$

(i) *There exists  $y \in W^{2,2}(\omega, \mathbb{R}^3)$  such that, up to a subsequence:*

$$y^h \rightarrow y \quad \text{strongly in } W^{1,2}(\Omega^1, \mathbb{R}^3).$$

*The deformation  $y$  realizes the compatibility of metrics  $G$  and  $\tilde{G}$  on the midplate  $\omega$ :*

$$(3.3) \quad (\nabla y)^T G \nabla y = \tilde{G}_{2 \times 2}.$$

*The unit normal  $\vec{N}$  to the surface  $y(\omega)$  and the metric  $G$ -induced normal  $\vec{M}$  below have the regularity  $\vec{N}, \vec{M} \in W^{1,2} \cap L^\infty(\omega, \mathbb{R}^3)$ :*

$$\vec{N} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|} \quad \vec{M} = \frac{\sqrt{\det G}}{\sqrt{\det \tilde{G}_{2 \times 2}}} G^{-1}(\partial_1 y \times \partial_2 y),$$

*where we observe that  $\langle \partial_i y, G\vec{M} \rangle = 0$  for  $i = 1, 2$  and  $\langle \vec{M}, G\vec{M} \rangle = 1$ .*

(ii) *Up to a subsequence, we have the convergence:*

$$\frac{1}{h} \partial_3 y^h \rightarrow \vec{b} \quad \text{strongly in } L^2(\Omega^1, \mathbb{R}^3),$$

*where the Cosserat vector  $\vec{b} \in W^{1,2} \cap L^\infty(\omega, \mathbb{R}^3)$  is given by:*

$$(3.4) \quad \vec{b} = (\nabla y)(\tilde{G}_{2 \times 2})^{-1} \begin{bmatrix} \tilde{G}_{13} \\ \tilde{G}_{23} \end{bmatrix} + \frac{\sqrt{\det \tilde{G}}}{\sqrt{\det \tilde{G}_{2 \times 2}}} \vec{M}.$$

(iii) *Define the quadratic forms:*

$$\mathcal{Q}_3(F) = D^2 W(\text{Id})(F, F),$$

$$\mathcal{Q}_2(x', F_{2 \times 2}) = \min \left\{ \mathcal{Q}_3 \left( \tilde{G}(x)^{-1/2} \tilde{F} \tilde{G}(x)^{-1/2} \right); \tilde{F} \in \mathbb{R}^{3 \times 3} \text{ with } \tilde{F}_{2 \times 2} = F_{2 \times 2} \right\}.$$

*The form  $\mathcal{Q}_3$  is defined for all  $F \in \mathbb{R}^{3 \times 3}$ , while  $\mathcal{Q}_2(x', \cdot)$  are defined on  $F_{2 \times 2} \in \mathbb{R}^{2 \times 2}$ . Both forms  $\mathcal{Q}_3$  and all  $\mathcal{Q}_2$  are nonnegative definite and depend only on the symmetric parts of their arguments. In particular, when the energy density  $W$  is isotropic, i.e.:*

$$\forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3) \quad W(RF) = W(F),$$

*then  $\mathcal{Q}_2(x', \cdot)$  is given in terms of the Lamé coefficients  $\lambda, \mu > 0$  by:*

$$\mathcal{Q}_2(x', F_{2 \times 2}) = \mu \left| (\tilde{G}_{2 \times 2})^{-1/2} F_{2 \times 2} (\tilde{G}_{2 \times 2})^{-1/2} \right|^2 + \frac{\lambda \mu}{\lambda + \mu} \left| \text{tr} \left( (\tilde{G}_{2 \times 2})^{-1/2} F_{2 \times 2} (\tilde{G}_{2 \times 2})^{-1/2} \right) \right|^2,$$

*for all  $F_{2 \times 2} \in \mathbb{R}_{sym}^{2 \times 2}$ .*

(iii) *We have the lower bound:*

$$(3.5) \quad \liminf_{h \rightarrow 0} \frac{1}{h^2} E^h(\xi^h) \geq \mathcal{I}_{G, \tilde{G}}(y) := \frac{1}{24} \int_{\Omega} \mathcal{Q}_2 \left( x', (\nabla y)^T G \nabla \vec{b} \right) dx'.$$



*Proof.* The convergences in (i) and (ii) rely on a version of an approximation result from [11]; there exists matrix fields  $Q^h \in W^{1,2}(\omega, \mathbb{R}^{3 \times 3})$  and a constant  $C$  uniform in  $h$ , i.e. depending only on the geometry of  $\omega$  and on  $G, \tilde{G}$ , such that:

$$\begin{aligned} \frac{1}{h} \int_{\Omega^h} |\nabla \xi^h(x', x_3) - Q^h(x')|^2 dx &\leq C \left( h^2 + \frac{1}{h} \int_{\Omega^h} \text{dist}^2(G^{1/2} \nabla \xi^h \tilde{G}^{-1/2}, SO(3)) dx \right), \\ \int_{\Omega} |\nabla Q^h(x')|^2 dx' &\leq C \left( 1 + \frac{1}{h^3} \int_{\Omega^h} \text{dist}^2(G^{1/2} \nabla \xi^h \tilde{G}^{-1/2}, SO(3)) dx \right). \end{aligned}$$

Further ingredients of the proof follow exactly as in [3], so we suppress the details.  $\blacksquare$

**Theorem 3.2.** *For every compatible immersion  $y \in W^{2,2}(\Omega, \mathbb{R}^3)$  satisfying (3.3), there exists a sequence of recovery deformations  $\xi^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ , such that:*

- (i) *The rescaled sequence  $y^h(x', x_3) = \xi^h(x', hx_3)$  converges in  $W^{1,2}(\Omega^1, \mathbb{R}^3)$  to  $y$ .*
- (ii) *One has:*

$$\lim_{h \rightarrow 0} \frac{1}{h^2} E^h(\xi^h) = \mathcal{I}_{G, \tilde{G}}(y),$$

where the Cosserat vector  $\vec{b}$  in the definition (3.5) of  $\mathcal{I}_{G, \tilde{G}}$  is derived by (3.4).

*Proof.* Let  $y \in W^{2,2}(\Omega, \mathbb{R}^3)$  satisfy (3.3). Define  $\vec{b}$  according to (3.4) and let:

$$Q = \begin{bmatrix} \partial_1 y & \partial_2 y & \vec{b} \end{bmatrix} \in W^{1,2} \cap L^\infty(\omega, \mathbb{R}^{3 \times 3}).$$

By Theorem 3.1, it follows that:

$$G^{1/2} Q \tilde{G}^{-1/2} \in SO(3) \quad \forall \text{a.e. } x' \in \omega.$$

Define the limiting warping field  $\vec{d} \in L^2(\Omega, \mathbb{R}^3)$ :

$$\vec{d}(x') = G^{-1} Q^{T,-1} \left( c(x', (\nabla y)^T G \nabla \vec{b}) - \begin{bmatrix} \langle \partial_1 \vec{b}, G \vec{b} \rangle \\ \langle \partial_2 \vec{b}, G \vec{b} \rangle \\ 0 \end{bmatrix} \right),$$

where  $c(x', F_{2 \times 2})$  denotes the unique minimizer of the problem in:

$$\forall F_{2 \times 2} \in \mathbb{R}_{sym}^{2 \times 2} \quad \mathcal{Q}_2(x', F_{2 \times 2}) = \min \left\{ \mathcal{Q}_3(\tilde{G}^{-1/2}(F_{2 \times 2}^* + \text{sym}(c \otimes e_3))\tilde{G}^{-1/2}); c \in \mathbb{R}^3 \right\}.$$

Let  $\{d^h\}$  be a approximating sequence in  $W^{1,\infty}(\Omega, \mathbb{R}^3)$ , satisfying:

$$(3.6) \quad d^h \rightarrow \vec{d} \quad \text{strongly in } L^2(\omega, \mathbb{R}^3), \quad \text{and} \quad h^2 \|d^h\|_{W^{1,\infty}} \rightarrow 0.$$

Note that such sequence can always be derived by reparametrizing (slowing down) a sequence of smooth approximations of  $\vec{d}$ . Similarly, consider the approximations  $y^h \in W^{2,\infty}(\omega, \mathbb{R}^3)$  and  $b^h \in W^{1,\infty}(\omega, \mathbb{R}^3)$ , with the following properties:

$$(3.7) \quad \begin{aligned} &y^h \rightarrow y \quad \text{strongly in } W^{2,2}(\omega, \mathbb{R}^3), \quad \text{and} \quad b^h \rightarrow \vec{b} \quad \text{strongly in } W^{1,2}(\omega, \mathbb{R}^3) \\ &h \left( \|y^h\|_{W^{2,\infty}} + \|b^h\|_{W^{1,\infty}} \right) \leq \epsilon \\ &\frac{1}{h^2} |\omega \setminus \omega_h| \rightarrow 0, \quad \text{where } \omega_h = \left\{ x' \in \omega; y^h(x') = y(x') \text{ and } b^h(x') = \vec{b}(x') \right\} \end{aligned}$$

for some appropriately small  $\epsilon > 0$ . Existence of approximations with the claimed properties follows by partition of unity and truncation arguments, as a special case of the Lusin-type result for Sobolev functions (see Proposition 2 in [12]).

We now define the recovery sequence  $\xi^h \in W^{1,\infty}(\Omega^h, \mathbb{R}^3)$  by:

$$\xi^h(x', x_3) = y^h(x') + x_3 b^h(x') + \frac{x_3^2}{2} d^h(x').$$

Consequently, the rescalings  $y^h \in W^{1,\infty}(\Omega^1, \mathbb{R}^3)$  are:

$$y^h(x', x_3) = y^h(x') + h x_3 b^h(x') + \frac{h^2}{2} x_3^2 d^h(x'),$$

and so in view of (3.6) and (3.7), Theorem 3.2 (i) follows directly. The remaining convergence in (ii) is achieved via standard calculations exactly as in [3]. We suppress the details.  $\blacksquare$

It now immediately follows that:

**Corollary 3.3.** *Existence of a  $W^{2,2}$  regular immersion satisfying (3.3) is equivalent to the upper bound on the energy scaling at minimizers:*

$$\exists C > 0 \quad \inf_{\xi \in W^{1,2}(\Omega^h, \mathbb{R}^3)} E^h(\xi) \leq Ch^2.$$

The following corollary is a standard conclusion of the established  $\Gamma$ -convergence (see e.g. [4]). It indicates that sequences of approximate solutions to the original problem on a thin shell are in one-one correspondence to the minimizers of the thin limit variational model  $\mathcal{I}_{G, \tilde{G}}$ .

**Corollary 3.4.** *Assume that (3.3) admits a  $W^{2,2}$ -regular solution  $y$ . Then any sequence of approximate minimizers  $\xi^h$  of (3.1), satisfying the property:*

$$\lim_{h \rightarrow 0} \frac{1}{h^2} (E^h(\xi^h) - \inf E^h) = 0,$$

*converges, after the proper rescaling and up to a subsequence (see Theorem 3.1), to a minimizer of the functional  $\mathcal{I}_{G, \tilde{G}}$ . In particular,  $\mathcal{I}_{G, \tilde{G}}$  attains its minimum. Conversely, any minimizer  $y$  of  $\mathcal{I}_{G, \tilde{G}}$  is a limit of approximate minimizers  $\xi^h$  to (3.1).*

On a final note, observe that the defect  $\kappa(G, \tilde{G})$ , as defined in (2.8), is of the order  $h^2 \min \mathcal{I}_{G, \tilde{G}} + o(h^2)$ . It would be hence of interest to discuss the necessary and sufficient conditions for having  $\min \mathcal{I}_{G, \tilde{G}} = 0$ , which in case of  $G = \text{Id}_3$  have been precisely derived in [3].

#### 4. AN EQUIVALENT SYSTEM OF PDES FOR (1.3)

In this section, we investigate the integrability conditions of the system (1.3). Firstly, recall that since the Levi-Civita connection is metric-compatible, we have:

$$(4.1) \quad \partial_i G_{jk} = G_{mk} \Gamma_{ij}^m + G_{mj} \Gamma_{ik}^m,$$

where the Christoffel symbols (of second kind) of the metric  $G$  are:

$$\Gamma_{kl}^i = \frac{1}{2} G^{im} (\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl}).$$

Above, we used the Einstein summation over the repeated upper and lower indices from 1 to  $n$ .

**Lemma 4.1.** *Assume that there is a bilipschitz map  $\xi : \Omega \rightarrow U := \xi(\Omega)$ , between two open, bounded subsets:  $\Omega, U$  of  $\mathbb{R}^n$ , satisfying (1.3) and such that  $\xi \in W^{2,2}(\Omega, U)$ .*

*Then, denoting by  $\tilde{\Gamma}_{ij}^k$  the Christoffel symbols of  $\tilde{G}$ , and by  $\Gamma_{ij}^k$  the Christoffel symbols of the metric  $G \circ \xi^{-1}$  on  $U$ , there holds, in  $\Omega$ :*

$$(4.2) \quad \forall i, j, s : 1 \dots n \quad \partial_{ij}\xi^s = \partial_m \xi^s \tilde{\Gamma}_{ij}^m - \partial_i \xi^p \partial_j \xi^q (\Gamma_{pq}^s \circ \xi).$$

*In particular  $\xi$  automatically enjoys higher regularity:  $\xi \in W^{2,\infty}(\Omega, U)$ .*

*Proof.* By (4.1) we obtain:

$$\partial_i \tilde{G}_{jk} = ((\nabla \xi)^T G (\nabla \xi))_{mk} \tilde{\Gamma}_{ij}^m + ((\nabla \xi)^T G (\nabla \xi))_{mj} \tilde{\Gamma}_{ik}^m,$$

while differentiating (1.3) directly, gives:

$$\partial_i \tilde{G}_{jk} = ((\nabla \partial_i \xi)^T G (\nabla \xi))_{jk} + \partial_i \xi^p ((\nabla \xi)^T \partial_p (G \circ \xi^{-1}) (\nabla \xi))_{jk} + ((\nabla \xi)^T G (\nabla \partial_i \xi))_{jk},$$

where we used [24, Theorem 2.2.2] to conclude that  $G \circ \xi^{-1} \in W^{1,\infty}$  and to apply the chain rule. Equating both sides above, and using (4.1) to the Lipschitz metric  $G \circ \xi^{-1}$ , we get:

$$\begin{aligned} \partial_m \xi^s G_{st} \partial_k \xi^t \tilde{\Gamma}_{ij}^m + \partial_m \xi^s G_{st} \partial_j \xi^t \tilde{\Gamma}_{ik}^m \\ = \partial_{ij} \xi^s G_{st} \partial_k \xi^t + \partial_j \xi^s \partial_i \xi^p \partial_k \xi^t (G_{tm} \Gamma_{ps}^m + G_{sm} \Gamma_{pt}^m) + \partial_{ik} \xi^s G_{st} \partial_j \xi^t, \end{aligned}$$

which we rewrite as:

$$(4.3) \quad \begin{aligned} \partial_m \xi^s \tilde{\Gamma}_{ij}^m (G \nabla \xi)_{sk} + \partial_m \xi^s \tilde{\Gamma}_{ik}^m (G \nabla \xi)_{sj} \\ = \partial_{ij} \xi^s (G \nabla \xi)_{sk} + \partial_j \xi^q \partial_i \xi^p \Gamma_{pq}^s (G \nabla \xi)_{sk} + \partial_j \xi^p \partial_k \xi^t \Gamma_{pt}^s (G \nabla \xi)_{sj} + \partial_{ik} \xi^s (G \nabla \xi)_{sj}. \end{aligned}$$

For each  $i, j : 1 \dots n$  we now define the vector  $P_{ij} \in \mathbb{R}^n$  with components:

$$P_{ij}^s = \partial_{ij} \xi^s - \partial_m \xi^s \tilde{\Gamma}_{ij}^m + \partial_j \xi^q \partial_i \xi^p \Gamma_{pq}^s, \quad s : 1 \dots n.$$

By (4.3), we see that:

$$\forall i, j, k : 1 \dots n \quad \langle P_{ij}, (G \nabla \xi)_{k-col} \rangle = -\langle P_{ik}, (G \nabla \xi)_{j-col} \rangle,$$

where  $(G \nabla \xi)_{k-col}$  is the vector denoting the  $k$ -th column of the matrix  $G \nabla \xi$ .

Since  $P_{ij} = P_{ji}$ , it now follows that:

$$(4.4) \quad \forall i, j, k : 1 \dots n \quad \langle P_{ij}, (G \nabla \xi)_{k-col} \rangle = 0.$$

But the columns of the invertible  $G \nabla \xi$  form a linearly independent system, so there must be  $P_{ij} = 0$ , which completes the proof of (4.2).  $\blacksquare$

**Corollary 4.2.** *Let  $\xi : \Omega \rightarrow U$  be a bilipschitz map satisfying (1.3) as in Lemma 4.1. Assume that  $\xi \in W^{2,2}(\Omega, U)$ . Then  $\xi$  and  $\zeta = \xi^{-1}$  are both smooth and bounded, together with all their derivatives. In particular, (4.2) holds everywhere in  $\Omega$ .*

*Proof.* It was already established that  $\nabla^2 \xi \in L^\infty$ . We have  $F := \nabla \zeta = (\nabla \xi \circ \zeta)^{-1} \in L^\infty$ . Notice that  $F^{-1}$  is the composition of two Lipschitz mappings and hence it is Lipschitz. We conclude that for all  $i$ ,  $\partial_i F = -F \partial_i (F^{-1}) F \in L^\infty$ , which implies that  $\nabla \zeta \in W^{1,\infty}$ , and hence  $\zeta \in W^{2,\infty}(U, \Omega)$ .

By (1.3) we get the following formula:  $(\nabla \zeta)^T (\tilde{G} \circ \zeta) (\nabla \zeta) = G \circ \zeta$ , valid in  $U$ . By the same calculations as in Lemma 4.1, it results in:

$$\forall i, j, s : 1 \dots n \quad \partial_{ij} \zeta^s = \partial_m \zeta^s \Gamma_{ij}^m - \partial_i \zeta^p \partial_j \zeta^q \tilde{\Gamma}_{pq}^s.$$

In view of (4.2) and by a bootstrap argument, we obtain that  $\xi, \zeta \in W^{k,\infty}$  for every  $k \geq 1$ . Hence the result follows.  $\blacksquare$

**Corollary 4.3.** *Equivalently, (4.2) can be written as:*

$$(4.5) \quad \forall i, j, s : 1 \dots n \quad \partial_{ij} \xi^s = \partial_m \xi^s \tilde{\Gamma}_{ij}^m - \frac{1}{2} G^{sm} \left( \partial_i \xi^q \partial_j G_{mq} + \partial_j \xi^p \partial_i G_{mp} - \partial_j \xi^p \partial_i \xi^q \partial_t G_{pq} ((\nabla \xi)^{-1})_{tm} \right).$$

*Proof.* Denoting  $\zeta = \xi^{-1}$  as before, we obtain:

$$\Gamma_{pq}^s = \frac{1}{2} G^{sm} \left( \partial_p \zeta^t \partial_t G_{mq} + \partial_q \zeta^t \partial_t G_{mp} - \partial_m \zeta^t \partial_t G_{pq} \right).$$

Inserting in (4.2) and contracting  $\partial_p \zeta^t \partial_j \xi^p$  to the Kronecker delta  $\delta_j^t$ , we obtain (4.5).  $\blacksquare$

**Theorem 4.4.** *Consider the following system of the algebraic-differential equations in the unknowns  $\xi, w_i : \Omega \rightarrow \mathbb{R}^n$ ,  $i : 1 \dots n$ :*

$$(4.6a) \quad \tilde{G}_{ij} = w_i^t G_{st} w_j^s$$

$$(4.6b) \quad w_i^t = \partial_i \xi^t$$

$$(4.6c) \quad \partial_j w_i^s = w_m^s \tilde{\Gamma}_{ij}^m - \frac{1}{2} G^{sm} \left( w_i^p \partial_j G_{mp} + w_j^p \partial_i G_{mp} - w_j^p w_i^q \partial_t G_{pq} W_m^t \right),$$

where the matrix  $[W_m^t]_{t,m=1..n}$  is defined as the inverse of the matrix field  $w : \Omega \rightarrow \mathbb{R}^{n \times n}$ , whose columns are the vectors  $w_i$ , i.e.:  $W_m^t = \left( \sum_{i=1}^n w_i \otimes e_i \right)^{tm}$ . Then we have the following:

- (i) *Problem (1.3) has a solution given by a bilipschitz map  $\xi \in W^{2,2}(\Omega, U)$  (as in Corollary 4.2), if and only if (4.6) has a solution  $(\xi, w)$  given by a bilipschitz  $\xi : \Omega \rightarrow U$  and  $w \in W^{1,2}(\Omega, \mathbb{R}^{n \times n})$ .*
- (ii) *Problem (4.6) has a solution, understood as in (i) above, if and only if (4.6c) has a solution. This statement should be understood in the following sense.*

*Assume that (4.6c) is solved in the sense of distributions, by the vector fields  $w_i \in L^\infty(\Omega, \mathbb{R}^n)$ ,  $i : 1 \dots n$ , such that  $W_m^t$  are well defined and  $W_m^t \in L^\infty(\Omega)$ . Then there exists a smooth  $\xi : \Omega \rightarrow U$  such that (4.6b) holds. Moreover,  $\xi$  is locally invertible to a smooth vector field  $\zeta$ , and the Christoffel symbols of the following metrics:*

$$\mathcal{G} = G \circ \zeta \quad \text{and} \quad \tilde{\mathcal{G}} = (\nabla \zeta)^T (\tilde{G} \circ \zeta) (\nabla \zeta)$$

*are the same. If additionally  $\xi$  is globally invertible to  $\zeta : U \rightarrow \Omega$ , and if we have:*

$$(4.7) \quad \exists x_0 \in \Omega \quad \tilde{G}(x_0) = w^T G w(x_0),$$

*then (4.6a) holds in  $\Omega$ .*

*Proof.* Clearly, the equivalence in (i) follows from Lemma 4.1 and Lemma 4.3. For the equivalence in (ii) note first that, by a bootstrap argument, an  $L^\infty$  vector field  $w$  satisfying (4.6c) is automatically smooth in  $\Omega$ , together with  $W$ . Further, in a simply connected domain  $\Omega$ , the condition (4.6b) is the same as:

$$\forall i, j, t : 1 \dots n \quad \partial_j w_i^t = \partial_i w_j^t,$$

which is implied by (4.6c), by the symmetry of its right hand side (in  $i, j$ ).

We now prove that the metrics  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  have the same Christoffel symbols on the subdomain of  $U$  where the local inverse  $\zeta$  is defined. Note that both  $\xi$  and  $\zeta$  are smooth. We first compute:

$$(4.8) \quad \partial_m \left( \partial_i \zeta^p \tilde{\mathcal{G}}_{ps} \partial_j \zeta^s \right) = \partial_{im} \zeta^p \tilde{\mathcal{G}}_{ps} \partial_j \zeta^s + \partial_i \zeta^p \tilde{\mathcal{G}}_{ps} \partial_{jm} \zeta^s + \partial_i \zeta^p \partial_j \zeta^s \partial_m \zeta^q \partial_q \tilde{\mathcal{G}}_{ps}.$$

By Lemma 4.1, we also obtain:

$$\partial_j w_i^s = w_m^s \tilde{\Gamma}_{ij}^m - w_j^p w_i^q \Gamma_{pq}^s,$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols of  $\mathcal{G}$ . Now, since:  $\partial_p \zeta^s w_i^p = \delta_i^s = \partial_i \zeta^p w_p^s$ , it follows that:

$$\partial_{pq} \zeta^s w_j^q w_i^p = -\partial_p \zeta^s \partial_j w_i^p = -\partial_p \zeta^s \left( w_m^p \tilde{\Gamma}_{ij}^m - w_j^\alpha w_i^\beta \Gamma_{\alpha\beta}^p \right) = -\tilde{\Gamma}_{ij}^s + \partial_p \zeta^s w_j^\alpha w_i^\beta \Gamma_{\alpha\beta}^p.$$

Consequently, (4.8) becomes:

$$(4.9) \quad \begin{aligned} \partial_m \tilde{\mathcal{G}}_{ij} &= \partial_{im} \zeta^p \tilde{\mathcal{G}}_{ps} \partial_j \zeta^s + \partial_i \zeta^p \tilde{\mathcal{G}}_{ps} \partial_{jm} \zeta^s \\ &\quad + \partial_i \zeta^p \partial_j \zeta^s \partial_m \zeta^q \tilde{\mathcal{G}}_{\alpha p} \left( -\partial_{\gamma\delta} \zeta^\alpha w_q^\gamma w_s^\delta + \partial_t \zeta^\alpha w_q^\gamma w_s^\delta \Gamma_{\gamma\delta}^t \right) \\ &\quad + \partial_i \zeta^p \partial_j \zeta^s \partial_m \zeta^q \tilde{\mathcal{G}}_{\alpha s} \left( -\partial_{\gamma\delta} \zeta^\alpha w_q^\gamma w_p^\delta + \partial_t \zeta^\alpha w_q^\gamma w_p^\delta \Gamma_{\gamma\delta}^t \right) \\ &= \partial_i \zeta^p \partial_t \zeta^\alpha \tilde{\mathcal{G}}_{\alpha p} \Gamma_{mj}^t + \partial_j \zeta^s \partial_t \zeta^\alpha \tilde{\mathcal{G}}_{\alpha s} \Gamma_{im}^t \\ &= \tilde{\mathcal{G}}_{ti} \Gamma_{mj}^t + \tilde{\mathcal{G}}_{tj} \Gamma_{mi}^t. \end{aligned}$$

Call  $\gamma_{ij}^k$  the Christoffel symbols of  $\tilde{\mathcal{G}}$ , so that:  $\partial_m \tilde{\mathcal{G}}_{ij} = \tilde{\mathcal{G}}_{ti} \gamma_{mj}^t + \tilde{\mathcal{G}}_{tj} \gamma_{mi}^t$ . Therefore:

$$\forall i, j, m : 1 \dots n \quad \tilde{\mathcal{G}}_{tj} (\Gamma_{mi}^t - \gamma_{mi}^t) = -\tilde{\mathcal{G}}_{ti} (\Gamma_{mj}^t - \gamma_{mj}^t).$$

Using the same reasoning as in the proof of (4.4) we get, as claimed:

$$\forall i, j, m : 1 \dots n \quad \Gamma_{ji}^m = \gamma_{ij}^m.$$

This concludes the proof of (ii) in view of Lemma 4.5 below, and since  $\tilde{\mathcal{G}}(\xi(x_0)) = \mathcal{G}(\xi(x_0))$ .  $\blacksquare$

**Lemma 4.5.** *Assume that the Christoffel symbols of two smooth metrics  $\mathcal{G}_1, \mathcal{G}_2$  on a connected domain  $U \subset \mathbb{R}^n$  coincide, and that for some  $x_0 \in U$ ,  $\mathcal{G}_1(x_0) = \mathcal{G}_2(x_0)$ . Then  $\mathcal{G}_1 = \mathcal{G}_2$ .*

*Proof.* Let  $\nabla_i$  represent the covariant derivative associated with the metric  $\mathcal{G}_i$  through the Levi-Civita connection [17, p. 114 and Theorem 2.2, p. 158]. An immediate consequence of the equivalence of the Christoffel symbols (in the identity coordinate system of  $U$ ) is that  $\nabla := \nabla_1 = \nabla_2$  [17, p. 144]. By [17, Proposition 2.1, p. 158], we have  $\nabla_i \mathcal{G}_i = 0$ , i.e. both  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are parallel tensor fields on  $U$  with respect to  $\nabla$  [17, pages 88 and 124]. Let  $x \in U$  and let  $\gamma$  be any piece-wise  $\mathcal{C}^1$  curve connecting  $x_0$  and  $x$  in  $U$ . Let  $\tau_\gamma$  be the parallel transport from the point  $u_0$  to  $u$  along  $\gamma$ . It follows that:

$$\tau_\gamma(\mathcal{G}_i(x_0)) = \mathcal{G}_i(x) \quad \forall i = 1, 2,$$

which concludes the proof.  $\blacksquare$

**Remark 4.6.** Assume that the Christoffel symbols of two smooth 2d metrics  $\mathcal{G}_1, \mathcal{G}_2$  on a connected domain  $U \subset \mathbb{R}^2$  coincide. Assume moreover that the Gaussian curvature of  $\mathcal{G}_1$  (and hence of  $\mathcal{G}_2$ ) does not vanish identically in  $U$ . Then, one can directly prove that there exists a positive constant  $\lambda$  such that  $\mathcal{G}_1 = \lambda \mathcal{G}_2$  in  $U$ .

Namely, let  $\nabla_i$  represent the covariant derivative associated with the metric  $\mathcal{G}_i$  through the Levi-Civita connection, as in the proof of Lemma 4.5. Again, equivalence of the Christoffel symbols implies that  $\nabla := \nabla_1 = \nabla_2$ . Let  $x_0 \in U$  and let  $H = \text{Hol}(U, x_0)$  be the holonomy group associated with  $\nabla$  at  $x_0$ . By [5], (see also [2, Theorem 392]),  $H$  is a connected subgroup of the special orthogonal group associated with the scalar product  $\mathcal{G}_i(x_0)$ . Since  $(\Omega, \mathcal{G}_i)$  is not flat,  $H$  cannot be the trivial subgroup. The only other possibility (when  $n = 2$ ) is that  $H$  is the entire  $SO(2, \mathcal{G}_i(x_0))$ . This implies that the  $SO(2, \mathcal{G}_1(x_0)) = SO(2, \mathcal{G}_2(x_0))$ , which by the transitivity of the action of the special orthogonal group over the unit sphere, results in  $\mathcal{G}_1(x_0) = \lambda \mathcal{G}_2(x_0)$ .

Note that for  $n > 2$ , the holonomy group  $H$  generically coincides with the full special orthogonal group [2, p.643]. In this case again  $SO(n, \mathcal{G}_1(x_0)) = SO(n, \mathcal{G}_2(x_0))$  for some  $x_0 \in U$ , and so the result is established for generic metrics  $\mathcal{G}_i$ .

## 5. SOME REMARKS AND EXAMPLES

**Remark 5.1.** Note that in problem (1.3) one can, without loss of generality, assume that:

$$(5.1) \quad \tilde{G}(x_0) = G(x_0) = \text{Id}_n$$

Indeed, denote  $\tilde{A} = \tilde{G}(x_0)^{-1/2}$  and  $A = G(x_0)^{-1/2}$  the inverses of the unique symmetric square roots of the metrics at  $x_0$ , and let  $\Omega_1 = \tilde{A}^{-1}(\Omega)$  be the preimage of  $\Omega$  under the linear transformation  $x \mapsto \tilde{A}x$ . Then  $\tilde{G}_1 = \tilde{A}(\tilde{G} \circ \tilde{A})\tilde{A}$  and  $G_2 = A(G \circ A)A$  are two metrics on  $\Omega_1$ , that equal  $\text{Id}$  at  $x_1 = \tilde{A}^{-1}x_0$ . Further, (1.3) can be written as:

$$(5.2) \quad \tilde{G}_1(x) = \left( A^{-1}(\nabla \xi(\tilde{A}x))\tilde{A} \right)^T G_1(x) \left( A^{-1}(\nabla \xi(\tilde{A}x))\tilde{A} \right) = \nabla \xi_1 G_1 \nabla \xi_1(x),$$

where  $\xi_1 = A^{-1}\xi \circ \tilde{A}$ . Clearly, existence of a solution  $\xi_1$  to (5.2) on  $\Omega_1$  is completely equivalent with existence of  $\xi$  solving (1.3) on  $\Omega$ , with the same required regularity.

Notice also that, in view of the compensated regularity for  $\xi$  in Corollary 4.2, any solution to (1.3) satisfies, up to a global reflection, the orientation preserving condition (2.1). In view of (5.1), (4.7) now becomes:  $w(x_0) \in SO(n)$ . For the particular case of  $w(x_0) = \text{Id}$ , it is easy to notice that (4.6c) at  $x_0$  reduces to:  $\partial_j w_i^s(x_0) = \tilde{\Gamma}_{ij}^s - \chi_{ij}^s$ , with  $\chi_{ij}^s$  denoting the Christoffel symbols of the metric  $G$  on  $\Omega$ .

**Remark 5.2.** We now present alternative calculations leading to the system (4.6a) – (4.6c), adapting classical ideas [20]. These calculations do not require knowledge of the metric compatibility of the connection.

1. Assume that there exists a bilipschitz map  $\xi : \Omega \rightarrow U$ , whose global inverse we denote by  $\zeta : U \rightarrow \Omega$ , such that (1.3) holds in  $\Omega$ . Let  $\tilde{\Gamma}_{ijk}$  the Christoffel symbols of the first kind of the metric  $\tilde{G}$  on  $\Omega$ :

$$\tilde{\Gamma}_{ijk} = \frac{1}{2}(\partial_i \tilde{G}_{jk} + \partial_j \tilde{G}_{ik} - \partial_k \tilde{G}_{ij}),$$

while let  $\Gamma_{\alpha\beta\gamma}$  stand for the Christoffel symbols of the metric  $\mathcal{G} = G \circ \zeta$  on  $U$ :

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\partial_\alpha \mathcal{G}_{\beta\gamma} + \partial_\beta \mathcal{G}_{\alpha\gamma} - \partial_\gamma \mathcal{G}_{\alpha\beta}).$$

As above, we will use the Latin indices for components of vectors in  $\Omega$ , and Greek indices in  $U$ .

Since by (1.3) we have  $\tilde{G} = (\nabla\xi)^T(\mathcal{G} \circ \xi)(\nabla\xi)$ , we obtain:

$$\begin{aligned}\partial_k \tilde{G}_{ij} &= \partial_k \left( \partial_i \xi^\alpha \partial_j \xi^\beta (\mathcal{G}_{\alpha\beta} \circ \xi) \right) \\ &= \left( \partial_{ki} \xi^\alpha \partial_j \xi^\beta + \partial_{kj} \xi^\alpha \partial_i \xi^\beta \right) (\mathcal{G}_{\alpha\beta} \circ \xi) + \partial_i \xi^\alpha \partial_j \xi^\beta \left( (\partial_\gamma \mathcal{G}_{\alpha\beta}) \circ \xi \right) \partial_k \xi^\gamma.\end{aligned}$$

Similarly, exchanging the following pairs of indices:  $\gamma$  with  $\alpha$  and  $\beta$  with  $\gamma$  in the first equation, and  $\gamma$  with  $\beta$  in the second equation below, we get:

$$\begin{aligned}\partial_i \tilde{G}_{jk} &= \left( \partial_{ij} \xi^\alpha \partial_k \xi^\beta + \partial_{ik} \xi^\alpha \partial_j \xi^\beta \right) (\mathcal{G}_{\alpha\beta} \circ \xi) + \partial_i \xi^\alpha \partial_j \xi^\beta \left( (\partial_\alpha \mathcal{G}_{\gamma\beta}) \circ \xi \right) \partial_k \xi^\gamma, \\ \partial_j \tilde{G}_{ik} &= \left( \partial_{ij} \xi^\alpha \partial_k \xi^\beta + \partial_{jk} \xi^\alpha \partial_i \xi^\beta \right) (\mathcal{G}_{\alpha\beta} \circ \xi) + \partial_i \xi^\alpha \partial_j \xi^\beta \left( (\partial_\beta \mathcal{G}_{\alpha\gamma}) \circ \xi \right) \partial_k \xi^\gamma.\end{aligned}$$

Consequently:

$$(5.3) \quad \tilde{\Gamma}_{ijk} = \partial_i \xi^\alpha \partial_j \xi^\beta \partial_k \xi^\gamma \left( \Gamma_{\alpha\beta\gamma} \circ \xi \right) + \partial_{ij} \xi^\alpha \partial_k \xi^\beta \left( \mathcal{G}_{\alpha\beta} \circ \xi \right).$$

**2.** We now compute the Christoffel symbols of the second kind for the metric  $\tilde{G}$ :

$$\tilde{\Gamma}_{ij}^k = \tilde{G}^{km} \tilde{\Gamma}_{ijm}.$$

Since  $\tilde{G}^{-1} = ((\nabla\zeta)\mathcal{G}^{-1}(\nabla\zeta)^T) \circ \xi$ , it follows that:  $\tilde{G}^{km} = (\partial_\alpha \zeta^k \partial_\beta \zeta^m \mathcal{G}^{\alpha\beta}) \circ \xi$ . By (5.3) we get:

$$\begin{aligned}(5.4) \quad \tilde{\Gamma}_{ij}^k &= \left( (\partial_\rho \zeta^k \partial_\nu \zeta^m \mathcal{G}^{\rho\nu}) \circ \xi \right) \partial_i \xi^\alpha \partial_j \xi^\beta \partial_m \xi^\gamma \left( \Gamma_{\alpha\beta\gamma} \circ \xi \right) + \left( (\partial_\rho \zeta^k \partial_\nu \zeta^m \mathcal{G}^{\rho\nu}) \circ \xi \right) (\mathcal{G}_{\alpha\beta} \circ \xi) \partial_{ij} \xi^\alpha \partial_m \xi^\beta \\ &= \left( (\mathcal{G}^{\rho\gamma} \Gamma_{\alpha\beta\gamma}) \circ \xi \right) \partial_i \xi^\alpha \partial_j \xi^\beta (\partial_\rho \zeta^k \circ \xi) + \left( (\mathcal{G}^{\rho\beta} \mathcal{G}_{\alpha\beta}) \circ \xi \right) \partial_{ij} \xi^\alpha (\partial_\rho \zeta^k \circ \xi) \\ &= (\Gamma_{\alpha\beta}^\rho \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta (\partial_\rho \zeta^k \circ \xi) + (\partial_\alpha \zeta^k \circ \xi) \partial_{ij} \xi^\alpha,\end{aligned}$$

where we contracted  $\partial_m \xi^\gamma (\partial_\nu \zeta^m \circ \xi)$  to the Kronecker delta  $\delta_\nu^\gamma$  and  $\partial_m \xi^\beta (\partial_\nu \zeta^m \circ \xi)$  to  $\delta_\nu^\beta$  in the second equality, and  $\mathcal{G}^{\rho\beta} \mathcal{G}_{\alpha\beta}$  to  $\delta_\alpha^\rho$  in the third equality, where we also used the definition of the Christoffel symbols  $\Gamma_{\alpha\beta}^\rho$  of the second kind of the metric  $\mathcal{G}$ . Further:

$$\begin{aligned}\tilde{\Gamma}_{ij}^k \partial_k \xi^\mu &= (\partial_\alpha \zeta^k \circ \xi) \partial_k \xi^\mu \partial_{ij} \xi^\alpha + (\Gamma_{\alpha\beta}^\rho \circ \xi) \partial_k \xi^\mu \partial_i \xi^\alpha \partial_j \xi^\beta (\partial_\rho \zeta^k \circ \xi) \\ &= \delta_\alpha^\mu \partial_{ij} \xi^\alpha + (\Gamma_{\alpha\beta}^\rho \circ \xi) \delta_\rho^\mu \partial_i \xi^\alpha \partial_j \xi^\beta,\end{aligned}$$

which implies the same formula as in (4.2):

$$(5.5) \quad \partial_{ij} \xi^\mu = \tilde{\Gamma}_{ij}^k \partial_k \xi^\mu - (\Gamma_{\alpha\beta}^\mu \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta,$$

**3.** We now proceed as in Corollary 4.3:

$$\begin{aligned}\Gamma_{\alpha\beta}^\mu &= \frac{1}{2} \mathcal{G}^{\mu\gamma} \left( \partial_\alpha \mathcal{G}_{\beta\gamma} + \partial_\beta \mathcal{G}_{\alpha\gamma} - \partial_\gamma \mathcal{G}_{\alpha\beta} \right) \\ &= \frac{1}{2} \mathcal{G}^{\mu\gamma} \left( (\partial_s \mathcal{G}_{\beta\gamma} \circ \zeta) \partial_\alpha \zeta^s + (\partial_r \mathcal{G}_{\alpha\gamma} \circ \zeta) \partial_\beta \zeta^r - (\partial_q \mathcal{G}_{\alpha\beta} \circ \zeta) \partial_\gamma \zeta^q \right).\end{aligned}$$

Consequently, and in view of (5.5):

$$\begin{aligned}\tilde{\Gamma}_{ij}^k \partial_k \xi^\mu - \partial_{ij} \xi^\mu &= (\Gamma_{\alpha\beta}^\mu \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta \\ &= \frac{1}{2} (\mathcal{G}^{\mu\gamma} \circ \xi) \left( \partial_s G_{\beta\gamma} (\partial_\alpha \zeta^s \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta + \partial_r G_{\alpha\gamma} (\partial_\beta \zeta^r \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta \right. \\ &\quad \left. - \partial_q G_{\alpha\beta} (\partial_\gamma \zeta^q \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta \right) \\ &= \frac{1}{2} G^{\mu\gamma} \left( \partial_s G_{\beta\gamma} \delta_i^s \partial_j \xi^\beta + \partial_r G_{\alpha\gamma} \delta_j^r \partial_i \xi^\alpha - \partial_q G_{\alpha\beta} W_\gamma^q \partial_i \xi^\alpha \partial_j \xi^\beta \right),\end{aligned}$$

which directly implies (4.6c). ■

**Remark 5.3.** We further observe that if (4.6c) holds, then defining  $\xi$  by (4.6b) we may again obtain (5.4) simply by reversing steps in Remark 5.2. Letting  $\tilde{G} = (\nabla \xi)^T (\mathcal{G} \circ \xi) (\nabla \xi)$  and going through the same calculations, the formula (5.4) follows, this time for the Christoffel symbols  $\tilde{\Gamma}_{ij}^k$  of  $\tilde{G}$ , i.e.:

$$\tilde{\Gamma}_{ij}^k = (\Gamma_{\alpha\beta}^\rho \circ \xi) \partial_i \xi^\alpha \partial_j \xi^\beta (\partial_\rho \zeta^k \circ \xi) + (\partial_\alpha \zeta^k \circ \xi) \partial_{ij} \xi^\alpha.$$

Hence, we see that  $\tilde{\Gamma}_{ij}^k = \bar{\Gamma}_{ij}^k$  in  $\Omega$  and again, in view of Lemma 4.5, it follows that  $\tilde{G} = \bar{G} = (\nabla \xi)^T G (\nabla \xi)$  in  $\Omega$ , provided that we have this identity at a given point  $x_0$ , as required in (4.7).

**Example 5.4.** Assume that  $G$  is constant. Then the equations in (4.6c) become:

$$(5.6) \quad \partial_j w_i^s = w_m^s \tilde{\Gamma}_{ij}^m,$$

whereas the Thomas condition ([21], see also next section) for the above system of differential equations is:

$$w_m^s \partial_k \tilde{\Gamma}_{ij}^m + w_p^s \tilde{\Gamma}_{km}^p \tilde{\Gamma}_{ij}^m = w_m^s \partial_j \tilde{\Gamma}_{ik}^m + w_p^s \tilde{\Gamma}_{jm}^p \tilde{\Gamma}_{ik}^m \quad \forall i, j, s, k : 1 \dots n.$$

Equivalently, the following should hold:

$$w_m^s \left( \partial_k \tilde{\Gamma}_{ij}^m - \partial_j \tilde{\Gamma}_{ik}^m + \tilde{\Gamma}_{kp}^m \tilde{\Gamma}_{ij}^p - \tilde{\Gamma}_{jp}^m \tilde{\Gamma}_{ik}^p \right) = 0 \quad \forall i, j, s, k : 1 \dots n,$$

at all points  $x$  in a neighborhood of  $x_0$ , and for all  $w$  in a neighborhood of a given invertible  $w_0 \in \mathbb{R}^{n \times n}$  which satisfies  $\tilde{G}(x_0) = w_0^T G w_0$ . Since the expression in parentheses above equals  $\tilde{R}_{ijk}^m(x)$ , it follows that the Thomas condition for (5.6) is precisely the vanishing of the whole Riemann curvature tensor  $\tilde{R}_{\dots}$  of the metric  $\tilde{G}$ , or equivalently that  $\tilde{G}$  be immersible in  $\mathbb{R}^n$ .

On the other hand, letting  $A = \sqrt{G}$  denote the unique positive definite symmetric square root of the matrix  $G$ , we see that the problem (1.3) becomes:

$$\tilde{G} = (\nabla \xi)^T G \nabla \xi = (A \nabla \xi)^T (A \nabla \xi) = (\nabla \xi_1)^T \nabla \xi_1,$$

with  $\xi_1(x) = A \xi(x)$ . Since  $A$  is invertible, we easily deduce that (1.3) has a solution iff  $\tilde{G}$  is immersible in  $\mathbb{R}^n$ . Consequently, in this example the Thomas condition for (4.6c) is necessary and sufficient for solvability of (1.3).

**Example 5.5.** Assume that  $G(x) = \mu(x) \text{Id}_3$ . Then the problem (1.3) becomes:

$$(5.7) \quad (\nabla \xi)^T \nabla \xi = \frac{1}{\mu} \tilde{G} = e^{2f} \tilde{G}, \quad \text{with } f = -\frac{1}{2} \log \mu.$$

Solution to (5.7) exists if and only if the Ricci curvature of the metric  $e^{2f} \tilde{G}$ , which is conformally equivalent to the metric  $\tilde{G}$ , is equal to 0 in  $\Omega$ . More precisely, denoting:  $\nabla_{\tilde{G}}^2 f =$



$[\partial_{ij}f - \tilde{\Gamma}_{ij}^k \partial_k f]_{i,j:1..3} \in \mathbb{R}^{3 \times 3}$ ,  $\Delta_{\tilde{G}}f = \tilde{G}^{jk} \partial_{jk}f - \tilde{G}^{jk} \tilde{\Gamma}_{jk}^l \partial_l f \in \mathbb{R}$ , and  $\|\nabla f\|_{\tilde{G}}^2 = \tilde{G}^{jk} \partial_j f \partial_k f \in \mathbb{R}$ , the condition reads:

$$(5.8) \quad 0 = \text{Ric}(e^{2f}\tilde{G}) = \text{Ric}(\tilde{G}) - \left(\nabla_{\tilde{G}}^2 f - \nabla f \otimes \nabla f\right) - \left(\Delta_{\tilde{G}}f + \|\nabla f\|_{\tilde{G}}^2\right)\tilde{G}.$$

When also  $\tilde{G} = \lambda(x)\text{Id}_3$ , then (5.8) after setting  $h = \frac{1}{2} \log \frac{\lambda}{\mu}$ , reduces to:

$$(5.9) \quad 0 = -(\nabla^2 h - \nabla h \otimes \nabla h) - (\Delta h + \|\nabla h\|^2)\text{Id}_3.$$

An immediate calculation shows that the only solutions of (5.9) are:

$$h(x) = -2\log|x - a| + c \quad \text{or} \quad h = c,$$

with arbitrary constants  $c \in \mathbb{R}$  and  $a \in \mathbb{R}^n$ . Indeed, in this case the solutions to (5.7) are conformal, and so by Liouville's theorem they are given by the Möbius transformations in  $\mathbb{R}^3$ , as compositions of rotations, dilations, inversions and translations of the form:

$$\xi(x) = b + \alpha \frac{R(x - a)}{|x - a|^\beta}, \quad R \in SO(3), \quad a, b \in \mathbb{R}^3, \quad \alpha \in \mathbb{R}, \quad \beta \in \{0, 2\}.$$

**Example 5.6.** In dimension  $n = 2$ , existence of a solution to (5.7) is equivalent to the vanishing of the Gauss curvature:

$$(5.10) \quad 0 = \kappa(e^{2f}\tilde{G}) = e^{-2f}(\kappa(\tilde{G}) - \Delta_{\tilde{G}}f),$$

which further becomes:  $\kappa(\tilde{G}) = \Delta_{\tilde{G}}f$ . When also  $\tilde{G} = \lambda(x)\text{Id}_3$ , this reduces to:  $\Delta h = 0$ , which is also equivalent to the following compatibility of the Gauss curvatures of  $G$  and  $\tilde{G}$ :  $\frac{\kappa(\lambda\text{Id}_2)}{\kappa(\mu\text{Id}_2)} = \frac{\mu}{\lambda}$ .

## 6. SOME FURTHER REMARKS ON SYSTEMS OF TOTAL DIFFERENTIAL EQUATIONS

Systems of total differential equations have been extensively studied in the literature [21, 14, 6, 10, 22]. They are over-determined systems in the unknown  $w : \Omega \rightarrow \mathbb{R}^N$ , of the form:

$$(6.1) \quad \partial_i w(x) = f_i(x, w(x)), \quad \forall i = 1 \dots n \quad \forall x \in \Omega,$$

where all the partial derivatives of  $w$  are given by functions  $f_i : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ . Note that if the latter are assumed of sufficient regularity, the uniqueness of solutions to (6.1) with a given initial data  $w(x_0) = w_0$  is immediate. For existence, observe that any solution must satisfy the compatibility conditions:

$$\partial_{ij}w = \partial_{ji}w.$$

This leads, under sufficient regularity assumptions, to the necessary condition for existence of  $w$ :

$$(6.2) \quad \left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} + \left(\frac{\partial f_i}{\partial w}\right)f_j - \left(\frac{\partial f_j}{\partial w}\right)f_i\right)(x, w(x)) = 0, \quad \forall i, j : 1 \dots n \quad \forall x \in \Omega.$$

The advantage of the system (6.2) is that it does not involve any partial derivatives of the a-priori unknown solution  $w$ , and hence, if certain conditions are satisfied, it can be used to obtain the candidates for  $w$  by solving for  $w(x)$  at each  $x \in \Omega$ .

Naturally, the more solutions (6.2) has, the more there is a chance to find a solution to (6.1). A plausible strategy, possibly adaptable as practical numerical schemes, is to find all the candidates  $w$  from (6.2) and check whether they satisfy (6.1). This insight, combined with the observation

about the over-determination of the original system, implies its rigidity, and leads to non-existence of solutions in generic situations. On the other hand, the ideal situation is to have:

$$(6.3) \quad \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} + \left(\frac{\partial f_i}{\partial w}\right)f_j - \left(\frac{\partial f_j}{\partial w}\right)f_i \equiv 0, \quad \forall i, j : 1 \dots n.$$

satisfied for all  $x \in \Omega$  and all  $w \in \mathbb{R}^N$ . All functions  $w : \Omega \rightarrow \mathbb{R}^N$  obtained this way are solutions to (6.2) and, as shown in [21], this leads to existence of a family of local solutions to (6.1), parametrized by the initial values at a given point  $x_0 \in \Omega$ . If (6.3) is not satisfied, one comes short of having an ample set of solutions  $w$  and might be content for other intermediate scenarios, where solutions to (6.1) may still exist but with less liberty in choosing initial values.

In section 4 we showed that (1.3) is equivalent to a system of total differential equations. Note that even though our problem shares some familiar features with the isometric immersion problem, it is of a fundamentally different nature in as much as we cannot establish the equivalence of (6.1) and (6.3). Not being able to close the system (6.2) as in the isometric immersion case, in order to find necessary and sufficient conditions, we need to study the above mentioned possible intermediate scenarios when the Thomas condition (6.3) is not satisfied. We will carry out this plan in this sections 7 and 8. Below, we begin by a simple example of (6.1), whose conditions (6.3) are far from optimal.

**Example 6.1.** For  $n = 2, N = 1$ , consider the system:

$$(6.4) \quad \nabla w = w^2 \vec{a} + w \vec{b} + \vec{c} \quad \text{in } \Omega \subset \mathbb{R}^2,$$

where  $\vec{a}, \vec{b} : \Omega \rightarrow \mathbb{R}^2$  are given smooth vector fields and  $\vec{c} \in \mathbb{R}^2$ . In order to find the Thomas condition (6.3) for existence of a solution  $w : \Omega \rightarrow \mathbb{R}$ , we take:

$$0 = \text{curl}(w^2 \vec{a} + w \vec{b} + \vec{c}) = w^2 \text{curl } \vec{a} + 2w \langle \nabla^\perp w, \vec{a} \rangle + w \text{curl } \vec{b} + \langle \nabla^\perp w, \vec{b} \rangle.$$

Substituting (6.4), we obtain the counterpart of (6.2) in the present case:

$$(6.5) \quad (\text{curl } \vec{a} + \langle \vec{a}, \vec{b}^\perp \rangle) w^2 + (\text{curl } \vec{b}) w = 0.$$

The satisfaction of the above in the  $(x, w)$  space is precisely the condition (6.3):

$$(6.6) \quad \text{curl } \vec{a} + \langle \vec{a}, \vec{b}^\perp \rangle \equiv 0 \quad \text{and} \quad \text{curl } \vec{b} \equiv 0 \quad \text{in } \Omega.$$

**Lemma 6.2.** *Condition (6.6) is not necessary for the existence of solutions for (6.4). Also, existence of a solution to (6.5) does not guarantee the existence of a solution to (6.4).*

*Proof. 1.* Let  $w : \Omega \rightarrow \mathbb{R}$  be any positive smooth function. Let  $\vec{c} = 0$  and let  $\vec{b} : \Omega \rightarrow \mathbb{R}^2$  be a vector field which is not curl free. Define:

$$\vec{a} = \frac{1}{w^2} (\nabla w - w \vec{b}).$$

Then (6.4) is satisfied but not (6.6).

**2.** Let  $\vec{a}, \vec{b} : \Omega \rightarrow \mathbb{R}^2$  be two smooth vector fields such that the first condition in (6.6) does not hold at any point  $x \in \Omega$ . For example, one may take:

$$\vec{a}(x) = x \quad \text{and} \quad \vec{b}(x) = x^\perp,$$

in any domain  $\Omega$  which avoids 0 in its closure. Then (6.5) becomes:

$$|x|^2 w^2(x) + 2w(x) = 0,$$

and it has only two smooth solutions:  $w \equiv 0$  and  $w(x) = -2/|x|^2$ . On the other hand:

$$\nabla\left(-\frac{2}{|x|^2}\right) = \frac{4}{|x|^4}x \neq -\left(\frac{2}{|x|^2}\right)^2x - \frac{2}{|x|^2}x^\perp + \vec{c},$$

so none of these functions is a solution to (6.4) when  $\vec{c} \neq 0$ . ■

Further, observe that augmenting (6.4) to a system of (decoupled) total differential equations:

$$(6.7) \quad \nabla w_I = w_I^2 \vec{a}_I + w_I \vec{b}_I + \vec{c}_I, \quad I = 1 \dots N,$$

gives the condition:

$$(6.8) \quad (\operatorname{curl} \vec{a}_I + \langle \vec{a}_I, \vec{b}_I^\perp \rangle) w_I^2 + (\operatorname{curl} \vec{b}_I) w_I = 0 \quad \forall I = 1 \dots N,$$

resulting in:

$$(6.9) \quad \operatorname{curl} \vec{a}_I + \langle a_I, \vec{b}_I^\perp \rangle \equiv 0 \quad \text{and} \quad \operatorname{curl} \vec{b}_I \equiv 0, \quad \text{in } \Omega.$$

As in Lemma 6.2, we can set up the data such that, there exists a solution  $w_1(x)$  to the first equation in the system (6.8) and that all the equations in (6.9) for  $I = 2 \dots N$  are satisfied in such a manner that (6.7) still has no solutions.

## 7. THE THOMAS CONDITION IN THE 2-DIMENSIONAL CASE OF (1.3)

In this section, we assume that  $n = 2$  and follow the approach of [7]. Recalling (2.3) as the equivalent form of (1.3), we note that it has a solution (on a simply connected  $\Omega$ ) if and only if:

$$(7.1) \quad \operatorname{curl}(G^{-1/2}R\tilde{G}^{1/2}) \equiv 0 \quad \text{in } \Omega,$$

for some rotation valued field  $R : \Omega \rightarrow SO(2)$ . Denote  $V = G^{1/2}$ ,  $\tilde{V} = \tilde{G}^{1/2}$  and represent  $R$  by a function  $\theta : \Omega \rightarrow \mathbb{R}$ , so that  $R(x) = R(\theta(x))$  is the rotation of angle  $\theta$ . Note that:

$$\partial_j R(\theta) = (\partial_j \theta) R W, \quad \text{where} \quad W = R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since  $W$  is a rotation, it commutes with all  $R \in SO(2)$ . Also:  $W^T = W^{-1} = -W$ , and:

$$(7.2) \quad \forall F \in GL(2) \quad W F W = -\operatorname{cof} F = -(\det F) F^{-1}.$$

We finally need to recall the conformal–anticonformal decomposition of  $2 \times 2$  matrices. Let  $\mathbb{R}_c^{2 \times 2}$  and  $\mathbb{R}_a^{2 \times 2}$  denote, respectively, the spaces of conformal and anticonformal matrices:

$$\mathbb{R}_c^{2 \times 2} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix}; a, b \in \mathbb{R} \right\}, \quad \mathbb{R}_a^{2 \times 2} = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix}; a, b \in \mathbb{R} \right\}.$$

It is easy to see that  $\mathbb{R}^{2 \times 2} = \mathbb{R}_c^{2 \times 2} \oplus \mathbb{R}_a^{2 \times 2}$  because both spaces have dimension 2 and they are mutually orthogonal:  $A : B = 0$  for all  $A \in \mathbb{R}_c^{2 \times 2}$  and  $B \in \mathbb{R}_a^{2 \times 2}$ .

For  $F = [F_{ij}]_{i,j:1,2} \in \mathbb{R}^{2 \times 2}$ , its projections  $F^c$  on  $\mathbb{R}_c^{2 \times 2}$ , and  $F^a$  on  $\mathbb{R}_a^{2 \times 2}$  are:

$$F^c = \frac{1}{2} \begin{bmatrix} F_{11} + F_{22} & F_{12} - F_{21} \\ F_{21} - F_{12} & F_{11} + F_{22} \end{bmatrix}, \quad F^a = \frac{1}{2} \begin{bmatrix} F_{11} - F_{22} & F_{12} + F_{21} \\ F_{12} + F_{21} & F_{22} - F_{11} \end{bmatrix}.$$

We easily obtain that:

$$(7.3) \quad F = F^c + F^a, \quad |F|^2 = |F^c|^2 + |F^a|^2 \quad \text{and} \quad \det F = 2(|F^c|^2 - |F^a|^2).$$

Also:

$$(7.4) \quad \forall R \in SO(2) \quad F^c R = R F^c \quad \text{and} \quad R^T F^a = F^a R.$$

**Lemma 7.1.** *The following system of total differential equations in  $\theta$  is equivalent with (1.3):*

$$(7.5) \quad \nabla\theta = \frac{1}{\det \tilde{V}} \tilde{V} \operatorname{curl} \tilde{V} + \frac{1}{\det \tilde{V}} \tilde{V} \left( A_1 \tilde{V} e_2 - A_2 \tilde{V} e_1 \right) + \frac{1}{\det \tilde{V}} \tilde{V} R(-2\theta) \left( B_1 \tilde{V} e_2 - B_2 \tilde{V} e_1 \right),$$

where we have denoted:

$$(7.6) \quad A_i = (V \partial_i V^{-1})^c, \quad B_i = (V \partial_i V^{-1})^a \quad \forall i = 1, 2.$$

*Proof.* We calculate the expression in (7.1):

$$\begin{aligned} \operatorname{curl}(V^{-1} R \tilde{V}) &= \partial_1(V^{-1} R \tilde{V} e_2) - \partial_2(V^{-1} R \tilde{V} e_1) \\ &= (\partial_1 V^{-1} R \tilde{V} + V^{-1} R \partial_1 \tilde{V}) e_2 - (\partial_2 V^{-1} R \tilde{V} + V^{-1} R \partial_2 \tilde{V}) e_1 \\ &\quad + (\partial_1 \theta) V^{-1} R W \tilde{V} e_2 - (\partial_2 \theta) V^{-1} R W \tilde{V} e_1 \end{aligned}$$

The two last terms above equal, in view of (7.2):

$$(\partial_1 \theta) V^{-1} R W \tilde{V} e_1 + (\partial_2 \theta) V^{-1} R W \tilde{V} e_2 = (V^{-1} R W \tilde{V} W)(\nabla \theta)^T = (\det \tilde{V})(V^{-1} R \tilde{V}^{-1})(\nabla \theta)^T.$$

Therefore, by (7.1) and (7.4):

$$\begin{aligned} \nabla \theta &= -\frac{1}{\det \tilde{V}} (\tilde{V} R^T \tilde{V}) \left( -(\partial_1 V^{-1} R \tilde{V} + V^{-1} R \partial_1 \tilde{V}) e_2 + (\partial_2 V^{-1} R \tilde{V} + V^{-1} R \partial_2 \tilde{V}) e_1 \right) \\ &= \frac{1}{\det \tilde{V}} \left( \tilde{V} R^T V \partial_1 V^{-1} R \tilde{V} e_2 - \tilde{V}^{-1} R^T V \partial_2 V^{-1} R \tilde{V} e_1 + \tilde{V}^{-1} \partial_1 \tilde{V} e_2 - \tilde{V}^{-1} \partial_2 \tilde{V} e_1 \right). \end{aligned}$$

We simplify the equation for  $\theta$  as follows:

$$\begin{aligned} \nabla \theta &= \frac{1}{\det \tilde{V}} \left( \tilde{V} A_1 \tilde{V} e_2 - \tilde{V} A_2 \tilde{V} e_1 + \tilde{V} R(-2\theta) B_1 \tilde{V} e_2 - \tilde{V} R(-2\theta) B_2 \tilde{V} e_1 \right. \\ &\quad \left. + \tilde{V} \partial_1 \tilde{V} e_2 - \tilde{V} \partial_2 \tilde{V} e_1 \right), \end{aligned}$$

which clearly implies (7.5). ■

If the Thomas condition for (7.5) is satisfied, there exists a unique solution to (7.5) for all initial values  $\theta(x_0) = \theta_0$ . The original problem (1.3) has then a unique solution for all initial values of the form  $w(x_0) = \nabla \xi(x_0) = G^{-1/2} R_0 \tilde{G}^{1/2}(x_0)$  with  $R_0 \in SO(2)$ .

**Example 7.2.** Assume that  $G = \operatorname{Id}_2$ . Then  $A_i = B_i = 0$ , so that (7.5) becomes:

$$(7.7) \quad \nabla \theta = h_{\tilde{V}} \quad \text{where} \quad h_{\tilde{V}} = \frac{1}{\det \tilde{V}} \tilde{V} \operatorname{curl} \tilde{V}.$$

The condition for solvability of (7.7) is:  $\operatorname{curl} h_{\tilde{V}} \equiv 0$ , which is expected from Example 5.4. Indeed, a direct calculation shows that (see also [8, Remark 2, page 113]) this condition is equivalent to the vanishing of the Gaussian curvature of  $\tilde{G}$ :

$$(7.8) \quad \kappa(\tilde{G}) = -\frac{1}{\det \tilde{V}} \operatorname{curl} h_{\tilde{V}}.$$

**Example 7.3.** As in Examples 5.5 and 5.6, assume that  $G = \mu(x) \operatorname{Id}_2 = e^{-2f} \operatorname{Id}_2$ , with  $f$  given in (5.7). Then:

$$(7.9) \quad V \partial_i V^{-1} = -\left( \frac{1}{2\mu} \partial_i \mu \right) \operatorname{Id} = (\partial_i f) \operatorname{Id}_2.$$

This implies that  $A_i = (\partial_i f) \text{Id}_2$  and  $B_i = 0$ . Consequently, (7.5) becomes:

$$\nabla\theta = \frac{1}{\det \tilde{V}} \tilde{V} \text{curl} \tilde{V} + \frac{1}{\det \tilde{V}} \tilde{V}^2 ((\partial_1 f)e_2 - (\partial_2 f)e_1) = h_{\tilde{V}} + \frac{1}{\sqrt{\det \tilde{G}}} \tilde{G} \nabla^\perp f.$$

But by (7.2) it follows that:

$$\begin{aligned} \frac{1}{\sqrt{\det \tilde{G}}} \tilde{G} \nabla^\perp f &= \frac{1}{\sqrt{\det \tilde{G}}} \tilde{G} W \nabla f = \frac{1}{\sqrt{\det \tilde{G}}} W^T W \tilde{G} W \nabla f = \sqrt{\det \tilde{G}} W \tilde{G}^{-1} \nabla f \\ &= (\sqrt{\det \tilde{G}} \tilde{G}^{-1} \nabla f)^\perp, \end{aligned}$$

and hence:

$$\text{curl} \left( \frac{1}{\sqrt{\det \tilde{G}}} \tilde{G} \nabla^\perp f \right) = \text{div} (\sqrt{\det \tilde{G}} \tilde{G}^{-1} \nabla f) = \sqrt{\det \tilde{G}} \Delta_{\tilde{G}} f.$$

Thus, the Thomas condition to (1.3) is here:

$$\text{curl} h_{\tilde{V}} + \sqrt{\det \tilde{G}} \Delta_{\tilde{G}} f = 0,$$

and we see that, in view of (7.8), it coincides with the equivalent condition (5.10) for existence of solutions to (1.3).

**Lemma 7.4.** *Without loss of generality and through a change of variable, we can assume that:*

$$\tilde{G} = \lambda(x) \text{Id}_2 = e^{2g} \text{Id}_2 \quad \text{with} \quad g = \frac{1}{2} \log \lambda.$$

Then, for an arbitrary metric  $G$ , condition (7.5) becomes:

$$(7.10) \quad \nabla\theta = \mathbf{m} + R(-2\theta)\mathbf{n},$$

where  $\mathbf{m} = \nabla^\perp g + (A_1 e_2 - A_2 e_1)$  and  $\mathbf{n} = B_1 e_2 - B_2 e_1$ . The Thomas condition of (7.10) reads:

$$(7.11) \quad \begin{cases} \text{curl} \mathbf{m} - 2|\mathbf{n}|^2 = 0, \\ \text{div} \mathbf{n} - 2\langle \mathbf{n}^\perp, \mathbf{m} \rangle = 0, \\ \text{curl} \mathbf{n} - 2\langle \mathbf{n}, \mathbf{m} \rangle = 0. \end{cases}$$

*Proof.* As in (7.9), we observe that  $\tilde{V} \text{curl} \tilde{V} = e^{2g} \nabla^\perp g$ , and hence:

$$\nabla\theta = \nabla^\perp g + (A_1 e_2 - A_2 e_1) + R(-2\theta)(B_1 e_2 - B_2 e_1).$$

The above equation has a similar structure to (24) in [7], even though the two original problems are different. In the present case, both  $G$  and  $\tilde{G}$  are involved in defining  $\mathbf{m}$ , while in [7] the vector fields  $\mathbf{m}, \mathbf{n}$  depend on the matrix field  $G$  in the Left Cauchy-Green equation:  $(\nabla\eta)(\nabla\eta)^T = G$ .

In order to derive the Thomas condition for (7.10), note that:

$$\begin{aligned} \text{curl} \nabla\theta &= \text{curl} \mathbf{m} + \partial_1 \langle R(-2\theta)\mathbf{n}, e_2 \rangle - \partial_2 \langle R(-2\theta)\mathbf{n}, e_1 \rangle \\ &= \text{curl} \mathbf{m} + \langle R(-2\theta)\partial_1 \mathbf{n}, e_2 \rangle - \langle R(-2\theta)\partial_2 \mathbf{n}, e_1 \rangle - 2\partial_1 \theta \langle R(-2\theta)W\mathbf{n}, e_2 \rangle + 2\partial_2 \theta \langle R(-2\theta)W\mathbf{n}, e_1 \rangle. \end{aligned}$$

Substituting  $\partial_i \theta$  from (7.10) we arrive at:

$$\begin{aligned}
0 &= \operatorname{curl} \mathbf{m} + \langle R(-2\theta) \partial_1 \mathbf{n}, e_2 \rangle - \langle R(-2\theta) \partial_2 \mathbf{n}, e_1 \rangle \\
&\quad - 2 \langle \mathbf{m} + R(-2\theta) \mathbf{n}, e_1 \rangle \langle R(-2\theta) W \mathbf{n}, W e_1 \rangle - 2 \langle \mathbf{m} + R(-2\theta) \mathbf{n}, e_2 \rangle \langle R(-2\theta) W \mathbf{n}, W e_2 \rangle \\
&= \operatorname{curl} \mathbf{m} + \langle R(-2\theta) \partial_1 \mathbf{n}, e_2 \rangle - \langle R(-2\theta) \partial_2 \mathbf{n}, e_1 \rangle - 2 \langle \mathbf{m} + R(-2\theta) \mathbf{n}, R(-2\theta) \mathbf{n} \rangle \\
&= \operatorname{curl} \mathbf{m} + \langle \partial_1 \mathbf{n}, R(2\theta) W e_1 \rangle + \langle \partial_2 \mathbf{n}, R(2\theta) W e_2 \rangle - 2 |\mathbf{n}|^2 - 2 \langle \mathbf{m}, R(-2\theta) \mathbf{n} \rangle \\
&= \operatorname{curl} \mathbf{m} - 2 |\mathbf{n}|^2 + \langle \nabla \mathbf{n} : R(2\theta) W \rangle - 2 \langle \mathbf{m}, R(-2\theta) \mathbf{n} \rangle \\
&= \operatorname{curl} \mathbf{m} - 2 |\mathbf{n}|^2 + \langle \nabla \mathbf{n} - 2 W \mathbf{n} \otimes \mathbf{m} : R(2\theta) W \rangle.
\end{aligned}$$

Finally, writing  $R(2\theta)W = R(2\theta + \pi/2) = -\sin(2\theta)\operatorname{Id}_2 + \cos(2\theta)W$ , we obtain:

$$0 = \operatorname{curl} \mathbf{m} - 2 |\mathbf{n}|^2 - \sin(2\theta)(\operatorname{div} \mathbf{n} - 2 \langle \mathbf{n}^\perp, \mathbf{m} \rangle) + \cos(2\theta)(\operatorname{curl} \mathbf{n} - 2 \langle \mathbf{n}, \mathbf{m} \rangle),$$

which should be satisfied for all  $x \in \Omega$  and all  $\theta \in [0, 2\pi)$ , implying hence (7.10).  $\blacksquare$

**Example 7.5.** In the setting and using notation of Example 5.6, we see that:  $\mathbf{m} = \nabla^\perp g + \nabla^\perp f = \nabla^\perp(f + g) = \nabla^\perp h$  where  $h = \frac{1}{2} \log \frac{\lambda}{\mu}$ , and  $\mathbf{n} = 0$ . The Thomas condition for (1.3) here is hence:  $\operatorname{curl} \mathbf{m} = 0$ , that is equivalent to:

$$\Delta h = \Delta \log\left(\frac{\lambda}{\mu}\right) = 0,$$

which is further exactly equivalent to existence of solutions in Example 5.6.

**Example 7.6.** We will now provide an example, where the Thomas condition (7.11) is not satisfied, but a solution to (1.3) exists. We start with requesting that  $\tilde{G} = \operatorname{Id}_2$ , which yields:  $\mathbf{m} = A_1 e_2 - A_2 e_1$  and  $\mathbf{n} = B_1 e_2 - B_2 e_1$ . Consider the general form of the diagonal metric  $G$ :

$$G(x) = \begin{bmatrix} e^{2a(x)} & 0 \\ 0 & e^{2b(x)} \end{bmatrix},$$

for some smooth functions  $a, b : \bar{\Omega} \rightarrow \mathbb{R}$ . Then:  $V \partial_i V^{-1} = -\operatorname{diag}\{\partial_i a, \partial_i b\}$ , and hence:

$$A_i = -\frac{1}{2} \partial_i(a + b) \operatorname{Id}_2, \quad B_i = \frac{1}{2} \operatorname{diag}\{\partial_i(b - a), \partial_i(a - b)\},$$

which leads to:

$$\mathbf{m} = -\frac{1}{2} \nabla^\perp(a + b) \quad \text{and} \quad \mathbf{n} = \frac{1}{2} (\partial_2(a - b), \partial_1(a - b)).$$

Therefore, the Thomas condition (7.11) becomes:

$$\begin{cases} \Delta(a + b) + |\nabla(a - b)|^2 = 0, \\ \partial_{12}(a - b) + \frac{1}{2} \partial_1(a - b) \partial_2(a + b) + \frac{1}{2} \partial_2(a - b) \partial_1(a + b) = 0, \\ \partial_{11}(a - b) - \partial_{22}(a - b) + \partial_1(a - b) \partial_1(a + b) - \partial_2(a - b) \partial_2(a + b) = 0. \end{cases}$$

It can be simplified to the following form, symmetric in  $a$  and  $b$ :

$$(7.12) \quad \begin{cases} \Delta(a + b) + |\nabla(a - b)|^2 = 0, \\ 2\partial_{12}a + \partial_1 a \partial_2 a = 2\partial_{12}b + \partial_1 b \partial_2 b, \\ \partial_{11}a - \partial_{22}a + (\partial_1 a)^2 - (\partial_2 a)^2 = \partial_{11}b - \partial_{22}b + (\partial_1 b)^2 - (\partial_2 b)^2. \end{cases}$$

We now specify to the claimed example. Take  $a = -\log x_1$  and  $b = -\log x_2$ , so that:

$$G(x_1, x_2) = \begin{bmatrix} x_1^{-2} & 0 \\ 0 & x_2^{-2} \end{bmatrix},$$

defined on a domain  $\Omega \subset \mathbb{R}^2$  in the positive quadrant, whose closure avoids 0. We easily check that the first condition in (7.12) does not hold, since:

$$\Delta \log(x_1 x_2) + |\nabla \log(\frac{x_1}{x_2})|^2 = -\partial_{11} \log x_1 + |\partial_1 \log x_1|^2 - \partial_{22} \log x_2 + |\partial_2 \log x_2|^2 = 2(\frac{1}{x_1^2} + \frac{1}{x_2^2}).$$

On the other hand,  $\xi_0(x_1, x_2) = \frac{1}{2}(x_1^2, x_2^2)$  solves (1.3), because:

$$(\nabla \xi_0)^T G \nabla \xi_0 = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \begin{bmatrix} x_1^{-2} & 0 \\ 0 & x_2^{-2} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} = \text{Id}_2 = \tilde{G}.$$

In fact, by reversing the calculations in the proof of Lemma 7.4, it can be checked that  $\xi_0$  and  $-\xi_0$  are the only two solutions to the problem (1.3).

## 8. A SUFFICIENT CONDITION FOR THE SOLVABILITY OF THE PROBLEM (4.6c) (4.7)

In this section we go back to the general setting of  $n \geq 2$  dimensional problem (1.3) and its equivalent formulation (4.6c). Let  $w_0 \in \mathbb{R}^{n \times n}$  be an invertible matrix, such that:

$$\tilde{G}(x_0) = w_0^T G(x_0) w_0.$$

For every  $s, i, j : 1 \dots n$  let  $f_j^{s,i} : \mathcal{O} \rightarrow \mathbb{R}$  be the smooth functions of  $(x, w)$ , defined in a small neighborhood  $\mathcal{O}$  of  $(x_0, w_0)$ , and coinciding with the right hand side of (4.6c):

$$f_j^{s,i}(x, w) = w_m^s \tilde{\Gamma}_{ij}^m - \frac{1}{2} G^{sm} \left( w_i^p \partial_j G_{mp} + w_j^p \partial_i G_{mp} - w_j^p w_i^q \partial_t G_{pq} (w^{-1})_m^t \right),$$

where  $w_s^m$  denotes the element in the  $s$ -th row and  $m$ -th column of the matrix  $w$ .

Differentiate  $f_j^{s,i}$  formally in  $\partial/\partial x_k$ , treating  $w$  as a function of  $x$ , and substitute each partial derivative  $\partial_k w_q^p$  by the expression in  $f_k^{p,q}$ . We call this new function  $\tilde{F}_{j,k}^{s,i}$  and note that for  $w$  satisfying (4.6c) one has:  $\partial_k \partial_j w_i^s(x) = \tilde{F}_{j,k}^{s,i}(x, w(x))$  for every  $x \in \Omega$ .

Now, Thomas' sufficient condition [21] for the local solvability of the total differential equation (4.6c) with the initial condition (4.7) where  $w(x_0) = w_0$ , is that:

$$(8.1) \quad \forall s, i, k, j : 1 \dots n \quad F_{j,k}^{s,i} = \tilde{F}_{j,k}^{s,i} - \tilde{F}_{k,j}^{s,i} \equiv 0 \quad \text{in } \mathcal{O}.$$

Following [1], we now derive another sufficient local solvability condition. For convenience of the reader, we present the whole argument as in section 4 [1].

For every  $\alpha_1 = 1 \dots n$ , let  $F_{j,k,\alpha_1}^{s,i} = 0$  be the equation obtained after formally differentiating  $F_{j,k}^{s,i} = 0$  in  $\partial/\partial x_{\alpha_1}$  and replacing each partial derivative  $\partial_{\alpha_1} w_l^r$  by  $f_{\alpha_1}^{r,l}$  as before. Inductively, for every  $m \geq 0$  and all  $\alpha_1, \alpha_2 \dots \alpha_m : 1 \dots n$ , we define the equations  $F_{j,k,\alpha_1 \dots \alpha_m}^{s,i} = 0$ .

**Theorem 8.1.** *Fix  $N \geq 0$ . Assume that there exists a set  $S \subset \{1 \dots n\}^2$  of cardinality  $1 \leq \#S = M \leq n^2$ , and there exist  $M$  equations  $\bar{F}^1 = 0, \bar{F}^2 = 0, \dots, \bar{F}^M = 0$ , among the equations:*

$$(8.2) \quad \{F_{j,k}^{s,i} = 0\}_{s,i,j,k} \cup \{F_{j,k,\alpha_1}^{s,i} = 0\}_{s,i,j,k,\alpha_1} \cup \dots \cup \{F_{j,k,\alpha_1 \dots \alpha_N}^{s,i} = 0\}_{s,i,j,k,\alpha_1 \dots \alpha_N},$$

such that:

$$(8.3) \quad \det \left[ \frac{\partial \bar{F}^r}{\partial w_q^p}(x_0, w_0) \right]_{\substack{r:1\dots M \\ (p,q) \in S}} \neq 0.$$

From now on, we will denote by  $w_q^p$  the coefficients in the given matrix  $w$  with  $(p, q) \in S$  and by  $\bar{w}_t^l$  the coefficients with  $(l, t) \notin S$ . By the implicit function theorem, (8.3) guarantees existence of smooth functions  $v_q^p : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$  for every  $(p, q) \in S$ , where  $\{v_q^p(x, \{\bar{w}_t^l\}_{(l,t) \notin S})\}_{(p,q) \in S} \in \mathbb{R}^M$  is defined for  $x$  in a small neighborhood  $\mathcal{U}$  of  $x_0$  and for  $\{\bar{v}_t^l\}$  in a small neighborhood  $\mathcal{V}$  of  $\{(\bar{w}_0)_t^l\}_{(l,t) \notin S}$ , satisfying:

$$(8.4) \quad \begin{aligned} \forall (p, q) \in S \quad v_q^p(x_0, \{(\bar{w}_0)_t^l\}) &= (\bar{w}_0)_q^p, \\ \forall r : 1 \dots M \quad \forall x \in \mathcal{U} \quad \forall \{\bar{v}_t^l\} \in \mathcal{V} \quad \bar{F}^r \left( x, \{\bar{v}_t^l\}_{(l,t) \notin S}, \{v_q^p(x, \{\bar{v}_t^l\}_{(l,t) \notin S})\}_{(p,q) \in S} \right) &= 0. \end{aligned}$$

Assume further that:

$$(8.5) \quad \begin{aligned} \forall m : 0 \dots N+1 \quad \forall i, s, j, \alpha_1 \dots \alpha_m \quad \forall x \in \mathcal{U} \quad \forall \{\bar{v}_t^l\} \in \mathcal{V} \\ F_{j,k,\alpha_1 \dots \alpha_m}^{s,i} \left( x, \{\bar{v}_t^l\}, \{v_q^p(x, \{\bar{v}_t^l\})\} \right) = 0. \end{aligned}$$

Then the problem (4.6c) (4.7) has a solution, defined in some small neighborhood  $\mathcal{U}$  of  $x_0$ , such that  $w(x_0) = w_0$ .

*Proof. 1.* Below, we drop the Einstein summation convention and use the  $\sum$  sign instead, for more clarity. By  $\mathcal{U}$  and  $\mathcal{V}$  we always denote appropriately small neighborhoods of  $x_0 \in \mathbb{R}^n$  and  $\{(\bar{w}_0)_t^l\}_{(l,t) \notin S} \in \mathbb{R}^{n^2-M}$ , respectively, although the sets  $\mathcal{U}$  and  $\mathcal{V}$  may vary from step to step.

We will seek for a solution in the form  $\left( \{\bar{w}_t^l(x)\}, \{w_q^p(x) = v_q^p(x, \{\bar{w}_t^l(x)\})\} \right)$ , where the functions  $v_q^p$  are defined in the statement of the theorem. Note first that by (8.5), for every  $\alpha : 1 \dots n$ , every  $m : 0 \dots N$ , and  $s, i, j, k, \alpha_1 \dots \alpha_m$ , after denoting:  $F = F_{j,k,\alpha_1 \dots \alpha_m}^{s,i}$  we have:

$$(8.6) \quad \frac{\partial F}{\partial x_\alpha}(x, v) + \sum_{(l,t) \notin S} \frac{\partial F}{\partial \bar{w}_t^l}(x, v) f_\alpha^{l,t}(x, v) + \sum_{(p,q) \in S} \frac{\partial F}{\partial w_q^p}(x, v) f_\alpha^{p,q}(x, v) = 0,$$

for all  $x \in \mathcal{U}$  and all  $\{\bar{v}_t^l\}_{(l,t) \notin S} \in \mathcal{V}$ , where above  $v = \left\{ \{\bar{v}_t^l\}_{(l,t) \notin S}, \{v_q^p\}_{(p,q) \in S} \right\}$  and  $v_q^p(x) = v_q^p(x, \{\bar{v}_t^l(x)\})$  for all  $(p, q) \in S$ .

**2.** Fix now any point  $x \in \mathcal{U}$  and let  $[0, \eta] \ni \tau \mapsto g(\tau) = \{g^\alpha(\tau)\}_{\alpha=1 \dots n} \in \mathcal{U}$  be a smooth path with constant speed, such that:  $g(0) = x$ . Consider the solution  $[0, \eta] \ni \tau \mapsto h(\tau) = \{h_t^l(\tau)\}_{(l,t) \notin S} \in \mathcal{V}$  of the following initial value problem:

$$(8.7) \quad \begin{cases} \frac{d}{d\tau} h_t^l(\tau) = \sum_{\alpha=1 \dots n} f_\alpha^{l,t} \left( g(\tau), h(\tau), \{v_q^p(g(\tau), h(\tau))\} \right) \frac{dg^\alpha}{d\tau}(\tau) & \forall \tau \in [0, 1] \quad \forall (l, t) \notin S \\ h_t^l(0) = \bar{v}_t^l. \end{cases}$$

Applying (8.5), we easily see that for  $F$  denoting, as in step 1, any function in the set of equation (8.2), there holds:

$$F \left( g(\tau), h(\tau), \{v_q^p(g(\tau), h(\tau))\} \right) = 0 \quad \forall \tau \in [0, \eta].$$



Differentiating in  $\tau$  yields:

$$\begin{aligned} \sum_{\alpha=1\dots n} \frac{\partial F}{\partial x_\alpha}(g, h, \{v_q^p(g, h)\}) \frac{dg^\alpha}{d\tau} + \sum_{(l,t) \notin S} \frac{\partial F}{\partial \bar{w}_t^l}(g, h, \{v_q^p(g, h)\}) \frac{dh_t^l}{d\tau} \\ + \sum_{(p,q) \in S} \frac{\partial F}{\partial w_q^p}(g, h, \{v_q^p(g, h)\}) \left( \frac{\partial v_q^p}{\partial x_\alpha}(g, h) \frac{dg^\alpha}{d\tau} + \frac{\partial v_q^p}{\partial \bar{v}_t^l}(g, h) \frac{dh_t^l}{d\tau} \right) = 0. \end{aligned}$$

We now use (8.6) to equate the first term in the expression above. In view of (8.7), the second term cancels out and we obtain, for the particular choice of  $F = \bar{F}^r$ ,  $r : 1 \dots M$ :

$$\begin{aligned} \forall \tau \in [0, \eta] \quad \forall r : 1 \dots M \\ \sum_{\alpha=1\dots n} \frac{dg^\alpha}{d\tau}(\tau) \sum_{(p,q) \in S} \frac{\partial \bar{F}^r}{\partial w_q^p}(g, h, \{v_q^p(g, h)\}) \left( \sum_{(l,t) \notin S} \frac{\partial v_q^p}{\partial \bar{v}_t^l}(g, h) f_\alpha^{l,t}(g, h, \{v_q^p(g, h)\}) \right. \\ \left. + \frac{\partial v_q^p}{\partial x_\alpha}(g, h) - f_\alpha^{p,q}(g, h, \{v_q^p(g, h)\}) \right) = 0. \end{aligned}$$

We proceed by evaluating the obtained formula at  $\tau = 0$ . Since  $dg/d\tau \neq 0$  is an arbitrary vector in  $\mathbb{R}^n$ , and since the matrix  $\left[ \frac{\partial \bar{F}^r}{\partial w_q^p}(x, \{\bar{v}_t^l\}) \right]$  is invertible by (8.3), it follows that:

$$(8.8) \quad \forall \alpha : 1 \dots n \quad \forall (p,q) \in S \quad \left( \sum_{(l,t) \notin S} \frac{\partial v_q^p}{\partial \bar{v}_t^l}(x, \{\bar{v}_t^l\}) f_\alpha^{l,t}(x, \{\bar{v}_t^l\}, \{v_q^p(x, \{\bar{v}_t^l\})\}) \right) \\ + \frac{\partial v_q^p}{\partial x_\alpha}(x, \{\bar{v}_t^l\}) - f_\alpha^{p,q}(x, \{\bar{v}_t^l\}, \{v_q^p(x, \{\bar{v}_t^l\})\}) = 0,$$

for all  $x \in \mathcal{U}$  and  $\{\bar{v}_t^l\} \in \mathcal{V}$ .

**3.** Consider the following system of total differential equations:

$$(8.9) \quad \forall \alpha : 1 \dots n \quad \forall (l,t) \notin S \quad \frac{\partial \bar{w}_t^l}{\partial x_\alpha} = f_\alpha^{l,t}(x, \{\bar{w}_t^l\}_{(l,t) \notin S}, \{v_q^p(x, \{\bar{w}_t^l\})\}_{(p,q) \in S}).$$

We now verify the Thomas condition for the system (8.9). It requires [21] the following expressions to be symmetric in  $j, k : 1 \dots n$ , for all  $(i, s) \notin S$  on  $\mathcal{U} \times \mathcal{V}$ :

$$\begin{aligned} \frac{\partial f_k^{i,s}}{\partial x_j}(x, \{\bar{w}_t^l\}, \{v_q^p\}) + \sum_{(l,t) \notin S} \frac{\partial f_k^{i,s}}{\partial \bar{w}_t^l}(x, \{\bar{w}_t^l\}, \{v_q^p\}) f_j^{l,t}(x, \{\bar{w}_t^l\}, \{v_q^p\}) \\ + \sum_{(p,q) \in S} \frac{\partial f_k^{i,s}}{\partial w_q^p}(x, \{\bar{w}_t^l\}, \{v_q^p\}) \left( \frac{\partial v_q^p}{\partial x_j}(x, \{\bar{w}_t^l\}) + \sum_{(l,t) \notin S} \frac{\partial v_q^p}{\partial \bar{v}_t^l}(x, \{\bar{w}_t^l\}) f_k^{l,t}(x, \{\bar{w}_t^l\}, \{v_q^p\}) \right). \end{aligned}$$

By (8.8), the above expression equals:

$$\begin{aligned} \frac{\partial f_k^{i,s}}{\partial x_j}(x, \{\bar{w}_t^l\}, \{v_q^p\}) + \sum_{(l,t) \notin S} \frac{\partial f_k^{i,s}}{\partial \bar{w}_t^l}(x, \{\bar{w}_t^l\}, \{v_q^p\}) f_j^{l,t}(x, \{\bar{w}_t^l\}, \{v_q^p\}) \\ + \sum_{(p,q) \in S} \frac{\partial f_k^{i,s}}{\partial w_q^p}(x, \{\bar{w}_t^l\}, \{v_q^p\}) f_j^{p,q}(x, \{\bar{w}_t^l\}, \{v_q^p\}), \end{aligned}$$

where as usual  $v_q^p = v_q^p(x, \bar{w}_t^l)$ . We see that the required symmetry follows exactly by (8.6).

Consequently, the problem (8.9) with the initial condition  $\bar{w}_t^l(x_0) = (w_0)_t^l$  has a unique solution  $\{\bar{w}_t^l\}_{(l,t) \notin S}$  on  $\mathcal{U}$ .

4. Let now  $\{w_q^p\}_{(p,q) \in S}$  be defined on  $\mathcal{U}$  through the formula:  $w_q^p(x) = v_q^p(x, \{\bar{w}_t^l(x)\}_{(l,t) \notin S})$ . We will prove that:

$$(8.10) \quad \forall \alpha : 1 \dots n \quad \forall (p, q) \in S \quad \frac{\partial w_q^p}{\partial x_\alpha} = f_\alpha^{p,q} \left( x, \{\bar{w}_t^l\}_{(l,t) \notin S}, \{w_q^p\}_{(p,q) \in S} \right).$$

Together with (8.9), this will establish the result claimed in the theorem.

Differentiating (8.4) where we set  $v = w$ , and applying (8.9) yields:

$$\forall r : 1 \dots M \quad \forall \alpha : 1 \dots n \quad \forall x \in \mathcal{U} \\ \frac{\partial \bar{F}^r}{\partial x_\alpha}(x, w(x)) + \sum_{(l,t) \notin S} \frac{\partial \bar{F}^r}{\partial \bar{w}_t^l}(x, w(x)) f_\alpha^{l,t}(x, w(x)) + \sum_{(p,q) \in S} \frac{\partial \bar{F}^r}{\partial w_q^p}(x, w(x)) \frac{\partial w_q^p}{\partial x_\alpha}(x) = 0.$$

In view of (8.6), applied to  $F = \bar{F}^r$ ,  $r = 1 \dots M$  and  $v = w$ , the desired equality in (8.10) follows directly by the invertibility of the matrix  $\left[ \frac{\partial \bar{F}^r}{\partial w_q^p}(x, w) \right]_{\substack{r: 1 \dots M \\ (p,q) \in S}}$ , which by (8.3) is valid in a sufficiently small neighborhood  $\mathcal{O}$  of  $(x_0, w_0)$ . ■

**Remark 8.2.** Another similar approach [1], would be as follows. In case the Thomas condition (8.1) is not satisfied, one could relax the initial condition (4.7) and instead add the set of equations (4.6a) to those in (8.1), before proceeding to derive the collections of equations in (8.2) by successive differentiation and substitution, as described above. Assume then that (8.3) holds in a large domain rather than a given point  $(x_0, w_0)$ . The advantage of this method is that one does not have to be concerned about satisfying the initial condition potentially limiting the choices of the functions  $\bar{F}^r$  in (8.3), and hence one has a larger set of equations among (8.2) to choose from. Through this method, existence of an  $n^2 - M$  parameter family of solutions to (4.6c) follows, as one has the freedom to set up the initial data in step 3. The disadvantage is that a larger set of equations in (8.5) must be satisfied for the sufficient condition to hold true.

## 9. REMARKS ON A RELAXED REFORMULATION OF ALGEBRAIC CONDITIONS FOR INTEGRABILITY OF (1.3)

While Theorem 8.1 does provide exact algebraic conditions of integrability for (1.3), these conditions are by no means easy to verify for specific problems. With a view towards a more practical, if only approximate, solution procedure, we consider the following alternative. In the spirit of section 2, that sets up an infinite-dimensional optimization problem whose zero-cost solutions correspond to exact solutions of (1.3) (and whose non-zero cost solutions may be considered as approximate solutions of (1.3)), we now construct a family of finite-dimensional optimization problems parametrized by  $x \in \Omega$  (uncoupled from one  $x$  to another), whose pointwise zero-cost solutions (for each  $x$ ) form the ingredients of an exact solution of (1.3), as outlined in section 6. As before, we use the Einstein summation convention for all lowercase Latin indices from 1 to  $n$ .

Let  $w \in \mathbb{R}^{n \times n}$ ,  $x \in \Omega \subset \mathbb{R}^n$  and define:

$$\begin{aligned} A_{ij}(w, x) &= \tilde{G}_{ij}(x) - w_i^t G_{st}(x) w_j^s \\ \tilde{B}_{ij}^s(w, x) &= w_m^s \tilde{\Gamma}_{ij}^m(x) - \frac{1}{2} G^{sm}(x) \left( w_i^p \partial_j G_{mp}(x) + w_j^p \partial_i G_{mp}(x) - w_j^p w_i^q \partial_t G_{pq}(x) W_m^t(w) \right) \\ C_{ijk}^s(w, x) &= \frac{\partial \tilde{B}_{ij}^s}{\partial x^k}(w, x) - \frac{\partial \tilde{B}_{ik}^s}{\partial x^j}(w, x) + \frac{\partial \tilde{B}_{ij}^s}{\partial w_p^r}(w, x) \tilde{B}_{pk}^r(w, x) - \frac{\partial \tilde{B}_{ik}^s}{\partial w_p^r}(w, x) \tilde{B}_{pj}^r(w, x). \end{aligned}$$

The symmetry of  $\tilde{B}_{ij}^s$  in the lower indices and  $C = 0$  represent the integrability conditions of (4.6b) and (4.6c), while  $A = 0$  is equivalent to (4.6a). Define:

$$(9.2) \quad \mathcal{E}(w; x) = K_1 \sum_{i,j=1}^n |A_{ij}(w, x)|^2 + K_2 \sum_{i,j,k,s=1}^n |C_{ijk}^s(w, x)|^2,$$

with  $K_i > 0, i = 1, 2$  as nondimensional constants, where we assume that  $A, C$  in (9.2) have been appropriately nondimensionalized.

We now seek minimizers  $w(x)$  of  $\mathcal{E}(\cdot; x)$  for each  $x \in \Omega$ . Each set  $\{w(x); x \in \Omega\}$  such that  $\mathcal{E}(w(x); x) = 0$  on  $\Omega$ , defines a function  $x \mapsto w(x)$ ,  $x \in \Omega$ . Each such function can be checked to see if it satisfies (4.6c), and any that passes this test allows for the construction of a solution to (4.6b) (because of the symmetry in  $i, j$ ) that in turn is a solution of (1.3) in view of  $A = 0$ .

If the system (4.6b) and (4.6c) were completely integrable (i.e. satisfying the Thomas condition) then there would be a  $(n^2 + n)$ -parameter family of solutions to (1.3). While the situation here is likely to be more constrained (with non-existence expected in the generic case), it is not possible to rule out, *a-priori*, the existence of solution families with fewer parameters. Thus, it is natural to expect nonuniqueness in looking for minimizers of  $\mathcal{E}(\cdot; x)$ . An algorithm for generating approximate solutions may be as follows:

- (1) Define  $S_x := \{w(x)\}$  as the set of minimizers of  $\mathcal{E}(\cdot; x)$  for each  $x$ . Pick one element, say  $\mathcal{W}(x)$  from  $S_x$  for each  $x \in \Omega$  that generates a function  $\mathcal{W}$  on  $\Omega$ .
- (2) For given  $\mathcal{W}$ , solve for a function  $\xi^h : \Omega \rightarrow \mathbb{R}^n$  as the minimizer of the quadratic functional

$$\frac{1}{2} \int_{\Omega} |\nabla \xi^h - \mathcal{W}|^2 dx.$$

- (3) Calculate the error  $\sqrt{\int_{\Omega} |A(\nabla \xi^h, x)|^2 dx}$ .

- (4) Repeat for candidate  $\mathcal{W}$ s and accept approximate solution as one with minimum error.

In a finite element setting, a tessellation of  $\Omega$  would be invoked, and  $\xi^h$  would be represented by a piecewise  $\mathcal{C}^1$  and globally  $\mathcal{C}^0$  finite-dimensional approximation, resulting in a  $W^{1,2}(\Omega)$  function. Step 1 above would be executed at each integration point of the finite element mesh, utilizing any of a variety of available techniques for algebraic function minimization - at each integration point this would be a problem in  $n^2$  variables. Step 2 would then result in a linear system of algebraic equations, to be solved uniquely for  $\nabla \xi^h$  by specifying the value of  $\xi^h$  at an arbitrarily chosen node of the finite element mesh. Note that the least squares minimization is ideally suited for the problem at hand as it reflects a projection of  $\mathcal{W}$  on the space of gradients, thus not requiring it to be an exact gradient which would be difficult to guarantee in Step 1.

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