Applications of Algebraic Topology in Elasticity*

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Abstract In this chapter we discuss some applications of algebraic topology in elasticity. This includes the necessary and sufficient compatibility equations of nonlinear elasticity for non-simply-connected bodies when the ambient space is Euclidean. Algebraic topology is the natural tool to understand the topological obstructions to compatibility for both the deformation gradient $\mathbf{F}$ and the right Cauchy-Green strain $\mathbf{C}$. We will investigate the relevance of homology, cohomology, and homotopy groups in elasticity. We will also use the relative homology groups in order to derive the compatibility equations in the presence of boundary conditions. The differential complex of nonlinear elasticity written in terms of the deformation gradient and the first Piola-Kirchhoff stress is also discussed.

1 Introduction

Compatibility equations of elasticity are more than 150 years old and according to Love [34] were first studied by Saint Venant in 1864. In nonlinear elasticity a given distribution of strain on a body $\mathcal{B}$ may not correspond to a deformation mapping. Similarly, in linear elasticity a given distribution of linearized strains may not correspond to a well-defined displacement field. Strain has to satisfy a set of integrability equations in order to correspond to some deformation field. These integrability equations are called compatibility equations in continuum mechanics. We provided a detailed history of the

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compatibility equations in nonlinear and linear elasticity in [62] and will not repeat it here. Compatibility equations for simply-connected bodies are well understood and are a set of PDEs that depend on the measure of strain. For non-simply connected bodies these "bulk" compatibility equations are only necessary. In other words, when the bulk compatibility equations are satisfied in a non-simply connected body the strain field may still be incompatible; there may be topological obstructions to compatibility. A classical example of incompatible strain fields that satisfy the bulk compatibility equations are Volterra's "distortions" (dislocations and disclinations) [55]. For a strain field on a non-simply connected body to be compatible, in addition to the bulk compatibility equations, some extra compatibility equations that explicitly depend on the topology of the body are needed [39, 14, 55, 30, 49] . We call these extra compatibility equations the complementary compatibility equations [52] or the auxiliary compatibility equations.

The natural mathematical tool for understanding the topological obstruction to compatibility is algebraic topology. Topological methods, and particularly algebraic topology have been used in fluid mechanics [7], and electromagnetism [27] for quite sometime. In the case of electromagnetism this goes back to the work of Maxwell [38] before the formal developments of algebraic topology that started in the work of Poincaré [43]. Algebraic topology has not been used systematically in solid mechanics until recently [62]. To motivate the present study consider the following problem. Having a solid sphere (a ball) with the different types of holes shown in Fig. 1, what are the compatibility equations for $\mathbf{F}$ and $\mathbf{C}$? The necessary compatibility equations ("bulk" compatibility equations) are well understood and our focus will be on the sufficient conditions. We will see that in case (a) of a spherical hole no extra compatibility equations are needed. For (b), (c), and (d) one needs to impose some extra constraints on the (red) loops (generators of the first homology group) to ensure compatibility.

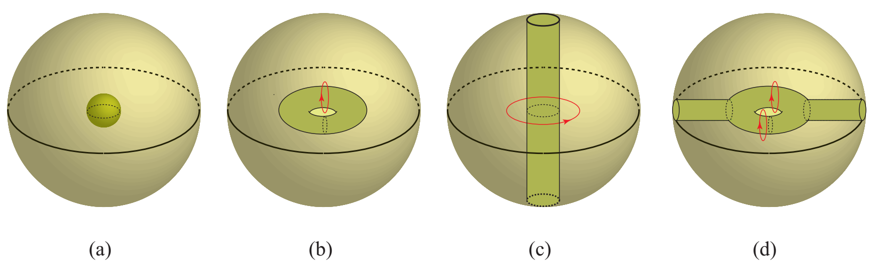

Fig. 1 Balls with (a) spherical, (b) toroidal, and (c) cylindrical holes. (d) A ball with a hole consistening of a solid torus attached to two solid cylinders. Betti numbers of these sets are zero, one, one, and two, respectively.

This chapter is structured as follows. In §2 we tersely review differential geometry. This follows by short discussions of presentation of groups, homology and cohomology groups, relative homology groups, the idea of homotopy and the fundamental group, classification of 2-manifolds with boundary, knot theory and the fundamental group of their complements in $\mathbb{R}^3$, and the topology of 3-manifolds in §3. In §4 we discuss the kinematics of nonlinear elasticity. In §5, **F**-compatibility equations for non-simply connected bodies are discussed. **F**-compatibility equations in the presence of essential (Dirichlet) boundary conditions are also derived. **C**-compatibility equations for non-simply connected bodies are derived. Several examples are presented. Finally, the necessary and sufficient compatibility equations of linearized elasticity are derived. In §6, the differential complex of nonlinear elasticity written in terms of the deformation gradient and the first Piola-Kirchhoff stress is discussed. Some applications are also briefly mentioned.

2 Differential Geometry

In this section, we briefly review the differential geometry background needed in the kinematic description of nonlinear elasticity.

Consider a map $\pi:\mathcal{E}\to\mathcal{B}$, where $\mathcal{E}$ and $\mathcal{B}$ are sets. The fiber over $X\in\mathcal{B}$ is defined to be the set $\mathcal{E}_X := \pi^{-1}(X)\subset\mathcal{E}$. If the map π is onto, fibers are non-empty and $\mathcal{E}=\sqcup_{X\in\mathcal{B}}\mathcal{E}_X$, where $\sqcup$ denotes disjoint union of sets. Now assume that $\mathcal{E}$ and $\mathcal{B}$ are manifolds and for any $X\in\mathcal{B}$, there exists a neighborhood $\mathcal{U}\subset\mathcal{B}$ of X, a manifold $\mathcal{F}$, and a diffeomorphism $\psi:\pi^{-1}(\mathcal{U})\to\mathcal{U}\times\mathcal{F}$ such that $\pi=\mathrm{pr}_1\circ\psi$, where $\mathrm{pr}_1:\mathcal{U}\times\mathcal{F}\to\mathcal{U}$ is projection onto the first factor. The triplet $(\mathcal{E},\pi,\mathcal{B})$ is called a fiber bundle and $\mathcal{E}$, π, and $\mathcal{B}$ are called the total space, the projection, and the base space, respectively. If $\pi^{-1}(X)$ is a vector space, for any $X\in\mathcal{B}$, then $(\mathcal{E},\pi,\mathcal{B})$ is called a vector bundle. The set of all smooth maps $\sigma:\mathcal{B}\to\mathcal{E}$ such that $\sigma(X)\in\mathcal{E}_X,\ \forall\ X\in\mathcal{B}$, is called the set of sections of this bundle, and is denoted by $\Gamma(\mathcal{E})$. The tangent bundle of a manifold is an example of a vector bundle for which $\mathcal{E}=T\mathcal{B}$.

A vector field on a manifold $\mathcal{B}$ is a section of the tangent bundle $T\mathcal{B}$ of $\mathcal{B}$. The set of all C^r vector fields on $\mathcal{B}$ is denoted by $\mathfrak{X}^r(\mathcal{B})$ and the set of all C^∞ vector fields by $\mathfrak{X}(\mathcal{B})$. A vector field on $\mathcal{B}$ is an assignment, to each $X\in\mathcal{B}$, of a tangent vector $\mathbf{W}_X\in T_X\mathcal{B}$. Note that for an N-dimensional manifold $\mathcal{B}$, $T_X\mathcal{B}$ is an N-dimensional vector space with a local basis $\left\{\frac{\partial}{\partial X^1},\dots,\frac{\partial}{\partial X^N}\right\}$ induced from a local chart $\{X^A\}$. Given a vector field $\mathbf{W}$, for each point $X\in\mathcal{B}$, $\mathbf{W}$ is locally described as

$$\mathbf{W}(X)=\sum_{A=1}^{N}W^A(X)\frac{\partial}{\partial X^A}, \tag{1}$$

where W^A are C^∞ maps. One important role of tangent vectors is the directional differentiation of functions. In other words, a vector field acts on functions by taking their directional derivative, i.e.,

$$\mathbf{W}[f] := \sum_{A=1}^{N} W^A(X) \frac{\partial f(X)}{\partial X^A}. \tag{2}$$

This is the directional or Lie derivative of f along $\mathbf{W}$ and is denoted by $\mathfrak{L}_{\mathbf{W}} f$. Thus, $\mathfrak{L}_{\mathbf{W}} f(X) := \mathbf{W}[f](X) = df(X) \cdot \mathbf{W}(X)$. This is the reason $\mathfrak{L} f = df$ belongs to the cotangent space of $\mathcal{B}$, where the cotangent space $T^*\mathcal{B}$ is defined as $T^*\mathcal{B} := \{\varphi : T\mathcal{B} \to \mathbb{R},\ \varphi \text{ is linear and bounded}\}$.

A linear (affine) connection on a manifold $\mathcal{B}$ is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall\ \mathbf{X}, \mathbf{Y}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{X}(\mathcal{B}), \forall\ f, f_1, f_2 \in C^\infty(\mathcal{B}), \forall\ a_1, a_2 \in \mathbb{R}$:

i) $\nabla_{f_1 \mathbf{X}_1 + f_2 \mathbf{X}_2} \mathbf{Y} = f_1 \nabla_{\mathbf{X}_1} \mathbf{Y} + f_2 \nabla_{\mathbf{X}_2} \mathbf{Y}$,
ii) $\nabla_{\mathbf{X}}(a_1 \mathbf{Y}_1 + a_2 \mathbf{Y}_2) = a_1 \nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2 \nabla_{\mathbf{X}}(\mathbf{Y}_2)$,
iii) $\nabla_{\mathbf{X}}(f\mathbf{Y}) = f \nabla_{\mathbf{X}} \mathbf{Y} + (\mathbf{X} f)\mathbf{Y}$.

$\nabla_{\mathbf{X}} \mathbf{Y}$ is called the covariant derivative of $\mathbf{Y}$ along $\mathbf{X}$. In a local chart $\{X^A\}$, $\nabla_{\partial_A} \partial_B = \Gamma^C{}_{AB} \partial_C$, where $\Gamma^C{}_{AB}$ are the Christoffel symbols of the connection, and $\partial_A = \frac{\partial}{\partial x^A}$ are the natural bases for the tangent space corresponding to a coordinate chart $\{x^A\}$. A linear connection is said to be compatible with a metric $\mathbf{G}$ of the manifold if

$$\nabla_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \nabla_{\mathbf{X}} \mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{Y}, \nabla_{\mathbf{X}} \mathbf{Z} \rangle\!\rangle_{\mathbf{G}}, \tag{3}$$

where $\langle\!\langle ., . \rangle\!\rangle_{\mathbf{G}}$ is the inner product induced by the metric $\mathbf{G}$. A connection ∇ is $\mathbf{G}$-compatible if and only if $\nabla \mathbf{G} = \mathbf{0}$, or in components, $G_{AB|C} = G_{AB,C} - \Gamma^D{}_{CA} G_{DB} - \Gamma^D{}_{CB} G_{AD} = 0$. We consider an N-dimensional manifold $\mathcal{B}$ with the metric $\mathbf{G}$ and a $\mathbf{G}$-compatible connection ∇. The torsion of a connection is a map $T : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by

$$T(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}], \tag{4}$$

where $[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$ is the commutator of the vector fields $\mathbf{X}$ and $\mathbf{Y}$. For an arbitrary scalar field f, $[\mathbf{X}, \mathbf{Y}][f] = \mathbf{X}[f]\mathbf{Y} - \mathbf{Y}[f]\mathbf{X}$. In components in a local chart $\{X^A\}$, $T^A{}_{BC} = \Gamma^A{}_{BC} - \Gamma^A{}_{CB}$. The connection ∇ is symmetric if it is torsion-free, i.e., $\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. It can be shown that on any Riemannian manifold $(\mathcal{B}, \mathbf{G})$ there is a unique linear connection (the Levi-Civita connection) ∇, which is compatible with $\mathbf{G}$ and is torsion-free with the Christoffel symbols $\Gamma^C{}_{AB} = \frac{1}{2} G^{CD}(G_{BD,A} + G_{AD,B} - G_{AB,D})$. In a manifold with a connection, the curvature is a map $\boldsymbol{\mathcal{R}} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by

$$\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} \mathbf{Z} - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} \mathbf{Z} - \nabla_{[\mathbf{X}, \mathbf{Y}]} \mathbf{Z}, \tag{5}$$

or in components, $\mathcal{R}^A{}_{BCD} = \Gamma^A{}_{CD,B} - \Gamma^A{}_{BD,C} + \Gamma^A{}_{BM} \Gamma^M{}_{CD} - \Gamma^A{}_{CM} \Gamma^M{}_{BD}$.

An N-dimensional Riemannian manifold is locally flat if it is isometric to Euclidean space. This is equivalent to vanishing of the curvature tensor

[31, 9]. Ricci curvature is defined as $R_{AB} = \mathcal{R}^C{}_{ACB}$. The trace of Ricci curvature is called scalar curvature: $\mathrm{R} = R_{AB}G^{AB}$. In dimensions two and three Ricci curvature algebraically determines the entire curvature tensor. In dimension three [28]:

$$\mathcal{R}_{ABCD} = G_{AC}R_{BD} - G_{AD}R_{BC} - G_{BC}R_{AD} + G_{BD}R_{AC} - \frac{1}{2}\mathrm{R}\left(G_{AC}G_{BD} - G_{AD}G_{BC}\right). \quad (6)$$

In dimension two $R_{AB} = \mathsf{R}g_{AB}$, and hence, scalar curvature completely characterizes the curvature tensor and is twice the Gauss curvature.[3]

2.1 Exterior calculus

We introduce differential forms on an arbitrary manifold $\mathcal{B}$ following [1]. The permutation group on N elements consists of all bijections $\sigma : \{1,...,N\} \to \{1,...,N\}$ and is denoted by S_N. For Banach spaces $\mathbb{E}$ and $\mathbb{F}$, a k-multilinear mapping $t \in L^k(\mathbb{E};\mathbb{F})$, i.e., $t : \mathbb{E}\times\mathbb{E}\times...\times\mathbb{E} \to \mathbb{F}$ is called skew symmetric if

$$t(e_1,...,e_k) = (\mathrm{sign}\ \sigma)t(e_{\sigma(1)},...,e_{\sigma(k)}), \qquad \forall e_1,...,e_k \in \mathbb{E},\ \sigma \in S_k, \quad (7)$$

where sign σ is $+1$ (-1) if σ is an even (odd) permutation. The subspace of skew-symmetric elements of $L^k(\mathbb{E};\mathbb{F})$ is denoted by $\Lambda^k(\mathbb{E},\mathbb{F})$. Elements of $\Lambda^k(\mathbb{E},\mathbb{F})$ are called exterior k-forms. Wedge product of two exterior forms $\alpha \in \Lambda^k(\mathbb{E},\mathbb{F})$ and $\beta \in \Lambda^l(\mathbb{E},\mathbb{F})$ is a $(k+l)$-form $\alpha\wedge\beta \in \Lambda^{k+l}(\mathbb{E},\mathbb{F})$, which is defined in components as

$$(\alpha\wedge\beta)_{i_1...i_{k+l}} = \sum_{(k,l)\in S_{K+l}} (\mathrm{sign}\ \sigma)\alpha_{\sigma(i_1)...\sigma(i_k)}\beta_{\sigma(i_{k+1})...\sigma(i_{k+l})}. \quad (8)$$

For a manifold $\mathcal{B}$, the vector bundle of exterior k-forms on $T\mathcal{B}$ is denoted by $\Lambda^k_{\mathcal{B}} : \Lambda^k(\mathcal{B}) \to \mathcal{B}$. In a local coordinate chart a differential k-form α has the following representation

$$\omega = \sum_{I_1<I_2<...<I_k} \omega_{I_1I_2...I_k} dX^{I_1}\wedge...\wedge dX^{I_k}\ , \qquad I_1,I_2...,I_k \in \{1,2,...,N\}, \quad (9)$$

where $\omega_{I_1I_2...I_k}$ are C^∞ maps. The space of k-forms on $\mathcal{B}$ is denoted $\Omega^k(\mathcal{B})$. Let

$$\Omega(\mathcal{B}) = \bigoplus_{k=0,1,...} \Omega^k(\mathcal{B}), \quad (10)$$

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[3] It is known that the necessary compatibility equations for the right Cauchy-Green strain $\mathbf{C}^\flat$ in 2D and 3D are written as $\mathbf{R}(\mathbf{C}^\flat) = \mathbf{0}$ and $R(\mathbf{C}^\flat) = \mathbf{0}$, respectively, i.e., in 2D there is only one compatibility equation while in 3D there are six. Note also that the Bianchi identities do not reduce the number of compatibility equations.

with its structure as a real vector space and multiplication $\wedge$. $\Omega(\mathcal{B})$ is called the algebra of exterior differential forms on $\mathcal{B}$.

Let U be an open subset of an N-manifold $\mathcal{B}$. Consider the unique family of mappings $d^k(U):\Omega^k(U)\to\Omega^{k+1}(U)$ $(k=0,1,\dots,N)$ merely denoted d with the following properties: (i) $d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta$, $\forall\alpha\in\Omega^k(U)$, $\beta\in\Omega^l(U)$, ii) If $f\in\Omega^0(U)$, df is the (usual) differential of f, iii) $d^2=d\circ d=0$ (i.e., $d^{k+1}(U)\circ d^k(U)=0$), iv) d is a local operator (natural with respect to restrictions), i.e., if $U\subset V\subset\mathcal{B}$ are open and $\alpha\in\Omega^k(V)$, then $d(\alpha|_U)=(d\alpha)|_U$. In component form, for the differential form in (9) one writes

$$d\omega=\frac{\partial\ \omega_{I_1I_2\dots I_k}}{\partial X^J}dX^J\wedge dX^{I_1}\wedge\dots\wedge dX^{I_k}, \tag{11}$$

where summation over repeated indices is implied.

For an N-manifold $\mathcal{B}$, $\dim[\Lambda^k(\mathcal{B})]=\binom{N}{k}=\binom{N}{N-k}=\dim[\Lambda^{N-k}(\mathcal{B})]$. This shows that $\Lambda^k(\mathcal{B})$ and $\Lambda^{N-k}(\mathcal{B})$ should be isomorphic to each other. The natural isomorphism is the Hodge star operator. Hodge star is the unique isomorphism $*:\Lambda^k(\mathcal{B})\to\Lambda^{N-k}(\mathcal{B})$ satisfying

$$\alpha\wedge*\beta=\langle\!\langle\alpha,\beta\rangle\!\rangle_{\mathbf{G}}\,\mu,\quad\forall\ \alpha,\beta\in\Lambda^k(\mathcal{B}), \tag{12}$$

where $\langle\!\langle,\rangle\!\rangle_{\mathbf{G}}$ and μ are the standard Riemannian inner product and the standard volume element on $\mathcal{B}$, respectively. As an example, $\Lambda^1(\mathbb{R}^3)$ and $\Lambda^2(\mathbb{R}^3)$ are both three dimensional and $*:\Lambda^1(\mathbb{R}^3)\to\Lambda^2(\mathbb{R}^3)$ is defined by

$$e^1\mapsto e^2\wedge e^3,\ \ e^2\mapsto e^3\wedge e^1,\ \text{and}\ \ e^3\mapsto e^1\wedge e^2. \tag{13}$$

The codifferential operator $\delta:\Omega^{k+1}(\mathcal{B})\to\Omega^k(\mathcal{B})$ is defined by

$$\begin{aligned}&\delta(\Omega^0(\mathcal{B}))=0,\\&\delta\alpha=(-1)^{Nk+1}*d*\alpha,\qquad\forall\ \alpha\in\Omega^{k+1}(\mathcal{B}),\ k=0,1,\dots,N-1.\end{aligned} \tag{14}$$

This is the adjoint of d with respect to $\langle\!\langle,\rangle\!\rangle_{\mathbf{G}}$. For an oriented smooth N-manifold $\mathcal{B}$ with boundary $\partial\mathcal{B}$ and $\alpha\in\Omega^{N-1}(\mathcal{B})$, Stokes' theorem tells us that

$$\int_{\partial\mathcal{B}}\alpha=\int_{\mathcal{B}}d\alpha, \tag{15}$$

assuming that both integrals exist.

3 Algebraic Topology

To make this chapter self-contained, we tersely review some notation and facts from algebraic topology and also refer the reader to the relevant literature for more details.

3.1 Homology and cohomology groups

An r-form ω is closed if $d\omega = 0$ and it is exact if there exists an $(r-1)$-form α such that $\omega = d\alpha$. An exact differential form is always closed, and from Poincaré's lemma a closed form is locally exact. However, globally a closed differential form may not be exact. Cohomology aims in finding the topological obstructions to exactness. This turns out to be directly related to the compatibility equations of elasticity. In the following we mainly follow [40, 20, 24, 27, 53].

3.1.1 Group theory

For two Abelian groups $(G_1,.)$ and $(G_2,.)$, a map $f : G_1 \to G_2$ is a homomorphism if

$$f(x.y) = f(x).f(y), \quad \forall x,y \in G_1. \tag{1}$$

Our notation is flexible here; we use $x.y$ and xy interchangeably. If in addition f is a bijection, it is an isomorphism, G_1 and G_2 are said to be isomorphic, and this is denoted by $G_1 \cong G_2$. Let $H \subset G$ be a subgroup. If $xy^{-1} \in H$, then $x,y \in G$ are called equivalent and we write $x \sim y$. The equivalence class of x is denoted by $[x]$. G/H is the quotient space — the set of equivalence classes — and $[x].[y] = [xy]$. If $ghg^{-1} \in H, \forall g \in G, h \in H$, H is called a normal subgroup. For a normal subgroup H, G/H is always a subgroup called the quotient group. For a homomorphism $f : G_1 \to G_2$, $\mathrm{Ker} f$ and $\mathrm{Im} f$ are subgroups of G_1 and G_2, respectively, where

$$\mathrm{Ker} f = \{x \in G_1 | f(x) = 1\}, \quad \mathrm{Im} f = \{x \in G_2 | x \in f(G_1) \subset G_2\}, \tag{2}$$

and 1 is the identity element of G_2. The isomorphism theorem of group theory tells us that $G_1/\mathrm{Ker} f \cong \mathrm{Im} f$.

Let $(G,.)$ be an Abelian group, i.e., $x.y = y.x, \ \forall \ x,y \in G$. If there exist $g_1,...,g_n \in G$ such that

$$g = g_1^{\lambda_1} ... g_n^{\lambda_n}, \quad \forall \ g \in G, \lambda_i \in \mathbb{Z}, \tag{3}$$

then G is called a finitely-generated Abelian group with generators $g_1,...,g_n$. If in addition

$$g = g_1^{\lambda_1} ... g_n^{\lambda_n} = 1 \quad \Rightarrow \quad \lambda_1 = ... = \lambda_n = 0, \tag{4}$$

G is called a free finitely-generated Abelian group, and $g_1,...,g_n$ are called free generators or a basis. It can be shown that $(G,.)$ is a free finitely-generated Abelian group if and only if every g has a unique representation with respect to the basis $\{g_1,...,g_n\}$.

Suppose $S = \{s_1,...,s_k\}$ is a set of distinct elements. Let $\tilde{S}$ be the set of expressions of the form $\tilde{s} = \prod_{i=1}^k s_i^{\lambda_i}$, where $\lambda_i \in \mathbb{Z}$. Then $\prod_{i=1}^k s_i^{\lambda_i} = \prod_{i=1}^k s_i^{\mu_i}$ if and only if $\lambda_i = \mu_i,\ i = 1,...,k$. Multiplication is defined as

$$\prod_i s_i^{\lambda_i} \prod_i s_i^{\mu_i} = \prod_i s_i^{\lambda_i+\mu_i}. \tag{5}$$

$\tilde{S}$ is a free finitely-generated Abelian group with basis $\{s_1^1 s_2^0 ... s_k^0, ..., s_1^0 ... s_{k-1}^0 s_k^1\}$. $\tilde{S}$ is called the free finitely-generated Abelian group on S. If G is an Abelian group, $g \in G$ has finite order if $g^n = 1$ for some $n \in \mathbb{N}$. The set of all elements of finite order in G is a subgroup called the torsion subgroup T of G. If T is trivial, i.e., $T = \{1\}$, G is called torsion-free. Any free Abelian group is torsion-free. For $x, y \in G$, and G a group, $[x,y] = xyx^{-1}y^{-1} \in G$ is called the commutator of x and y. $[G,G]$ is a normal subgroup of G generated by all commutators. Note that $G/[G,G]$ is an Abelian group.

The direct sum of two groups A and B is the set of pairs $(a,b), a \in A, b \in B$ and is denoted by $A \oplus B$. Group multiplication in $A \oplus B$ is defined as

$$(a_1,b_1).(a_2,b_2) = (a_1a_2, b_1b_2), \quad \forall a_1, a_2 \in A,\ \forall b_1, b_2 \in B. \tag{6}$$

Generalization of this to any finite number of groups is straightforward.

3.1.2 Combinatorial group theory

In combinatorial group theory one studies groups that are described by generators and some defining relations. Here we mainly follow [8] and [53]. If $X \subset G$, the smallest subgroup of G containing X is denoted by $\langle X \rangle$ and is characterized as

$$\langle X \rangle = \{g \in G |\ g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} ... x_k^{\varepsilon_k},\ x_i \in X, \varepsilon_i = \pm 1\}. \tag{7}$$

$x_1^{\varepsilon_1} x_2^{\varepsilon_2} ... x_k^{\varepsilon_k}$ is called an X-word or simply a word. A word is reduced if $x_i = x_{i+1}$ implies that $\varepsilon_i + \varepsilon_{i+1} \neq 0,\ i = 1,...,k-1$. For example, the word $x_1^{-1}x_1^{-1}x_2x_2^{-1}x_1x_1x_1x_2$ is not reduced while x_1x_2 is reduced. If $G = \langle X \rangle$ and every non-empty reduced X-word $w \neq_G 1$, X is called a free group. In this case, two reduced X-words have equal values in G if and only if they are identical. A group is finitely generated if it can be generated by a finite set. If G is a freely generated group by X, then for any group H and map $\psi : X \to H$, there is a unique homomorphism $\varphi : G \to H$ such that $\varphi|_X = \psi$. For a group G, and $X \subset G$, the normal closure of X in G (the smallest normal subgroup of G containing X) is defined as

$$\mathsf{gp}_G(X) = \left\langle \{g^{-1}xg |\ g \in G, x \in X\} \right\rangle. \tag{8}$$

If F is a free group on $X \subset G$ and $\psi : X \to G$, a map such that $G = \langle \psi(X) \rangle$, then the extension of this map $\varphi : F \to G$ has kernel $K = \mathsf{gp}_F(R)$, where $R \subset F$. Then one writes $G = \langle X; R \rangle$ and this is called a presentation for G, which comes with an implicit map $\psi : X \to G$, the presentation map. Elements of R are called defining relators. A group is finitely-presented if it has a finite presentation, i.e., if both X and R are finite.

Any normal subgroup of a group G consists of elements expressed by words of the following form

$$\prod_{i=1}^{n} g_i x_{j_i}^{\varepsilon_i} g_i^{-1}, \quad g_i, x_{j_i} \in G, \ \varepsilon_i = \pm 1. \tag{9}$$

This normal subgroup is said to be generated by $x_1, x_2, \ldots \in G$ and is denoted by $\mathsf{gp}_G(\{x_1, x_2, \ldots\})$ as in (8). Dyck's theorem says that the group $\langle X, R\rangle$ is the quotient of $F = \langle X\rangle$ by its normal subgroup $\mathsf{gp}_G(R)$.

3.1.3 Chain complexes and homology groups

Let $\{v_0, \ldots, v_k\}$ be a geometrically independent set in $\mathbb{R}^N$, i.e., $\{v_1 - v_0, \ldots, v_k - v_0\}$ is a set of linearly independent vectors in $\mathbb{R}^N$. A k-simplex σ^k is defined as

$$\sigma^k = \left\{ x \in \mathbb{R}^N \middle| x = \sum_{i=0}^{k} t_i v_i, \text{ where } 0 \le t_i \le 1, \sum_{i=0}^{k} t_i = 1 \right\}. \tag{10}$$

The numbers t_i are uniquely determined by x and are called barycentric coordinates of the point x of σ with respect to vertices $v_0, \ldots, v_k$. The number k is the dimension of σ^k. A simplical complex K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that (i) every face of a simplex of K is in K, and (ii) the intersection of any two simplices is either empty or a face of each of them. The largest dimension of the simplices of K is called the dimension of K. A subcomplex of K is a subcollection of K that contains all faces of its elements.

Suppose K is an oriented simlicial complex of dimension n. Let α_p be the number of p-simplices of K, $0 \le p \le n$. Let $\{\sigma_p^1, \ldots, \sigma_p^{\alpha_p}\}$ be the set of p-simplices of K. The pth chain group of K with integer coefficients is denoted by $C_p(K)$ and is a free Abelian group on the set $\{\sigma_p^1, \ldots, \sigma_p^{\alpha_p}\}$, i.e.,[4]

$$\sigma \in C_p(K), \quad \sigma = \sum_{i=1}^{\alpha_p} \lambda_i \sigma_p^i, \quad \lambda_i \in \mathbb{Z}. \tag{11}$$

For $p > n$ or $p < 0$, $C_p(K) = 0$. Let $\sigma = (v^0, \ldots, v^p)$ be an oriented p-simplex of K. Then, the boundary of σ is defined as

$$\partial\sigma = \partial_p \sigma = \sum_{i=0}^{p} (-1)^i (v^0, \ldots, \hat{v^i}, \ldots, v^p), \tag{12}$$

[4] Here, we find it more convenient to use an additive notation. Also, to be more specific we should denote the group by $C_p(K;\mathbb{Z})$ to emphasis that it has integer coefficients.

where hat over v^i indicates omission of v^i. The boundary homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ is defined as

$$\partial_p \left(\sum \lambda_i \sigma_p^i \right) = \sum_i \lambda_i \partial_p(\sigma_p^i). \tag{13}$$

Note that for any p, $\partial \circ \partial = \partial_{p-1} \circ \partial_p = 0$. Note also that $\operatorname{Im} \partial_{p+1} \subset \operatorname{Ker} \partial_p$. $Z_p = \operatorname{Ker} \partial_p$ is the set of p-cycles and $B_p = \operatorname{Im} \partial_{p+1}$ is the set of p-boundaries. $H_p(K) = Z_p(K)/B_p(K)$ is a finitely generated Abelian group and quantifies the non-bounding p-cycles of K. This is called the pth homology group of K (with integer coefficients). Note that $H_n(K) = Z_n(K)$ is free Abelian. Two p-cycles z and $z' \in Z_p(K)$ are homologous ($z \sim z'$) if $z - z' \in B_p(K)$. It is a fact that homology groups are topological invariants, i.e., two homeomeorphic topological spaces have isomorphic homology groups. For a simplicial complex, the set of simplices as subsets of $\mathbb{R}^m$ ($m \le n$) is called the polyhedron $|K|$ of K. For a topological space X, if there exists a simplicial complex K and a homeomorphism $f : |K| \to X$, X is said to be triangulable and (K, f) is called a triangulation of X. For a triangulable topological space X, given an arbitrary triangulation (K, f), $H_r(X) := H_r(K), r = 0, 1, \ldots$.[5]

Example 3.1 Circle S^1 is not the boundary of any 2-chain, and hence, $H_1(S^1)$ is generated by the circle itself (only one generator), i.e., $H_1(S^1) = \mathbb{Z}$. S^1 is connected, and hence, $H_0(S^1) = \mathbb{Z}$. A similar example is the punctured plane $\mathbb{R}^2 \backslash (0,0)$, which is connected and its first homology group is generated by any simple closed curve circling the origin once.

Example 3.2 Torus T^2 is not a boundary of any 3-chain. Thus, $H_2(T^2)$ is freely generated by one generator, the surface itself, i.e., $H_2(T^2) \cong \mathbb{Z}$. T^2 is connected, and hence, $H_0(T^2) \cong \mathbb{Z}$. $H_1(T^2)$ is freely generated by the loops γ_1 and γ_2 (see Fig. 2), and hence, $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. The group presentation can be written as $\pi_1(T^2) = \langle \gamma_1, \gamma_2 \rangle$. For a torus of genus g (the number of closed cuts that leave the torus path-connected)

$$H_1(\Sigma_g) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{2g}. \tag{14}$$

Example 3.3 Möbius band is constructed from a square by the identification shown in Fig. 3. z is a generator of the first homology group $H_1(\mathcal{M}, \mathbb{Z})$, i.e., $H_1(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$.

Remark 3.4 Note that $Z_r(K)$ and $B_r(K)$ are both free Abelian groups as they are both subgroups of a free Abelian group $C_r(K)$. However, this does not

[5] Note that the homology groups are independent of triangulations. Note also that not every space can be triangulated. For such spaces one can still define homology, e.g., singular and Čech homologies.

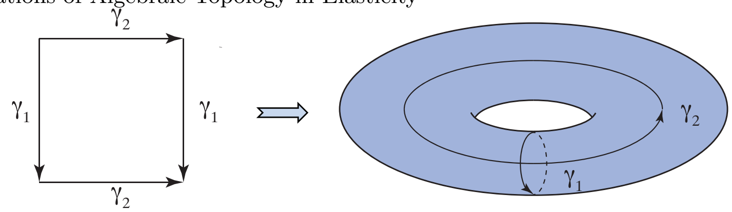

Fig. 2 A torus can be constructed from a square by the identifications shown above. γ_1 and γ_2 are generators of the first homology and first homotopy groups.

imply that $H_r(K)$ is also free Abelian. From the fundamental theorem of finitely-generated Abelian groups one has

$$H_1(K;\mathbb{Z}) \cong \underbrace{\mathbb{Z}\oplus\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}}_{f}\oplus \underbrace{\mathbb{Z}_{k_1}\oplus\ldots\oplus\mathbb{Z}_{k_p}}_{\text{torsion subgroup}}, \tag{15}$$

where $k_1,\ldots,k_p$ are integers, k_{i+1} divides k_i $(i=1,\ldots,p-1)$, and $\mathbb{Z}_{k_i}=\mathbb{Z}/k_i\mathbb{Z}$ is the set of integers modulo k_i. f is called the rank of $H_1(K;\mathbb{Z})$ or the first Betti number and p is called the torsion number. The torsion subgroup contains all the elements of the first homology group that have finite order.

Let M be an m-dimensional manifold and let σ_r be an r-simplex in $\mathbb{R}^m$, and $f:\sigma_r\to M$ a smooth map, not necessarily invertible. $s_r=f(\sigma_r)\subset M$ is called a singular r-simplex in M (these simplices do not provide a triangulation of M). Given the set of r-simplices $\{s_i^r\}$ in M, an r-chain in M is defined as

$$c=\sum_i a_i s_i^r,\quad a_i\in\mathbb{R}. \tag{16}$$

The r-chains in M form the chain group $C_r(M)$ with real coefficients. The boundary of a singular r-simplex s_r is defined as $\partial s_r := f(\partial\sigma_r)$. The boundary and cycle groups $B_r(M)$ and $Z_r(M)$ are defined similarly to those of simplicial complexes. The singular homology group is defined as $H_r(M):=Z_r(M)/B_r(M)$. The singular homology group is isomorphic to the corresponding simplicial homology group with $\mathbb{R}$-coefficients.

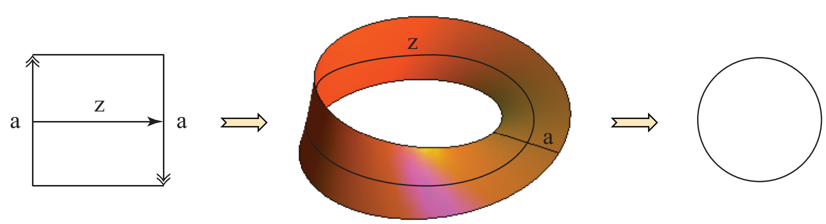

Fig. 3 Möbius band and its deformation retract to a circle.

3.1.4 Cohomology groups

Integration of an r-form ω over an r-chain in M is defined as

$$\int_{s_r} \omega = \int_{\sigma_r} f^* \omega, \tag{17}$$

where $f^*\omega$ is the pull-back of ω under f. For $c = \sum_i a_i s_i^r \in C_r(M)$:

$$\int_c \omega = \sum_i a_i \int_{s_i^r} \omega. \tag{18}$$

The set of closed r-forms (rth cocycle group) is denoted by $Z^r(M)$. The set of exact r-forms (the rth coboundary group with real coefficients) is denoted by $B^r(M)$. The rth de Rham cohomology group of M is defined as

$$H^r(M;\mathbb{R}) := Z^r(M)/B^r(M). \tag{19}$$

For $\omega \in Z^r(M)$, $[w] \in H^r(M)$ (the equivalence class of ω) is defined as

$$[\omega] = \{\omega' \in Z^r(M) | \omega' = \omega + d\psi,\ \psi \in \Omega^{r-1}(M)\}, \tag{20}$$

where $\Omega^{r-1}(M)$ is the set of $(r-1)$-forms on M.

Example 3.5 The first cohomology group of the unit circle $S^1 = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$ is calculated as follows. Let ω and ω' be closed forms ($d\omega = d\omega' = 0$) that are not exact. Note that $\omega' - a\omega$ is exact when $a = \int_0^{2\pi} \omega' / \int_0^{2\pi} \omega$. Thus, given ω such that $d\omega = 0$, any closed 1-form ω' is cohomogolous to $a\omega$ for some $a \in \mathbb{R}$. Hence, each cohomology class is given by a real number a. Therefore, $H^1(S^1) = \mathbb{R}$.

The period of a closed r-form ω over a cycle c is defined as $(c,\omega) = \int_c \omega$. For $[c] \in H_r(M), [\omega] \in H^r(M)$ define

$$\Lambda([c],[\omega]) := (c,\omega) = \int_c \omega. \tag{21}$$

We note that both $\Lambda(.,[\omega]) : H_r(M) \to \mathbb{R}$, and $\Lambda([c],.) : H^r(M) \to \mathbb{R}$ are linear maps. De Rham's theorem [18, 25] says that if M is a compact manifold, $H_r(M)$ and $H^r(M)$ are finite-dimensional and the map $\Lambda : H_r(M) \times H^r(M) \to \mathbb{R}$ is bilinear and non-degenerate. Hence, $H^r(M)$ is the dual vector space of $H_r(M)$. As a corollary of de Rham's theorem, for a compact manifold M, let $b_r = \dim H_r(M;\mathbb{R})$ be its rth Betti number. Let $c_1, \ldots, c_{b_r}$ be generators of $Z_r(M)$. Then, a closed r-form ψ is exact if and only if[6]

[6] This was conjectured by Cartan in 1928 and was proved later on by de Rham [20]. This theorem can be summarized as follows. If for a closed form ω, $(c,\omega) = 0$ for all p-cycles, then ω is exact. If for a p-cycle c, $(c,\omega) = 0$ for all closed p-forms, then c is a boundary.

$$\int_{c_i} \psi = 0, \quad i = 1, \ldots, b_r. \tag{22}$$

Note that $\Lambda([c_i], .) : H^r(M) \to \mathbb{R}$ is non-degenerate, and hence, $\Lambda([c_i], [\omega]) = 0$ implies $[\omega] = 0$, i.e., the cohomology class of exact forms. Duff [22] generalized this theorem to manifolds with boundary.[7]

3.1.5 Relative homology groups

The relative homology groups were introduced by S. Lefschetz [32]. These are important in problems with boundary conditions and also appear in duality theorems. Let K be an oriented simplicial complex of dimension n and $L \subset K$. The pth chain group of K modulo L (the pth relative chain group) is the subgroup of $C_p(K)$ in which the coefficient of every simplex of L is zero. This is denoted by $C_p(K, L) \subset C_p(K)$. Let us define a homomorphism $j = j_q : C_q(K) \to C_q(K, L)$, which changes to zero the coefficient of every simplex in L. The relative boundary homomorphism $\tilde{\partial} = \tilde{\partial}_p : C_p(K, L) \to C_{p-1}(K, L)$ is defined as

$$\tilde{\partial} c = j_{p-1}(\partial_p c), \qquad \forall c \in C_p(K, L). \tag{23}$$

Note that $\tilde{\partial}_p = j_{p-1} \circ \partial_p \circ i_p$, where $i_p : C_p \to C_p(K)$ is the inclusion map. Note also that for any p, $\tilde{\partial} \circ \tilde{\partial} = \tilde{\partial}_{p-1} \circ \tilde{\partial}_p = 0$.

Let Ω be a compact manifold and $S \subset \Omega$ a compact subset. $C_*(\Omega) = \{C_p(\Omega), \partial_p\}$ is the chain complex corresponding to Ω and for $S \subset \Omega, C_*(S) = \{C_p(S), \partial'_p\}$, where $C_p(S) \subset C_p(\Omega)$, $\forall p$, is the chain complex associated with S. The relative chain group is defined as

$$C_p(\Omega, S) := C_p(\Omega)/C_p(S) = \{c + C_p(S)\},\ c \in C_p(\Omega). \tag{24}$$

The induced boundary operator $\partial''_p : C_p(\Omega)/C_p(S) \to C_{p-1}(\Omega)/C_{p-1}(S)$ is defined the obvious way. $Z_p(\Omega, S) = \operatorname{Ker} \partial''_p$ is the group of relative p-cycles modulo S and $B_p(\Omega, S) = \operatorname{Im} \partial''_{p+1}$ is the group of relative p-boundaries of Ω modulo S. Note that z is a relative p-cycle if its boundary lies in S and b is a relative p-boundary if it is homologous to some p-chain in S. In Fig. 4, four paths on a cylinder are shown. c_1 and c_2 are relative boundaries, i.e., are elements of $B_1(\Omega, \partial\Omega)$, $c_3 \in H_1(\Omega)$, and $c_4 \in H_1(\Omega, \partial\Omega)$.

$C^p(\Omega, S)$ is defined to be the set of linear combinations of p-forms whose support lies in $\Omega \backslash S$. For $z \in Z^p_c(\Omega \backslash S)$, $\int_z \omega$ is the relative period of ω on z, where $Z^p_c(\Omega \backslash S)$ is the set of closed p-forms with compact support in $\Omega \backslash S$. Suppose M is a manifold with boundary ∂M. If a closed $p-$form has zero relative periods in M, then α is an exact relative p-form [22].

---

[7] Duff [22] showed that a closed form with zero relative periods in $H_1(M, \partial M)$ is a closed relative form, i.e., a closed form with compact support in M.

3.1.6 Duality theorems in algebraic topology

The following duality theorems are useful in nonlinear elasticity applications.

- Poincaré duality: For an orientable n-manifold M without boundary, $H^p_c(M) \cong H_{n-p}(M)$, where $H^p_c(M) := Z^p_c(M)/B^p_c(M)$, and $Z^p_c(M)$ and $B^p_c(M)$ are the closed and exact p-forms with compact supports in M, respectively. For compact manifolds from de Rham's theorem $H_p(M) \cong H_{n-p}(M)$.
- Lefschetz duality: For a compact n-manifold M, $H^{n-p}_c(M) \cong H_p(M, \partial M)$. From de Rham's theorem, $H_{n-p}(M) \cong H^p_c(M \backslash \partial M)$. Therefore, $H_{n-p}(M) \cong H_p(M, \partial M)$.[8] Thus, $b_{n-p}(M) = b_p(M, \partial M)$.
- Poincaré-Lefschetz duality: For a compact, orientable n-manifold M with boundary (for $0 \leq k \leq n$), $H^k(M;\mathbb{Z}) \cong H_{n-k}(M, \partial M; \mathbb{Z})$. This holds for any Abelian coefficient group as well.
- Alexander duality: For a closed subset M of an n-manifold Q, $H^p(M) \cong H_{n-p}(Q, Q \backslash M)$. In elasticity applications, $Q = \mathbb{R}^3$. It can be shown that for $p \neq 2$, $H^p(M) \cong H_{2-p}(\mathbb{R}^3 \backslash M)$, and $\mathbb{R} \otimes H^2(M) \cong H_0(\mathbb{R}^3 \backslash M)$ [27]. Thus, for $p \neq 2$, $b_p(M) = b_{2-p}(\mathbb{R}^3 \backslash M)$, and $1 + b_2(M) = b_0(\mathbb{R}^3 \backslash M)$.

Let us now restrict ourselves to embedded 3-submanicolds of $\mathbb{R}^3$,[9] which model our three-dimensional deformable bodies in elasticity. $H_0(M)$ is generated by equivalence classes of points in M; two points are in the same equivalence class if they can be connected to each other by a continuous path in M. $H_1(M)$ is generated by equivalent classes of oriented loops; two loops are in the same equivalence class if their "difference" is the boundary of an oriented surface in M. $H_1(M, \partial M)$ is generated by the equivalence class

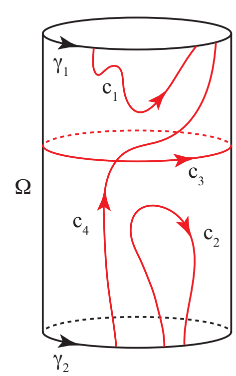

Fig. 4 A cylinder $\Omega = S^1 \times [0,1]$. $S = \partial\Omega = \gamma_1 \cup \gamma_2$ has two components. c_1 and c_2 are relative boundaries, c_3 generates $H_1(\Omega)$, and c_4 is a relative cycle but not a relative boundary; it generates $H_1(\Omega, \partial\Omega)$.

[8] Love [34] in Article 156 writes: "Now suppose the multiply-connected region to be reduced to a simply-connected one by means of a system of barriers." A "barrier" Ω in a three-dimensional body $\mathcal{B}$ is a generator of $H_2(\mathcal{B}, \partial\mathcal{B}) \cong H_1(\mathcal{B})$, and in a two-dimensional body it is a generator of $H_2(\mathcal{B}, \partial\mathcal{B}) \cong H_1(\mathcal{B})$.

[9] Cantarella, et al. [12] present an elementary exposition of homology theory with applications to vector calculus. The reader may find their exposition useful.

of oriented paths with end points on ∂M; two paths are equivalent if their "difference" (augmented by paths on ∂M if necessary) is the boundary of an oriented surfaces in M. From Poincaré duality we know that

$$H_0(M) \cong H_3(M, \partial M), \tag{25}$$
$$H_1(M) \cong H_2(M, \partial M), \tag{26}$$
$$H_2(M) \cong H_1(M, \partial M), \tag{27}$$
$$H_3(M) \cong H_0(M, \partial M). \tag{28}$$

Define $M^c = \mathbb{R}^3 \backslash M$. From Alexander duality one has

$$H_0(M) \cong H_2(M^c), \quad H_1(M) \cong H_1(M^c), \quad H_0(M^c) \cong \mathbb{R} \otimes H_2(M). \tag{29}$$

Let $\varSigma_1, ..., \varSigma_k$ be a family of surfaces in M with boundaries on ∂M such that they generate $H_2(M, \partial M)$. As an example, consider the solid torus with two holes shown in Fig. 5 for which $k = 2$. Let $\gamma_1, ..., \gamma_k$ be loops in the interior of M that generate $H_1(M)$ chosen such that intersection number of c_i with $\varSigma_j$ is δ_{ij}.[10] These loops can be chosen to be disjoint. If one pushes the boundaries of $\varSigma_1, ..., \varSigma_k$ slightly into M^c, one obtains the loops $\varGamma_1, ..., \varGamma_k$ that generate $H_1(M^c)$.

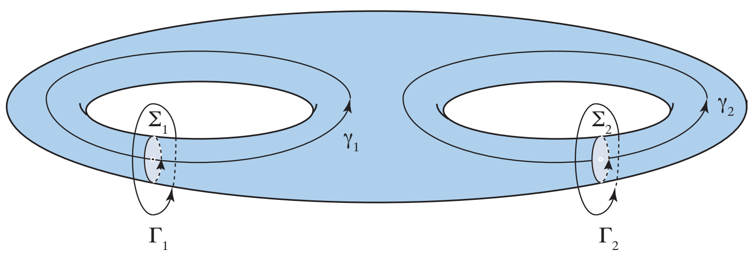

Fig. 5 A two-hole solid torus M. The closed curves γ_1 and γ_2 are generators of $H_1(M)$. $\varGamma_1$ and $\varGamma_2$ are generators of $H_1(\mathbb{R}^3 \backslash M)$.

3.2 Homotopy and the fundamental group

Fundamental group was introduced by Poincaré in 1895 and plays an important role in understanding compatibility equations. It is much easier to define compared to homology groups but it is much harder to calculate, in general. A path in a topological space X is a map $c : [0,1] \to X$. It is simple if it is one-to-one. A closed path (loop) has the same end points, i.e., $c(0) = c(1)$, which

[10] This is possible as a consequence of Poincaré duality.

is called the base point of the loop. A cycle is a continuous map $\gamma: S^1 \to X$. It is different from a loop because in a cycle no end points are distinguished. Two paths c_1 and c_2 having the same end points are homotopic if there is a continuous family of paths whose end points are the same as those of c_1 and c_2. Roughly speaking, the set of equivalent paths based at x_0 constitute the fundamental group $\pi_1(X, x_0)$. An isotopy between c_1 and c_2 is a homotopy for which the curves remain simple during the whole deformation process from c_1 to c_2. Note that two homotopic simple paths are not necessarily isotopic. We make these notions more precise in the following.

Consider a topological space X and a base point $x_0 \in X$. Two loops based at x_0 are equivalent if one loop can be continuously deformed to the other loop. A loop based at x_0 is a continuous map $f: I = [0,1] \to X$ such that $f(0) = f(1) = x_0$. Two loops f, g are called homotopic if there is a continuos function $F: I \times I \to X$ such that $F(s,0) = f(s), F(s,1) = g(s)$, $F(0,t) = F(1,t) = x_0$. F is a homotopy between f and g and this is denoted by $f \sim_F g$. It can be shown that homotopy gives an equivalence relation on loops based at x_0. The equivalence class of f is denoted by $[f]$ and the equivalence classes are elements of the fundamental group $\pi_1(X, x_0)$. Group multiplication is defined as $[f][g] = [fg]$, where fg is defined by first going along the loop f and then along the loop g. Inverse of a loop f, f^{-1} is the same loop with the opposite orientation and $[f]^{-1} = [f^{-1}]$. Identity loop at x_0 is a loop $f: [0,1] \to X$ such that $f(s) = x_0, \forall\ s \in [0,1]$. For a path-connected topological space X fundamental groups at two distinct points x_0 and x are isomorphic. A path α connecting x_0 to x $(\alpha(0) = x_0, \alpha(1) = x)$, induces an isomorphism $\alpha_*: \pi_1(X, x) \to \pi_1(X, x_0)$ defined as $\alpha_*([f]) = [\alpha f \alpha^{-1}]$ (see Fig. 6).

A path-connected space X is simply-connected if $\pi_1(X, x_0) \cong \{1\}$. $\mathbb{R}^n$ is an example of a simply-connected space. Another example is the 2-sphere S^2. In a simply-connected and path-connected space any closed path can be continuously shrunk to any point in the space.

Example 3.6 The fundamental group of the unit circle S^1 is $\pi_1(S^1) = \mathbb{Z}$. Homotopy class of a loop is determined by the number of times it winds around. In other words, any closed path in the circle can be tightened through homotopy into the product of n standard circular paths. Torus $T^2 = S^1 \times S^1$ has the fundamental group $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ and is Abelian.

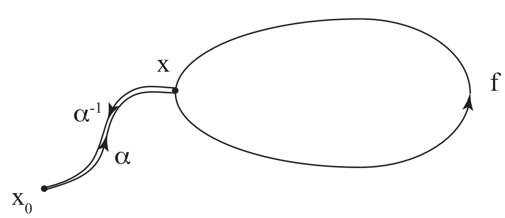

Fig. 6 Having a loop f based at x a loop $\alpha\gamma\alpha^{-1}$ based at x_0 is constructed.

Consider two paths $f,g:I\to X$, $f(0)=a_0, f(1)=a_1$, and $g(0)=b_0, g(1)=b_1$. f and g are said to be freely homotopic if there exists a continuous map $F:I\times I\to X$ such that $F(s,0)=f(s), F(s,1)=g(s)$. In addition to this if $F(0,t)=a_0, F(1,t)=b_0$, f and g are called homotopic. Two loops $f,g:I\to X$ are freely homotopic if there is a continuos map $F:I\times I\to X$ such that $F(s,0)=f(s), F(s,1)=g(s)$ and $F(0,t)=F(1,t)=\alpha(t)$ is a path between $f(0)=f(1)=a$ and $g(0)=g(1)=b$.

Let Y be a topological space. $X\subset Y$ is a retract of Y if there exists a continuous map $r:Y\to X$ such that $r(x)=x$ for all $x\in X$. X is a deformation retract of Y if it is a retract of Y and there is a continuous map $h:[0,1]\times Y\to Y$ such that: i) $h(0,y)=y$, $h(1,y)=r(y)$, $\forall y\in Y$, and ii) $h(t,x)=x$, $\forall x\in X$, $\forall t\in [0,1]$. A deformation retract $r:Y\to X$ induces an isomorphism $r_*:\pi_1(Y)\to \pi_1(X)$. One can visualize deformation retraction as a continuous collapse of Y onto X in such a way that each point of X remains fixed during the deformation process.

Example 3.7 The Möbius band $\mathcal{M}$ is constructed from a square by the identification shown in Fig. 3. This is an example of a non-orientable surface. The circle S^1 is a deformation retract of the Möbius band, and hence, $\pi_1(\mathcal{M})=\mathbb{Z}$.

Example 3.8 Consider the solid cylinder Ω with four tubular holes shown in Fig. 7. As is shown schematically Ω has a deformation retract to a bouquet of four circles, and hence, $\pi_1(\Omega)=\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}\otimes\mathbb{Z}$, i.e., the free group with four generators. If this is a solid body, e.g., a hollow bar under torsion and bending, we will see in the next section that because c_i's are free generators of the fundamental group, each would require an additional (vectorial) compatibility equation.

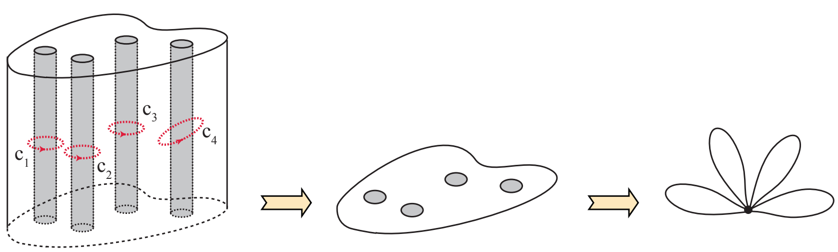

Fig. 7 A solid cylinder with four tubular holes and its deformation retract to a bouquet of four circles. c_1,c_2,c_3,c_4 are the generators of the fundamental group.

H. F. F. Tietze (1908) showed that the fundamental group of any compact, finite-dimensional, path-connected manifold is finitely presented. One forms the Abelization of a group by taking the quotient over the subgroup generated by all commutators $g^{-1}h^{-1}gh$. Pioncaré isomorphism theorem tells us that

(Pioncaré, 1895)[11]

$$\pi_1(M)/[\pi_1(M),\pi_1(M)] \cong H_1(M,\mathbb{Z}). \tag{30}$$

Given a group G with the presentation

$$G = \langle a_1,\ldots,a_m;\ r_1,\ldots,r_n\rangle, \tag{31}$$

its Abelianization is obtained by adding the relations $a_i a_j = a_j a_i$ and it is independent of the presentation of G.

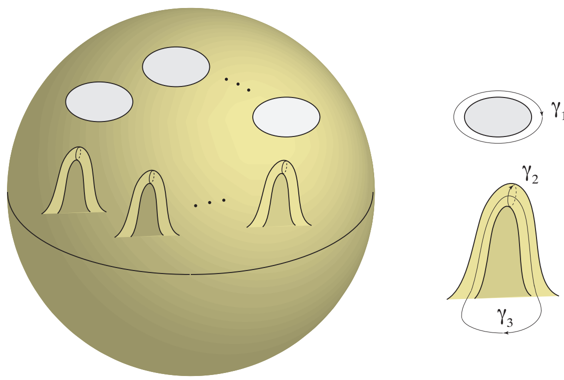

Fig. 8 A sphere with k holes and n handles. γ_1, γ_2, and γ_3 are typical generators for the first homologgy group. Note that a sphere with a single hole is simply-connected, i.e., there are $k-1$ generators corresponding to the k holes.

3.3 Classification of compact 2-manifolds with boundary

Let M_1 and M_2 be compact manifolds with boundary. Assume that their boundaries have the same number of components. M_1 and M_2 are homeomorphic if and only if the manifolds M_1^* and M_2^* obtained by gluing a disk to each boundary component are homeomorphic. Any compact surface is either homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes. Any compact orientable two-manifold with boundary is homeomorphic to a sphere with n handles and k holes, see Fig. 8.

[11] If $\gamma_1^{n_1}\gamma_2^{n_2}\ldots\gamma_k^{n_k} = 1$, Poincaré observed that $n_1\gamma_1 + n_2\gamma_2 + \ldots + n_k\gamma_k$ is null-homologous [20].

3.4 Curves on oriented surfaces

R. Baer in 1928 showed that simple closed curves on a 2-manifold are isotopic if and only if they are homotopic [53]. Epstein [23] showed that any two simple, homotopic, non-contractible loops on an orientable surface are isotopic. If c is a simple, null-homotopic (contractible) loop on a surface, then it is the boundary of a topological disk (a genus zero surface with one boundary curve) [33, 35], see Fig. 9. A zero-genus surface with two boundary curves is called a cylinder. Any two non-contractible, non-intersecting, and freely homotopic curves on a closed surface bound a cylinder [33]. We will use these facts to derive the "bulk" compatibility equations.[12]

3.5 Theory of knots

Topology of subsets of $\mathbb{R}^3$ with tubular holes can, at least partially, be understood using the complementary spaces of knots. For the background in knot theory we mainly follow [3, 17, 53]. A knot $\mathcal{K}$ is a simple closed curve in $\mathbb{R}^3$. A knot $\mathcal{K}$ is trivial if it is isotopic to the circle in $\mathbb{R}^3$. The fundamental group of a trivial knot $\mathbb{R}^3\setminus\mathcal{K}$ is infinite cyclic. Any knot $\mathcal{K}$ can be represented by a projection on a plane with no multiple points higher than double, with an indication of the upper branch of each crossing point (each of the double points). A projection of the trefoil knot (the simplest non-trivial knot) is shown in Fig. 10. A link is a set of knots tangled up together.

If the lower branch (under crossing) of each crossing is broken, one obtains a finite number of arcs α_i. It turns out that $\pi_1(\mathbb{R}^3\setminus\mathcal{K})$ is generated by loops c_i that pass around these arcs (this is rigorously proved using the Seifert-van Kampen theorem). This means that the number of generators of $\pi_1(\mathbb{R}^3\setminus\mathcal{K})$ is equal to the number of crossing points. Given the crossing point shown in Fig. 11a, the three generators of the fundamental group corresponding to the

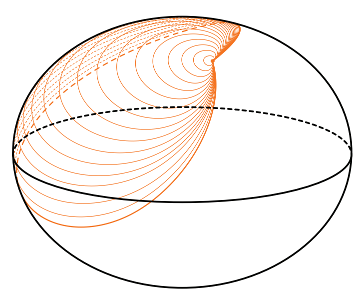

Fig. 9 A null-homotopic curve on an orientable surface bounds a region.

12 These topological results are implicitly assumed in the literature of compatibility equations.

arcs α_i, α_{i+1}, and α_j are c_i, c_{i+1}, and c_j, respectively, and are oriented using the right-hand rule. It can be shown that $c_i c_j^{-1} c_{i+1}^{-1} c_j$ is null-homotopic, or equivalently, at this crossing point we have the relation $c_{i+1} c_j = c_j c_i$ [53]. All the four possibilities and their corresponding group relations are shown in Fig. 11b.

Next, as examples, we find the fundamental groups of the complements of the two-crossing link, and the trefoil knot (see Fig. 12). Using the diagrams of Fig. 11b, it is straightforward to see that the fundamental groups of the complements of the two-crossing link $\mathcal{T}_1$ and the trefoil knot $\mathcal{T}_2$ are, respectively:

$$\begin{aligned} &\pi_1\left(\mathbb{R}^3\backslash\mathcal{T}_1\right) = \langle c_1, c_2;\ c_1c_2 = c_2c_1\rangle, \\ &\pi_1\left(\mathbb{R}^3\backslash\mathcal{T}_2\right) = \langle c_1, c_2, c_3;\ c_3c_1 = c_2c_3, c_2c_3 = c_1c_2, c_1c_3 = c_2c_1\rangle. \end{aligned} \tag{32}$$

Note that if the two circles are unlinked then $\pi_1\left(\mathbb{R}^3\backslash\mathcal{T}_1\right) = \langle c_1, c_2\rangle$, i.e., a free group with c_1 and c_2 as generators.

Remark 3.9 In the case of knots, Abelianization always gives an infinite cyclic group [53]. A handlebody $\mathcal{H}_n$ is a solid body bounded by an orientable surface of genus n embedded in $\mathbb{R}^3$. $\pi_1\left(\mathbb{R}^3\backslash\mathcal{H}_n\right)$ is the free group of rank n.

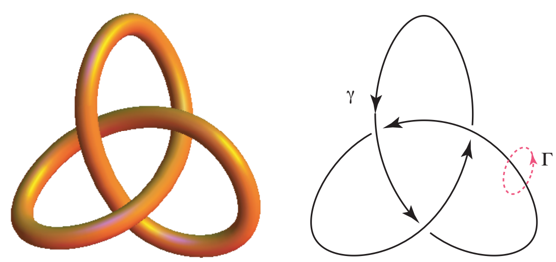

Fig. 10 A trefoil knot and its projection. γ is the generator of the first homology group of the "thickened" trefoil and Γ is the generator of the first homology group of $\mathbb{R}^3\backslash\mathcal{T}$.

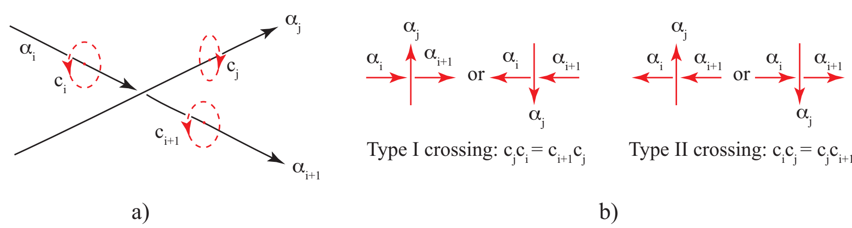

Fig. 11 a) A crossing point. α_j corresponds to the over crossing and α_i and α_{i+1} correspond to the under crossing. Their corresponding loops c_j, c_i, and c_{i+1} are oriented using the right-hand rule. b) Two types of crossing points and their corresponding group relations. Note that c_i is the fundamental group generator corresponding to the arc α_i, etc.

3.6 Topology of 3-manifolds

Material manifold — the natural configuration of a body — may be non-Euclidean in many applications [41, 58, 59, 60, 61, 51]. However, for most applications the ambient space is the Euclidean 3-space. We consider a body that has a non-trivial topology, i.e., it has "holes". We assume that the body is elastic and the material manifold is an embedded 3-submanifold of $\mathbb{R}^3$. There is a complete classification of 3-manifolds [29, 37], but it is not known what 3-manifolds can be embedded in $\mathbb{R}^3$. However, a large class of embedded 3-submanifolds can be constructed by thickened knots and their complements in $\mathbb{R}^3$. The important thing to note is the complexity of embedded 3-manifolds and the importance of algebraic topology in deriving their necessary and sufficient compatibility equations for non-simply connected bodies.

For an embedded 3-manifold with boundary in $\mathbb{R}^3$, its boundary is an embedded closed (orientable) 2-manifold, which has a complete classification. If the boundary of the 3-manifold is the two-sphere, then its topology is uniquely determined by the genus of the boundary, i.e., the manifold is simply the compact region bounded by the boundary in $\mathbb{R}^3$ (by the generalized Jordan-Brower separation theorem, any closed embedded 2-manifold in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into a pair of regions, and precisely one of these regions has compact closure). If the boundary is not connected, then things are more complicated. For instance even when the boundary consists of a single torus, the compact region that it bounds in $\mathbb{R}^3$ is not uniquely determined, but it is known that it must be either a solid torus or a knot complement. Things get more complicated when the boundary has genus larger than one. The only simple case is when the boundary is a sphere, in which case the manifold must necessarily be a ball by Jordan's theorem. To summarize, while 3-manifolds with boundary have been completely classified, it is not known which ones can be embedded in $\mathbb{R}^3$. The answer certainly depends on both the topology of the boundary, as well as its isotopy (or knotting) in $\mathbb{R}^3$. As to what types of "holes" can occur in a 3-dimensional solid, consider the following example: Put a knot in the solid body, then "thicken" it to obtain a (knotted) solid torus, and then remove the interior of that torus. This way one can construct as many different types of holes (or topological types for the solid) as there are knots. Now consider doing the same construction with multiple tori or higher genus surfaces, which may be linked with each other.

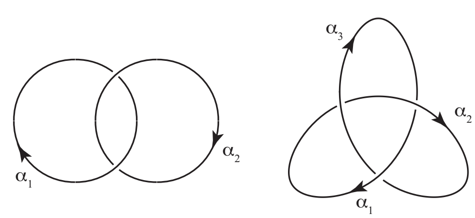

Fig. 12 The double link and the trefoil knot and their corresponding arcs α_i.

4 Kinematics of Nonlinear Elasticity

In this section we review the kinematics of nonlinear elasticity. A body $\mathcal{B}$ is identified with a Riemannian manifold $(\mathcal{B},\mathbf{G})$[13] and a configuration of $\mathcal{B}$ is a mapping $\varphi:\mathcal{B}\to\mathcal{S}$, where $(\mathcal{S},\mathbf{g})$ is another Riemannian manifold. The set of all configurations of $\mathcal{B}$ is denoted by $\mathcal{C}$. A motion is a curve $c:\mathbb{R}\to\mathcal{C};t\mapsto\varphi_t$ in $\mathcal{C}$. The material manifold is, by construction, the natural configuration of the body. For a fixed t, $\varphi_t(X)=\varphi(X,t)$ and for a fixed X, $\varphi_X(t)=\varphi(X,t)$, where X is the position of a material point in the undeformed configuration $\mathcal{B}$. The material velocity is given by $\mathbf{V}_t(X)=\mathbf{V}(X,t)=\frac{\partial\varphi(X,t)}{\partial t}$. Similarly, the material acceleration is defined by $\mathbf{A}_t(X)=\mathbf{A}(X,t)=\frac{\partial\mathbf{V}(X,t)}{\partial t}$. In components, $A^a=\frac{\partial V^a}{\partial t}+\gamma^a{}_{bc}V^bV^c$, where $\gamma^a{}_{bc}$ is the Christoffel symbol of the local coordinate chart $\{x^a\}$. The spatial velocity of a regular motion φ_t is defined as $\mathbf{v}_t:\varphi_t(\mathcal{B})\to T_{\varphi_t(X)}\mathcal{S}$, $\mathbf{v}_t=\mathbf{V}_t\circ\varphi_t^{-1}$, and the spatial acceleration $\mathbf{a}_t$ is defined as $\mathbf{a}=\dot{\mathbf{v}}=\frac{\partial\mathbf{v}}{\partial t}+\nabla_{\mathbf{v}}\mathbf{v}$. In components, $a^a=\frac{\partial v^a}{\partial t}+\frac{\partial v^a}{\partial x^b}v^b+\gamma^a{}_{bc}v^bv^c$.

Let $\varphi:\mathcal{B}\to\mathcal{S}$ be a C^1 configuration of $\mathcal{B}$ in $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{S}$ are manifolds. The deformation gradient is the tangent map of φ and is denoted by $\mathbf{F}=T\varphi$. Thus, at each point $X\in\mathcal{B}$, it is a linear map $\mathbf{F}(X):T_X\mathcal{B}\to T_{\varphi(X)}\mathcal{S}$. If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ are $F^a{}_A(X)=\frac{\partial\varphi^a}{\partial X^A}(X)$. $\mathbf{F}$ has the following local representation $\mathbf{F}=F^a{}_A\frac{\partial}{\partial x^a}\otimes dX^A$. $\mathbf{F}$ can be thought of a vector-valued 1-form with the representation $\mathbf{F}=\frac{\partial}{\partial x^a}\otimes\vartheta^a$, with the coframes $\vartheta^a=F^a{}_AdX^A$. The adjoint of $\mathbf{F}$ is defined by

$$\mathbf{F}^{\mathrm{T}}:T_x\mathcal{S}\to T_X\mathcal{B},\quad \langle\!\langle\mathbf{F}\mathbf{W},\mathbf{w}\rangle\!\rangle_{\mathbf{g}}=\langle\!\langle\mathbf{W},\mathbf{F}^{\mathrm{T}}\mathbf{w}\rangle\!\rangle_{\mathbf{G}},\quad \forall\mathbf{W}\in T_X\mathcal{B},\ \mathbf{w}\in T_X\mathcal{S}. \tag{1}$$

In components, $(F^{\mathrm{T}}(X))^A{}_a=g_{ab}(x)F^b{}_B(X)G^{AB}(X)$, where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively. The right Cauchy-Green deformation tensor is defined as

$$\mathbf{C}(X):T_X\mathcal{B}\to T_X\mathcal{B},\qquad \mathbf{C}(X)=\mathbf{F}^{\mathrm{T}}(X)\mathbf{F}(X). \tag{2}$$

In components, $C^A{}_B=(F^{\mathrm{T}})^A{}_aF^a{}_B$. It is straightforward to show that, $\mathbf{C}^\flat=\varphi^*(\mathbf{g})=\mathbf{F}^\star\mathbf{g}\mathbf{F}$, i.e., $C_{AB}=F^a{}_A(g_{ab}\circ\varphi)F^b{}_B$, where the dual of the deformation gradient is defined as $\mathbf{F}^\star=F^a{}_AdX^A\otimes\frac{\partial}{\partial x^a}$. The Finger tensor is defined as $\mathbf{b}=\mathbf{c}^{-1}$, where $\mathbf{c}=\varphi_{t*}\mathbf{G}$. In components, $b^a{}_b=F^a{}_Ag_{bc}F^c{}_BG^{AB}=F^a{}_A\left(F^{\mathsf{T}}\right)^A{}_b$. Thus

$$\mathbf{b}(x):T_x\mathcal{S}\to T_x\mathcal{S},\qquad \mathbf{b}(x)=\mathbf{F}(X)\mathbf{F}^{\mathrm{T}}(X). \tag{3}$$

Polar decomposition theorem states that $\mathbf{F}=\mathbf{R}\mathbf{U}$ [48]. In components it reads $F^a{}_A=R^a{}_BU^B{}_A$, where $\mathbf{R}(X):T_X\mathcal{B}\to T_{\varphi_t(X)}\mathcal{S}$ is a $(\mathbf{G},\mathbf{g})$-orthogonal transformation, i.e., $G_{AB}=R^a{}_AR^b{}_Bg_{ab}$, and $\mathbf{U}(X):T_X\mathcal{B}\to T_X\mathcal{B}$ is the material stretch tensor. Note that $\mathbf{G}=\mathbf{R}^*\mathbf{g}$ and $\mathbf{C}=\mathbf{U}^*\mathbf{G}$.

[13] In general, $(\mathcal{B},\mathbf{G})$ is the underlying Riemannian manifold of the material manifold, i.e., its natural state. See [58, 59, 60, 61] for more details.

5 Compatibility Equations in Nonlinear Elasticity

In this section we summarize the results of [62], [4], and [5]. We assume a finite body, and hence, the material manifold $(\mathcal{B},\mathbf{G})$ is a compact Riemannian manifold. We also assume that the first homology and homotopy groups $H_1(\mathcal{B})$ and $\pi_1(\mathcal{B})$ are given. In the presence of boundary conditions we will use the relative homology groups, which are also assumed to be given. In [62] we derived the compatibility equations for the deformation gradient **F** using a generalization of de Rham's theorem. The **F**-compatibility equations can be derived using the fundamental group as well. It turns out that understanding the role of homotopy in compatibility equations is crucial in formulating the **C**-compatibility equations [62].

5.1 Compatibility equations for the deformation gradient F

The following old questions in vector calculus are relevant to the compatibility equations: i) Given a vector field defined on some bounded domain in the Euclidean 3-space, is it the gradient of some function defined on the same domain? ii) Is it the curl of another vector field? It turns out that the topology of the domain of definition of the vector field plays a crucial role. The **F**-compatibility problem is stated as: Given a body $\mathcal{B}\subset\mathbb{R}^3$, find the condition(s) that guarantee existence of a map $\varphi:\mathcal{B}\to\mathbb{R}^3$ such that $\mathbf{F}=T\varphi$. Question i) is related to compatibility equations while question ii) is related to the existence of stress functions in elasticity. The following proposition summarizes the **F**-compatibility equations, which is a simple extension of de Rham's theorem to $\mathbb{R}^3$-valued forms.

Proposition 5.1 (Yavari [62]) The necessary and sufficient **F**-compatibility equations are[14]

$$d\mathbf{F}=\mathbf{0}, \quad \text{and} \quad \int_{c_i}\mathbf{F}d\mathbf{X}=\mathbf{0}, \quad i=1,\dots,b_1(\mathcal{B}), \tag{1}$$

where $c_i,\ i=1,\dots,b_1(\mathcal{B})$ are the generators of $H_1(\mathcal{B};\mathbb{R})$.

Instead of using de Rham's theorem, one may follow a different path using the fundamental group. Let us assume that the position of a point $X_0\in\mathcal{B}$ in the deformed configuration $x_0\in\mathcal{S}$ is given. The position of an arbitrary point $X\in\mathcal{B}$ in the deformed configuration is given as

[14] The exterior derivative of the deformation gradient $d\mathbf{F}$ can be identified with $\mathsf{Curl}\,\mathbf{F}$. Note that $d\mathbf{F}=\mathbf{0}$ is equivalent to $\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}$, where $\nabla^{\mathbf{G}}$ is the Levi-Civita connection corresponding to the material metric $\mathbf{G}$ [63]. In components, $F^a{}_{A,B}=F^a{}_{B,A}$, or equivalently, $F^a{}_{A|B}=F^a{}_{B|A}$.

$$\mathbf{x} = \mathbf{x}_0 + \int_\gamma \mathbf{F} d\mathbf{X}. \tag{2}$$

Note that the ambient space is Euclidean, and hence, integrating vector fields makes sense. $\mathbf{F}$ is compatible if and only if the above integral is path-independent, which is equivalent to

$$\int_\gamma \mathbf{F} d\mathbf{X} = \mathbf{0}, \tag{3}$$

for any closed path γ based at X_0.

Suppose $\pi_1(\mathcal{B})$ has the generators $\{\gamma_i\}_{i=1,\dots,m}$. For a compact material manifold $\mathcal{B}$, i.e., a finite body, the fundamental group has a finite presentation [53]

$$\pi_1(\mathcal{B}) = \langle \gamma_1, \dots, \gamma_m; r_1, \dots, r_n \rangle, \tag{4}$$

where

$$r_i = \gamma_{i_1}^{\varepsilon_{i_1}} \dots \gamma_{j_i}^{\varepsilon_{j_i}} = 1, \quad i = 1, \dots, n, \; \varepsilon_k = \pm 1, \tag{5}$$

are the relators of the fundamental group. If γ is a contractible (null homotopic) curve that lies on a 2-submanifold $\mathcal{P} \subset \mathcal{B}$, then

$$\int_\gamma \mathbf{F} d\mathbf{X} = \int_{\partial \mathcal{U}} \mathbf{F} d\mathbf{X} = \int_{\mathcal{U}} d(\mathbf{F} d\mathbf{X}) = \mathbf{0}, \tag{6}$$

where $\gamma = \partial \mathcal{U} \subset \mathcal{P}$ [33, 23]. Because $\mathcal{P}$ is arbitrary one concludes that $d\mathbf{F} = \mathbf{0}$ in $\mathcal{B}$, which is a necessary compatibility condition. Note that from (3)

$$\int_{\gamma_i} \mathbf{F} d\mathbf{X} = \mathbf{0}, \quad i = 1, \dots, m. \tag{7}$$

Therefore, $d\mathbf{F} = \mathbf{0}$, and $\int_{\gamma_i} \mathbf{F} d\mathbf{X} = \mathbf{0}, \; i = 1, \dots, m$ subjected to $\int_{r_i} \mathbf{F} d\mathbf{X} = \mathbf{0}, \; i = 1, \dots, n$ are necessary for compatibility of $\mathbf{F}$. It turns out that they are sufficient as well. Given a null-homotopic curve γ, $\gamma = \partial \Omega$, and hence, from $d\mathbf{F} = \mathbf{0}$, one can write

$$\int_\gamma \mathbf{F} d\mathbf{X} = \int_{\partial \mathcal{U}} \mathbf{F} d\mathbf{X} = \int_{\mathcal{U}} d(\mathbf{F} d\mathbf{X}) = \int_{\mathcal{U}} d\mathbf{F} \wedge d\mathbf{X} = \mathbf{0}. \tag{8}$$

If γ is non-contractible, in terms of the group generators it has the representation $\gamma = \left(w_1 \gamma_1 w_1^{-1}\right)^{\varepsilon_1} \dots \left(w_p \gamma_p w_p^{-1}\right)^{\varepsilon_p}$, where w_i is a curve joining X_0 to a point on γ_i and $\{\gamma_1, \dots, \gamma_p\}$ is a subset of the group generators with possible relabelings. Using the relations $\int_{w_i \gamma_i w_i^{-1}} \mathbf{F} d\mathbf{X} = \int_{\gamma_i} \mathbf{F} d\mathbf{X}$, one has

$$\int_\gamma \mathbf{F} d\mathbf{X} = \varepsilon_1 \int_{\gamma_1} \mathbf{F} d\mathbf{X} + \dots + \varepsilon_p \int_{\gamma_p} \mathbf{F} d\mathbf{X} = \mathbf{0}. \tag{9}$$

The relators of the group representation impose the following constraints

$$\int_{r_i} \mathbf{F} d\mathbf{X} = \mathbf{0}, \quad i = 1, \dots, n. \tag{10}$$

This implies that the conditions $\int_{\gamma_i} \mathbf{F}d\mathbf{X} = \mathbf{0}$, $i = 1,...,m$, may not all be independent.

Proposition 5.2 (Yavari [62]) The necessary and sufficient $\mathbf{F}$-compatibility conditions are:

i) $d\mathbf{F} = \mathbf{0}$ in $\mathcal{B}$,
ii) If $\pi_1(\mathcal{B}) = \langle \gamma_1,...,\gamma_m; r_1,...,r_n \rangle$, then

$$\int_{\gamma_i} \mathbf{F}d\mathbf{X} = \mathbf{0}, \quad i = 1,...,m, \tag{11}$$

$$\text{subjected to} \int_{r_i} \mathbf{F}d\mathbf{X} = \mathbf{0}, \quad i = 1,...,n. \tag{12}$$

For a path-connected set $\mathcal{B}$ the first homology group is the Abelianization of the fundamental group [11]. One Abelianizes $\pi_1(\mathcal{B})$ by adding the relations $\gamma_i\gamma_j = \gamma_j\gamma_i$, which do not lead to any new compatibility equations.

One should note that the generators of the torsion subgroup do not contribute to the $\mathbf{F}$-compatibility equations because for γ an element of the torsion subgroup $\gamma^n = 1$ for some $n \in \mathbb{N}$, and thus, $\int_\gamma \mathbf{F}d\mathbf{X} = \mathbf{0}$ is trivially satisfied. This means that it is sufficient to have $\int_\gamma \mathbf{F}dX = \mathbf{0}$ only on each generator of the first homology group with real coefficients. Therefore, the number of the complementary compatibility equations is $Nb_1(\mathcal{B})$, where $N = \dim \mathcal{S}$.

Example 5.3 Let us assume that $\dim \mathcal{B} = 1$ and $\mathcal{S} = \mathbb{R}^2$. The bulk compatibility equations are trivially satisfied. It is known that when $\mathcal{B}$ is a graph its fundamental group is freely generated. Assuming that $\gamma_1,...,\gamma_k$ are the free generators of $\pi_1(\mathcal{B})$, there are $2k$ compatibility equations. As an example, let us assume that $\mathcal{B} = S^1(R)$, i.e., the circle with radius R and let $X = \Theta$ be the standard parametrization of S^1. Compatibility equations read

$$\int_0^{2\pi} \mathbf{F}d\Theta = \mathbf{0}. \tag{13}$$

As examples, $\mathbf{F} = (\kappa_1\Theta, \kappa_2)^\mathsf{T}$, where κ_1 and κ_2 are arbitrary constants, is not compatible, while $\mathbf{F} = (\kappa_1 \sin\Theta, \kappa_2 \cos\Theta)^\mathsf{T}$ is compatible.

Remark 5.4 One should note that the $\mathbf{F}$-compatibility equations derived here are valid for anelastic bodies as well. In other words, we have not assumed a flat material manifold $(\mathcal{B}, \mathbf{G})$; the compatibility equations have the same form even in problems for which the material manifold is non-flat. As an example, see the discussion of universal deformations and eigenetrains in compressible solids in [63].

5.2 Examples of non-simply-connected bodies and their **F**-compatibility equations

We next look at a few examples of 2D and 3D non-simply-connected bodies and derive their compatibility equations.

5.2.1 2D elasticity on a torus and a punctured torus

The first homology groups of both torus and punctured torus (handle) are generated by the loops γ_1 and γ_2 in Figs. 2 and 13. Hence, the **F**-compatibility equations read

$$d\mathbf{F} = \mathbf{0}, \qquad \int_{\gamma_1} \mathbf{F} d\mathbf{X} = \int_{\gamma_2} \mathbf{F} d\mathbf{X} = \mathbf{0}. \tag{14}$$

The fundamental group of torus (see Fig. 2) has the presentation

$$\pi_1(T^2) = \langle \gamma_1, \gamma_2; \gamma_1\gamma_2 = \gamma_2\gamma_1 \rangle. \tag{15}$$

Therefore, the group relator is written as $r_1 = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} = 1$. Note that

$$\int_{r_1} \mathbf{F} d\mathbf{X} = \int_{\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}} \mathbf{F} d\mathbf{X} = \int_{\gamma_1} \mathbf{F} d\mathbf{X} + \int_{\gamma_2} \mathbf{F} d\mathbf{X} - \int_{\gamma_1} \mathbf{F} d\mathbf{X} - \int_{\gamma_2} \mathbf{F} d\mathbf{X} = \mathbf{0}. \tag{16}$$

This means that (14) are the necessary and sufficient **F**-compatibility equations.

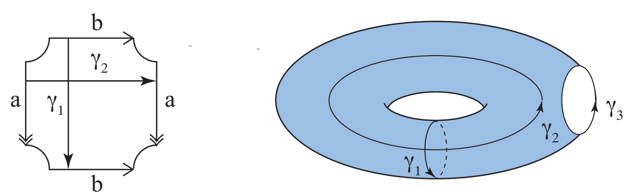

Fig. 13 A punctured torus. γ_1, γ_2, and γ_3 are generators of the fundamental group.

For a punctured torus (see Fig. 13) the fundamental group has three generators and the following presentation [53]

$$\pi_1(\mathcal{H}) = \langle \gamma_1, \gamma_2, \gamma_3; \gamma_3 = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} \rangle. \tag{17}$$

Therefore, the group relator is written as $r_1 = \gamma_3\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1} = 1$. Note that

$$\begin{aligned}
\mathbf{0} &= \int_{r_1} \mathbf{F} d\mathbf{X} \\
&= \int_{\gamma_3\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}} \mathbf{F} d\mathbf{X} \\
&= \int_{\gamma_3} \mathbf{F} d\mathbf{X} + \int_{\gamma_2} \mathbf{F} d\mathbf{X} + \int_{\gamma_1} \mathbf{F} d\mathbf{X} - \int_{\gamma_2} \mathbf{F} d\mathbf{X} - \int_{\gamma_1} \mathbf{F} d\mathbf{X} \\
&= \int_{\gamma_3} \mathbf{F} d\mathbf{X}.
\end{aligned} \tag{18}$$

Therefore, the necessary and sufficient **F**-compatibility equations read

$$d\mathbf{F} = \mathbf{0}, \qquad \int_{\gamma_1} \mathbf{F} d\mathbf{X} = \int_{\gamma_2} \mathbf{F} d\mathbf{X} = \mathbf{0}. \tag{19}$$

One observes that γ_3 is a generator of the fundamental group but does not have a corresponding complementary compatibility equation. The boundary of the hole in a punctured torus is null-homologous path but not null-homotopic.

5.2.2 2D elasticity on arbitrary compact orientable 2-manifolds

When $\mathcal{B}$ is an arbitrary compact orientable 2-manifold, it is homeomorphic to a sphere with n handles. Each handle corresponds to two generators of the first homology group, and hence, there are $3 \times 2n$ complementary compatibility equations. If the body manifold has boundaries they correspond to k holes, which introduce another $k-1$ generators of the first homology group (see Fig. 8). The total number of complementary compatibility equations are $3(2n+k-1)$.

5.2.3 3D elastic bodies with holes

A 3D solid with internal cavities has a trivial $H_1(\mathcal{B})$. As an example, a solid with a spherical hole (see Fig. 1a) has a trivial first homology group, and hence, $d\mathbf{F} = \mathbf{0}$ is both necessary and sufficient for compatibility of **F**. The First homology group of a solid torus has only one generator. The body shown in Fig. 1c is homeomorphic to a solid torus and the (red) closed curve generates its first homology group. The body shown in Fig. 1d has Betti number two and the two (red) loops generate its first homology group. The complement of a solid torus has Betti number one (see Fig. 1b). The First homology group of a 2-holed solid torus has the two generators γ_1 and γ_2 shown in Fig. 5. Its complement has Betti number two and the generators Γ_1 and Γ_2 are shown in Fig. 5. A thick torus with two tubular holes has Betti number three. The generators of the first homology group are shown in Fig. 14.

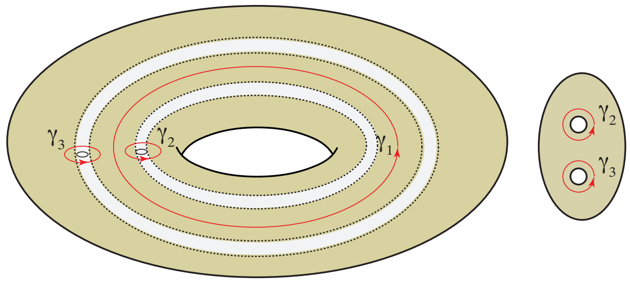

Fig. 14 A solid torus with two tubular holes.

A solid trefoil knot has Betti number one and a generator of its first homology group is $\boldsymbol{\gamma}$ shown in Fig. 10. Its complement in $\mathbb{R}^3$ has Betti number one as well and $\boldsymbol{\Gamma}$ in Fig. 10 is its generator. A cylinder and an annulus are homeomorohic. The first homology group is generated by the loop c_3 in Fig. 4. A solid with tubular holes shown in Fig. 7 has a fundamental group freely generated by the four loops $c_i, i = 1,2,3,4$. Each c_i corresponds to three extra compatibility equations for $\mathbf{F}$ (six extra compatibility equations for $\mathbf{C}^\flat$). A thick hollow cylinder is the special case of this example when there is only one hole. The Betti number of both the Möbius band $\mathcal{M}$ and the thick Möbius band $\mathcal{M} \times [0,1]$ are one. The knotted ball shown in Fig. 15(left) has Betti number one. Note that its fundamental group has four generators but only one requires complementary compatibility equations. The ball shown in Fig. 15(right) has a hole, which is a genus four handlebody. Its Betti number is four.

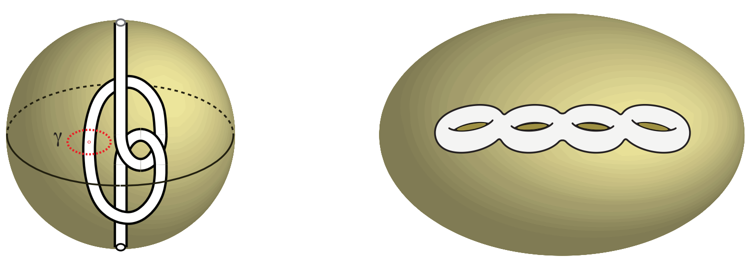

Fig. 15 Left: A knotted ball. $\boldsymbol{\gamma}$ is a generator of the first homology group. Right: A ball with a toridal hole of genus four. This body has Betti number four.

5.3 **F**-compatibility equations in the presence of Dirichlet boundary conditions

In [5], the compatibility equations in the presence of Dirichlet boundary conditions were derived using some Hodge-type orthogonal decompositions. Here, we follow [22] and find the **F**-compatibility equations when deformation mapping (or displacement field) is prescribed on part of the boundary $\partial_D\mathcal{B} \subset \partial\mathcal{B}$.

Consider a k-form $\boldsymbol{\omega}$ ($k \geq 1$) on $\mathcal{B}$. Using the inclusion map $\iota : \partial\mathcal{B} \hookrightarrow \mathcal{B}$, the tangential component of $\boldsymbol{\omega}$ is defined as $\mathsf{t}\boldsymbol{\omega} = \iota^*\boldsymbol{\omega}$ [44]. This can equivalently be defined using the decomposition of vector fields on $\partial\mathcal{B}$ into tangential and normal parts. Given $\mathbf{X} \in \Gamma(T\mathcal{B}|_{\partial\mathcal{B}})$, $\mathbf{X} = \mathbf{X}^{\parallel} + \mathbf{X}^{\perp}$, and the tangential part of the k-form $\boldsymbol{\omega}$ is defined as

$$\mathsf{t}\boldsymbol{\omega}(\mathbf{X}_1,\ldots,\mathbf{X}_k) = \boldsymbol{\omega}(\mathbf{X}_1^{\parallel},\ldots,\mathbf{X}_k^{\parallel}), \qquad \forall \mathbf{X}_1,\ldots,\mathbf{X}_k \in \Gamma(T\mathcal{B}|_{\partial\mathcal{B}}). \tag{20}$$

The normal part is defined as $\mathsf{n}\boldsymbol{\omega} = \boldsymbol{\omega} - \mathsf{t}\boldsymbol{\omega}$. For $k = 0$, $\mathsf{t}\boldsymbol{\omega} = \boldsymbol{\omega}$. The deformation mapping can be thought of an $\mathbb{R}^3$-valued 0-form. The Dirichlet boundary conditions are given as $\varphi^a|_{\partial_D\mathcal{B}} = \hat{\varphi}^a$, where φ^a, $a = 1,2,3$, are 0-forms defined on $\mathcal{B}$, and $\hat{\varphi}^a$, $a = 1,2,3$, are 0-forms defined on $\partial_D\mathcal{B}$.

The following result is a simple corollary of [22, Theorem 6].

Proposition 5.5 Suppose **F** is an $\mathbb{R}^3$-valued 1-form in $\mathcal{B}$. Also assume that $\mathsf{t}\mathbf{F} = d\hat{\varphi}$ on $\partial_D\mathcal{B}$.[15] The necessary and sufficient conditions for compatibility of **F**, i.e., the existence of an $\mathbb{R}^3$-valued 0-form φ such that $\mathbf{F} = d\varphi$, and $\varphi|_{\partial_D\mathcal{B}} = \hat{\varphi}$ are:

$$d\mathbf{F} = \mathbf{0}, \quad \text{and} \quad \int_{c_i} \mathbf{F}d\mathbf{X} = \int_{\partial c_i} \hat{\varphi} = \hat{\varphi}(X_2^i) - \hat{\varphi}(X_1^i), \quad i = 1,\ldots,b_1(\mathcal{B},\partial_D\mathcal{B}), \tag{21}$$

where c_i's are the generators of the first relative singular homology group $H_1(\mathcal{B},\partial_D\mathcal{B};\mathbb{R})$. Note that each $\partial c_i = [X_1^i, X_2^i]$ is an oriented pair of points (X_1^i, X_2^i) such that X_1^i and X_2^i lie on $\partial_D\mathcal{B}$.

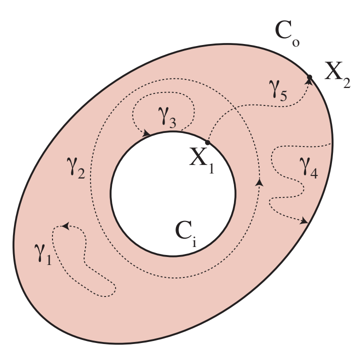

Fig. 16 The boundary of $\mathcal{B}$ is the union of the inner circle C_i and the outer ellipse C_o.

[15] $\mathsf{t}\mathbf{F} = d\hat{\varphi}$ means that for any vector $\mathbf{W} \in T\mathcal{B}|_{\partial_D\mathcal{B}}$, $\mathbf{F}\mathbf{W}^{\parallel} = \langle d\hat{\varphi}, \mathbf{W}^{\parallel}\rangle$.

Example 5.6 Let us consider the body shown in Fig. 16. We consider the following four cases of boundary conditions.

- $\partial_D\mathcal{B} = \varnothing$: In this case the auxiliary compatibility equation reads
$$\int_{\gamma_2} \mathbf{F}d\mathbf{X} = \mathbf{0}, \tag{22}$$
where γ_2 is the generator of the first de Rham cohomology group (see Fig. 16).
- $\partial_D\mathcal{B} = C_i$: In this case there are no auxiliary compatibility equations. Note that γ_2 and γ_3 are relative boundaries.
- $\partial_D\mathcal{B} = C_o$: In this case there are no auxiliary compatibility equations. Note that γ_2 and γ_4 are relative boundaries.
- $\partial_D\mathcal{B} = \partial\mathcal{B}$: In this case a generator of $H_1(\mathcal{B}, \partial_D\mathcal{B}; \mathbb{R})$ is γ_5, and the auxiliary compatibility equation reads
$$\int_{\gamma_5} \mathbf{F}d\mathbf{X} = \hat{\varphi}(X_2) - \hat{\varphi}(X_1). \tag{23}$$
Note that γ_2, γ_3, and γ_4 are relative boundaries.

Our calculations in this example are consistent with [5, Example 10] in which the Dirichlet boundary was assumed fixed.

5.4 Compatibility equations for the right Cauchy-Green strain $\mathbf{C}^\flat$

Consider a motion of a body $\varphi_t : \mathcal{B} \to \mathcal{S}$ and assume that $\dim\mathcal{B} = \dim\mathcal{S}$. The right Cauchy-Green deformation tensor is defined as $\mathbf{C}^\flat = \varphi_t^*\mathbf{g}$. For a Euclidean ambient space $\boldsymbol{\mathcal{R}}(\mathbf{g}) = \mathbf{0}$. Thus

$$\mathbf{0} = \varphi_t^* \boldsymbol{\mathcal{R}}(\mathbf{g}) = \boldsymbol{\mathcal{R}}(\varphi_t^*\mathbf{g}) = \boldsymbol{\mathcal{R}}(\mathbf{C}^\flat), \tag{24}$$

i.e., a necessary condition for $\mathbf{C}^\flat$ to be compatible is vanishing of its Riemann curvature, or equivalently local flatness of the Riemannian manifold $(\mathcal{B}, \mathbf{C}^\flat)$. Note that this is independent of the geometry of $(\mathcal{B}, \mathbf{G})$. In other words, even for a non-flat material manifold $\boldsymbol{\mathcal{R}}(\mathbf{C}^\flat) = \mathbf{0}$ is a necessary compatibility equation for $\mathbf{C}^\flat$. Marsden and Hughes [36] showed that this condition is locally sufficient as well. In the case of simply-connected elastic bodies this condition guarantees the existence of a global deformation mapping [16].

Suppose $\{X^A\}, \{x^a\}$ are coordinate charts for $\mathcal{B}$, and $\mathcal{S}$, respectively. The Levi-Civita connection coefficients of $\mathbf{g}$ and $\mathbf{C}^\flat = \varphi^*\mathbf{g}$ are denoted by $\gamma^a{}_{bc}$ and $\Gamma^A{}_{BC}$, respectively. They are related as

$$\Gamma^A{}_{BC} = \frac{\partial X^A}{\partial x^a}\frac{\partial x^b}{\partial X^B}\frac{\partial x^c}{\partial X^C}\gamma^a{}_{bc} + \frac{\partial^2 x^m}{\partial X^B \partial X^C}\frac{\partial X^A}{\partial x^m}. \tag{25}$$

Assuming that $\{x^a\}$ is a Cartesian coordinate chart for the Euclidean ambient space, $\gamma^a{}_{bc} = 0$, and hence

$$\Gamma^A{}_{BC} = \frac{\partial^2 x^m}{\partial X^B \partial X^C}\frac{\partial X^A}{\partial x^m}. \tag{26}$$

Therefore

$$\frac{\partial^2 x^a}{\partial X^B \partial X^C} = \frac{\partial}{\partial X^C}F^a{}_B = F^a{}_A \Gamma^A{}_{BC}. \tag{27}$$

Using the polar decomposition in (27) one obtains[16]

$$R^a{}_{A,B} = R^a{}_C \Omega^C{}_{AB}, \tag{28}$$

where

$$\Omega^C{}_{AB} = \left(\Gamma^M{}_{BN}U^C{}_M - U^C{}_{N,B}\right)U_A{}^N, \quad \Gamma^C{}_{AB} = \frac{1}{2}C^{CD}\left(C_{BD,A} + C_{AD,B} - C_{AB,D}\right), \tag{29}$$

and $U_A{}^N$ are components of $\mathbf{U}^{-1}$. Note that the material manifold is assumed to be embedded in the Euclidean ambient space. Choosing Cartesian coordinates for $\mathcal{B}$, $G_{AB} = \delta_{AB}$. For a path γ that connects X_0, $X \in \mathcal{B}$ and is parametrized by $s \in I$, one obtains the following system of linear ODEs governing the rotation tensor

$$\frac{d}{ds}\mathbf{R} = \mathbf{R}\mathbf{K}, \tag{30}$$

where

$$K^C{}_A(s) = \Omega^C{}_{AB}(s)\dot{X}^B(s). \tag{31}$$

Note that $K_{BA} = -K_{AB}$. Therefore, (30) is a linear ODE for $\mathbf{R} \in SO(3)$, and $\mathbf{K} \in \mathfrak{so}(3)$, the Lie algebra of the Lie group $SO(3)$. For each a

$$\frac{dR^a{}_A}{ds} - \Omega^C{}_{AB}R^a{}_C\dot{X}^B(s) = 0. \tag{32}$$

This is the equation of parallel transport of $R^a{}_A$ along the curve γ when $\mathcal{B}$ is equipped with the connection Ω. Let us assume that $\mathbf{R}(0) = \mathbf{R}_0$. We see that rotation tensor at s is the parallel transport of $\mathbf{R}_0$. It can be shown that in a simply-connected body the integrality conditions of (32) are equivalent to vanishing of curvature tensor of $\mathbf{C}^\flat$ [42]. For solving (30) in [62] we used product integration and wrote the solution as

$$\mathbf{R}(s) = \mathbf{R}_0 \prod_0^s (\gamma)\, e^{\mathbf{K}(\xi)d\xi}, \tag{33}$$

[16] Note that Eq. (28) is identical to Shield [45]'s Eq. (8).

where $\mathbf{R}_0 = \mathbf{R}(s)$ is assumed to be given and $\prod_0^s(\gamma)e^{\mathbf{K}(\xi)d\xi}$ is the product integral of $\mathbf{K}$ along the path γ from 0 to s. For more details on product integration see [62], and [21, 50].

For a compatible $\mathbf{C}^\flat$, the rotation tensor calculated from (33) must be independent of the path γ. Therefore, for any closed path γ in $\mathcal{B}$

$$\prod_\gamma e^{\mathbf{K}(s)ds} = \mathbf{I}. \tag{34}$$

It was shown in [62] that a necessary and sufficient condition is

$$\int_0^1 \mathbf{K}(s)ds = \mathbf{0}, \tag{35}$$

where $\gamma : [0,1] \to \mathcal{B}$ is any closed path.

$\mathbf{C}^\flat$-compatibility is formulated as follows. Given $\mathbf{C}$, $\mathbf{U} = \sqrt{\mathbf{C}}$ is determined uniquely. The system of ODEs (28) govern the rotation $\mathbf{R}$. The calculated rotation is path independent if and only if the curvature tensor of $\mathbf{C}^\flat$ vanishes, and (35) are satisfied over each generator of the first homology group.

Proposition 5.7 (Yavari [62]) The necessary and sufficient $\mathbf{C}^\flat$-compatibility conditions in a non-simply-connected body $\mathcal{B}$ are:

i) $\mathcal{R}(\mathbf{C}^\flat) = \mathbf{0}$ in $\mathcal{B}$,
ii) $\int_{c_i} \mathbf{K}(s)ds = \mathbf{0}$, $i = 1, \dots, b_1(\mathcal{B})$, where c_i's are generators of $H_1(\mathcal{B};\mathbb{R})$,
iii) The above two conditions guarantee that deformation gradient $\mathbf{F} = \mathbf{R}\sqrt{\mathbf{C}}$ is uniquely determined. For the deformation gradient to be compatible, one must have, $\int_{\gamma_i} \mathbf{F}d\mathbf{X} = \mathbf{0}$, $i = 1, \dots, b_1(\mathcal{B})$.

5.5 Compatibility equations in linearized elasticity

Suppose φ_ϵ is a 1-parameter family of deformations around a reference motion $\mathring{\varphi}$, and let $\epsilon = 0$ correspond to the reference motion. The displacement field is defined as [57, 64]:

$$\mathbf{U}(X) = \delta\varphi(X) = \frac{d\varphi_\epsilon(X)}{d\epsilon}\Big|_{\epsilon=0}. \tag{36}$$

The linearization of the deformation gradient is written as [36, 64]: $\mathscr{L}(\mathbf{F}) = \nabla\mathbf{U}$. In components, $\mathscr{L}(\mathbf{F})^a{}_A = U^a{}_{|A} = \frac{\partial U^a}{\partial X^A} + \gamma^a{}_{bc}F^b{}_A U^c$, where $\gamma^a{}_{bc}$ are the connection coefficients of the Riemannian manifold $(\mathcal{S}, \mathbf{g})$. The spatial and material strain tensors are defined, respectively, as $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \varphi_{t*}\mathbf{G})$, and $\mathbf{E} = \frac{1}{2}(\varphi_t^*\mathbf{g} - \mathbf{G})$ [36]. In components, $e_{ab} = \frac{1}{2}\left(g_{ab} - G_{AB}F_a{}^A F_b{}^B\right)$, and $E_{AB} = \frac{1}{2}(C_{AB} - G_{AB})$. It can be shown that $\mathscr{L}(\mathbf{C})_{AB} = 2F^a{}_A F^b{}_B\ \epsilon_{ab}$, where $\epsilon_{ab} = \frac{1}{2}(u_{a|b} + u_{b|a})$ is the linearized strain, and $\mathbf{u} = \mathbf{U} \circ \varphi^{-1}$. Thus, $\mathscr{L}(\mathbf{C}) = 2\varphi_t^*\epsilon$, and hence, $\epsilon = \varphi_{t*}\mathscr{L}(\mathbf{E})$.

When the ambient space is Euclidean and one uses Cartesian coordinates the covariant derivatives reduce to partial derivatives and the classical definition of linear strain in terms of partial derivatives is recovered, i.e.,

$$\varepsilon_{ab} = \frac{1}{2}\left(\frac{\partial u_a}{\partial x^b} + \frac{\partial u_b}{\partial x^a}\right). \tag{37}$$

The necessary and sufficient conditions for compatibility in terms of $\mathbf{F}$ are $\int_\gamma \mathbf{F} d\mathbf{X} = \mathbf{0}$, for every loop γ in $\mathcal{B}$. The linearization of this condition reads $\int_\gamma \nabla \mathbf{U} d\mathbf{X} = \mathbf{0}$. In components, $\int_\gamma u^a{}_{,B} dX^B = 0$, where $\{X^A\}$ and $\{x^a\}$ are coordinate charts for $\mathcal{B}$ and $\mathcal{S}$, respectively. Linearization is assumed about the standard embedding of $\mathcal{B}$ in $\mathbb{R}^N$, i.e., $F^a{}_A = \delta^a_A$. This implies that $dX^B = \frac{\partial X^B}{\partial x^b} dx^b = \delta^B_b dx^b$, and thus

$$\int_\gamma u_{a,B} dX^B = \int_\gamma u_{a,b} dx^b = \int_\gamma \left(e_{ab} + \omega_{ab}\right) dx^b = 0, \tag{38}$$

where $e_{ab} = u_{(a,b)} = \frac{1}{2}(u_{a,b} + u_{b,a})$, and $\omega_{ab} = u_{[a,b]} = \frac{1}{2}(u_{a,b} - u_{b,a})$, are the linearized strain and rotation tensors, respectively. Note that

$$\int_\gamma \omega_{ab} dx^b = \int_\gamma \left[(x^c \omega_{ac})_{,b} - x^c \omega_{ac,b}\right] dx^b = -\int_\gamma x^c \omega_{ac,b} dx^b. \tag{39}$$

The gradient of the rotation tensor is rewritten as

$$\begin{aligned} \omega_{ac,b} &= \frac{1}{2}\left(u_{a,cb} - u_{c,ab}\right) + \frac{1}{2}\left(u_{b,ac} - u_{b,ac}\right) \\ &= \frac{1}{2}\left(u_{a,bc} + u_{b,ac}\right) - \frac{1}{2}\left(u_{c,abc} + u_{b,ac}\right) \\ &= e_{ab,c} - e_{bc,a}. \end{aligned} \tag{40}$$

For a given e_{ab}, ω_{ab} is calculated by integrating $\omega_{ab,c} = e_{ac,b} - e_{cb,a}$ along an arbitrary curve. To ensure that the rotation field is well-defined one must have $\int_\gamma \left(e_{ac,b} - e_{cb,a}\right) dx^c = 0$, for any closed path $\gamma \in \mathcal{B}$. When γ is null-homotopic, $\gamma = \partial\Omega$, on a 2-submanifold of $\mathcal{B}$, and hence

$$\begin{aligned} \int_\gamma \left(e_{ac,b} - e_{cb,a}\right) dx^c &= \int_\Omega d\left(e_{ac,b} - e_{cb,a}\right) \wedge dx^c \\ &= \int_\Omega \left(e_{ad,bc} + e_{bc,ad} - e_{ac,bd} - e_{bd,ac}\right)(dx^c \wedge dx^d) = 0, \end{aligned} \tag{41}$$

where $\{(dx^c \wedge dx^d)\} = \{dx^c \wedge dx^d\}_{c<d}$ is a basis of 2-forms. Note that (41) is equivalent to $\mathsf{curl} \circ \mathsf{curl}\, \mathbf{e} = \mathbf{0}$, which is the classical bulk compatibility equation of linear elasticity [34]. From (40), one writes

$$\int_\gamma u_{a,b} dx^b = \int_\gamma \mathsf{C}_{ab} dx^b = 0, \tag{42}$$

where $\mathsf{C}_{ab} = e_{ab} - x^c(e_{ab,c} - e_{bc,a})$ is called the Cesàro tensor. The above representation is called the Cesàro integral [14]. For a null-homotopic path γ that lies on a surface $\mathcal{P} \subset \mathcal{B}$, $\gamma = \partial\Omega$ for some $\Omega \subset \mathcal{P}$. Hence, using Stokes' theorem one writes

$$\begin{aligned}\int_\gamma \mathsf{C}_{ab} dx^b &= \int_\Omega d\mathsf{C}_{ab} \wedge dx^b = \int_\Omega \mathsf{C}_{ab,c} dx^c \wedge dx^b \\ &= \int_\Omega \left[e_{bc,a} - x^d \left(e_{ab,cd} - e_{bd,ac} \right) \right] dx^c \wedge dx^b.\end{aligned} \tag{43}$$

Due to symmetry of strain $e_{bc,a} dx^c \wedge dx^b = 0$, and hence

$$\begin{aligned}\int_\gamma u_{a,b} dx^b &= \int_\Omega x^d \left(e_{bd,ac} - e_{ab,cd} \right) dx^c \wedge dx^b \\ &= \int_\Omega x^d \left(e_{ab,cd} + e_{cd,ab} - e_{ac,bd} - e_{bd,ac} \right) dx^b \wedge dx^c = 0.\end{aligned} \tag{44}$$

One should note that (44) are equivalent to $\mathsf{curl} \circ \mathsf{curl}\, \mathbf{e} = \mathbf{0}$, i.e, the classical bulk compatibility equations [34].

Proposition 5.8 (Yavari [62]) The necessary and sufficient conditions for compatibility conditions for the linearized strain $\mathbf{e} = \mathfrak{L}_{\mathbf{u}}\mathbf{g}$ in $\mathcal{B}$ are:

i) $\mathsf{curl} \circ \mathsf{curl}\, \mathbf{e} = \mathbf{0}$ in $\mathcal{B}$,
ii) For each generator of $H_1(\mathcal{B};\mathbb{R})$

$$\int_{c_i} \mathsf{C} d\mathbf{X} = \mathbf{0}, \quad \& \quad \int_{c_i} \left(e_{ac,b} - e_{cb,a} \right) dx^c = 0, \qquad i = 1, \ldots, b_1(\mathcal{B}). \tag{45}$$

We call $(45)_1$ and $(45)_2$ the Cesàro and the rotation compatibility equations, respectively. Note that in dimension n ($n = 2$ or 3) for each c_i, there are n Cesàro and $n(n-1)/2$ rotation compatibility equations. Hence, each c_i has $n(n+1)$ complementary compatibility equations. In dimension three, there are six bulk compatibility equations, and six auxiliary compatibility equations for each generator of the first homology group. We should mention that this is consistent with Weingarten's theorem [56] that says that if a body is cut along a surface the jump in the displacement field is a rigid-body motion, see Love [34] for a detailed discussion (he calls homotopic paths, "reconcilable circuits" and a null-homotopic path, a "evanescible circuit"). Zubov [65] and Casey [13] demonstrated the validity of Weingarten's theorem for finite strains (see also Acharya [2]). In [62] it was pointed out that the discussion in [49] regarding sufficient compatibility equations in linear elasticity is flawed as Skalak, et al. missed the rotation compatibility conditions $(45)_2$. In [62, Example 29] the rotation compatibility conditions were trivially satisfied. Next, we provide an example of an incompatible strain field for which the Cesàro compatibility conditions are satisfied while the rotation compatibility conditions are not satisfied.

Example 5.9 Consider a single wedge disclination [19, 60] along the z-axis in an infinite linear elastic body. The linearized strain field of the disclination in the Cartesian coordinates (x,y,z) reads [19]

$$
\begin{aligned}
e_{11} &= \frac{\Omega}{4\pi(1-\nu)}\left[(1-2\nu)\ln\sqrt{x^2+y^2}+\frac{y^2}{x^2+y^2}\right],\\
e_{22} &= \frac{\Omega}{4\pi(1-\nu)}\left[(1-2\nu)\ln\sqrt{x^2+y^2}+\frac{x^2}{x^2+y^2}\right],\\
e_{12} &= -\frac{\Omega}{4\pi(1-\nu)}\frac{xy}{x^2+y^2},\quad e_{33}=e_{13}=e_{23}=0,
\end{aligned}
\tag{46}
$$

where Ω is the Frank vector of the disclination. For this strain field the bulk compatibility equation $e_{11,yy}+e_{22,xx}-2e_{12,xy}=0$ is satisfied in $\mathbb{R}^3\setminus$ z–axis. Eq. $(45)_1$ gives the following two Cesàro compatibility equations

$$
\begin{aligned}
&\int_\gamma \left[e_{11}-y(e_{11,y}-e_{12,x})\right]dx+\left[e_{12}-y(e_{12,y}-e_{22,x})\right]dy=0,\\
&\int_\gamma \left[e_{12}-x(e_{12,x}-e_{11,y})\right]dx+\left[e_{22}-x(e_{22,x}-e_{12,y})\right]dy=0,
\end{aligned}
\tag{47}
$$

where γ is any closed curve lying in a plane normal to the z-axis and enclosing the origin. Using a square path with corners $(a,-a,0)$, $(a,a,0)$, $(-a,a,0)$, and $(-a,-a,0)$, where $a>0$, it is straightforward to show that the two Cesàro compatibility equations are trivially satisfied. For this strain field there is only one rotation compatibly equation, which is not satisfied, namely

$$
\int_\gamma (e_{11,y}-e_{12,x})dx+(e_{12,y}-e_{22,x})dy=-\Omega\neq 0. \tag{48}
$$

6 Differential Complexes in Nonlinear Elasticity

For a flat 3-manifold $\mathcal{B}$, let $\Omega^k(\mathcal{B})$ be the space of smooth k-forms on $\mathcal{B}$, i.e., $\alpha\in\Omega^k(\mathcal{B})$ is an anti-symmetric $\binom{0}{k}$-tensor with smooth components $\alpha_{i_1\cdots i_k}$. The exterior derivatives $d_k:\Omega^k(\mathcal{B})\to\Omega^{k+1}(\mathcal{B})$ are linear differential operators that satisfy $d_{k+1}\circ d_k=0$. In order to simplify the notation we drop the subscript k in d_k. The following sequence of spaces and operators

$$
0\longrightarrow \Omega^0(\mathcal{B})\xrightarrow{d}\Omega^1(\mathcal{B})\xrightarrow{d}\Omega^2(\mathcal{B})\xrightarrow{d}\Omega^3(\mathcal{B})\longrightarrow 0, \tag{1}
$$

is denoted by $(\Omega(\mathcal{B}),d)$ and is called the de Rham complex. Note that each operator is linear and the composition of any two successive operators vanishes. Also the first operator on the left sends 0 to the zero function and the last operator on the right sends $\Omega^3(\mathcal{B})$ to zero. The property $d\circ d=0$, implies that $\operatorname{im} d_k\subset\ker d_{k+1}$, where $\operatorname{im} d_k$ is the image of d_k and $\ker d_{k+1}$ is the kernel of d_{k+1}. If $\operatorname{im} d_k=\ker d_{k+1}$, the complex is exact. The de Rham cohomology groups are defined as $\mathrm{H}^k_{dR}(\mathcal{B}) = \ker d_k / \mathrm{im}\, d_{k-1}$. A complex is exact if and only if all $\mathrm{H}^k_{dR}(\mathcal{B})$ are the trivial group $\{0\}$. Cohomology groups quantify non-exactness of a complex.

For $\beta \in \Omega^k(\mathcal{B})$, the necessary and sufficient condition for the existence of a solution for the PDE $d\alpha = \beta$ is $\beta \in \mathrm{im}\, d$. If $(\Omega(\mathcal{B}), d)$ is exact, $\beta \in \mathrm{im}\, d$ if and only if $d\beta = 0$. Assuming that $\mathrm{H}^k_{dR}(\mathcal{B})$ is finite dimensional, de Rham's theorem states that $\dim \mathrm{H}^k_{dR}(\mathcal{B}) = b_k(\mathcal{B})$, where $b_k(\mathcal{B})$ is the k-th Betti number — a purely topological property of $\mathcal{B}$. Thus, $\beta \in \mathrm{im}\, d$ if and only if

$$d\beta = 0, \text{ and } \int_{c_k} \beta = 0, \quad k = 1, \ldots, b_k(\mathcal{B}), \tag{2}$$

where c_k are the generators of the kth homology group of $\mathcal{B}$.

Sometimes one may be able to establish a connection between a given complex and the de Rham complex. A complex closely related to the de Rham complex is the grad-curl-div complex of vector analysis. Let $C^\infty(\mathcal{B})$ and $\mathfrak{X}(\mathcal{B})$ be the spaces of smooth real-valued functions and smooth vector fields on $\mathcal{B}$, an open subset of $\mathbb{R}^3$. Consider the three operators $\mathsf{grad} : C^\infty(\mathcal{B}) \to \mathfrak{X}(\mathcal{B})$, $\mathsf{curl} : \mathfrak{X}(\mathcal{B}) \to \mathfrak{X}(\mathcal{B})$, and $\mathsf{div} : \mathfrak{X}(\mathcal{B}) \to C^\infty(\mathcal{B})$. The classical identities $\mathsf{curl} \circ \mathsf{grad} = 0$, and $\mathsf{div} \circ \mathsf{curl} = 0$, allow one to write the following complex

$$0 \longrightarrow C^\infty(\mathcal{B}) \xrightarrow{\mathrm{grad}} \mathfrak{X}(\mathcal{B}) \xrightarrow{\mathrm{curl}} \mathfrak{X}(\mathcal{B}) \xrightarrow{\mathrm{div}} C^\infty(\mathcal{B}) \longrightarrow 0, \tag{3}$$

which is called the grad-curl-div complex or simply the gcd complex. One can show that the gcd complex is equivalent to the de Rham complex, or more precisely is isomorphic to the de Rham complex [4]. As an example of the application of this isomorphism, one can show that a vector field $\mathbf{w}$ is the gradient of a function if and only if

$$\mathsf{curl}\, \mathbf{w} = \mathbf{0}, \text{ and } \quad \int_\gamma \mathbf{w} \cdot \mathbf{t}_\gamma \, ds = 0, \quad \forall \gamma \subset \mathcal{B}, \tag{4}$$

where γ is an arbitrary closed curve in $\mathcal{B}$, $\mathbf{t}_\gamma$ is the unit tangent vector field along γ, and $\mathbf{w} \cdot \mathbf{t}_\gamma$ is the standard inner product of $\mathbf{w}$ and $\mathbf{t}_\gamma$ in $\mathbb{R}^3$.

It turns out that when using deformation gradient $\mathbf{F}$ and its corresponding stress, i.e., the first Piola-Kirchhoff stress $\mathbf{P}$, the differential complex of nonlinear elasticity is isomorphic to the $\mathbb{R}^3$-valued de Rham complex [4]. Let us assume that the ambient space is Euclidean, i.e., $\mathcal{S} = \mathbb{R}^3$ with Cartesian coordinates $\{x^i\}$. Suppose $\varphi : \mathcal{B} \to \mathcal{S}$ is a smooth map and define the following operators for two-point tensors in $\Gamma(T\varphi(\mathcal{B}))$ and $\Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$:

$$\begin{aligned}
\mathbf{Grad} &: \Gamma(T\varphi(\mathcal{B})) \to \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}), & (\mathbf{Grad}\,\mathbf{U})^{iI} &= U^i{}_{,I}, \\
\mathbf{Curl} &: \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \to \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}), & (\mathbf{Curl}\mathbf{F})^{iI} &= \varepsilon_{IKL} F^{iL}{}_{,K}, \\
\mathbf{Div} &: \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \to \Gamma(T\varphi(\mathcal{B})), & (\mathbf{Div}\,\mathbf{F})^i &= F^{iI}{}_{,I}.
\end{aligned}$$

Note that $\mathbf{Curl} \circ \mathbf{Grad} = \mathbf{0}$, and $\mathbf{Div} \circ \mathbf{Curl} = \mathbf{0}$. Therefore, the **GCD** complex or the nonlinear elasticity complex is written as:

$$\mathbf{0} \longrightarrow \Gamma(T\varphi(\mathcal{B})) \xrightarrow{\mathbf{Grad}} \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \xrightarrow{\mathbf{Curl}} \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \xrightarrow{\mathbf{Div}} \Gamma(T\varphi(\mathcal{B})) \longrightarrow \mathbf{0}.$$

Any $\mathbb{R}^3$-valued k-form $\alpha \in \Omega^k(\mathcal{B};\mathbb{R}^3)$ can be written as $\alpha = (\alpha^1, \alpha^2, \alpha^3)$, where $\alpha^i \in \Omega^k(\mathcal{B})$, $i = 1,2,3$. The exterior derivative $\mathbf{d} : \Omega^k(\mathcal{B};\mathbb{R}^3) \to \Omega^{k+1}(\mathcal{B};\mathbb{R}^3)$ is defined as $\mathbf{d}\alpha = (d\alpha^1, d\alpha^2, d\alpha^3)$. From $d \circ d = 0$, one concludes that $\mathbf{d} \circ \mathbf{d} = \mathbf{0}$, which gives the $\mathbb{R}^3$-valued de Rham complex $\big(\Omega(\mathcal{B};\mathbb{R}^3), \mathbf{d}\big)$.

Let us define the following isomorphisms

$$\begin{aligned}
&I_0 : \Gamma(T\varphi(\mathcal{B})) \to \Omega^0(\mathcal{B};\mathbb{R}^3), && [I_0(\mathbf{U})]^i = U^i,\\
&I_1 : \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \to \Omega^1(\mathcal{B};\mathbb{R}^3), && [I_1(\mathbf{F})]^i{}_J = F^{iJ},\\
&I_2 : \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \to \Omega^2(\mathcal{B};\mathbb{R}^3), && [I_2(\mathbf{F})]^i{}_{JK} = \varepsilon_{JKL} F^{iL},\\
&I_3 : \Gamma(T\varphi(\mathcal{B})) \to \Omega^3(\mathcal{B};\mathbb{R}^3), && [I_3(\mathbf{U})]^i{}_{123} = U^i,
\end{aligned}$$

where ε_{JKL} is the permutation symbol. The following diagram commutes [4].

$$\begin{array}{ccccccccc}
\mathbf{0} \longrightarrow & \Gamma(T\varphi(\mathcal{B})) & \xrightarrow{\mathbf{Grad}} & \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) & \xrightarrow{\mathbf{Curl}} & \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) & \xrightarrow{\mathbf{Div}} & \Gamma(T\varphi(\mathcal{B})) & \longrightarrow \mathbf{0}\\
& \big\downarrow I_0 & & \big\downarrow I_1 & & \big\downarrow I_2 & & \big\downarrow I_3 & \\
\mathbf{0} \longrightarrow & \Omega^0(\mathcal{B};\mathbb{R}^3) & \xrightarrow{\mathbf{d}} & \Omega^1(\mathcal{B};\mathbb{R}^3) & \xrightarrow{\mathbf{d}} & \Omega^2(\mathcal{B};\mathbb{R}^3) & \xrightarrow{\mathbf{d}} & \Omega^3(\mathcal{B};\mathbb{R}^3) & \longrightarrow \mathbf{0}
\end{array}$$

The above isomorphisms induce a cohomology isomorphism $\mathrm{H}^k_{\mathbf{GCD}}(\mathcal{B}) \approx \oplus_{i=1}^3 \mathrm{H}^k_{dR}(\mathcal{B})$, where $\mathrm{H}^k_{\mathbf{GCD}}(\mathcal{B})$ is the k-th cohomology group of the **GCD** complex. Let $\langle \mathbf{F}, \mathbf{W} \rangle := \sum_{i,I=1}^3 F^{iI} W^I \mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is the standard basis of $\mathbb{R}^3$. The following result can be proved using the fact that the nonlinear elasticity and the $\mathbb{R}^3$-valued de Rham complexes are isomorphic.

Theorem 6.1 (Angoshtari and Yavari [4]) Given $\mathbf{F} \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$, there exists $\mathbf{U} \in \Gamma(T\varphi(\mathcal{B}))$ such that $\mathbf{F} = \mathbf{Grad}\,\mathbf{U}$, if and only if

$$\mathbf{Curl}\ \mathbf{F} = \mathbf{0}, \quad \text{and} \qquad \int_\gamma \langle \mathbf{F}, \mathbf{t}_\gamma \rangle dS = \mathbf{0}, \quad \forall \gamma \subset \mathcal{B}, \tag{5}$$

where γ is any closed curve in $\mathcal{B}$, and $\mathbf{t}_\gamma$ is the unit tangent vector field along γ. Moreover, there exists $\Psi \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$ such that $\mathbf{F} = \mathbf{Curl}\ \Psi$, if and only if

$$\mathbf{Div}\,\mathbf{F} = \mathbf{0}, \quad \text{and} \qquad \int_{\mathcal{C}} \langle \mathbf{F}, \mathbf{N}_{\mathcal{C}} \rangle dA = \mathbf{0}, \quad \forall \mathcal{C} \subset \mathcal{B}, \tag{6}$$

where $\mathcal{C}$ is any closed surface in $\mathcal{B}$ and $N_{\mathcal{C}}$ is the unit outward normal vector field of $\mathcal{C}$.

Using the first Piola-Kirchhoff stress $\mathbf{P}$, one defines a complex that describes both the kinematics and the kinetics of motion. The displacement field

$\mathbf{U} \in \Gamma(T\varphi(\mathcal{B}))$ is defined as $\mathbf{U}(X) = \varphi(X) - X \in T_{\varphi(X)}\mathcal{S}$, $\forall X \in \mathcal{B}$. Then, $\mathbf{Grad}\,\mathbf{U}$ is the displacement gradient, and $\mathbf{Curl} \circ \mathbf{Grad}\,\mathbf{U} = \mathbf{0}$ expresses the compatibility of the displacement gradient. On the other hand, $\mathbf{P} = \mathbf{Curl}\ \boldsymbol{\Psi}$, where $\boldsymbol{\Psi}$ is a stress function. $\mathbf{Div}\mathbf{P} = \mathbf{0}$ are the equilibrium equations. Therefore, the **GCD** complex or the nonlinear elasticity complex contains both the kinematics and the kinetics of motion as schematically shown below.

$$
\begin{array}{ccccccccc}
 & & \text{displacements} & \longrightarrow & \text{disp. gradients} & \longrightarrow & \text{compatibility} & & \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\mathbf{0} & \longrightarrow & \Gamma(T\varphi(\mathcal{B})) & \xrightarrow{\mathbf{Grad}} & \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B}) & \xrightarrow{\mathbf{Curl}} & \Gamma(T\varphi(\mathcal{B})\otimes T\mathcal{B}) & \xrightarrow{\mathbf{Div}} & \Gamma(T\varphi(\mathcal{B})) \longrightarrow \mathbf{0} \\
 & & & & \updownarrow & & \updownarrow & & \updownarrow \\
 & & & & \text{stress functions} & \longrightarrow & \text{first PK stresses} & \longrightarrow & \text{equilibrium}
\end{array}
$$

Using the differential complex of nonlinear elasticity, a new family of mixed finite elements – compatible-strain mixed finite element methods (CSFEMs) – has been introduced for both compressible and incompressible nonlinear elasticity [6, 46, 47]. CSFEMs are capable of capturing very large strains and accurately approximating stresses. CSFEMs, by construction, satisfy both the Hadamard jump conditions, and the continuity of traction at the discrete level. This makes them quite efficient for modeling heterogeneous solids. Moreover, CSFEMs seem to be free from numerical artifacts such as checkerboarding of pressure, hourglass instability, and locking.

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