Stability of Media and Structures

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1 Introduction

Any structure that survives after construction must be stable, in the sense that it has already demonstrated its ability to withstand a range of loads without undergoing unacceptable deflection or distortion. It is, however, necessary to consider the question of how great a load a structure can support, before its performance is compromised – that is, before it will collapse. Examples may be of several types: for instance, an exceptional fall of snow could result in a shell roof supporting more weight than was envisaged in its design. As the snow continues to fall, the load increases until the roof collapses. Conceptually, at the instant before the last snowflake landed, the roof was in a state of unstable equilibrium, so that the small load represented by the last snowflake caused the large deflection and the onset of the failure. Excessive loads from other sources are easy to envisage. It is also possible that a structure may be stable but that, under some exceptional conditions, one of its resonant vibrations is stimulated. In the absence of sufficient damping, this can result in the build-up of a vibration of large amplitude and consequent failure. Earthquake damage can (but does not always) fall into this category. The famous collapse of the Tacoma Narrows suspension bridge involved the development of an oscillation of large amplitude, induced by wind, by the mechanism of "flutter".

Evidently, the safe design of a structure must take into account its possible modes and frequencies of vibration, and must incorporate sufficient margins of safety. Stability (in the sense that a "small" disturbance induces only a small response in the structure) is necessary but not sufficient: what about a "moderate" disturbance, for instance? These questions must be addressed quantitatively, in relation to the types of load that a structure will experience during service.

This course provides an introductory account of the concepts and methodology required for the assessment of structural stability. Structural collapse can be a "global" event, involving the whole structure, or it can result from a large deformation occurring locally, because the material from which the structure is built reaches a critical condition. Some attention is also devoted to this topic, though specialized aspects such as the development and propagation of cracks are not addressed; any such topic would require a whole course by itself.

The remainder of this introduction is devoted to a simple example, which requires no specialized knowledge and yet illustrates many of the features present in the analysis of the stability of any structure.

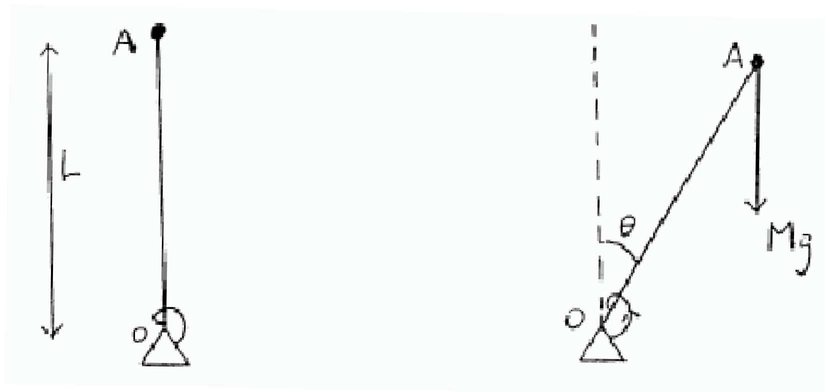

Fig. 1.1. Model structure.

1.1 An elementary one-dimensional example

The configuration shown in Fig. 1.1 displays most of the features of the buckling of a strut or a column under compression. A rigid rod OA of zero mass and length L is pivoted at a point O, and its deflection from the vertical (with A above O) is resisted by a nonlinear spring, which exerts a restoring couple

$$C = f(\theta); \quad f(0) = 0 \tag{1.1}$$

when OA makes an angle θ with the upward vertical. It is assumed that $f'(\theta) > 0$. A point mass M is attached at A. A force acts vertically downwards at A. This could be due to gravity acting on the mass M, in which case the force would have magnitude Mg. However, to preserve generality, we let its magnitude be λ. The equation of motion of this system follows from the balance of moment of momentum:

$$ML^2\ddot{\theta} = \lambda L \sin\theta - f(\theta). \tag{1.2}$$

Any equilibrium position is defined by $\dot{\theta} = \ddot{\theta} = 0$, and must therefore satisfy

$$\lambda L \sin\theta = f(\theta). \tag{1.3}$$

The number of equilibria depends on the form of the function f and the value of λL. The vertical configuration $\theta = 0$ is in equilibrium for any value of λL (though it need not be stable). Consider the length L of the rod to be fixed but suppose that the load λ

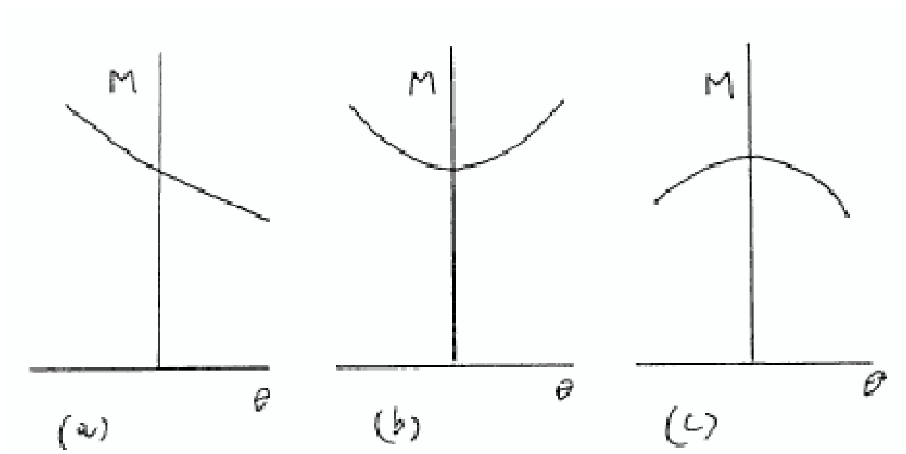

Fig. 1.2. Equilibrium paths. (a) $\lambda_1 < 0$; (b) $\lambda_1 = 0$, $\lambda_2 > 0$; (c) $\lambda_1 = 0$, $\lambda_2 < 0$.

is open to choice. Equation (1.3) then defines an *equilibrium path* in the θ-λ plane. The path $\theta = 0$ will be called the fundamental path. If $f(\theta)$ has the expansion

$$f(\theta) = K_1\theta + K_2\theta^2 + K_3\theta^3 + \cdots \quad (K_1 > 0), \tag{1.4}$$

another path defining a "buckled state" is connected to the fundamental path $\theta = 0$ at the critical mass λ_c defined by

$$\lambda_c = K_1/L = f'(0)/L. \tag{1.5}$$

The buckled state, in the vicinity of $\theta = 0$, lies on the path

$$\lambda = \lambda_c + \lambda_1\theta + \lambda_2\theta^2 + \cdots, \tag{1.6}$$

where

$$\lambda_1/\lambda_c = K_2/K_1, \quad \lambda_2/\lambda_c = (K_3/K_1 + \tfrac{1}{6}), \quad \cdots \tag{1.7}$$

The point $(0, \lambda_c)$ is called a *point of bifurcation.* The path (1.6) defines the initial post-bifurcation (or post-buckling) response. Figure 1.2 illustrates possible paths. Figure 1.2(a) illustrates an *asymmetric bifurcation* (the case $\lambda_1 > 0$ just reverses the slope of the bifurcated path). Figures 1.2(b) and (c) illustrate symmetric bifurcations. The response of an actual structure depends in part on the form of the post-bifurcation response. Two aspects are investigated below.

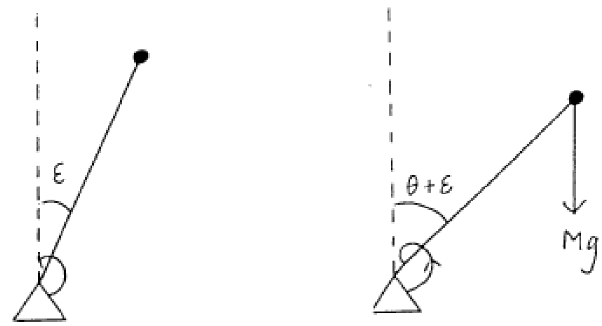

Fig. 1.3. Imperfect model structure.

Effect of an imperfection

Real structures are never perfect. The effect of an imperfection may be illustrated by introducing a small offset into the restoring spring. This is modelled by measuring θ from the configuration in which the spring exerts no couple, which occurs when the rod makes a small angle ε to the vertical, as shown in Fig. 1.3. The equation of motion now becomes

$$ML^2\ddot{\theta} = \lambda L \sin(\theta + \varepsilon) - f(\theta), \tag{1.8}$$

and any equilibrium configuration must satisfy

$$\lambda L \sin(\theta + \varepsilon) = f(\theta). \tag{1.9}$$

Thus, near $\theta = 0$, and since $|\varepsilon| \ll 1$,

$$\lambda = [K_1\theta + K_2\theta^2 + K_3\theta^3 + \cdots]/[(\theta + \varepsilon) - \tfrac{1}{6}(\theta + \varepsilon)^3 + \cdots]L. \tag{1.10}$$

The path $\theta = 0$ for the perfect structure is altered, to lowest order when λ is sufficiently below λ_c, to

$$\theta \sim \frac{\varepsilon}{(\lambda_c/\lambda - 1)}. \tag{1.11}$$

However, when λ is close to λ_c, this approximation breaks down. If $K_2 \neq 0$, a better approximation is given by the solution of the quadratic equation

$$(K_1 - \lambda L)\theta + K_2\theta^2 = \lambda L\varepsilon, \tag{1.12}$$

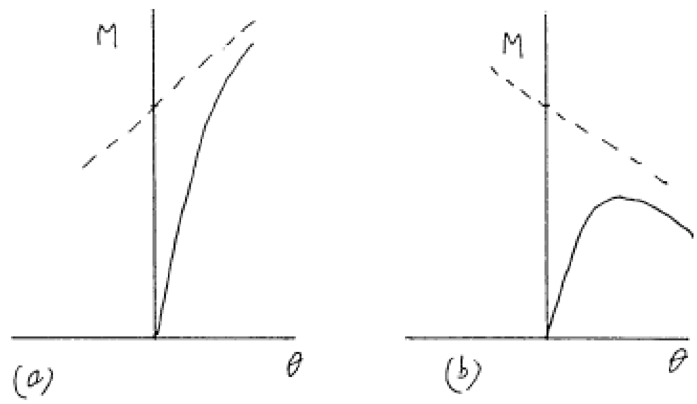

Fig. 1.4. Perturbed equilibrium paths. (a) $K_2 > 0$; (b) $K_2 < 0$.

or

$$(K_2/K_1)\theta^2 + (1 - \lambda/\lambda_c)\theta = (\lambda/\lambda_c)\varepsilon. \tag{1.13}$$

The resulting equilibrium paths are sketched (for $\varepsilon > 0$) in Fig. 1.4. The branch that passes through the origin has the equation

$$\theta = \frac{-(1 - \lambda/\lambda_c) + [(1 - \lambda/\lambda_c)^2 + 4(K_2/K_1)(\lambda/\lambda_c)\varepsilon]^{1/2}}{2(K_2/K_1)}. \tag{1.14}$$

When λ/λ_c is sufficiently smaller than 1, this reproduces the formula (1.11). The more interesting of the two cases shown is $K_2 < 0$. The branch passing through the origin displays a maximum allowed λ: increasing λ from zero would generate the deflection shown, until the maximum is reached. Any further attempt to increase λ must require the structure to deform substantially. The deformation would be dynamic (limited by inertia). Such a situation is termed a snap-through buckle.

Elementary analysis shows that the maximum λ admitted by (1.14) is given asymptotically, for small ε, by the formula

$$(1 - \lambda_{max}/\lambda_c) \sim 2(-K_2/K_1)^{1/2}\varepsilon^{1/2}. \tag{1.15}$$

Thus, the presence of the imperfection has reduced the buckling load by a quantity proportional to $\sqrt{\varepsilon}$.

This conclusion can also be reached by introducing another small parameter ξ and writing

$$\theta = \theta_1\xi \tag{1.16}$$

and

$$\lambda/\lambda_c = 1 + \mu_1\xi + \mu_2\xi^2 + \cdots \tag{1.17}$$

Substituting these into (1.10) and expanding, but keeping only the term of lowest order in ε, gives

$$\begin{aligned}(1 + \mu_1\xi + \mu_2\xi^2 + \cdots)(\theta_1\xi - \tfrac{1}{6}\theta_1^3\xi^3 + \varepsilon + \cdots)\\ = \theta_1\xi + (K_2/K_1)\theta_1^2\xi^2 + (K_3/K_1)\theta_1^3\xi^3 + \cdots\end{aligned} \tag{1.18}$$

Simplifying, therefore,

$$\begin{aligned}\varepsilon + \mu_1\theta_1\xi^2 + (\mu_2\theta_1 - \tfrac{1}{6}\theta_1^3)\xi^3 + \cdots\\ = (K_2/K_1)\theta_1\xi^2 + (K_3/K_1)\theta_1^3\xi^3 + \cdots\end{aligned} \tag{1.19}$$

All relevant formulae can now be obtained.

First, if $\varepsilon = 0$, equating terms of like order in ξ gives

$$\mu_1 = (K_2/K_1)\theta_1, \quad \mu_2 = [(K_3/K_1) + \tfrac{1}{6}]\theta_1^2, \tag{1.20}$$

exactly consistent with formulae (1.7).

Next, suppose that $\varepsilon \neq 0$ and $K_2 \neq 0$. Retaining just terms of lowest order gives

$$\mu_1 = \frac{K_2\theta_1}{K_1} - \frac{\varepsilon}{\theta_1\xi^2}, \tag{1.21}$$

and then, for consistency, it is necessary to choose $\xi = \varepsilon^{1/2}$.

If $K_2 = 0$, equation (1.21) contains no interaction between the imperfection and the parameters defining the spring. This suggests that the procedure as so far given is unsuitable when $K_2 = 0$. A balance is obtained at order ξ^3 if we take $\mu_1 = 0$. Then,

$$\mu_2 = \left(\frac{K_3}{K_1} + \tfrac{1}{6}\right)\theta_1^2 - \frac{\varepsilon}{\theta_1\xi^3} \tag{1.22}$$

and for consistency the choice $\xi = \varepsilon^{1/3}$ must be made.

The most interesting cases correspond to $K_2 < 0$ and $(K_3/K_1) + (1/6) < 0$, respectively. It follows then from (1.21) that

$$(\lambda/\lambda_c - 1) \approx \mu_1\xi \leq -2(-K_2/K_1)^{1/2}\epsilon^{1/2} \tag{1.23}$$

and from (1.22) that

$$\mu_1 = 0, \quad \mu_2\xi^2 \leq -3\{-[(K_3/K_1) + \tfrac{1}{6}]/4\}^{1/3}\epsilon^{2/3}. \tag{1.24}$$

The first of these results implies (1.15), while the second gives

$$\lambda_{max}/\lambda_c \sim 1 - 3\{-[(K_3/K_1) + \tfrac{1}{6}]\}^{1/3} (\varepsilon/2)^{2/3} \quad (K_2 = 0,\ [(K_3/K_1) + \tfrac{1}{6}] < 0). \tag{1.25}$$

The pattern for cases in which more μ's vanish should now be apparent.

Dynamics and stability

By definition, the structure under consideration is fully described by the ordinary differential equation (1.2) (or (1.8) if the imperfect structure were considered). This is simple enough to allow exact analysis. However, the present purpose is to introduce procedures that can be applied more generally.

First, considering the perfect structure, $\theta = 0$ defines an equilibrium configuration for any value of λ. Stability of equilibrium is addressed, in the first instance, via analysis of the differential equation, linearized about the equilibrium solution. In the present case, the linear equation is simply

$$ML^2\ddot{\theta} = (\lambda L - K_1)\theta. \tag{1.26}$$

Its general solution has the form[1]

$$\theta(t) = A\mathrm{e}^{\mathrm{i}\omega t} + B\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1.27}$$

where

$$\omega^2 = (K_1 - \lambda L)/ML^2 = [\lambda_c - \lambda]/ML. \tag{1.28}$$

Thus, the solution is bounded – and therefore remains small if it was initiated by a small disturbance – so long as $\lambda < \lambda_c$. Conversely, if $\lambda > \lambda_c$, ω becomes imaginary, the solution grows exponentially (except for very special initial conditions that generate only the negative exponential). The linearization under which it was derived becomes invalid and study of what actually happens requires a return to the original nonlinear differential equation. In the former situation ($\lambda < \lambda_c$), the equilibrium configuration $\theta = 0$ is described as stable; in the latter it is unstable[2].

The nonlinear dynamics may be investigated, in the vicinity of the critical point $(0, \lambda_c)$, by retention of the "next" terms in the differential equation (1.2). Thus, now,

$$ML^2\ddot{\theta} = \lambda L(\theta - \theta^3/6 + \cdots) - (K_1\theta + K_2\theta^2 + K_3\theta^3 + \cdots). \tag{1.29}$$

[1] In the case that ω is real, the most general real solution is obtained by taking A and B to be complex conjugates.

[2] Stability will be defined formally later

If $K_2 \neq 0$, retaining just terms up to order θ^2 gives

$$\ddot{\theta} = \left(\frac{\lambda}{ML}\right)\left[\left(\frac{\lambda - \lambda_c}{\lambda}\right)\theta - \left(\frac{K_2\lambda_c}{K_1\lambda}\right)\theta^2\right]. \tag{1.30}$$

The procedure underlying the derivation of (1.30) can be formalised by writing

$$\lambda/\lambda_c = 1 + \mu_1\xi, \quad \theta = \theta_1\xi, \quad \text{and} \quad \tau = \xi^{1/2}t. \tag{1.31}$$

The variable θ_1 is regarded as a function of the "slow" time variable τ. Substituting into (1.2) then gives

$$\begin{aligned} ML^2\xi^2\theta_1'' = \; & \lambda_c L[(1 + \mu_1\xi)(\theta_1\xi - \theta_1^3\xi^3/6 + \cdots) \\ & -(\theta_1\xi + (K_2/K_1)\theta_1^2\xi^2 + (K_3/K_1)\theta_1^3\xi^3 + \cdots)], \end{aligned} \tag{1.32}$$

the prime denoting differentiation with respect to τ. The terms of order ξ cancel. Equating those of order ξ^2 gives

$$\theta_1'' = A\theta_1 + B\theta_1^2, \tag{1.33}$$

where

$$A = \mu_1\lambda_c/ML, \quad B = -(K_2/K_1)\lambda_c/ML. \tag{1.34}$$

The differential equation (1.33) admits constant solutions $\theta_1 = 0$ and $\theta_1 = -A/B$. The latter corresponds exactly to the initial post-buckling path [*c.f.* (1.6) with (1.7)]. Linearizing about $\theta_1 = 0$ shows that this solution is stable so long as $A < 0$. Linearizing about $\theta_1 = -A/B$ gives the equation

$$\varphi_1'' = -A\varphi_1, \tag{1.35}$$

having written $\theta_1 = -A/B + \varphi_1$. Thus, the solution $\theta_1 = -A/B$ is stable if $A > 0$. Phase portraits (plots in a θ_1-θ_1' plane) are sketched in Fig. 1.5. When $A < 0$, so that $\theta_1 = 0$ is stable against an infinitesimal perturbation, study of (1.33) permits an assessment of exactly how large a perturbation would be allowed, before the solution would deviate far from $\theta_1 = \theta_1' = 0$. The interest of equation (1.33) is that it will be seen to emerge *generically* from weakly-nonlinear stability analysis in the vicinity of a non-symmetric bifurcation.

If $K_2 = 0$, a different parametrization is required. A balance is obtained if

$$\lambda/\lambda_c = 1 + \mu_2\xi^2, \quad \theta = \theta_1\xi, \quad \tau = \xi t. \tag{1.36}$$

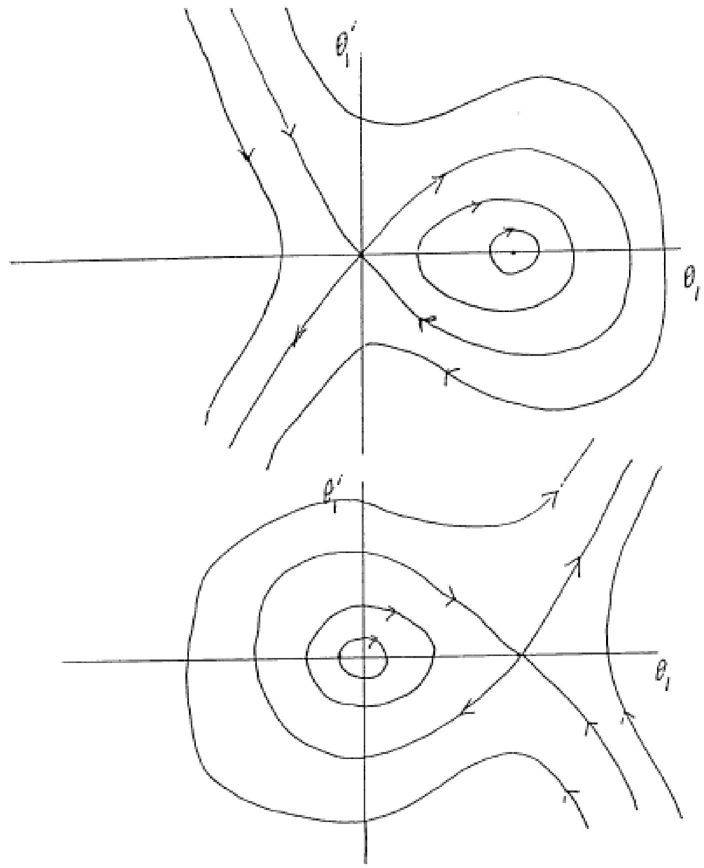

Fig. 1.5. Phase portraits for the differential equation (1.33).

The equation that results is

$$\theta_1'' = A\theta_1 + B\theta_1^3, \tag{1.37}$$

where

$$A = \mu_2\lambda_c/ML, \quad B = -(K_3/K_1 + \tfrac{1}{6})\lambda_c/ML. \tag{1.38}$$

The constant solutions $\theta_1 = \pm(-A/B)^{1/2}$ (which exist if $A/B < 0$) correspond to the initial post-buckling path [*c.f.* (1.6) and (1.7) with $K_2 = 0$]. Equation (1.37) will emerge as a generic feature associated with a symmetric bifurcation.

The effect of an imperfection (ε) can be incorporated by adding its lowest-order contribution to the governing differential equation. This has the effect of replacing (1.33) by

$$\theta_1'' = A\theta_1 + B\theta_1^2 + C(\varepsilon/\xi^2), \tag{1.39}$$

where

$$C = \lambda_c/ML. \tag{1.40}$$

[For consistency, it is necessary that $\varepsilon/\xi^2 = O(1)$]. The equilibrium point of (1.39) agrees with the approximation given by (1.12). If $K_2 = 0$, an analogous modification follows for (1.37).

2 Stability of systems: general discussion

This section discusses the stability of systems with any finite number of degrees of freedom. Although real structures are continua, they are almost always modelled as discrete for the purpose of stress analysis – for example by the use of finite elements – and hence in practice this discussion will apply, at least at a formal level, to virtually all structures, as well as to other dynamical systems. It is usual to consider a first-order system,

$$\dot{u} = f(u, t; \lambda), \tag{2.1}$$

where $u : R \to R^n$ is a vector-valued function of time t. The function f has the arguments shown and takes values in R^n. The system (1.2) fits this pattern. With the definitions $u_1 = \theta$, $u_2 = \dot{\theta}$, it can be written

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= [\lambda L \sin u_1 - f(u_1)]/(ML^2). \end{aligned} \tag{2.2}$$

In the general equation (2.1), λ represents any number m of scalar parameters; that is, it could be an m-dimensional vector. The most significant difference between the general system (2.1) and the realisation (2.2) is that (2.1) contains time t explicitly: it is not autonomous. In fact, in the sequel only autonomous systems will be considered. However, some formal definitions of stability will be given for the system (2.1).

Suppose that $u_0(t)$ is a particular solution of (2.1). It is called *stable* if the solution of the initial-value problem comprising (2.1) for $t > t_0$, together with the initial condition

$$u(t_0) = u_0(t_0) + \epsilon v_0 \tag{2.3}$$

has the property

$$\|u(t) - u_0(t)\| \to 0 \ \text{ as } \ \epsilon \to 0 \tag{2.4}$$

uniformly for all $t \geq t_0$, for all v_0 with $\|v_0\| = 1$. Here, $\| \cdot \|$ can be taken as the Euclidean norm. This choice of norm is not important, however, because all norms on a finite-dimensional vector space are equivalent.

If the solution u satisfies the stronger requirement that

$$\|u(t) - u_0(t)\| \to 0 \ \text{ as } \ t \to \infty \tag{2.5}$$

for all sufficiently small ϵ ($|\epsilon| < \delta$ for some $\delta > 0$), the system (2.1) is called *asymptotically stable.*

The discussion of the preceding section strongly suggests that the solution $u = 0$ of the system (2.2) is stable but not asymptotically stable, when $\lambda < \lambda_c$. This can be verified from its original form (1.2) by noting that $\ddot{\theta} = \dot{\theta} d(\dot{\theta})/d\theta$ and integrating, to obtain the energy integral

$$\tfrac{1}{2} M L^2 \dot{\theta}^2 + \int_0^\theta f(\theta')d\theta' - \lambda L(1 - \cos\theta) = E, \text{ constant.} \tag{2.6}$$

The condition $\lambda < \lambda_c$ ensures that the function on the left side of (2.6) is a convex function of $(\dot{\theta}, \theta)$ in a neighbourhood of $(0, 0)$ and hence, when the constant E on the right side, which is fixed by the initial conditions, is sufficiently small, $(\dot{\theta}, \theta)$ remains close to $(0, 0)$ for all t.

Before proceeding, it is appropriate to express a word of caution in relation to continuous systems. All norms are *not* equivalent for such systems, and so not only the definition of stability, but also its relevance or utility, will depend upon the appropriate choice of norm. A related concern is the possibility that any chosen finite-dimensional approximation of a continuous system simply might not contain some instability of the original system, though such a feature would be likely to show up in practice in the form of strong sensitivity of predictions made from the discrete system to the precise detail, such as the finite element mesh. A pragmatic approach is adopted throughout these lectures: formal methods, such as the perturbation theory already used in Section 1, will be employed. Such methods do not establish rigorously when instability is reached but they have the virtue of providing fully explicit indications of the nature of likely instabilities, and associated estimates for quantities such as critical loads.

Having made these general remarks, we now consider a system of the form[3]

$$M\ddot{u} = F(u, \lambda). \tag{2.7}$$

Here, u is a vector, F is vector-valued and M is a matrix. λ is a scalar parameter that represents the intensity of the loading on the system.

Various aspects of the system (2.7) are now considered, in turn.

Equilibrium

Equilibrium configurations of the system must satisfy the equation

$$F(u, \lambda) = 0. \tag{2.8}$$

[3]Complications will be considered later.

This equation may have several solutions (it will be assumed to have at least one, at least for some range of values of λ). Their number may change with the value of λ. The functional dependence upon λ of any one solution may be investigated by regarding λ to be an increasing function of time (or any time-like parameter), while insisting that the equilibrium condition (2.7) remains satisfied. Then, differentiating (2.7) gives

$$F_u(u,\lambda)\dot{u} + F_\lambda(u,\lambda)\dot{\lambda} = 0. \tag{2.9}$$

Just to explain the notation employed here, regard F as a vector with r-component F_r. Equation (2.8) is equivalent to

$$F_{r,u_s}(u,\lambda)\dot{u}_s + F_{r,\lambda}(u,\lambda)\dot{\lambda} = 0, \tag{2.10}$$

the comma denoting a partial derivative with respect to the following variable, and summation over the repeated suffix s is implied. Assuming that the matrix F_u is not singular, this equation has the unique solution

$$\dot{u} = -[F_u]^{-1}F_\lambda\dot{\lambda}. \tag{2.11}$$

This is equivalent to the differential equation

$$du/d\lambda = -[F_u]^{-1}F_\lambda,$$

which defines the branch of the solution of (2.7) that is being followed. Any such branch is called an *equilibrium path*.

Suppose, now, that an equilibrium path is followed, with λ increasing, until a critical value λ_c is reached, with corresponding equilibrium configuration u_c, at which F_u becomes singular. There are two possibilities:
(i) Equation (2.8) has no solution. In this case, λ cannot be increased beyond λ_c for this branch. Any attempt to increase λ would have to result in motion, in which the inertia $m\ddot{u}$ is important.
(ii) Equation (2.8) has non-unique solution. The point (u_c, λ_c) is then a *point of bifurcation.*

In either case, (u_c, λ_c) is a *critical point.*

Uniqueness

Another perspective on the same phenomenon is gained by considering directly whether equation (2.7) has a unique solution. Suppose that there are two solution branches, $u_1(\lambda)$ and $u_2(\lambda)$, and suppose that for some set of values of λ they are close together. It follows that

$$0 = F(u_1, \lambda) - F(u_2, \lambda) \sim F_u(u_2, \lambda)(u_1 - u_2), \tag{2.12}$$

which implies that F_u must be singular, and that u_1 and u_2 may differ only by a multiple of the eigenvector of F_u. In case (ii) above, the two distinct branches cross at (u_c, λ_c). In case (i), it may occur that u_1 and u_2 are different parts of a single branch that "turns over", as illustrated in Fig. 1.3(b).[4] However, it is conceivable that the branch simply terminates, the nonlinear terms omitted from (2.12) preventing its continuation. Thus, by itself, linearized analysis provides an indication of what *may* happen but does not predict what *will* happen.

Influence of an imperfection

Suppose that an imperfection is present, whose magnitude is described by the parameter ϵ. Recall that, in the example given in Section 1, the imperfection was a small tilt of the bar away from the vertical, when in equilibrium under zero load. More generally, an imperfection could be any geometrical feature, or perhaps some variation in stiffness, or perhaps both. In the present general formulation, the presence of the imperfection is represented by replacing the function $F(u, \lambda)$ in (2.8) by $F(u, \lambda, \epsilon)$. It is possible, in fact, to let ϵ be a vector of any finite dimension, so that it represents the effect of several types of imperfections. Equilibrium is now governed by the equation

$$F(u, \lambda, \epsilon) = 0, \tag{2.13}$$

with $\epsilon = 0$ defining the structure with no imperfection. The primary solution branch (that is, the one in which we are interested) is denoted u_0 when $\epsilon = 0$. That is,

$$F(u_0, \lambda, 0) = 0. \tag{2.14}$$

It is convenient to re-define variables so that $u_0 \equiv 0$. Thus,

$$F(0, \lambda, 0) = 0 \tag{2.15}$$

for all λ. Now call the perturbed solution (for $\epsilon \neq 0$) u, and assume that u is small and $u \to 0$ as $\epsilon \to 0$. Then,

$$F_u(0, \lambda, 0)u + F_\epsilon(0, \lambda, 0)\epsilon \sim 0, \tag{2.16}$$

[4]This figure is for a structure regarded as imperfect but the phenomenon can occur generally.

which implies that

$$u \sim -[F_u(0, \lambda, 0)]^{-1} F_\epsilon(0, \lambda, 0)\epsilon, \tag{2.17}$$

except when λ is close to λ_c (where the matrix F_u is singular).

The initial post-bifurcation path

Before considering the perturbed path when λ is close to λ_c, it is useful to examine further the response of the unperturbed system in this vicinity. Within the present framework, the primary solution branch is $u = u_0 \equiv 0$, so the critical point (u_c, λ_c) becomes $(0, \lambda_c)$. Assume that this is a point of (simple) bifurcation. Then, when λ is close to λ_c, there is one other solution v say, and $v \to 0$ as $\lambda \to \lambda_c$. Therefore, expanding the equation

$$F(v, \lambda, 0) = 0$$

about $(0, \lambda_c, 0)$,

$$F_u v + \tfrac{1}{2} F_{uu} v^2 + F_{u\lambda} v(\lambda - \lambda_c) + \tfrac{1}{6} F_{uuu} v^3 + \tfrac{1}{2} F_{uu\lambda} v^2 (\lambda - \lambda_c) + \tfrac{1}{2} F_{u\lambda\lambda} v(\lambda - \lambda_c)^2 + \cdots = 0. \tag{2.18}$$

Here, for example, $F_{uu}v^2$ represents the vector whose i-component is $F_{i,u_r u_s} v_r v_s$ (summation over r and s implied). All derivatives of F are here evaluated at $(0, \lambda_c, 0)$. Derivatives with respect to λ by itself (such as $F_{\lambda\lambda}$) are not included because they are zero, on account of (2.15).

Now clearly, to lowest order, v is a multiple of the right eigenvector of $F_u(0, \lambda_c, 0)$, as found earlier[5]. Suppose, therefore, that u_1^* and u_1 are respectively left and right eigenvectors:

$$u_1^* F_u = 0 \ \text{ and } \ F_u u_1 = 0. \tag{2.19}$$

Now when λ is close to λ_c, v is small and (asymptotically) parallel to u_1. Therefore, introduce a small parameter ξ and write

$$\begin{aligned} v &= \xi v_1 + \xi^2 v_2 + \cdots, \\ \lambda &= \lambda_c + \xi\lambda_1 + \xi^2 \lambda_2 + \cdots. \end{aligned} \tag{2.20}$$

Substituting these into (2.17) gives

$$\xi F_u v_1 + \xi^2 \{\tfrac{1}{2} F_{uu} v_1^2 + \lambda_1 F_{u\lambda} v_1 + F_u v_2\} + \xi^3 \{\tfrac{1}{6} F_{uuu} v_1^3 + \tfrac{1}{2}\lambda_1 F_{uu\lambda} v_1^2 + \tfrac{1}{2}\lambda_1^2 F_{u\lambda\lambda} v_1 + \lambda_2 F_{u\lambda} v_1 + \lambda_1 F_{u\lambda} v_2 + F_{uu} v_1 v_2 + F_u v_3\} + \cdots = 0. \tag{2.21}$$

[5]There is only one linearly independent right eigenvector, on account of the assumption that the bifurcation point is simple.

Therefore, by considering the coefficient of ξ,

$$F_u v_1 = 0, \tag{2.22}$$

implying that $v_1 = \alpha_1 u_1$ for some α_1. Now considering the coefficient of ξ^2,

$$\tfrac{1}{2} F_{uu} v_1^2 + \lambda_1 F_{u\lambda} v_1 + F_u v_2 = 0. \tag{2.23}$$

This implies necessarily (*c.f.* $(2.19)_1$) that

$$\lambda_1 = \frac{-\alpha_1 u_1^* F_{uu} u_1^2}{2 u_1^* F_{u\lambda} u_1}, \tag{2.24}$$

and then (2.23) can be solved for v_2. The solution is unique only up to a term $\alpha_2 u_1$.

The coefficient of ξ^3 can be treated similarly: it yields

$$\lambda_2 = \frac{-u_1^* \{ \frac{1}{6} F_{uuu} v_1^3 + \frac{1}{2} \lambda_1 F_{uu\lambda} v_1^2 + \frac{1}{2} \lambda_1^2 F_{u\lambda\lambda} v_1 + \lambda_1 F_{u\lambda} v_2 + F_{uu} v_1 v_2 \}}{\alpha_1 u_1^* F_{u\lambda} u_1}. \tag{2.25}$$

The simplest way to fix α_1 and α_2 (and corresponding constants in succeeding terms) is to define ξ in terms of v by insisting that $\xi = \hat{u}^T v$ for some vector $\hat{u}$, such that $\hat{u}^T u_1 = 1$. Then, $\alpha_1 = 1$ and the requirement that $\hat{u}^T v_2 = 0$ fixes α_2. Another possibility would be to insist that $\lambda = \lambda_c + \xi \lambda_1$ exactly, so that $\lambda_2 = \lambda_3 = \cdots = 0$. Then, equation (2.24) gives α_1 in terms of λ_1, and (2.25) fixes α_2. In any case, v_3 exists so long as (2.25) is satisfied.

If $u_1^* F_{u\lambda} u_1 = 0$, then (2.24) and (2.25) do not apply. A balance of terms is obtained if $v_1 = 0$ and $v_2 = \alpha_2 u_1$. This rather pathological case is not discussed further.

The imperfect system

Now revert to the imperfect system, with $\epsilon \neq 0$. Expanding the equation

$$F(v, \lambda, \epsilon) = 0$$

about the point $(0, \lambda_c, 0)$ gives

$$\begin{aligned} F_u v + \tfrac{1}{2} F_{uu} v^2 + F_{u\lambda} v (\lambda - \lambda_c) + \tfrac{1}{6} F_{uuu} v^3 + \tfrac{1}{2} F_{uu\lambda} v^2 (\lambda - \lambda_c) \\ + \tfrac{1}{2} F_{u\lambda\lambda} v (\lambda - \lambda_c)^2 + \cdots + F_\epsilon \epsilon = 0, \end{aligned} \tag{2.26}$$

having retained only the leading-order term with respect to ϵ. Again, set

$$v = \xi v_1 + \xi^2 v_2 + \cdots,$$

and let

$$\lambda = \lambda_c + (\lambda_1 + \hat{\lambda}_1)\xi + (\lambda_2 + \hat{\lambda}_2)\xi^2 + \cdots, \tag{2.27}$$

where λ_r are as before, so that the terms $\hat{\lambda}_r$ give the additional perturbation of λ due to ϵ.

Note first that an attempt to balance the term proportional to ξ with the term that is linear in ϵ generally cannot succeed, because the condition for consistency of the equation

$$\xi F_u v_1 + F_\epsilon \epsilon = 0$$

is $u_1^* F_\epsilon = 0$, which is not usually the case. Therefore, it is necessary to assume that ϵ is of order ξ^k for some $k > 1$. Then, as obtained previously, $F_u v_1 = 0$ and so $v_1 = \alpha_1 u_1$ for some α_1.

If ϵ is of order ξ^2, equating terms of order ξ^2 gives

$$F_u v_2 + \tfrac{1}{2}\alpha_1^2 F_{uu} u_1^2 + \alpha_1(\lambda_1 + \hat{\lambda}_1) F_{u\lambda} u_1 + F_\epsilon \epsilon/\xi^2 = 0. \tag{2.28}$$

The condition for consistency is

$$u_1^* \{\tfrac{1}{2}\alpha_1^2 F_{uu} u_1^2 + \alpha_1(\lambda_1 + \hat{\lambda}_1) F_{u\lambda} u_1 + F_\epsilon \epsilon/\xi^2\} = 0.$$

Therefore,

$$\hat{\lambda}_1 = \frac{-u_1^* F_\epsilon \epsilon}{\alpha_1 u_1^* F_{u\lambda} u_1 \xi^2}, \tag{2.29}$$

since λ_1 as given by (2.24) cancels the other terms. Thus, to this order,

$$\lambda \sim \lambda_c - \left\{ \frac{u_1^* F_{uu} u_1^2 \alpha_1}{2u_1^* F_{u\lambda} u_1} + \frac{u_1^* F_\epsilon \epsilon}{u_1^* F_{u\lambda} u_1 \alpha_1 \xi^2} \right\} \xi. \tag{2.30}$$

The perturbation is of the form

$$-(A\alpha_1 \xi + B/\alpha_1 \xi),$$

whose greatest value is $-2(AB)^{1/2}$ if $A > 0$ and $B > 0$. Thus, in this case,

$$\lambda \leq \lambda_c - \frac{\{2|(u_1^* F_{uu} u_1^2)(u_1^* F_\epsilon \epsilon)|\}^{1/2}}{|u_1^* F_{u\lambda} u_1|}. \tag{2.31}$$

A snap-through buckle is indicated at a level of λ an amount of order $\|\epsilon\|^{1/2}$ lower than λ_c.

If $\lambda_1 = 0$ (*i.e.* $u_1^* F_{uu} u_1^2 = 0$), a different scaling is needed to obtain the desired balance. It is appropriate, in fact, to take $k = 3$. Then, $\hat{\lambda}_1 = 0$ and equating terms of order ξ^3 gives

$$\hat{\lambda}_2 = \frac{-u_1^* F_\epsilon \epsilon}{\alpha_1 u_1^* F_{u\lambda} u_1 \xi^3}.$$

Then,

$$\lambda \sim \lambda_c + \lambda_2 \xi^2 - \frac{u_1^* F_\epsilon \epsilon}{\alpha_1 u_1^* F_{u\lambda} u_1 \xi}. \tag{2.32}$$

This has the form

$$\lambda \sim \lambda_c - \{A(\alpha_1 \xi)^2 + B/\alpha_1 \xi\},$$

and if $A > 0$ and $B > 0$, it follows that

$$\lambda \le \lambda_c - 3A^{1/3}(B/2)^{2/3}. \tag{2.33}$$

The reduction in the critical load is of order $\|\epsilon\|^{2/3}$.

Stability

We consider now the stability of the primary equilibrium path $u = 0$ for the perfect structure. This requires study of the system of differential equations

$$M\ddot{u} = F(u, \lambda), \quad \text{with} \quad F(0, \lambda) = 0. \tag{2.34}$$

First, linearizing gives

$$M\ddot{u} = F_u(0, \lambda) u, \tag{2.35}$$

for which a solution may be sought of the form

$$u(t) = v\mathrm{e}^{\mathrm{i}\omega t}.$$

This generates the eigenvalue problem

$$[F_u(0, \lambda) + \mu M] v = 0 \quad (\mu = \omega^2). \tag{2.36}$$

Since the system is real, complex eigenvalues must occur in complex conjugate pairs, so instability is inevitable unless all eigenvalues μ are real and positive. It is reasonable to assume that the system is stable at least for small loads (small λ), so assume that all eigenvalues are real and positive for $\lambda < \lambda_c$. Suppose, furthermore, that the smallest eigenvalue $\mu_1 = 0$ when $\lambda = \lambda_c$ and that it is simple. This gives

$$F_u(0, \lambda_c)u_1 = 0,$$

exactly as discussed already. The solution branch $u = 0$ will be unstable for some range of λ with $\lambda > \lambda_c$, if μ decreases below zero when λ increases beyond λ_c.

Now some weakly-nonlinear analysis can be developed assuming that λ is close to λ_c. Let

$$\lambda = \lambda_c + \lambda_1 \xi, \tag{2.37}$$

$$u = \xi v_1 + \xi^2 v_2 + \cdots, \tag{2.38}$$

scale the time so that

$$\tau = \xi^{1/2} t \tag{2.39}$$

and regard v_i as functions of τ. Substituting into (2.34) then gives

$$\xi F_u v_1 + \xi^2 \{\lambda_1 F_{u\lambda} v_1 + \tfrac{1}{2} F_{uu} v_1^2 - M v_1'' + F_u v_2\} + \cdots = 0, \tag{2.40}$$

the prime denoting differentiation with respect to τ. It follows that $v_1 = A(\tau)u_1$, and that the scalar-valued function $A(\tau)$ must satisfy the equation

$$(u_1^* M u_1)A'' = \lambda_1(u_1^* F_{u\lambda} u_1)A + \tfrac{1}{2}(u_1^* F_{uu} u_1^2)A^2. \tag{2.41}$$

In the presence of an imperfection, an additional term $F_\epsilon \epsilon$ is added to the left side of (2.40). If ϵ can be regarded as of order ξ^2, equation (2.41) simply becomes altered to

$$(u_1^* M u_1)A'' = \lambda_1(u_1^* F_{u\lambda} u_1)A + \tfrac{1}{2}(u_1^* F_{uu} u_1^2)A^2 + u_1^* F_\epsilon \epsilon / \xi^2. \tag{2.42}$$

It should be noted that equation (2.41) has equilibrium solutions corresponding to

$$A = 0 \;\text{ and }\; A = \frac{-2\lambda_1(u_1^* F_{u\lambda} u_1)}{u_1^* F_{uu} u_1^2}.$$

This is exactly consistent with the static post-bifurcation analysis performed earlier (*c.f.* (2.24)).

The main point about the equation (2.41) is that it arose *generically*, from a study of a fairly general system with several degrees of freedom, as was announced in the Introduction, where an equation of this type arose from study of a simple one-dimensional example. Equation (2.42) can be reduced to the form (2.41) by adding a suitable constant to A. Equation (2.41) provides immediately an estimate for how large a perturbation can be, when the system is stable but close to instability. That is, it provides an estimate for the margin of stability. The phase portraits illustrated in Fig. 1.5 provide this information in graphical form.

It is left as a relatively simple exercise to analyze the case in which $u_1^* F_{uu} u_1^2 = 0$.

Flutter

It is appropriate at least to mention a phenomenon that is intrinsically dynamic in nature. It cannot occur unless the matrix F_u is non-symmetric. In this case, the possibility exists that two eigenvalues, μ_1 and μ_2 say, coincide at a load lower than λ_c (where λ_c still denotes the load at which $F_u(0, \lambda)$ first becomes singular), so that they are still positive, but then, as λ increases further, they split and become complex conjugate pairs, having, at least initially, positive real parts. The elementary static bifurcation theory gives no hint of trouble: there is no equilibrium solution close to the primary one $u = 0$. However, even a linearized dynamical analysis predicts instability, because there are values of ω (*c.f.* (2.36)) which will have negative imaginary parts. This type of bifurcation is called a *Hopf bifurcation*. Weakly-nonlinear analysis is possible for this situation also: the motion is basically harmonic, but with an amplitude that evolves slowly in time. It is more complicated than the analysis presented above, because of the need to deal simultaneously with two time-scales (the "ordinary" one which appears in the simple harmonic motion and the "slow" time τ, upon which the amplitude depends), and is not pursued in detail.

Conservative systems

If, in fact, $F(u, \lambda)$ is derived from a scalar potential Φ, so that $F(u, \lambda) = -\Phi_u(u, \lambda)$, then automatically the matrix $F_u = -\Phi_{uu}$ is symmetric. Conversely, if F_u is symmetric, a potential Φ exists. Then, assuming also that the "mass matrix" M is symmetric and independent of u, the equation of motion (2.7) has the following first integral[6]

$$\tfrac{1}{2}\dot{u}^T M \dot{u} + \Phi(u, \lambda) = E, \quad \text{constant}, \tag{2.43}$$

which is an expression of conservation of energy. Now let $u = u_0 + v$, where u_0 is an equilibrium solution so that $F(u_0, \lambda) = -\Phi_u(u_0, \lambda) = 0$, and v is a small time-

[6]Exposure to a course on Lagrangian and Hamiltonian dynamics would permit the derivation of this result in greater generality.

dependent perturbation. Expanding (2.41) about u_0 to second order (which is equivalent to linearizing (2.7)) gives

$$\tfrac{1}{2}\dot{v}^T M \dot{v} + \tfrac{1}{2}\Phi_{uu} v^2 = E - \Phi(u_0, \lambda). \tag{2.44}$$

The kinetic energy quadratic form is positive-definite. It follows that if, also, the quadratic form $\Phi_{uu} v^2$ is positive-definite, a perturbation that is bounded initially remains bounded, and therefore that the equilibrium configuration u_0 is stable against a small perturbation. Conversely, if the form $\Phi_{uu} v^2$ is indefinite (or even negative-definite), there will exist disturbances for which linearized analysis would predict exponential growth, and thus instability. This is the Dirichlet condition for stability: u_0 is stable if it attains a local minimum for the potential energy function Φ.

3 The Euler column

The approach outlined in the preceding section can be applied also when the structure is a continuum (so having an infinite number of degrees of freedom). A strict discussion would entail the introduction of suitable function spaces and corresponding norms. Such machinery would be out of place here. It is, nevertheless, still possible to track the reasoning presented in the preceding section virtually line-by-line, to derive at least a formal (and physically credible) description of the static and dynamic characteristics of a structure, close to a critical point. This will now be illustrated by considering the classical example of the buckling of the Euler column. The column is modelled as a one-dimensional structure that can support tension or compression, and also has resistance to bending. Such a structure (which is called an *elastica*) can resist torsion as well, but this aspect is not needed for the present example.

3.1 Equations of motion

It is necessary first to set up equations of motion. For this purpose, with reference to Fig. 3.1, the column (or beam) is modelled as initially straight. Arc length s measured, in the undeformed configuration from one end (labelled O), is taken as Lagrangian coordinate. Only deformations in the x, y plane are envisaged. Therefore, the deformed configuration at time t is specified by the mapping $s \to (x(s,t), y(s,t))$. The parts of the beam on either side of any point P exert a resultant force and couple on each other. The part OP experiences, at P, a force with components T along the beam and N normal to it, and a couple of moment M, as illustrated. The complementary part experiences the opposite force and couple. The unit tangent at P has components $(x', y')/(x'^2 + y'^2)^{1/2}$, and the normal has components $(-y', x')/(x'^2 + y'^2)^{1/2}$, the prime denoting differentiation with respect to s. Let the beam have mass per unit length m. Equating the rate of change of linear momentum of the section OP to the forces applied to it gives

$$\frac{d}{dt}\int_0^s m\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} ds = \left[\left\{T\begin{pmatrix}x'\\ y'\end{pmatrix} + N\begin{pmatrix}-y'\\ x'\end{pmatrix}\right\}/(x'^2+y'^2)^{1/2}\right]_0^s. \tag{3.1}$$

Equating the rate of change of moment of momentum to the total applied moment gives

$$\frac{d}{dt}\int_0^s m(x\dot{y} - y\dot{x})ds = \left[\frac{T(xy' - yx') + N(xx' + yy')}{(x'^2+y'^2)^{1/2}} + M\right]_0^s. \tag{3.2}$$

It follows by differentiation of these equations with respect to s that

$$m\ddot{x} \;=\; \{(Tx' - Ny')/(x'^2+y'^2)^{1/2}\}', \tag{3.3}$$

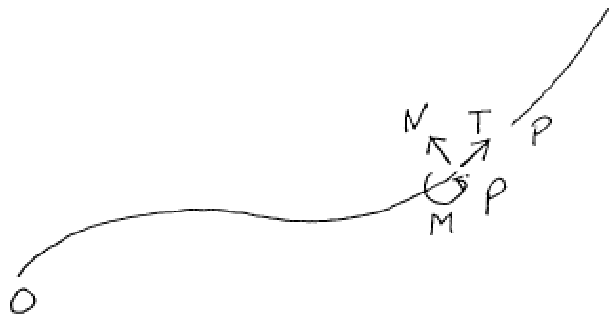

Fig. 3.1. Forces and couple on the section OP of the beam.

$$\begin{aligned} m\ddot{y} &= \{(Ty' + Nx')/(x'^2 + y'^2)^{1/2}\}', && (3.4)\\ 0 &= N(x'^2 + y'^2)^{1/2} + M', && (3.5) \end{aligned}$$

the last equation having been simplified by use of the first two.

The formulation is completed by appending constitutive equations, which characterise the response of the beam being considered. In general, T is related to the local stretch $(x'^2 + y'^2)^{1/2}$, and M is related to the "Lagrangian curvature" κ, where[7]

$$\kappa = \frac{y''x' - x''y'}{(x'^2 + y'^2)}. \qquad (3.6)$$

However, we will adopt the idealisation that the beam is inextensible. Then it is subject to the constraint

$$(x'^2 + y'^2) = 1 \qquad (3.7)$$

and T becomes an undetermined multiplier. The constitutive relation for M will be taken to be

$$M = f(\kappa) = B_1\kappa + B_3\kappa^3 + \cdots, \qquad (3.8)$$

where now the inextensibility constraint reduces κ to

$$\kappa = y''x' - x''y'. \qquad (3.9)$$

Equation (3.5) simplifies correspondingly.

[7]This choice is in the spirit of taking material derivatives. It is based on the formula $\kappa = d\{\tan^{-1}(y'/x')\}/ds$. More generally, M could depend on local stretch and κ. However, inextensibility will be assumed in any case.

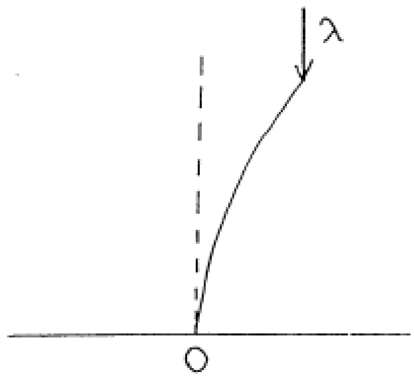

Fig. 3.2. The Euler column.

3.2 The problem

The problem to be studied is illustrated in Fig. 3.2. The column is initially vertical, and its end O is clamped so that $x = 0$ and $x' = 0$ when $s = 0$. A dead load of magnitude λ is applied, vertically downwards, at the upper end of the column, $s = l$. No moment is applied at the upper end. Therefore, the boundary conditions are that

$$\begin{aligned} x(0,t) = y(0,t) = 0, \quad & x'(0,t) = 0, \\ T(l,t) = -\lambda y'(l,t), \quad & N(l,t) = -\lambda x'(l,t), \quad M(l,t) = 0. \end{aligned} \tag{3.10}$$

Evidently, one solution of the equations of motion is

$$x = 0, \quad y = s, \quad \text{and} \quad T = -\lambda. \tag{3.11}$$

This corresponds to the fundamental equilibrium path u_0 of the preceding section; our basic objective is to examine its incremental uniqueness as λ increases, and its stability.

Static analysis

First, equilibrium configurations are studied by considering time-independent solutions. These satisfy the equations

$$(Tx' - Ny')' = 0, \tag{3.12}$$
$$(Ty' + Nx')' = 0, \tag{3.13}$$
$$N + M' = 0, \tag{3.14}$$
$$M = f(\kappa), \tag{3.15}$$
$$x'^2 + y'^2 = 1. \tag{3.16}$$

The first two of these equations may be integrated and the constants fixed from their known values at $s = l$. Solving the resulting two equations then gives

$$N = -\lambda x', \quad T = -\lambda y'. \tag{3.17}$$

Although this is not essential, it is convenient to satisfy the constraint (3.16) identically by setting

$$x' = \sin\phi, \quad y' = \cos\phi. \tag{3.18}$$

Then, $\kappa = -\phi'$. Substituting all of the relations so far established into (3.14) now yields

$$(B_1 + 3B_3\phi'^2 + \cdots)\phi'' + \lambda \sin\phi = 0. \tag{3.19}$$

The boundary conditions are $\phi(0,t) = \phi'(l,t) = 0$. As already observed, one solution is $\phi = 0$.

Bifurcation

To investigate bifurcation, suppose that there is another solution, close to $\phi = 0$. This can be investigated by linearizing (3.19):

$$B_1\phi'' + \lambda\phi = 0. \tag{3.20}$$

The solution for which $\phi(0) = 0$ is

$$\phi = A\sin[(\lambda/B_1)^{1/2}s], \tag{3.21}$$

and the smallest value of λ that satisfies the condition at $s = l$ (with $A \neq 0$) is

$$\lambda_c = \frac{B_1\pi^2}{4l^2}. \tag{3.22}$$

The post-bifurcation path can be studied asymptotically by setting

$$\phi = \xi\phi_1 + \xi^2\phi_2 + \cdots, \quad \lambda = \lambda_c + \xi\lambda_1 + \xi^2\lambda_2 + \cdots. \tag{3.23}$$

Then,

$$\begin{aligned}(B_1 + 3B_3\xi^2\phi_1'^2 + \cdots)(\xi\phi_1'' + \xi^2\phi_2'' + \cdots)\\ + (\lambda_c + \xi\lambda_1 + \xi^2\lambda_2 + \cdots)(\xi\phi_1 + \xi^2\phi_2 + \xi^3(\phi_3 - \tfrac{1}{6}\phi_1^3) + \cdots) = 0.\end{aligned} \tag{3.24}$$

Equating to zero the coefficient of ξ gives

$$B_1\phi_1'' + \lambda_c\phi_1 = 0, \tag{3.25}$$

which corresponds exactly to the linearization (3.20), and the associated boundary conditions. Thus,

$$\phi_1 = A\sin(\pi s/2l), \tag{3.26}$$

having taken into account the definition (3.22) of λ_c. The terms of order ξ^2 give

$$B_1\phi_2'' + \lambda_c\phi_2 + \lambda_1\phi_1 = 0. \tag{3.27}$$

There is no solution ϕ_2 that satisfies the boundary conditions unless $\lambda_1 = 0$. Then with this condition, ϕ_2 has the same form as ϕ_1 and nothing is lost if it is specified that $\phi_2 = 0$. The terms of order ξ^3 now give

$$B_1\phi_3'' + \lambda_c\phi_3 + 3B_3\phi_1'^2\phi_1'' + \lambda_2\phi_1 - \tfrac{1}{6}\lambda_c\phi_1^3 = 0. \tag{3.28}$$

The condition for consistency of this equation, with ϕ_3 satisfying the required boundary conditions, is obtained in exactly the same way as in Section 3. The analogue of the left eigenvector is the function $\sin(\pi s/2l)$. Multiplying equation (3.28) by $\sin(\pi s/2l)$ and integrating from 0 to l gives, necessarily,

$$\int_0^l \sin(\pi s/2l)\{3B_3\phi_1'^2\phi_1'' + \lambda_2\phi_1 - \tfrac{1}{6}\lambda_c\phi_1^3\}ds = 0, \tag{3.29}$$

since integration by parts and use of the boundary conditions cancels out the terms involving the still-unknown ϕ_3. Changing the variable of integration to $u = \pi s/2l$ and substituting explicitly for ϕ_1 gives

$$\int_0^{\pi/2} \sin u \left\{ A^3 \left[\frac{-3B_1\pi^4}{16l^4}\cos^2 u \sin u - \tfrac{1}{6}\lambda_c \sin^3 u \right] + A\lambda_2 \sin u \right\} du = 0. \tag{3.30}$$

The required integrals are

$$\int_0^{\pi/2} \sin^2 u du = \pi/4, \quad \int_0^{\pi/2} \sin^4 u\, du = 3\pi/16, \quad \int_0^{\pi/2} \cos^2 u \sin^2 u\, du = \pi/16. \tag{3.31}$$

Thus,

$$A\lambda_2 - A^3\left\{\frac{\lambda_c}{8} + \frac{3B_3\pi^4}{64l^4}\right\} = 0. \tag{3.32}$$

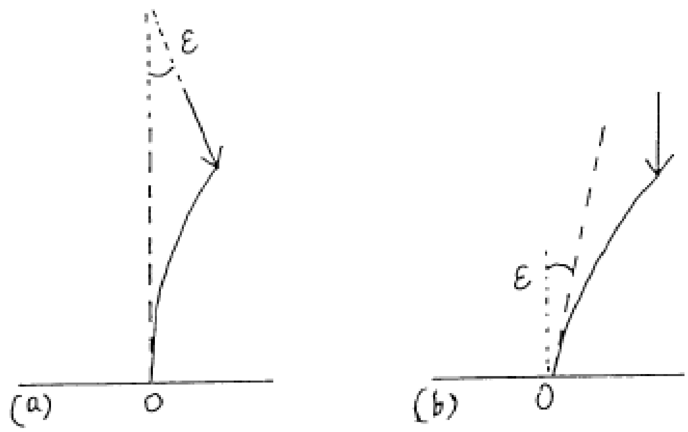

Fig. 3.3. Imperfect structure.
(a) Beam vertical, load off-vertical, (b) Beam off-vertical, load vertical.

Therefore, if $A \neq 0$, then A and λ_2 must be related so that

$$\lambda_2 = A^2 \left\{ \frac{\lambda_c}{8} + \frac{3B_3\pi^4}{64l^4} \right\}. \tag{3.33}$$

This equation is the analogue of (2.25) for the problem of the Euler column.

The effect of an imperfection

One obvious possible imperfection is that the beam may not be exactly straight. Analysis of this would require the development of equations of equilibrium (and also of motion) for such a beam. This is avoided here by considering an alternative simple model: the direction of the load λ is not exactly vertical but instead makes an angle ϵ with the downward vertical, as shown in Fig. 3.3(a). This is equivalent to vertical loading of a beam whose unloaded configuration is not quite vertical (as depicted in Fig. 3.3(b)), because gravity has already been disregarded in the discussion of the perfect structure, and will continue to be ignored here.

Equations (3.12) and (3.13) still apply, but now their integration in conjunction with specifying that the force at the end $s = l$ has horizonal and vertical components $\lambda \sin \epsilon$ and $-\lambda \cos \epsilon$ respectively gives

$$T = -\lambda(-x' \sin \epsilon + y' \cos \epsilon), \quad N = -\lambda(x' \cos \epsilon + y' \sin \epsilon). \tag{3.34}$$

Equivalently, with x' and y' expressed in terms of ϕ as in (3.18),

$$\begin{aligned} T &= -\lambda \cos(\phi + \epsilon) \sim -\lambda \cos \phi + \lambda \epsilon \sin \phi, \\ N &= -\lambda \sin(\phi + \epsilon) \sim -\lambda \sin \phi - \lambda \epsilon \cos \phi, \end{aligned} \tag{3.35}$$

having retained only the perturbation of order ϵ. This perturbation generates a term additional to those displayed in equation (3.19). The perturbed equation is

$$(B_1 + 3B_3\phi'^2 + \cdots)\phi'' + \lambda \sin\phi + \lambda\epsilon\cos\phi \sim 0. \tag{3.36}$$

The perturbation of the primary solution $\phi = 0$ due to the perturbation, when λ is close to λ_c, can be investigated by again postulating the expansions (3.23), and now substituting into (3.36). It is necessary to decide how to relate ϵ to ξ. If it is assumed that they are of the same order, then the term of order ξ in (3.37) gives

$$B_1\phi_1'' + \lambda_c\phi_1 + \lambda_c\epsilon/\xi = 0. \tag{3.37}$$

This equation has no solution ϕ_1 satisfying the boundary conditions unless the last term is zero. Equivalently, ϵ is (at least) of order ξ^2. Once this is assumed, it follows that $\phi_1 = \sin(\pi s/2l)$, exactly as before. The equation that results from considering the term of order ξ^2, subjected to similar reasoning, leads to the conclusion that ϵ should be of order ξ^3, if the perturbation expansion is to succeed. Then, ϕ_2 can be taken to be zero, without loss. The coefficient of ξ^3 now gives

$$B_1\phi_3'' + \lambda_c\phi_3 + 3B_3\phi_1'^2\phi_1'' + \lambda_2\phi_1 - \tfrac{1}{6}\lambda_c\phi_1^3 + \lambda_c\epsilon/\xi^3 = 0. \tag{3.38}$$

The condition for consistency is obtained by multiplying the equation by $\sin(\pi s/2l)$ and integrating from 0 to l. There is one additional term in comparison with (3.29). This is

$$\frac{\lambda_c\epsilon}{\xi^3}\int_0^l \sin(\pi s/2l)\,ds = \left(\frac{2l}{\pi}\right)\frac{\lambda_c\epsilon}{\xi^3}. \tag{3.39}$$

It follows that

$$A\lambda_2\frac{\pi}{4} - A^3\left\{\frac{\lambda_c\pi}{32} + \frac{3B_3\pi^5}{256l^4}\right\} + \frac{\lambda_c\epsilon}{\xi^3} = 0. \tag{3.40}$$

The equilibrium path for the imperfect structure, close to the critical point, therefore has the asymptotic form

$$\phi \sim (A\xi)\sin(\pi s/2l), \quad \lambda - \lambda_c \sim -\frac{4\lambda_c\epsilon}{\pi(A\xi)} + \left\{\frac{\lambda_c}{8} + \frac{3B_3\pi^4}{64l^4}\right\}(A\xi)^2. \tag{3.41}$$

A maximum load is indicated, if the term in curly brackets is negative. In this case, the maximum load is smaller than λ_c by an amount of order $\epsilon^{2/3}$.

3.3 Stability

The Dirichlet condition

The system under discussion is conservative. Therefore, one of the elementary ways to consider the stability of an equilibrium path is to investigate whether the configuration realises a local energy minimum (the Dirichlet condition). The energy is

$$E = \int_0^l [\tfrac{1}{2}B_1\phi'^2 + \tfrac{1}{4}B_3\phi'^4 + \cdots]\, ds + \lambda y(l), \tag{3.42}$$

where

$$y(l) = \int_0^l \cos\phi\, ds. \tag{3.43}$$

The requirement is to compare the energy evaluated at the solution with the energy evaluated for a neighbouring configuration. For a point on the primary path $\phi = 0$, the energy of a neighbouring configuration is given by (3.42), expanded to lowest non-trivial order when ϕ is small. The energy difference is, asymptotically,

$$\Delta E = E(\phi) - E(0) \sim \tfrac{1}{2}\int_0^l [B_1\phi'^2 - \lambda\phi^2]\, ds. \tag{3.44}$$

The only restriction on $\phi(s)$ is that $\phi(0) = 0$. Now we investigate whether there is any function ϕ (with $\phi(0) = 0$) for which ΔE is negative. Since ΔE is homogeneous of degree 2 it suffices to restrict ϕ further so that

$$\int_0^l \phi^2\, ds = 1.$$

It is necessary then to introduce a Lagrange multiplier, μ say, which has the effect of replacing λ by $\lambda + \mu$ in the functional (3.44). Candidate minimizers, subject to this constraint, can be found by perturbing ϕ to $\phi + \delta\phi$. The functional is stationary if

$$\int_0^l [B_1\phi'\delta\phi' - (\lambda + \mu)\phi\delta\phi]\, ds = 0 \tag{3.45}$$

for all allowed $\delta\phi$. Integrating the first term by parts and imposing the boundary condition gives

$$\phi'(l)\delta\phi(l) - \int_0^l [B_1\phi'' + (\lambda + \mu)\phi]\delta\phi\, ds = 0 \tag{3.46}$$

for all allowed $\delta\phi$. It follows that ϕ must satisfy

$$B_1\phi'' + (\lambda + \mu)\phi = 0,$$

and $\phi'(l) = 0$ in addition to $\phi(0) = 0$. There is no such stationary point (apart from $\phi = 0$) unless μ satisfies

$$(\lambda + \mu)/B_1 = (2k+1)^2\pi^2/4l^2$$

for some integer k. The corresponding ϕ is

$$\phi(s) = (2/l)^{1/2} \sin[(2k+1)\pi s/2l].$$

The stationary value of ΔE then follows as

$$\frac{1}{2}\left[\frac{\pi^2(2k+1)^2 B_1}{4l^2} - \lambda\right].$$

This is positive – and so the solution $\phi = 0$ is stable – for $0 \le \lambda < \lambda_c$.

Linearized dynamics

The study of dynamics requires a return to the system of equations (3.3)-(3.5) and (3.7)-(3.9). Linearized about the equilibrium solution $x = 0,\ \ y = s,\ \ T = -\lambda,\ \ N = 0$, they give

$$(-\lambda\hat{x}' - \hat{N})' \ = \ m\ddot{\hat{x}}, \tag{3.47}$$

$$\hat{N} - B_1\hat{x}''' \ = \ 0, \tag{3.48}$$

the quantities $\hat{x}$, etc. representing the perturbations. There is no equation for $\hat{y}$ because the constraint of inextensibility gives $\hat{y} \sim 0$. Elimination of $\hat{N}$ gives

$$-\lambda\hat{x}'' - B_1\hat{x}'''' = m\ddot{\hat{x}}. \tag{3.49}$$

The boundary conditions (3.10) imply that

$$\hat{x}(0,t) = \hat{x}'(0,t) = \hat{x}''(l,t) = \lambda\hat{x}'(l,t) + B_1\hat{x}'''(l,t) = 0. \tag{3.50}$$

Normal mode solutions may now be sought by assuming $\exp(i\omega t)$ time dependence. The partial differential equation (3.49) then implies

$$B_1\hat{x}'''' + \lambda\hat{x}'' - m\omega^2\hat{x} = 0. \tag{3.51}$$

The solution of this fourth order ordinary differential equation, together with the boundary conditions (3.50), is algebraically complicated. It will not be pursued further, except to remark that the problem so defined is an eigenvalue problem, since the differential equation and boundary conditions are homogeneous, and that any eigenvalue $m\omega^2$ must be real, because the problem is self-adjoint. To see this, multiply equation (3.51) by a function u and integrate from 0 to l. This gives, employing integration by parts,

$$\begin{aligned}\int_0^l uL\hat{x}\,ds &\equiv \int_0^l u[B_1\hat{x}'''' + \lambda\hat{x}'' - m\omega^2\hat{x}]\,ds \\ &= \int_0^l [B_1\hat{x}''u'' - \lambda\hat{x}'u' - m\omega^2\hat{x}u]\,ds - [B_1\hat{x}''u' - (\lambda\hat{x}' + B_1\hat{x}''')u]_0^l\,,\end{aligned} \tag{3.52}$$

having called the differential operator L. Thus, if both $\hat{x}$ and u satisfy the boundary conditions (3.50), the form on the right side of (3.52) is symmetric and it follows that

$$\int_0^l uL\hat{x}\,ds = \int_0^l \hat{x}Lu\,ds. \tag{3.53}$$

It is easy to deduce from this symmetry – just as for symmetric matrices – that eigenvalues $m\omega^2$ must be real. The primary solution is thus stable so long as all eigenvalues are positive, and this requirement is first lost when the smallest eigenvalue becomes zero. It is known already, from the preceding subsection, that this occurs when $\lambda = \lambda_c$.

Weakly-nonlinear dynamics

It is interesting to observe that, even though exact linearized analysis is complicated, asymptotic analysis close to the critical point is relatively easy, and furthermore nonlinear terms can be retained. The pattern follows that already established in Section 3. Let

$$\begin{aligned} x &= \xi x_1 + \xi^2 x_2 + \cdots, \\ y &= s + \xi y_1 + \xi^2 y_2 + \cdots, \\ T &= -\lambda_c + \xi T_1 + \xi^2 T_2 + \cdots, \\ N &= \xi N_1 + \xi^2 N_2 + \cdots, \\ \lambda &= \lambda_c + \xi^2\lambda_2, \\ \tau &= \xi t. \end{aligned} \tag{3.54}$$

Note that here, a decision has been taken from the outset to define the parameter ξ in terms of the departure of the load λ from its critical value. The functions x_r etc. are regarded as functions of s and τ. To save introducing more notation, in the equations to follow a superposed dot will mean differentiation with respect to τ.

The governing equations now become

$$
\begin{aligned}
\xi^3 m(\ddot{x}_1 + \xi\ddot{x}_2 + \cdots) &= \Big\{(-\lambda_c + \xi T_1 + \xi^2 T_2 + \cdots)(\xi x_1' + \xi^2 x_2' + \xi^3 x_3' + \cdots) \\
&\quad -(\xi N_1 + \xi^2 N_2 + \cdots)(1 + \xi y_1' + \xi^2 y_2' + \cdots)\Big\}', \qquad (3.55)\\
\xi^3 m(\ddot{y}_1 + \xi\ddot{y}_2 + \cdots) &= \Big\{(-\lambda_c + \xi T_1 + \xi^2 T_2 + \cdots)(1 + \xi y_1' + \xi^2 y_2' + \cdots) \\
&\quad +(\xi N_1 + \xi^2 N_2 + \cdots)(\xi x_1' + \xi^2 x_2' + \cdots)\Big\}', \qquad (3.56)\\
(\xi N_1 + \xi^2 N_2 + \xi^3 N_3 + \cdots) &+ B_1\Big\{(\xi y_1'' + \xi^2 y_2'' + \cdots)(\xi x_1' + \xi^2 x_2' + \cdots) \\
-(1 + \xi y_1' + \xi^2 y_2' + \cdots)&(\xi x_1'' + \xi^2 x_2'' + \cdots)\Big\}' + 3B_3\xi^3(x_1'')^2(-x_1''') = 0. \qquad (3.57)
\end{aligned}
$$

The boundary conditions (3.10) and the remaining equation (3.7) are expanded similarly. The coefficients of successive powers of ξ are now set to zero. First, the terms of order ξ give

$$
\begin{aligned}
(-\lambda_c x_1' - N_1)' &= 0, \\
(-\lambda_c y_1' + T_1)' &= 0, \\
N_1 - B_1 x_1''' &= 0. \qquad (3.58)
\end{aligned}
$$

The constraint (3.7) implies that $y_1' = 0$. Therefore,

$$T_1 = 0, \quad N_1 = -\lambda_c x_1', \quad -\lambda_c x_1' - B_1 x_1''' = 0. \qquad (3.59)$$

It follows (upon use of the boundary conditions) that

$$x_1' = A(\tau)\sin(\pi s/2l), \qquad (3.60)$$

the slowly-varying amplitude $A(\tau)$ being so far undetermined. Next, the terms of order ξ^2 give

$$
\begin{aligned}
(-\lambda_c x_2' - N_2)' &= 0, \\
(-\lambda_c y_2' + T_2 + N_1 x_1')' &= 0, \\
N_2 + B_1(-x_2'')' &= 0. \qquad (3.61)
\end{aligned}
$$

The constraint (3.7) gives

$$y_2' = -x_1'^2/2. \qquad (3.62)$$

It follows that $N_2 = -\lambda_c x_2'$, $T_2 = -\lambda_2 + \lambda_c x_1'^2/2$ and then $-\lambda_c x_2' - B_1 x_2''' = 0$. Thus, x_2' has the same form as x_1', and can without loss be set to zero. Now considering terms of order ξ^3,

$$
\begin{aligned}
m\ddot{x}_1 &= (-\lambda_c x_3' + T_2 x_1' - N_3 - N_2 y_1' - N_1 y_2')', \\
m\ddot{y}_1 &= (-\lambda_c y_3' + T_3 + N_2 x_1' + N_1 x_2')', \\
N_3 + B_1(y_2'' x_1' - x_3'' - y_2' x_1'')' &- 3B_3(x_1'')^2 x_1''' = 0. \qquad (3.63)
\end{aligned}
$$

The second of these equations implies that $T_3 = 0$, since $y_1 = 0$, $x_2' = 0$, so $N_2 = 0$ and the constraint gives $y_3' = 0$. The first and third equations simplify correspondingly:

$$\begin{aligned} m\ddot{x}_1 &= (-\lambda_c x_3' - \lambda_2 x_1' - N_3)', \\ N_3 - B_1 x_3''' - 3B_3(x_1'')^2 x_1''' &- \tfrac{1}{2} B_1 (x_1'^2 x_1'')' = 0. \end{aligned} \tag{3.64}$$

Therefore, eliminating N_3,

$$m\ddot{x}_1 + \lambda_2 x_1'' + \left\{ B_1 (x_1'^2 x_1'')'/2 + 3B_3 (x_1'')^2 x_1''' \right\}' = -\lambda_c x_3'' - B_1 x_3''''. \tag{3.65}$$

The boundary conditions for x_3 are

$$x_3(0,\tau) = x_3'(0,\tau) = x_3''(l,\tau) = 0, \quad \lambda_c x_3'(l,\tau) + B_1 x_3'''(l,\tau) = -\lambda_2 x_1'(l,\tau) - \tfrac{1}{2} B_1 [\{x_1'(l,\tau)\}^2 x_1''(l,\tau)]'. \tag{3.66}$$

The consistency condition for the existence of x_3 is obtained by multiplying by the eigenvector x_1 and integrating from 0 to l. Substituting the expression (3.60) for x_1' and integrating by parts as appropriate, the consistency condition becomes (with the change of variable $u = \pi s/2l$)

$$\begin{aligned} &\left(\frac{2l}{\pi}\right)^3 m\ddot{A} \int_0^{\pi/2} (1-\cos u)^2 \, du + \frac{\lambda_2 A l}{2} \\ &+ A^3 \left\{ \frac{\pi B_1}{4l} \int_0^{\pi/2} \sin u [2 \sin u \cos^2 u - \sin^3 u] \, du - 3B_3 \left(\frac{\pi}{2l}\right)^3 \int_0^{\pi/2} \sin^2 u \cos^2 u \, du \right\} = 0. \end{aligned} \tag{3.67}$$

Evaluating the integrals gives, finally,

$$\left(\frac{3\pi}{8} - 1\right) m\ddot{A} + \frac{\pi^3}{8l^2} A \left\{ \lambda_2 - A^2 \left[\frac{B_1 \pi^2}{32 l^2} + \frac{3B_3 \pi^4}{64 l^4} \right] \right\} = 0. \tag{3.68}$$

The equilibrium post-buckling relation (3.32) is recovered exactly by setting $\ddot{A} = 0$.

In conclusion of this discussion, it is remarked that the asymptotic analysis given above provides information on "slow dynamics" near a critical point but does not necessarily provide the complete picture. Depending on the details of the system, there could be other dynamical solution branches nearby, and nonlinear terms neglected in the low-order asymptotics could couple these to the motion calculated and introduce significant deviations, even where equation (3.68) predicts a periodic solution. Of course, if the equation predicts an unbounded solution, its validity is in any case restricted to the régime where $A(\tau)$ is of order unity. Equation (3.68) should therefore be interpreted as showing just how the system may first respond to a small departure from the primary solution path, close to the critical point.

4 Stability of continua

Considerations of the type described in Sections 2 and 3 apply also to bodies that have to be modelled as two- or three-dimensional continua. It has been remarked already that problems for continua are most usually approached by performing a discretization. There is, nevertheless, some advantage in discussing continua directly, for basic understanding and also because another phenomenon – that of *localisation* of deformation – is possible in a continuum. When this is likely to occur, it is important that any discretization should be *designed* so that it can track the deformation with sufficient accuracy. It is also important to understand when a problem is "ill-posed". Further comment will be made when localisation is discussed. First, however, the same basic sequence of reasoning as has already been seen in the preceding sections will be followed through.

4.1 Notation

A brief self-contained summary of nonlinear continuum mechanics – not, strictly, part of this course – is given in Section 6. This subsection simply records the main notation that is employed.

Under a deformation, a point initially at position $\mathbf{X}$, with Cartesian components $\{X_\alpha\}$, moves to $\mathbf{x}$, with Cartesian components $\{x_i\}$. The deformation gradient matrix $\mathbf{A}$ has components

$$A_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}.$$

Principal stretches λ_r, $r = 1, 2, 3$, are defined so that the symmetric matrix $\mathbf{A}^T\mathbf{A}$ has eigenvalues λ_r^2.

A strain measure $\mathbf{e}^f$ is defined, relative to a function f, to have the same principal axes as $\mathbf{A}^T\mathbf{A}$, and eigenvalues $f(\lambda_r)$. The function f is monotone increasing, $f(1) = 0$ and $f'(1) = 1$. Green strain corresponds to

$$f(\lambda) = \tfrac{1}{2}(\lambda^2 - 1).$$

The corresponding strain measure, denoted by $\mathbf{E}$, is

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{A}^T\mathbf{A} - \mathbf{I}),$$

where $\mathbf{I}$ denotes the identity.

The stress $\mathbf{T}^f$, conjugate to the strain $\mathbf{e}^f$, is defined so that $T^f_{\alpha\beta}\dot{e}^f_{\alpha\beta}$ is the rate of working of the stress per unit initial volume, during the deformation. For an elastic medium, with energy density function per unit initial volume W, expressed as a function of $\mathbf{e}^f$, it follows that

$$T^f_{\alpha\beta} = \frac{\partial W}{\partial e^f_{\alpha\beta}}.$$

The stress $\mathbf{T}^{(2)}$ that is conjugate to $\mathbf{E}$ is the second Piola–Kirchhoff stress tensor. It is also convenient to introduce $\mathbf{S}$, with components $S_{\alpha i}$, as the nominal stress tensor. Its transpose is also called the first Piola–Kirchhoff stress tensor, or the Boussinesq tensor. It has the property that the rate of working of the stress, per unit initial volume, is $S_{\alpha i}\dot{A}_{i\alpha}$, and it follows that

$$S_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}}.$$

4.2 Equilibrium

A three-dimensional body, in equilibrium under some system of loading, adopts a configuration that satisfies the equations of equilibrium

$$S_{\alpha i,\alpha} + \rho_0 b_i = 0, \quad \mathbf{X} \in \mathcal{B}_0. \tag{4.1}$$

This is simply the time-independent version of the equations of motion (*c.f.* (6.12)). The loading comprises the body-force $\mathbf{b}$, together with boundary conditions. At each point of $\partial\mathcal{B}_0$, three conditions must be given. For instance, all three components x_i of $\mathbf{x}$ may be prescribed, or all three components $N_\alpha S_{\alpha i}$ of surface traction may be given as functions of $\mathbf{X}$, or some mixture, such as the normal component of traction and the tangential components of $\mathbf{x}$. In addition, it may be that a component of traction is specified as a function not only of $\mathbf{X}$, but also of $\mathbf{x}$ and $\mathbf{A}$. It is possible, also, that the body-force $\mathbf{b}$ could depend on the current position $\mathbf{x}$ of the material point if, for example, it were applied via a non-uniform magnetic field.

In any case, it will be assumed here that body-force, and the given combination of surface displacements and surface tractions depend on a parameter λ, so that $\lambda = 0$ corresponds to no loading and the loading increases in some sense with λ. As in previous sections, there may be more than one solution branch, but for the branch being followed, first the question of uniqueness of the increment of solution associated with an increment of λ will be addressed. As previously, it is convenient to discuss rates of change in place of increments, while disregarding inertia. These must conform to the equilibrium condition

$$\dot{S}_{\alpha i,\alpha} + \rho_0 \dot{b}_i = 0, \quad \mathbf{X} \in \mathcal{B}_0, \tag{4.2}$$

which is the rate equation corresponding to (4.1), together with rate forms of the boundary conditions. In the most general configuration-dependent case (*c.f.* (6.73)), these would take the form

$$N_\alpha \dot{S}_{\alpha i} = f_i + k_{ij}\dot{x}_j + C_{i\beta j}\dot{x}_{j,\beta} \tag{4.3}$$

wherever $N_\alpha S_{\alpha i}$ is given. Here, $f_i = \partial\psi_i/\partial t$, $k_{ij} = \partial\psi_i/\partial x_j$ and $C_{i\beta j} = \partial\psi_i/\partial A_{j\beta}$. In addition, $\dot{b}_i$ could contain a term $(\partial b_i/\partial x_j)\dot{x}_j$.

It is also necessary to specify the constitutive relation of the body, in rate form.

Elastic body, simple boundary conditions

First, consider an elastic body, subjected to a combination of dead loading and given boundary displacements, as considered in Section 2.3. Thus, in (4.3), $k_{ij} = C_{i\beta j} = 0$. The body force $\mathbf{b}$ is similarly assumed to be of dead-loading type. The elastic constitutive relation (2.41), in rate form, gives

$$\dot{S}_{\alpha i} = c_{\alpha i\beta j}\dot{x}_{j,\beta}, \tag{4.4}$$

where

$$c_{\alpha i\beta j} = \frac{\partial^2 W}{\partial A_{i\alpha}\partial A_{j\beta}} \tag{4.5}$$

(*c.f.* (6.47)). We wish now to examine the uniqueness of the solution of the equilibrium equations (4.2) (in which $\dot{\mathbf{b}}$ is given as a function of $\mathbf{X}$), together with the constitutive relation (4.4) and boundary conditions.

The usual way to discuss uniqueness is to assume that there are two different solutions. Then their difference, denoted with the prefix Δ, satisfies the corresponding system of *homogeneous* equations. Thus,

$$\Delta\dot{S}_{\alpha i,\alpha} = 0, \quad \mathbf{X} \in \mathcal{B}_0, \tag{4.6}$$

where

$$\Delta\dot{S}_{\alpha i} = c_{\alpha i\beta j}\Delta\dot{x}_{j,\beta}, \tag{4.7}$$

together with homogeneous boundary conditions.

Now multiply equation (4.6) by $\Delta\dot{x}_i$, sum over i and integrate over $\mathcal{B}_0$. This gives

$$\begin{aligned} 0 &= \int_{\mathcal{B}_0} [\Delta\dot{x}_i c_{\alpha i\beta j}\Delta\dot{x}_{j,\beta}]_{,\alpha}\,\mathbf{dX} - \int_{\mathcal{B}_0} \Delta\dot{x}_{i,\alpha} c_{\alpha i\beta j}\Delta\dot{x}_{j,\beta}\mathbf{dX} \\ &= \int_{\partial\mathcal{B}_0} \Delta\dot{x}_i N_\alpha c_{\alpha i\beta j}\Delta\dot{x}_{j,\beta} dS_0 - \int_{\mathcal{B}_0} \Delta\dot{x}_{i,\alpha} c_{\alpha i\beta j}\Delta\dot{x}_{j,\beta}\mathbf{dX} \\ &= -\int_{\mathcal{B}_0} \Delta\dot{x}_{i,\alpha} c_{\alpha i\beta j}\Delta\dot{x}_{j,\beta}\mathbf{dX}, \end{aligned} \tag{4.8}$$

having employed the divergence theorem and made use of the fact that the boundary conditions are homogeneous.

Uniqueness of $\dot{x}_{i,\alpha}$ is guaranteed if

$$\int_{\mathcal{B}_0} \Delta \dot{x}_{i,\alpha} c_{\alpha i \beta j} \Delta \dot{x}_{j,\beta} \mathbf{dX} > 0 \tag{4.9}$$

for all $\Delta \dot{x}_i$ not identically zero, that are consistent with the boundary conditions. That is, $\Delta \dot{x}_i$ must be zero wherever on the boundary x_i is prescribed. Furthermore, $\dot{x}_i$ is unique provided it is prescribed at *some* point of the boundary. (Negative-definiteness of the quadratic form would do equally well, but it will be seen later that positive-definiteness is needed for stability).

A sufficient condition for (4.9) to hold is that

$$a_{i\alpha} c_{\alpha i \beta j} a_{j\beta} \geq 0, \forall \mathbf{X} \in \mathcal{B}_0, \tag{4.10}$$

with equality only if $a_{i\alpha} = 0$. It is also necessary for (4.9) in some cases. All-round dead loading, generating uniform stress and deformation, is an example. It is, however, possible that (4.9) may hold for all $\Delta \dot{x}_{i,\alpha}$ allowed by other boundary conditions, even if (4.10) does not hold. Suppose, conversely, that the minimum value of the quadratic form (4.9) is zero, and that it is attained for some $\Delta \dot{x}_i$ not identically zero. Then $\Delta \dot{x}_i$ satisfies the equilibrium equations (4.6), and hence the solution is *not* unique. Such a field is called an eigenmode.

Relation to work-conjugate variables

Insensitivity of the energy function W to rigid rotations implies that W can only depend on the deformation gradient $\mathbf{A}$ through some measure of strain, such as $\mathbf{e}^f$. Perhaps the simplest of these is the Green strain, $\mathbf{e}^{(2)} \equiv \mathbf{E} = \frac{1}{2}(\mathbf{A}^T\mathbf{A} - \mathbf{I})$. It follows from the chain rule for partial differentiation that

$$S_{\alpha i} = A_{i\gamma} \frac{\partial W}{\partial E_{\alpha\gamma}}. \tag{4.11}$$

Therefore,

$$\dot{S}_{\alpha i} = A_{i\gamma} \frac{\partial^2 W}{\partial E_{\alpha\gamma} \partial E_{\beta\delta}} A_{j\delta} \dot{A}_{j\beta} + \delta_{ij} \frac{\partial W}{\partial E_{\alpha\gamma}} \dot{A}_{i\gamma}. \tag{4.12}$$

Hence,

$$c_{\alpha i \beta j} = A_{i\gamma} L_{\alpha\gamma\beta\delta} A_{j\delta} + \delta_{ij} S_{\alpha k} B_{k\beta}, \tag{4.13}$$

where

$$L_{\alpha\gamma\beta\delta} = \frac{\partial^2 W}{\partial E_{\alpha\gamma}\partial E_{\beta\delta}} \tag{4.14}$$

and B^T is the inverse of A, so that $B_{k\alpha} = \partial\xi_\alpha/\partial x_k$, $A_{j\alpha}B_{k\alpha} = \delta_{kl}$. The formula (4.13) is derived in Section 6 in the wider context of inelastic deformations.

It is plausible, for example, that the quadratic form $e_{\alpha\gamma}L_{\alpha\gamma\beta\delta}e_{\beta\gamma}$ may be positive-definite with respect to *symmetric* $e_{\alpha\gamma}$. The form (4.13) can be expressed

$$a_{i\alpha}c_{\alpha i\beta j}a_{j\beta} = b_{ik}\{A_{i\gamma}A_{k\alpha}L_{\alpha\gamma\beta\delta}A_{j\delta}A_{l\beta} + \det(\mathbf{A})\delta_{ij}T_{kl}\}b_{jl}, \tag{4.15}$$

where $a_{i\alpha} = b_{ik}A_{k\alpha}$ or, equivalently, $b_{ik} = a_{i\alpha}B_{k\alpha}$. $\mathbf{T}$ is Cauchy stress. Evidently, the quadratic form (4.10), or (4.15), cannot be positive-definite if any of the principal Cauchy stresses are negative (this can be demonstrated by choosing b_{ik} to be skew-symmetric). Tensile stress, on the other hand, enhances the positive-definiteness. Therefore, in the presence of tensile stress, bifurcation from a uniform state of deformation, maintained by all-round dead loading, would need to be associated with the quadratic form generated from $L_{\alpha\gamma\beta\delta}$, becoming indefinite by a sufficiently large amount. This form is a property of the energy function W. It is remarked, however, that there is nothing special about the choice of the Green strain tensor $\mathbf{E}$, except that it made the calculations simple. If some other strain measure were employed, a formula of the same general form would result, but $L_{\alpha\gamma\beta\delta}$ would be different, and the difference would be accounted for by an additional term involving the current stress. The formula (6.35) provides this additional term.

General boundary conditions

If the boundary conditions are of the general configuration-dependent form discussed above, it is still possible to write down the system of linear partial differential equations and boundary conditions that govern any possible $\Delta\dot{x}_i$. Since the system is homogeneous, an eigenvalue problem for the loading parameter λ is defined, and any solution is again an eigenmode. No further discussion is given here.

Post-bifurcation behaviour

It is possible to study the initial post-bifurcation path, by following the pattern already established in Sections 2 and 3. An elementary example will be presented in the context of weakly-nonlinear dynamics.

4.3 Localisation

In contrast to the type of bifurcation which was envisaged above, we introduce now the notion that material may become locally unstable, in the sense of admitting the development of a discontinuity in the velocity field. If such a discontinuity survives for any finite time, a discontinuity in displacement ensues if the surface across which the discontinuity exists remains stationary; otherwise, it moves through the material, forming a shock. The formation of a stationary discontinuity is called *localisation*. In practice, discontinuities are not realised exactly. There will be some fine structure. This, however, is not captured by the simple constitutive description adopted so far. Although much is already known, understanding is still far from complete for inelastic solids. Here, we confine attention to the possible onset of localisation by identifying conditions under which the rate equations of equilibrium (4.2) permit the development of discontinuities in rates of stress and deformation gradient.

Suppose, therefore, that velocity is continuous but that stress-rate is discontinuous across a surface S_0 in the reference configuration, which maps onto the surface S in the current configuration. If velocity is continuous across S_0, the tangential components of its gradient must be continuous. Therefore, at most,

$$[\dot{x}_{i,\alpha}] = a_i N_\alpha \tag{4.16}$$

for some vector $\mathbf{a}$[8]. Here, $\mathbf{N}$ is the normal to S_0 and the square brackets denote the jump across S_0 of the quantity enclosed. The equations of equilibrium (4.2) cannot hold at S_0 but equilibrium still requires that

$$[N_\alpha \dot{S}_{\alpha i}] = 0. \tag{4.17}$$

This condition can be interpreted as an enforcement of (4.2) in the weak sense. Now substituting the constitutive relation (4.4) gives

$$c_{\alpha i \beta j} N_\alpha N_\beta a_j = 0. \tag{4.18}$$

This is the condition for localisation: equation (4.18) should have non-trivial solution $\mathbf{a}$ for some direction $\mathbf{N}$.

Note that if the quadratic form given in (4.10) is positive-definite, then

$$c_{\alpha i \beta j}(a_i N_\alpha)(a_j N_\beta) > 0 \tag{4.19}$$

[8] In fact, from (4.16), $\mathbf{a}$ is the jump in the normal derivative of $\dot{\mathbf{x}}$, $a_i = [N_\alpha \dot{x}_{i,\alpha}]$.

for all **a** and **N**, and localisation is impossible. The condition (4.19) is the condition for *strong ellipticity* of the system of partial differential equations (4.2) with (4.4). It is weaker than the condition (4.10) – and therefore it is possible that the onset of bifurcation may occur before the onset of localisation. There is, however, a result called Van Hove's theorem, that states that if displacements are prescribed over the whole of the boundary, then the solution of the rate problem is unique if (4.19) is satisfied. For this particular problem, therefore, bifurcation does not precede localisation.

Recall, again, that the constants $c_{\alpha i \beta j}$ that appear in (4.19) depend explicitly as well as implicitly on the current level of stress.

4.4 Linearized dynamics

Now suppose that the fundamental equilibrium solution, denoted with a superscript zero, is perturbed dynamically, with the loading held fixed. Denote the perturbed stress and perturbed position

$$S_{\alpha i} = S^0_{\alpha i} + s_{\alpha i}, \quad x_i = x^0_i + u_i. \tag{4.20}$$

Then, the linearized equations of motion give

$$s_{\alpha i,\alpha} = \rho_0 \ddot{u}_i, \tag{4.21}$$

together with homogeneous boundary conditions, and (still considering an elastic body) the constitutive relations

$$s_{\alpha i} = c_{\alpha i \beta j} u_{j,\beta}. \tag{4.22}$$

Normal mode solutions have the time-dependence $\exp(\mathrm{i}\omega t)$ and satisfy the system of equations

$$(c_{\alpha i \beta j} u_{j,\beta})_{,\alpha} + \rho_0 \omega^2 u_i = 0. \tag{4.23}$$

For the simple types of boundary conditions considered above, these equations are self-adjoint and all eigenvalues ω^2 are real[9]. For sufficiently small loads, it is reasonable to expect that the equilibrium configuration is stable, and therefore that the eigenvalues ω^2 are all positive. Instability first becomes possible when the smallest eigenvalue is zero. The usual argument, involving multiplication of the equation by u_i and integrating over $\mathcal{B}_0$, gives

$$\omega^2 \int_{\mathcal{B}_0} \rho_0 u_i u_i \mathbf{dX} = \int_{\mathcal{B}_0} u_{i,\alpha} c_{\alpha i \beta j} u_{j,\beta} \mathbf{dX}. \tag{4.24}$$

[9]The proof is very similar to that given in detail for the Euler column.

Thus, the smallest eigenvalue becomes zero at the value of λ for which

$$\min \int_{\mathcal{B}_0} u_{i,\alpha} c_{\alpha i\beta j} u_{j,\beta} \mathbf{dX} = 0, \tag{4.25}$$

the minimum being taken over fields u_i that are compatible with any given displacements on the boundary, and for which $\int_{\mathcal{B}_0} \rho_0 u_i u_i \, \mathbf{dX} = 1$. Equation (4.25) defines $\mathbf{u}$ as an eigenmode, as introduced in the context of static bifurcation.

It may be noted that the potential energy of the system is given by (6.51). Therefore, the difference in energy, between the configurations $\mathbf{x}$ and $\mathbf{x}^0$, is

$$\Delta E = \int_{\mathcal{B}_0} \left[W(x^0_{i,\alpha} + u_{i,\alpha}) - W(x^0_{i,\alpha}) - \rho_0 b_i u_i \right] \mathbf{dX} - \int_{\partial \mathcal{B}_0} N_\alpha S^0_{\alpha i} u_i \, dS_0. \tag{4.26}$$

Expanded to second order in $u_{i,\alpha}$, this gives

$$\Delta E \sim \int_{\mathcal{B}_0} \tfrac{1}{2} u_{i,\alpha} c_{\alpha i\beta j} u_{j,\beta} \, \mathbf{dX}. \tag{4.27}$$

The term linear in u_i and $u_{i,\alpha}$ vanishes because the energy is stationary at $\mathbf{x}^0$. The configuration $\mathbf{x}^0$ is *stable* (all eigenvalues ω^2 are positive) if ΔE, as given by (4.27), is positive-definite. This is the Dirichlet condition for stability; it coincides exactly with the condition (4.9).

Wave propagation

Consider now an infinitesimal plane wave disturbance, propagating through uniform material, uniformly pre-deformed to the level defined by $\mathbf{S}^0$ and $\mathbf{x}^0$. The general plane wave has the form

$$u_i = a_i f(t - N_\alpha X_\alpha / c), \tag{4.28}$$

where $\mathbf{a}$ is the amplitude of the wave and c is its speed. Substituting this form into the equations of motion (4.21) gives

$$[c_{\alpha i\beta j} N_\alpha N_\beta - \rho_0 c^2 \delta_{ij}] a_j f''(t - \mathbf{N} \cdot \mathbf{X}/c) = 0. \tag{4.29}$$

This is satisfied, for any wave-form f, if

$$[c_{\alpha i\beta j} N_\alpha N_\beta - \rho_0 c^2 \delta_{ij}] a_j = 0. \tag{4.30}$$

The matrix $c_{\alpha i\beta j} N_\alpha N_\beta$ is symmetric and therefore has real eigenvalues. The corresponding wave speeds are real so long as the eigenvalues are positive. This is precisely the condition for strong ellipticity of the static equations, which precludes localisation. There is an obvious sense in which the material can be regarded as locally stable: there should exist three real wave speeds[10]. Conversely, localisation of deformation occurs when some disturbance cannot propagate, and therefore has no alternative but to build up. $c_{\alpha i\beta j}N_\alpha N_\beta$ is also called the acoustic tensor, or the Christoffel tensor. If it has three positive eigenvalues, the equations of motion (4.21) are called totally hyperbolic. It should be noted that, when the equations of motion are *not* totally hyperbolic, the "usual" problem in which initial values of $\mathbf{u}$ and $\dot{\mathbf{u}}$ are prescribed, becomes *ill-posed*. Conversely, when the condition (4.30) for localisation is met, the corresponding problems for equilibrium fail to be elliptic, and problems with the usual kinds of boundary conditions become ill-posed. The correct resolution must be to admit into the physical model features so far neglected: viscosity, dependence of stress on higher-order gradients of deformation, etc. This *should* be reflected in any finite-element representation. If it is not, the discretized problem will have a solution but it is unavoidably mesh-dependent. Arbitrary choice of any particular mesh is equivalent to the injection of some additional physics. If this is not identified explicitly, there is no reason to suppose that the finite-element model will reflect physical reality. It has to be remarked that this expedient is nevertheless frequently adopted by practitioners!

4.5 Weakly-nonlinear dynamics

The dynamics of the system will now be investigated, at a load close to that which produces bifurcation. Towards this end, let the primary solution be $\mathbf{x}^0$, and let the given tractions and body-forces be t_i^0 and b_i^0. These all depend on the parameter λ. The bifurcation level is given the superscript c in place of 0. Now modify the applied loading, so that

$$b_i = b_i^c + \xi b_i^{(1)} + \xi^2 b_i^{(2)}, \tag{4.31}$$

and any given components of displacement or traction on the boundary have the forms

$$x_i = x_i^c + \xi x_i^{(1)} + \xi^2 x_i^{(2)}, \;\; t_i = t_i^c + \xi t_i^{(1)} + \xi^2 t_i^{(2)}, \;\; \mathbf{X} \in \partial\mathcal{B}_0. \tag{4.32}$$

It is important that the quantities with superscript "(1)" should be directed tangentially to the original loading path (that is, $b_i^{(1)}$ is proportional to $db_i^0/d\lambda$, etc.), but the quantities with superscript "(2)" are unrestricted. Let the perturbation $\mathbf{u}$ have the expansion

$$\mathbf{u} = \xi\mathbf{u}^{(1)} + \xi^2\mathbf{u}^{(2)} + \cdots. \tag{4.33}$$

[10]Counting multiplicity: it does not matter if two wave speeds coincide, as in the case of an unstressed isotropic material.

The perturbed stress satisfies

$$s_{\alpha_i} = c_{\alpha i\beta j}u_{j,\beta} + d_{\alpha i\beta j\gamma k}u_{j,\beta}u_{k,\gamma} + \cdots, \tag{4.34}$$

where

$$d_{\alpha i\beta j\gamma k} = \tfrac{1}{2}\frac{\partial^3 W}{\partial A_{i\alpha}\partial A_{j\beta}\partial A_{k\gamma}}(\mathbf{A}^0). \tag{4.35}$$

The equation of motion governing the perturbation is

$$s_{\alpha i,\alpha} + \rho_0(\xi b_i^{(1)} + \xi^2 b_i^{(2)}) = \rho_0\ddot{u}_i, \tag{4.36}$$

exactly. Assume that the fields with the superscripts "(1)" or "(2)" depend on time only through the "slow" variable

$$\tau = \xi^{1/2}t.$$

Then the equations of motion give, upon substituting the series,

$$[c_{\alpha i\beta j}(\xi u_{j,\beta}^{(1)} + \xi^2 u_{j,\beta}^{(2)}],_\alpha + \xi^2[d_{\alpha i\beta j\gamma k}u_{j\beta}^{(1)}u_{k,\gamma}^{(1)}],_\alpha + \rho_0(\xi b_i^{(1)} + \xi^2 b_i^{(2)}) = \rho_0\xi^2 u_i^{(1)\prime\prime} + O(\xi^3), \tag{4.37}$$

where the prime denotes differentiation with respect to τ. The boundary conditions are, to order ξ^2, that

$$\begin{aligned} &\textit{either} \quad \xi u_i^{(1)} + \xi^2 u_i^{(2)} = \xi x_i^{(1)} + \xi^2 x_i^{(2)} \\ &\textit{or} \quad N_\alpha[c_{\alpha i\beta j}(\xi u_{j,\beta}^{(1)} + \xi^2 u_{j,\beta}^{(2)}) + \xi^2 d_{\alpha i\beta j\gamma k}u_{j,\beta}^{(1)}u_{k,\gamma}^{(1)}] = \xi t_i^{(1)} + \xi^2 t_i^{(2)}, \;\; \mathbf{X} \in \partial\mathcal{B}_0. \end{aligned} \tag{4.38}$$

Equating terms of order ξ gives

$$[c_{\alpha i\beta j}u_{j,\beta}^{(1)}],_\alpha + b_i^{(1)} = 0, \tag{4.39}$$

together with the corresponding boundary conditions. Now *by hypothesis*, this system *does* have a solution, which continues the primary branch. Call this $\hat{u}_i^{(1)}$. It is not unique, however. Therefore,

$$u_i^{(1)} = \hat{u}_i^{(1)} + A(\tau)v_i^{(1)} \tag{4.40}$$

for some function $A(\tau)$, where here the eigenmode has been designated $\mathbf{v}^{(1)}$. Now equating terms of order ξ^2,

$$[c_{\alpha i\beta j}u_{j,\beta}^{(2)} + d_{\alpha i\beta j\gamma k}(\hat{u}_{j,\beta}^{(1)} + Av_{j,\beta}^{(1)})(\hat{u}_{k,\gamma}^{(1)} + Av_{k,\gamma}^{(1)})],_\alpha + b_i^{(2)} = \rho_0 A'' v_i^{(1)}. \tag{4.41}$$

Here, it has been assumed that the terms with superscript "(1)" are independent of τ. Differentiation with respect to τ is indicated by a prime. The corresponding boundary conditions are

$$\begin{aligned} &\textit{either} \quad u_i^{(2)} = x_i^{(2)} \\ &\textit{or} \quad N_\alpha[c_{\alpha i\beta j}u_{j,\beta}^{(2)} + d_{\alpha i\beta j\gamma k}(\hat{u}_{j,\beta}^{(1)} + Av_{j,\beta}^{(1)})(\hat{u}_{k,\gamma}^{(1)} + Av_{k,\gamma}^{(1)})] = t_i^{(2)}, \;\; \mathbf{X} \in \partial\mathcal{B}_0. \end{aligned} \tag{4.42}$$

The condition of consistency, for a solution $\mathbf{u}^{(2)}$ to exist, can be found by multiplying equation (4.41) by $v_i^{(1)}$ and integrating over $\mathcal{B}_0$. This gives, upon use of the divergence theorem,

$$\begin{aligned} &\int_{\mathcal{B}_0}\{v_i^{(1)}(b_i^{(2)} - \rho_0 A'' v_i^{(1)}) - v_{i,\alpha}^{(1)} d_{\alpha i\beta j\gamma k}(\hat{u}_{j,\beta}^{(1)} + Av_{j,\beta}^{(1)})(\hat{u}_{k,\gamma}^{(1)} + Av_{k,\gamma}^{(1)})\}\mathbf{dX} \\ &\qquad + \int_{\partial\mathcal{B}_0}[v_i^{(1)}t_i^{(2)} - v_{i,\alpha}^{(1)}c_{\alpha i\beta j}N_\beta x_j^{(2)}]dS_0 = 0. \end{aligned} \tag{4.43}$$

The first term in the surface integral only involves the given components $t_i^{(2)}$ because $v_i^{(1)}$ satisfies homogeneous boundary conditions and therefore is zero wherever $t_i^{(2)}$ is *not* given. The symmetry

$$c_{\alpha i\beta j} = c_{\beta j\alpha i}$$

of the elastic constants ensures that the second term similarly involves only the prescribed components $x_j^{(2)}$. This symmetry also ensured that $[v_{i,\alpha}^{(1)} c_{\alpha i\beta j}]_{,\beta} = 0$, which was also exploited in the derivation.

Equation (4.43) is written more tidily as

$$A'' \int_{\mathcal{B}_0} \rho_0 v_i^{(1)} v_i^{(1)} \, \mathbf{dX} + 2A \int_{\mathcal{B}_0} v_{i,\alpha}^{(1)} d_{\alpha i\beta j\gamma k} \hat{u}_{j,\beta}^{(1)} v_{k,\gamma}^{(1)} \, \mathbf{dX} + A^2 \int_{\mathcal{B}_0} v_{i,\alpha}^{(1)} d_{\alpha i\beta j\gamma k} v_{j,\beta}^{(1)} v_{k,\gamma}^{(1)} \, \mathbf{dX}$$
$$= \int_{\mathcal{B}_0} (v_i^{(1)} b_i^{(2)} - v_{i,\alpha}^{(1)} d_{\alpha i\beta j\gamma k} \hat{u}_{j,\beta}^{(1)} \hat{u}_{k,\gamma}^{(1)}) \, \mathbf{dX} + \int_{\partial\mathcal{B}_0} (v_i^{(1)} t_i^{(2)} - N_\alpha c_{\alpha i\beta j} v_{j,\beta}^{(1)} x_i^{(2)}) \, dS_0. \tag{4.44}$$

Equilibrium points are found by setting $A'' = 0$ in (4.44). The resulting algebraic equation embodies the static post-bifurcation response. It is slightly different in form from equation (3.20) which was derived for a system with a finite number of degrees of freedom for two reasons. One is that the fundamental solution is here not identified as zero. The other is that the extra loading, described by $\mathbf{b}^{(2)}$ etc., is somewhat more general, in that it may define some deviation from the loading path of the primary solution.

If the term containing A^2 should vanish (as it could if the system and its loading path had some symmetry), then a different parameterisation would be needed: the preceding Section which dealt with the Euler column provides a template.

The "health warning" given at the end of the corresponding discussion for the Euler column is repeated here: the "weakly nonlinear" dynamics developed here are interesting and relevant but, depending on the details of the problem, it is possible that there could be other solution branches nearby whose analysis would require more sophisticated methods.

4.6 Inelastic media

This Section is concluded with a very brief discussion of inelastic media. The discussion will be confined to equilibrium problems. The equation for continuing equilibrium remains (4.2). Now, however, a different constitutive relation is adopted. The relation (4.4) becomes nonlinear because the tangent moduli $c_{i\alpha j\beta}$ are homogeneous functions of degree zero in $\dot{\mathbf{A}}$. The tangent moduli are, in fact, often taken to be *piecewise constant* functions: for example, the plastic response of a single crystal is usually viewed as resulting from slip on a definite set of slip systems, and the tangent moduli take constant values which depend on which slip systems are activated. If, in addition, the moduli have the symmetry

$$c_{\alpha i\beta j} = c_{\beta j\alpha i},$$

the relation between stress-rate and deformation-rate can be given in the form

$$\dot{S}_{\alpha i} = \partial U / \partial \dot{A}_{i\alpha}, \tag{4.45}$$

where

$$U(\dot{\mathbf{A}}) = \tfrac{1}{2} \dot{A}_{i\alpha} c_{\alpha i \beta j} \dot{A}_{j\beta}. \tag{4.46}$$

so long as certain other conditions are also met, to ensure that the potential (4.46) is continuous and differentiable.

The constitutive response can be modelled for many materials by the relation (4.45), where U is any homogeneous function of degree 2 in $\dot{\mathbf{A}}$. This model will be assumed in the discussion to follow. It is justified at least for the usual model of single crystal response, and, in fact, then follows (from micromechanical considerations) for any polycrystalline material whose individual crystals respond in this way.

Incremental uniqueness and bifurcation

Suppose that the rate equations of equilibrium admit two different solutions, with superscripts 1 and 2. Then

$$\dot{S}^k_{\alpha i,\alpha} + \rho_0 \dot{b}_i = 0, \quad (k = 1, 2), \tag{4.47}$$

where

$$\dot{S}^k_{\alpha i} = \frac{\partial U}{\partial \dot{A}_{i\alpha}} (\dot{\mathbf{A}}^k). \tag{4.48}$$

The fields must also satisfy boundary conditions, which will be taken as a mixture of prescribed displacements or dead-load tractions, as before. It follows that

$$\int_{\mathcal{B}_0} (\dot{A}^1_{i\alpha} - \dot{A}^2_{i\alpha}) \left(\frac{\partial U}{\partial \dot{A}_{i\alpha}} (\dot{\mathbf{A}}^1) - \frac{\partial U}{\partial A_{i\alpha}} (\dot{\mathbf{A}}^2) \right) \mathbf{dX} = 0. \tag{4.49}$$

This result follows from the use of the divergence theorem and the equations of equilibrium, coupled with the fact that both fields satisfy the same boundary conditions.

Suppose first that U is a *strictly convex* function of $\dot{\mathbf{A}}$: this is a generalisation of the condition (4.10). It is a general property of a strictly convex function (of $\dot{\mathbf{A}}$) that, for any $\dot{\mathbf{A}}^1$ and $\dot{\mathbf{A}}^2$,

$$(\dot{A}^1_{i\alpha} - \dot{A}^2_{i\alpha}) \left(\frac{\partial U}{\partial \dot{A}_{i\alpha}} (\dot{\mathbf{A}}^1) - \frac{\partial U}{\partial A_{i\alpha}} (\dot{\mathbf{A}}^2) \right) \geq 0, \tag{4.50}$$

with equality only if $\dot{\mathbf{A}}^1 = \dot{\mathbf{A}}^2$. It follows that the solution of the rate problem is unique if U is strictly convex.

Now introduce a "comparison potential"

$$U^0(\dot{\mathbf{A}}) = \tfrac{1}{2}\dot{A}_{i\alpha}c^0_{\alpha i\beta j}\dot{A}_{j\beta}, \tag{4.51}$$

where the $c^0_{\alpha i\beta j}$ are constants, with the symmetry specified above. Suppose now that the function $U - U^0$ is convex. It follows that

$$(\dot{A}^1_{i\alpha} - \dot{A}^2_{i\alpha})\left(\frac{\partial(U-U^0)}{\partial\dot{A}_{i\alpha}}(\dot{\mathbf{A}}^1) - \frac{\partial(U-U^0)}{\partial\dot{A}_{i\alpha}}(\dot{\mathbf{A}}^2)\right) \geq 0, \tag{4.52}$$

and hence that

$$\int_{\mathcal{B}_0}(\dot{A}^1_{i\alpha} - \dot{A}^2_{i\alpha})\left(\frac{\partial U}{\partial\dot{A}_{i\alpha}}(\dot{\mathbf{A}}^1) - \frac{\partial U}{\partial A_{i\alpha}}(\dot{\mathbf{A}}^2)\right)\mathbf{dX} \geq \int_{\mathcal{B}_0}(\dot{A}^1_{i\alpha} - \dot{A}^2_{1\alpha})c^0_{\alpha i\beta j}(\dot{A}^1_{j\beta} - \dot{A}^2_{j\beta})\,\mathbf{dX}. \tag{4.53}$$

Uniqueness is assured if the quadratic form on the right side of (4.53) is positive-definite for all $\Delta\dot{\mathbf{A}} \equiv (\dot{\mathbf{A}}^1 - \dot{\mathbf{A}}^2)$ consistent with the boundary conditions. It is not necessary that the function U^0 should be convex. The advantage of the use of a comparison potential is that the form on the right side of (4.53) can be investigated by studying its "eigenmodes", using standard methods of linear analysis. Positive-definiteness of the form is sufficient for uniqueness of the rate problem for the actual material. The prediction of bifurcation for the comparison medium does not, however, automatically imply bifurcation for the actual medium. Whether or not it does depends (at least) on the choice of comparison medium, and needs to be considered case by case. The device of introducing a comparison potential was introduced by R. Hill, and the uniqueness result that follows from (4.53) is referred to as "Hill's comparison theorem".

Localisation

Now consider again the possibility that rates of stress and deformation gradient may be discontinuous across the surface which maps back to S_0 in the reference configuration. The relations (4.16) and (4.17) now imply that

$$N_\alpha\left(\frac{\partial U}{\partial\dot{A}_{i\alpha}}(\dot{\mathbf{A}}^2) - \frac{\partial U}{\partial\dot{A}_{i\alpha}}(\dot{\mathbf{A}}^1)\right) = 0, \tag{4.54}$$

where

$$\dot{A}^2_{i\alpha} = \dot{A}^1_{i\alpha} + a_i N_\alpha. \tag{4.55}$$

Multiplying (4.53) by a_i and summing over i then yields

$$(\dot{A}^2_{i\alpha} - \dot{A}^1_{i\alpha}) \left(\frac{\partial U}{\partial \dot{A}_{i\alpha}}(\dot{\mathbf{A}}^2) - \frac{\partial U}{\partial \dot{A}_{i\alpha}}(\dot{\mathbf{A}}^1) \right) = 0. \tag{4.56}$$

It follows immediately that localisation cannot occur if the potential U is convex, for then the convexity condition (4.50) together with (4.56) implies that $\mathbf{a} = 0$. It may be noted further that localisation is similarly excluded if (4.50) holds for all $\dot{\mathbf{A}}^1$ and $\dot{\mathbf{A}}^2$ that satisfy (4.55). This states that $\dot{\mathbf{A}}^1$ and $\dot{\mathbf{A}}^2$ differ by a matrix of rank one (a matrix with only one linearly independent row or column). The associated restriction on U is that U is rank-one convex. Of course if a comparison medium is introduced, and localisation cannot occur in that medium, then it cannot occur in the actual medium. However, there is no advantage in introducing a comparison potential U^0 just for discussing localisation, since the criterion for localisation is purely algebraic.

Media not satisfying normality

If the tangent moduli of a material do not possess the symmetry $c_{\alpha i \beta j} = c_{\beta j \alpha i}$, it is possible for two real eigenvalues of the acoustic tensor $c_{i\alpha j\beta} N_\alpha N_\beta$ to be real up to some level of deformation at which they coalesce and then, for increased deformation, move into the complex plane as complex conjugate pairs. It should be noted that the equations of continuing equilibrium remain elliptic, because the determinant of the acoustic tensor remains non-zero. Therefore, depending on the boundary conditions, it is possible that the equilibrium path suffers no bifurcation. However, if any dynamic disturbance is considered, the corresponding equations of motion must be employed, and these cease to be totally hyperbolic. The natural initial value problems become ill-posed. By analogy with flutter, this particular type of material condition is called the flutter ill-posedness. Exactly what to do about it is so far undecided. Analysis to date has been confined almost exclusively to identifying when the condition might occur. Clearly features of material response that are usually unimportant and so are neglected in the models discussed above need to be recognised and allowed for. These will include rate-dependence (for which systematic descriptions exist) and non-local response, upon which there is as yet no universal agreement.

4.7 Illustrative example

The formulae derived above will now be developed more explicitly, for the constitutive model (6.67). Thus, during plastic loading,

$$\dot{\mathbf{e}} = \mathcal{M}\dot{\mathbf{T}} + \mathbf{P}\mathbf{Q}^T\dot{\mathbf{T}}/h. \tag{4.57}$$

Here, $\mathbf{T}$ and $\mathbf{e}$ are conjugate measures of stress and strain; a particular choice will be made below.

The first task is to invert the relation (4.57). Elementary algebra yields

$$\dot{\mathbf{T}} = \mathcal{L}\dot{\mathbf{e}} - \frac{\mathcal{L}\mathbf{P}\mathbf{Q}^T\mathcal{L}\dot{\mathbf{e}}}{\mathbf{Q}^T\mathcal{L}\mathbf{P} + h}. \tag{4.58}$$

Thus, during loading,

$$\mathbf{L} = \mathcal{L} - \frac{\mathcal{L}\mathbf{P}\mathbf{Q}^T\mathcal{L}}{\mathbf{Q}^T\mathcal{L}\mathbf{P} + h}. \tag{4.59}$$

Now to be definite, take the conjugate stress and strain pair to be the second Piola-Kirchhoff stress $\mathbf{T}^{(2)}$ and the Green strain $\mathbf{e}^{(2)} \equiv \mathbf{E}$. Then, the corresponding moduli relating rates of nominal stress and deformation gradient are given by (6.71). If, in addition, the current state is chosen as reference configuration, so that $\mathbf{A} = \mathbf{I}$ at the present instant, it follows that

$$c_{kilj} = \mathcal{L}_{kilj} + \delta_{ij}T_{kl} - \frac{\mathcal{L}_{kipq}P_{qp}Q_{rs}\mathcal{L}_{srlj}}{P_{pq}\mathcal{L}_{qpsr}Q_{rs} + h}. \tag{4.60}$$

Here, since the reference configuration is the current one, Greek suffixes have been dispensed with. Also, since at this instant all measures of stress coincide, $\mathbf{T}$ can be viewed equally as second Piola-Kirchhoff stress, or nominal stress, or Cauchy stress. It is important to remember, though, that the moduli c_{kilj} relate the nominal stress-rate, given by (6.38) with (6.36), to the rate of deformation.

Now specialise further, by taking

$$\mathbf{P} = \mathbf{Q} = \mathbf{T}'/\|\mathbf{T}'\|, \tag{4.61}$$

where $\mathbf{T}'$ is the deviatoric stress

$$T'_{ij} = T_{ij} - \tfrac{1}{3}\delta_{ij}T_{kk} \tag{4.62}$$

and $\|\mathbf{T}'\| = (T'_{ij}T'_{ji})^{1/2}$. It is consistent with the implied isotropy to take

$$\mathcal{L}_{kilj} = \lambda\delta_{ki}\delta_{lj} + \mu(\delta_{kl}\delta_{ij} + \delta_{kj}\delta_{li}). \tag{4.63}$$

The relation (4.61) means that plastic strain-rate and deviatoric stress are "parallel" to one another; volume change and hydrostatic stress are related elastically. The constitutive relation so developed is a generalisation to finite deformations of the conventional J_2, or von Mises theory, which states that plastic deformation occurs when the magnitude of the shear stress, $\|\mathbf{T}'\|$, reaches a critical value. The coefficient h defines the amount of hardening: during simple shear, it relates the rate of increase of the shear stress to the rate of increase of the plastic part of the strain-rate[11]. It should be emphasised that the constitutive relation that has been developed depends on the choice of conjugate stress and strain measures. Any other choice would provide a different generalisation of the small-deformation theory. It should be noted, however, that this or any other choice of conjugate variables yields a tangent modulus tensor that displays the symmetry $c_{kilj} = c_{ljki}$ exactly[12].

The corresponding potential U now takes the form

$$U(\dot{\mathbf{A}}) = \tfrac{1}{2}\dot{A}_{ik}\left\{\mathcal{L}_{kilj} + \delta_{ij}T_{kl} - \frac{\mathcal{L}_{kipq}P_{qp}P_{rs}\mathcal{L}_{srlj}}{P_{pq}\mathcal{L}_{qpsr}P_{rs} + h}H(P_{ab}\mathcal{L}_{badc}\dot{A}_{cd})\right\}\dot{A}_{jl}, \tag{4.64}$$

where H denotes the Heaviside step function. Evidently, the potential U is rank-one convex if (4.19) is satisfied, with c_{kijl} taking their "inelastic" values. Substituting into this condition, with $\mathbf{a}$ orthogonal to $\mathbf{N}$ (because it is already known that this model of plasticity only allows for inelastic shear deformations), gives

$$a_iN_kc_{kilj}a_jN_l = \mu - \frac{4\mu^2(a_iT_{ik}N_k)(N_lT_{lj}a_j)}{(2\mu + h)T'_{rs}T'_{sr}} + N_kT_{kl}N_l > 0. \tag{4.65}$$

Now suppose that $\mathbf{T}$ has the form

$$\begin{pmatrix} -p & T_{12} & 0 \\ T_{12} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

[11] This description is accurate when the strains are infinitesimal. It is given only for the purpose of motivation.

[12] It can be noted that, in contrast, the common assumption that the co-rotational or Jaumann derivative of Kirchhoff stress is related to strain-rate through a set of moduli of the form (4.59) introduces some asymmetry; it is also not compatible *exactly* with the existence of an energy function, in the case of no plastic deformation.

Take $a_i = \delta_{i1}$ and $N_k = \delta_{k2}$. The condition for avoidance of localisation then reduces to $h > 0$. However, for more complicated patterns of stress, it is possible that localisation might not occur even for (some) negative values of h. It is equally possible that the onset of localisation could occur at some positive value of h.

Finally, an example of a simple model of "dilatant" plasticity is presented:

$$P_{ki} = T'_{ki}/\|\mathbf{T}'\| + \alpha\delta_{ki}, \quad Q_{ki} = T'_{ki}/\|\mathbf{T}'\| + \beta\delta_{ki}. \tag{4.66}$$

In this case, plastic yielding depends on the hydrostatic part of the stress as well as on the shear, and the inelastic deformation likewise has a "dilatant" component. Soils display this type of behaviour. There is, however, no universal agreement on the relative values of the parameters α and β, except that they are most unlikely to be equal, so that "flutter ill-posedness" is a potential problem.

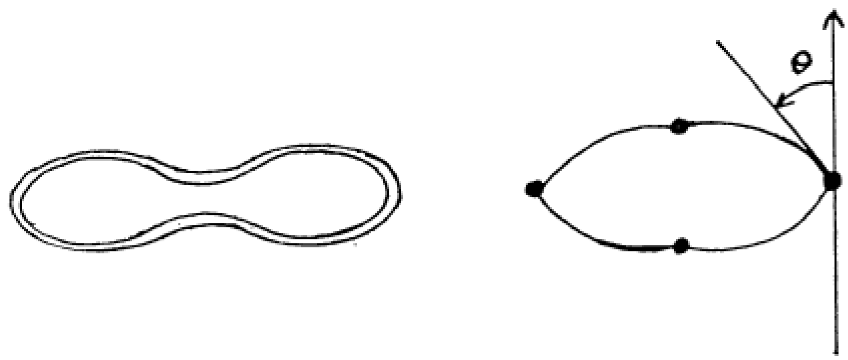

Fig. 5.1. (a) Sketch of the cross-section of a buckled pipe. (b) The one degree-of-freedom model, with four hinges.

5 Propagating Instabilities

Considering small small departures from some equilibrium path as in previous sections serves to identify bifurcation points, critical loads, and static or dynamic response close to such points. Achievement of instability is most likely to lead to a large departure from the configuration from which the instability commenced. Furthermore, there are situations in which a local large perturbation can act as a trigger for an instability that affects the whole structure, even though the loading that is applied is insufficient to generate instability in the absence of the local perturbation. An example is the propagation of a buckle along an undersea pipeline. The pipeline is modelled as a long circular cylinder, subjected to external pressure loading. The essence of the phenomenon is captured by a simple model, with just one degree of freedom. After buckling, the cross-section of the pipe usually has the form depicted in Fig. 5.1(a). This motivates considering the simple model shown in Fig. 5.1(b). Four quarter-circle segments of radius a are joined by nonlinear hinges, each of which resists bending through an angle φ by exerting a resisting couple of moment

$$M = f(\varphi). \tag{5.1}$$

The hinges may be elastoplastic but, since only monotonically increasing loads will be considered, the moment M can be considered to be a single-valued function of φ. The configuration of the cross-section is defined by the angle θ shown in Fig. 5.1(b). If the pipe is initially circular, then $\theta = 0$ defines its initial configuration. An initial imperfection is modelled by taking $\theta = \theta_0 \neq 0$ prior to deformation.

5.1 The critical pressure

The response modelled by (5.1) can be treated as though it is elastic, even if it is not; the difference would emerge only if unloading were considered. Therefore, it is possible to define a potential "energy" per unit length of pipe

$$U(\varphi) = \int_0^{\varphi} f(q)dq. \tag{5.2}$$

The external pressure has fixed magnitude p. The potential associated with this (per unit length of pipe) is p times the area of cross-section of the pipe. This area, $A(\theta)$ say, is 4 times the area enclosed by a curved segment, plus the area of the rhombus inside, whose sides are of length $\sqrt{2}a$. Thus,

$$A(\theta) = (\pi a^2 - 2a^2) + 2a^2 \cos(2\theta), \tag{5.3}$$

since the acute angle of the rhombus is $\pi/2 - 2\theta$. The total energy (per unit length) of the system is therefore

$$E(\theta) = pA(\theta) + 2U(2(\theta - \theta_0)) + 2U(-2(\theta - \theta_0)), \tag{5.4}$$

since two hinges undergo the deflection $\varphi = 2(\theta - \theta_0)$ while the other two undergo the deflection $\varphi = -2(\theta - \theta_0)$. Equilibrium requires that $E(\theta)$ should be stationary. Thus, by differentiating (5.4) with respect to θ,

$$4f(2(\theta - \theta_0)) - 4f(-2(\theta - \theta_0)) - 4pa^2 \sin(2\theta) = 0. \tag{5.5}$$

For the perfect structure, $\theta_0 = 0$. Therefore, since $f(0) = 0$, the symmetric configuration $\theta = 0$ is in equilibrium. It is stable so long as $d^2E(\theta)/d\theta^2 > 0$ when $\theta = 0$. That is,

$$p < p_c, \tag{5.6}$$

where

$$p_c = 2f'(0)/a^2. \tag{5.7}$$

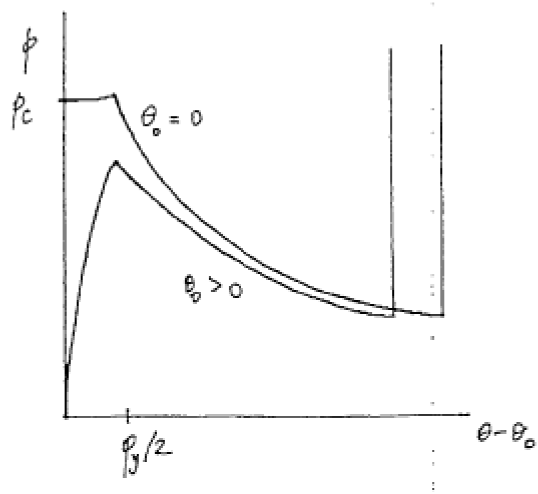

Fig. 5.2. Qualitative plots of pressure p versus deformation as measured by $\theta - \theta_0$, for the perfect structure ($\theta_0 = 0$) and an imperfect structure ($\theta_0 > 0$). The pressure can take any value when $\theta = \pi/4$, corresponding to contact of opposing faces.

Now to be more explicit, suppose that the hinges are elastic-perfectly plastic, so that

$$f(\varphi) = \begin{cases} k\varphi, & |\varphi| \le \varphi_y \\ k\varphi_y, & |\varphi| \ge \varphi_y. \end{cases} \tag{5.8}$$

Then,

$$U(\varphi) = \begin{cases} \frac{1}{2}k\varphi^2, & 0 \le \varphi \le \varphi_y \\ \frac{1}{2}k\varphi_y^2 + k\varphi_y(\varphi - \varphi_y), & \varphi \ge \varphi_y. \end{cases} \tag{5.9}$$

Also, $U(\varphi) = U(-\varphi)$. The equilibrium condition (5.5) now gives

$$\frac{p}{p_c} = \begin{cases} 2(\theta - \theta_0)/\sin(2\theta), & 0 \le (\theta - \theta_0) \le \varphi_y/2 \\ \varphi_y/\sin(2\theta), & (\theta - \theta_0) \ge \varphi_y/2. \end{cases} \tag{5.10}$$

Only the case $\theta - \theta_0 \ge 0$ is considered. The form of the relation (5.10) is sketched in Fig. 5.2, for the perfect structure ($\theta_0 = 0$) and an imperfect structure ($\theta_0 > 0$). It suggests that collapse will occur once the maximum pressure, defined by putting $\theta = \theta_0 + \varphi_y/2$ in (5.10), is attained. The maximum pressure is reduced by the presence of an imperfection. There is, in fact, an upper limit to θ, given by $\theta = \pi/4$. This corresponds to opposing

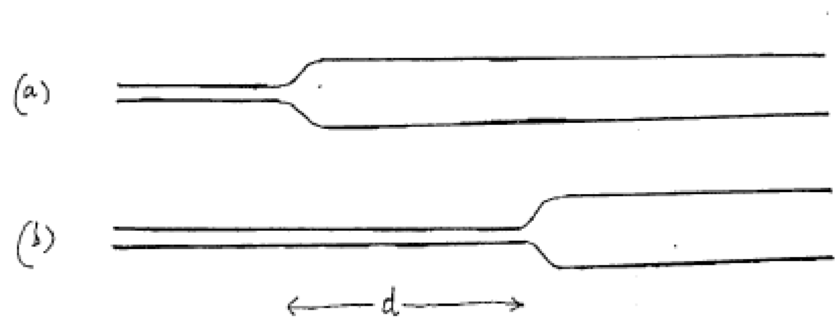

Fig. 5.3. Schematic picture of a pipe buckled over part of its length. The pipe buckles over an additional length d in going from configuration (a) to configuration (b). The transition region between buckled and unbuckled parts retains its form but is translated though distance d.

faces of the cylinder coming into contact, after which p may become arbitrarily large without inducing further deformation in this model. This is what we mean by collapse. Equivalently, the tube is said to have buckled.

5.2 Propagation of a buckle

If, for some reason, the pipe buckles (essentially as described above) over some limited portion of its whole length, the deformation must be three-dimensional, since there must be one or two regions of transition between the buckled and unbuckled configurations. There is the possibility that the buckle may spread along the pipe, even if the pressure is less than the critical pressure p_c that is required to initiate a buckle in a perfect section. Fortunately, the pressure required to propagate a buckle can be estimated, at least approximately, without recourse to three-dimensional analysis. If the buckled section is already long, the configurations shown in Fig. 5.3 may be considered. Figure 5.3(a) shows the buckle in a certain position and Fig. 5.3(b) depicts the configuration after the buckle has propagated through a distance d. The transition region retains its form but has translated through a distance d. The work done by the pressure between configurations (a) and (b) is just d times the difference in the energy per unit length for the undeformed and the buckled sections. Thus, from (5.3), with $\theta = 0$ for the undeformed section, the work done is

$$W = 2a^2pd, \tag{5.11}$$

since $\theta = \pi/4$ in the buckled configuration.

It is *impossible* for the buckle to have propagated, unless the work W is at least as great as the work required to extend the buckle through the distance d. If the material of the pipe is (nonlinearly) elastic (so that the work done within it is independent of the loading path), this latter work is exactly d times the work done (per unit length) against the hinges, in taking the tube from its initial state to its completely buckled state, since the region over which the deformation is three-dimensional has translated but the pattern of deformation within it is unchanged. Thus, for an elastic pipe, the work done within the tube is exactly

$$2[U(\pi/2) + U(-\pi/2)]d. \tag{5.12}$$

It is plausible, though no proof is offered, that, if the pipe is made of inelastic material, the work done against the stresses must be *at least* this value. Thus, a plausible *lower bound* for the pressure required to propagate the buckle is given by

$$p_p = [U(\pi/2) + U(-\pi/2)]/a^2. \tag{5.13}$$

When $U(\varphi)$ has the form given in equation (5.9),

$$p_p = k\varphi_y[\pi - \varphi_y]/a^2. \tag{5.14}$$

If σ_0 denotes yield stress of the pipe material in tension or compression, the most elementary estimate for the yield moment $M_y = k\varphi_y$ of a hinge is

$$M_y = 2\int_0^{t/2} \sigma_0 z\, dz = \sigma_0 t^2/4, \tag{5.15}$$

where t denotes the thickness of the tube. Specializing also to rigid-perfectly plastic response, so that $\varphi_y \to 0$, the "lower-bound" critical pressure for propagation p_p becomes

$$p_p = \pi\sigma_0(t/2a)^2. \tag{5.16}$$

This is very much smaller than the pressure p_c required to initiate a buckle in a perfect section of pipe, which formula (5.7) gives as twice the elastic bending stiffness, divided by a^2 and so is infinite in the rigid-perfectly plastic limit!

5.3 Other propagating instabilities

There are other problems in which an instability, once started, can propagate along a structure under a load much lower than that required to initiate it. One example, in common experience, is provided by blowing up a long thin balloon: significant pressure is required to achieve a large expansion over some portion of the balloon, but thereafter it becomes relatively easy to blow up, at moderate pressure, by the mechanism of lengthening of the inflated section. The equations are not presented, but again the key is to recognise that the pressure versus radius relation has a form similar to that given for the imperfect pipe in Fig. 5.2, with radius replacing the parameter $\theta - \theta_0$. The "vertical" part of the response shown in Fig. 5.2 is not vertical in the case of the balloon, but it does rise steeply as radius increases beyond a certain value.

Materials that can undergo phase transformation can display a related phenomenon. There is an impressive body of theory for phase-transforming materials that can be modelled as (nonlinearly) elastic[13]. Problems involving more than one spatial dimension are very far from trivial, and completely beyond the scope of this course. It is possible, however, to give a brief outline for the one-dimensional case, realised by the tension or compression of a bar. The stress-strain response is defined by the relation

$$\sigma = \partial W/\partial e, \tag{5.17}$$

where σ denotes nominal tensile stress (load divided by cross-sectional area prior to deformation) and e is the tensile strain, $e = \partial u/\partial x$, where u is displacement and x is the Lagrangian coordinate along the bar. The energy function W has two or more minima. The case of exactly two minima will be discussed, as depicted in Fig. 5.4(a). At a minimum, $\partial W/\partial e = 0$, and so there is zero stress. Different minima correspond to different phases of the material. If e is measured relative to the unstressed state in phase 0, then $e = 0$ defines the minimum in the energy associated with phase 0, and some other strain, $e = e_1$, defines the corresponding minimum for phase 1. The stress is zero at $e = 0$ and $e = e_1$, and each of these configurations is stable. There must be, however, a maximum for W in between, at $e = e^*$ say, at which the stress is also zero but the associated configuration is unstable. The stress-strain relation thus has the general character depicted in Fig. 5.4(b). The stress-strain curve passes through the origin, rises to a maximum value $\sigma_{\max}$ at some strain between 0 and e^*, passes through zero at $e = e^*$, then falls to a minimum value $\sigma_{\min}$ at a strain between e^* and e_1, and

[13] There are, of course, also materials that undergo plastic deformation in addition to phase transformation, for which theory is much less advanced.

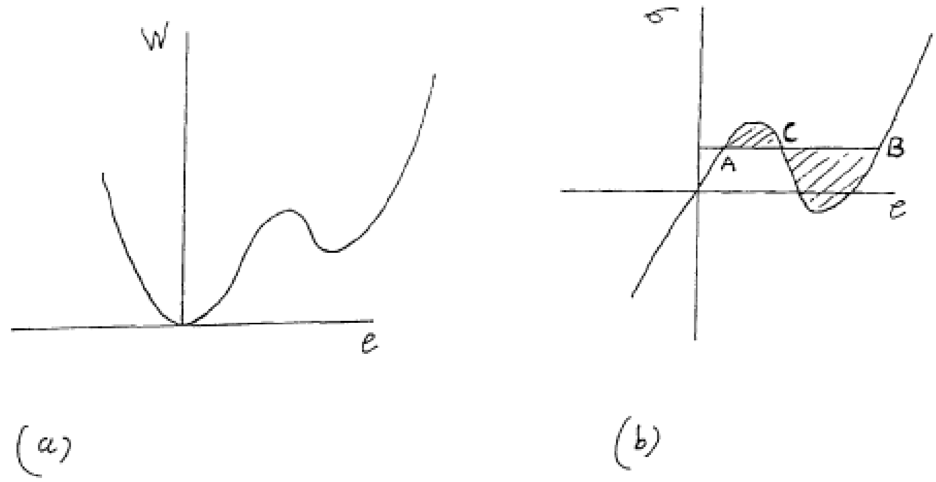

Fig. 5.4. (a) Energy function of a material that can exist in two phases, 0 and 1. (b) The corresponding stress-strain relation.

thereafter rises. Suppose that the bar is subjected to end displacements that produce nominal stress σ, and that the bar is initially in phase 0, with zero strain. The relative displacement of the ends of the bar is $\overline{e}$ times its length, where $\overline{e}$ is the mean tensile strain. As $\overline{e}$ increases from zero, the strain is uniform along the bar and has value $\overline{e}$, up to the level at which $\sigma = \sigma_{\max}$. Thereafter, the stress σ lies between $\sigma_{\max}$ and $\sigma_{\min}$, but the material has a choice: it can either adopt the uniform strain $\overline{e}$ along its entire length, corresponding to the point C in Fig. 5.4(b), or it can adopt the strain e_C over a fraction f of its length, and the strain e_A over the remaining fraction, $(1-f)$, so that

$$(1-f)e_A + f e_C = \overline{e}. \tag{5.18}$$

If it adopts a configuration of this latter type, then at least each part of the bar corresponds to a state of stress and strain that is stable. The material at strain e_A is in phase 0 and the material at strain e_C is in phase 1. There are infinitely many configurations of this type. Even the level of the line ABC is not fixed, except that σ must lie between $\sigma_{\max}$ and $\sigma_{\min}$. Exactly what happens will depend upon how deformation past the point corresponding to $\sigma_{\max}$ may be triggered.

Suppose now that, for some reason, the bar (which occupies the region $0 < x < L$ before deformation) adopts the strain e_C for $0 < x < fL$ and the strain e_A for $fL <$

$x < L$: the phase transformation has spread from the end $x = 0$. If the mean strain is increased by an amount $\delta\overline{e}$, the transformation will spread further along the bar, a distance $\delta x = \delta f L$, say, and e_A and e_C will undergo changes δe_A, δe_C. These changes are related so that they are consistent with (5.18):

$$\delta\overline{e} = \delta f(e_C - e_A) + (1 - f)\delta e_A + f\delta e_C. \tag{5.19}$$

Also, since the stress has to be constant along the bar, for equilibrium, $\sigma = W'(e_A) = W'(e_C)$.

The work done on the bar during this process must at least suffice to provide the additional strain energy in the bar. The former is $L\sigma\delta\overline{e}$, while the latter is

$$\begin{aligned} \delta U &= L\left\{\delta f[W(e_C) - W(e_A)] + (1 - f)W'(e_A)\delta e_A + fW'(e_C)\delta e_C\right\} \\ &= L\left\{\delta f[W(e_C) - W(e_A)] + \sigma[\delta\overline{e} - \delta f(e_C - e_A)]\right\}, \end{aligned} \tag{5.20}$$

having used (5.19). Thus, propagation is not possible unless

$$\sigma = W'(e_A) = W'(e_C) \geq \frac{W(e_C) - W(e_A)}{e_C - e_A} =: \sigma_M. \tag{5.21}$$

The level at which equality is achieved in (5.21) defines the Maxwell stress σ_M. It corresponds to equality of the areas shown hatched in Fig. 5.4(b). The inequality states that the stress required to propagate the phase transformation must equal or exceed the Maxwell stress. This is exactly analogous to the lower bound p_p for the pressure required to propagate a buckle along a pipe. Estimation of the precise value of the stress at which a phase transformation front may propagate would require a detailed model of the kinetics of the phase transformation process. This is analogous to the need to model the three-dimensional deformation in the transition region in the buckle propagation problem – but no microscopic model of the phase transformation process has yet found wide acceptance.

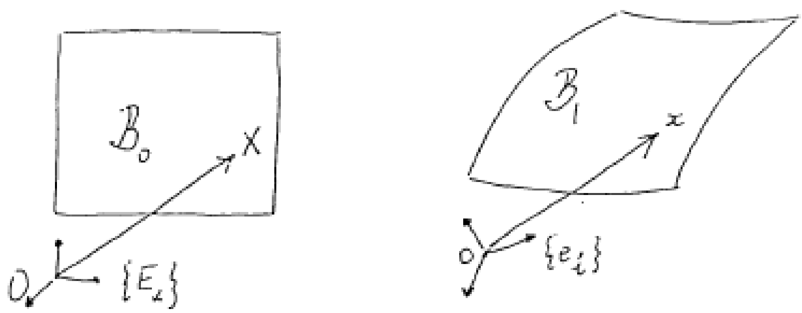

Fig. 6.1. Sketch of a body in undeformed and deformed configurations.

6 Review of Nonlinear Solid Mechanics

This section presents a quick overview of continuum mechanics, as applied to solids. It is not strictly part of the course but serves to make these notes self-contained.

6.1 Deformation and Stress

An outline of the basic notions of deformation and stress, relevant to all materials, is presented here. A wide variety of notations is in current use; here, the arbitrary choice has been made to follow that used in the book by R.W. Ogden (*Nonlinear Elastic Deformations*, Ellis Horwood, Chichester 1984).

Deformation

The deformation of a body is depicted in Fig. 6.1. It occupies a domain $\mathcal{B}_0$ in its reference configuration and $\mathcal{B}_1$ currently; if the deformation varies with time t, then $\mathcal{B}_1$ depends on t. A generic point of the body has position vector $\mathbf{X} \in \mathcal{B}_0$ initially, and $\mathbf{x} \in \mathcal{B}_1$ at time t, relative to origins $\mathbf{O}$ and $\mathbf{o}$ respectively. Relative to Cartesian bases $\{\mathbf{E}_\alpha\}$ for the initial configuration and $\{\mathbf{e}_i\}$ currently, the vectors $\mathbf{X}$ and $\mathbf{x}$ have coordinate representations

$$\mathbf{X} = X_\alpha \mathbf{E}_\alpha \quad \text{and} \quad \mathbf{x} = x_i \mathbf{e}_i, \tag{6.1}$$

with implied summation over the values 1,2,3 for the repeated suffixes.

The deformation is defined by an invertible map from $\mathcal{B}_0$ to $\mathcal{B}_1$. In terms of $\mathbf{X}$ and $\mathbf{x}$,

$$\mathbf{x} = \chi(\mathbf{X}, t) \tag{6.2a}$$

or, in components,

$$x_i = \chi_i(\mathbf{X}, t). \tag{6.2b}$$

The deformation gradient $\mathbf{A}$ is then defined as

$$\mathbf{A} = A_{i\alpha}\mathbf{e}_i \otimes \mathbf{E}_\alpha, \tag{6.3}$$

where

$$A_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}. \tag{6.4}$$

Then, an infinitesimal line segment $\mathbf{dX}$ deforms into the segment $\mathbf{dx}$, where

$$\mathbf{dx} = \mathbf{A}\,\mathbf{dX}, \qquad dx_i = A_{i\alpha}dX_\alpha. \tag{6.5}$$

Strain tensors relate lengths and angles before and after deformation. If infinitesimal line segments $\mathbf{dX}$ and $\mathbf{dY}$ transform respectively into $\mathbf{dx}$ and $\mathbf{dy}$, then

$$\mathbf{dx}.\mathbf{dy} = \mathbf{dx}^T\mathbf{dy} = \mathbf{dX}^T\mathbf{A}^T\mathbf{A}\mathbf{dY}. \tag{6.6}$$

All information on length and angle changes is thus contained in $\mathbf{A}^T\mathbf{A}$. Perhaps the simplest strain measure – the Green strain – is then

$$\mathbf{E} = \tfrac{1}{2}\left(\mathbf{A}^T\mathbf{A} - \mathbf{I}\right), \quad \text{or} \quad E_{\alpha\beta} = \tfrac{1}{2}\left(A_{i\alpha}A_{i\beta} - \delta_{\alpha\beta}\right). \tag{6.7}$$

A general class of strain measures is obtained by first defining the eigenvalues and normalized eigenvectors of $\mathbf{A}^T\mathbf{A}$ as λ_i^2 and $\mathbf{u}^{(i)}$, so that

$$\mathbf{A}^T\mathbf{A} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{6.8}$$

Then, if f is any monotone increasing function for which $f(1) = 0$ and $f'(1) = 1$, a strain tensor $\mathbf{e}$ is defined as

$$\mathbf{e} = \sum_{i=1}^{3} f(\lambda_i)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}. \tag{6.9}$$

The strain tensor (6.7) fits this pattern, with $f(\lambda) = \frac{1}{2}(\lambda^2 - 1)$.

The polar decomposition theorem

A result needed later, related to (6.8), is the polar decomposition theorem. Define

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$$

and then set

$$\mathbf{R} = \mathbf{A}\mathbf{U}^{-1} = \mathbf{A} \sum_{i=1}^{3} \lambda_i{}^{-1} \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}.$$

It follows that $\mathbf{R}$ represents a rotation (so that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$), while $\mathbf{U}$ is symmetric. The representation

$$\mathbf{A} = \mathbf{R}\mathbf{U}$$

is the required result.

Stress and equations of motion

Suppose the body is acted upon by surface and body forces, which may vary with time t. These forces can be represented *either* as functions of $\mathbf{x}$ and t, relative to the current configuration, *or* as functions of $\mathbf{X}$ and t, relative to the initial configuration, to which the current configuration is related by (6.2). Since the mapping χ is usually not known in advance of solving the problem, we choose to employ the latter representation. Thus, with the mass density of the body given as ρ_0 per unit volume in the reference configuration, the body force $\mathbf{b}$ per unit mass can equivalently be expressed as $\rho_0\mathbf{b}$ per unit initial volume. Since the force is actually applied to the body in its current configuration, it is usual to express $\mathbf{b}$, in components, as

$$\mathbf{b} = b_i \mathbf{e}_i.$$

The forces applied to the surface are defined similarly, per unit of surface area in the reference configuration. Thus, if an element of surface $\mathbf{dS}_0$ is mapped by (6.2) into a surface element $\mathbf{ds}$, the force applied to $\mathbf{ds}$ is

$$\mathbf{df} = \mathbf{t} dS_0,$$

where $\mathbf{t}$ is the force per unit reference area and dS_0 denotes the magnitude of $\mathbf{dS}_0$.

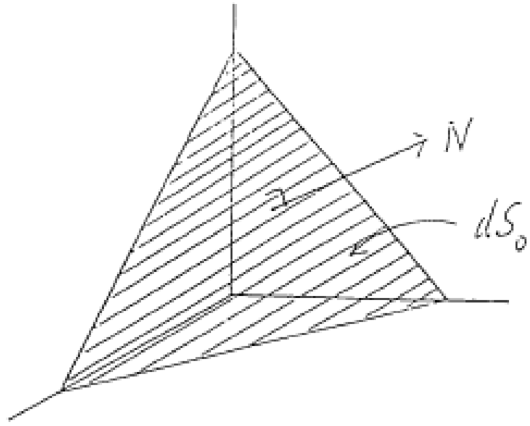

Fig. 6.2. Element of volume employed for deriving (6.11).

Balance of linear momentum then requires that

$$\frac{d}{dt}\int_{\mathcal{B}_0} \rho_0 \mathbf{v}\, \mathbf{dX} = \int_{\mathcal{B}_0} \rho_0 \mathbf{b}\, \mathbf{dX} + \int_{\partial\mathcal{B}_0} \mathbf{t}\, dS_0, \tag{6.10}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the particle velocity and $\partial\mathcal{B}_0$ denotes the surface of $\mathcal{B}_0$. A similar relation must apply to any part of $\mathcal{B}_0$. In particular, by taking a small volume element of the type shown in Fig. 6.2, it follows that $\mathbf{t}$ has the representation

$$\mathbf{t} = \mathbf{S}^T\mathbf{N}, \quad \text{or} \quad t_i = S_{\alpha i} N_\alpha, \tag{6.11}$$

where $\mathbf{N}$ denotes the unit normal to the surface element $\mathbf{dS}_0$[14]. If the representation (6.11) is now substituted into (6.10) (applied to *any* part of $\mathcal{B}_0$), the divergence theorem then yields the equation of motion

$$S_{\alpha i,\alpha} + \rho_0 b_i = \rho_0 dv_i/dt, \quad \mathbf{X} \in \mathcal{B}_0. \tag{6.12}$$

It may be noted that (6.12) is exactly like the more usual equation involving Cauchy stress, but ρ_0 and $\mathcal{B}_0$ are *known*; the equation is thus *linear*, even though it is exact.

Balance of moment of momentum requires that

$$\frac{d}{dt}\int_{\mathcal{B}_0} \epsilon_{ijk} x_j (\rho_0 v_k) \mathbf{dX} = \int_{\mathcal{B}_0} \epsilon_{ijk} x_j (\rho_0 b_k) \mathbf{dX} + \int_{\partial\mathcal{B}_0} \epsilon_{ijk} x_j (S_{\alpha k} N_\alpha) dS_0.$$

[14] The ‘derivation’ of (6.11) can, if preferred, be by-passed by treating (6.11) as a postulate.

Transformation of the surface integral to one over $\mathcal{B}_0$ by the divergence theorem, followed by use of (6.12) and the recognition that $\mathcal{B}_0$ may be chosen arbitrarily gives the result $\epsilon_{ijk}A_{j\alpha}S_{\alpha k} = 0$. Equivalently,

$$A_{i\alpha}S_{\alpha j} = A_{j\alpha}S_{\alpha i}, \quad \text{or} \quad \mathbf{AS} = (\mathbf{AS})^T. \tag{6.13}$$

In view of its reference to initial area, $\mathbf{S}$ is called the tensor of *nominal stress*; its transpose is called the first Piola-Kirchhoff stress tensor. It can be shown that Cauchy stress $\mathbf{T}$ is related to $\mathbf{S}$ by

$$\mathbf{T} = \mathbf{AS}/\det(\mathbf{A}).$$

Work-conjugate stresses and strains

The rate of working of the forces applied to the body is obtained by multiplying forces by velocities; thus, the total rate of working is

$$\int_{\mathcal{B}_0} \rho_0 b_i v_i \mathbf{dX} + \int_{\partial\mathcal{B}_0} N_\alpha S_{\alpha i} v_i \, dS_0 = \dot{w}, \quad \text{say}.$$

Application of the divergence theorem and use of the equation of motion (6.12) transforms this to

$$\dot{w} = \int_{\mathcal{B}_0} \left[\left(\frac{d}{dt} \right) \tfrac{1}{2}\rho_0 v_i v_i + S_{\alpha i} v_{i,\alpha} \right] \mathbf{dX}. \tag{6.14}$$

The last term in the integrand represents the rate of working, per unit reference volume, of the stresses. It can also be written

$$S_{\alpha i} v_{i,\alpha} = S_{\alpha i} \dot{A}_{i\alpha}. \tag{6.15}$$

(The superposed dot means d/dt.)

Now consider the strain tensor obtained by taking $f(\lambda) = \lambda - 1$, so that $\mathbf{e} = \mathbf{U} - \mathbf{I}$. Employing in (6.15) the polar decomposition of $\mathbf{A}$ gives

$$\begin{aligned} S_{\alpha i}\dot{A}_{i\alpha} &= S_{\alpha i}\dot{R}_{i\beta}U_{\beta\alpha} + S_{\alpha i}R_{i\beta}\dot{U}_{\beta\alpha} \\ &= S_{\alpha i}R_{i\beta}\dot{U}_{\beta\alpha}. \end{aligned}$$

The term involving $\dot{\mathbf{R}}$ is zero. This can be seen by noting that

$$S_{\alpha i}\dot{R}_{i\beta}U_{\beta\alpha} = S_{\alpha i}\dot{R}_{i\beta}R_{j\beta}R_{j\gamma}U_{\gamma\alpha} = A_{j\alpha}S_{\alpha i}\dot{R}_{i\beta}R_{j\beta},$$

and invoking the symmetry (6.13) together with the antisymmetry $\dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T$. Hence, exploiting the symmetry of $\mathbf{U}$,

$$S_{\alpha i}\dot{A}_{i\alpha} = T^{(1)}_{\alpha\beta}\dot{e}_{\beta\alpha}, \tag{6.16}$$

where

$$T^{(1)}_{\alpha\beta} = \tfrac{1}{2}\left[S_{\alpha i}R_{i\beta} + S_{\beta i}R_{i\alpha}\right]. \tag{6.17}$$

The tensor $\mathbf{T}^{(1)}$ is the stress which is conjugate to the strain $\mathbf{e}$.

The same idea applies to other measures of strain. Another simple example is provided by the Green strain (6.7). The associated conjugate stress is $\mathbf{T}^{(2)}$, where

$$T^{(2)}_{\alpha\beta} = S_{\alpha i}B_{i\beta}, \tag{6.18}$$

with the notation

$$\mathbf{B}^T = \mathbf{A}^{-1}. \tag{6.19}$$

The stress tensor $\mathbf{T}^{(2)}$ is the second Piola-Kirchhoff stress tensor; it is symmetric, since (6.13) gives

$$\mathbf{S}\mathbf{B} = \mathbf{B}^T\mathbf{A}\mathbf{S}\mathbf{B} = \mathbf{B}^T\mathbf{S}^T\mathbf{A}^T\mathbf{B} = \mathbf{B}^T\mathbf{S}^T.$$

Convected coordinates

Suppose a coordinate net is scribed into the initial configuration. It is rectilinear initially but, after deformation, an infinitesimal segment $\mathbf{E}_\alpha dX_\alpha$ is transformed into a segment $\mathbf{e}_i A_{i\alpha} dX_\alpha = \mathbf{e}_\alpha dX_\alpha$, say. The vectors $\{\mathbf{e}_\alpha\}$ form a basis, but this is not orthonormal; it is associated instead with *curvilinear coordinates* $\{X_\alpha\}$ in the current configuration. Now we can associate with $\mathbf{T}^{(2)}$, for example, a tensor

$$\hat{\mathbf{T}}^{(2)} = T^{(2)}_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \hat{T}^{(2)}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{6.20}$$

where

$$\hat{T}^{(2)}_{ij} = A_{i\alpha}A_{j\beta}T^{(2)}_{\alpha\beta}. \tag{6.21}$$

The components $T^{(2)}_{\alpha\beta}$ can be viewed as the *contravariant components* of the tensor $\hat{\mathbf{T}}^{(2)}$, relative to the basis $\{\mathbf{e}_\alpha\}$. It can be checked by calculation that $\hat{T}^{(2)}_{ij} = \det(\mathbf{A})T_{ij}$ ($\mathbf{T}$ denoting Cauchy stress). The tensor $\hat{\mathbf{T}}^{(2)}$ is called the Kirchhoff stress.

Similar constructions could be based upon the components of other stress tensors; there is no particular advantage to this, but the discussion presented perhaps explains the source of the wide variety of possible descriptions of stress.

It should be noted, finally, that *all of the measures of stress that have been discussed coincide, when the reference configuration is chosen as the current configuration at the instant of interest.*

Stress rates

The subject of stress rates will only be touched upon during the lectures. The account to follow, though itself only a sketch, is included so that the exposition in these notes is in a sense complete.

First we establish some notation. The strain measure (6.9), constructed with the function f, will be denoted $\mathbf{e}^f$:

$$\mathbf{e}^f = \sum_{r=1}^{3} f(\lambda_r)\mathbf{u}^{(r)} \otimes \mathbf{u}^{(r)} \text{ or in suffix notation, } e^f_{\alpha\beta} = \sum_{r=1}^{3} f(\lambda_r)u^{(r)}_\alpha u^{(r)}_\beta. \tag{6.22}$$

The conjugate stress is now denoted $\mathbf{T}^f$, with components $T^f_{\alpha\beta}$. The corresponding stress rate is nothing other than $\dot{\mathbf{T}}^f$, with components $\dot{T}^f_{\alpha\beta}$. As was discussed above for the second Piola-Kirchhoff stress $\mathbf{T}^{(2)}$, another tensor, $\hat{\mathbf{T}}^f$, can be formed by taking $T^f_{\alpha\beta}$ to be its contravariant components on the basis $\{\mathbf{e}_\alpha\}$, so that its Cartesian components are given by a formula exactly like (6.21). Then, the components $\dot{T}^f_{\alpha\beta}$ can be employed to form a certain rate for the tensor $\hat{\mathbf{T}}^f$, say

$$\frac{\delta\hat{\mathbf{T}}^f}{\delta t} = \dot{T}^f_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta, \text{ or } \frac{\delta\hat{T}^f_{ij}}{\delta t} = A_{i\alpha}A_{j\beta}\dot{T}^f_{\alpha\beta}. \tag{6.23}$$

It is of interest to express the stress-rate $\delta\hat{\mathbf{T}}^f/\delta t$ in terms of the components $\hat{T}^f_{ij}$ and their derivatives, particularly in the case when the current configuration is taken as the reference configuration, so that $\mathbf{A} = \mathbf{I}$. Relative to this configuration, it has already been remarked (and will be proved below) that all stress measures become identical. So also, do all strain-rates, but the same is *not* true of the stress-rates.

To begin, the strain-rate corresponding to $\mathbf{e}^f$ has components

$$\dot{e}^f_{\beta\gamma} = \sum_{r=1}^{3}[f'(\lambda_r)\dot{\lambda}_r u^{(r)}_\beta u^{(r)}_\gamma + f(\lambda_r)(d/dt)(u^{(r)}_\beta u^{(r)}_\gamma)]. \tag{6.24}$$

(Here and elsewhere, d/dt has the same meaning as a superposed dot, either representing the time derivative at a fixed material point, so that $\mathbf{X}$ is kept fixed.) The stress components $T^f_{\alpha\beta}$ are related to those of the nominal stress through

$$S_{\alpha i}\dot{A}_{i\alpha} = T^f_{\beta\gamma}\dot{e}^f_{\beta\gamma} = T^f_{\beta\gamma}\frac{\partial e^f_{\beta\gamma}}{\partial A_{i\alpha}}\dot{A}_{i\alpha}. \tag{6.25}$$

Thus,

$$S_{\alpha i} = T^f_{\beta\gamma}\frac{\partial e^f_{\beta\gamma}}{\partial A_{i\alpha}}. \tag{6.26}$$

An expression for $\partial e^f_{\beta\gamma}/\partial A_{i\alpha}$ is therefore required. A calculation summarised in Appendix 2A gives

$$\begin{aligned}\frac{\partial e^f_{\beta\gamma}}{\partial A_{i\alpha}} &= \sum_{r=1}^{3}\Bigg\{\frac{f'(\lambda_r)}{\lambda_r}u^{(r)}_\beta u^{(r)}_\gamma u^{(r)}_\alpha A_{i\mu}u^{(r)}_\mu \\ &\quad + \sum_{s\neq r}\frac{f(\lambda_r)}{\lambda_r^2-\lambda_s^2}(u^{(r)}_\beta u^{(s)}_\gamma + u^{(s)}_\beta u^{(r)}_\gamma)(u^{(r)}_\alpha A_{i\mu}u^{(s)}_\mu + u^{(s)}_\alpha A_{i\mu}u^{(r)}_\mu)\Bigg\}. \end{aligned} \tag{6.27}$$

This formula is somewhat inconvenient for investigating the case when $\mathbf{A} = \mathbf{I}$, because it requires a limiting operation. Appendix 2A derives this limit directly. It involves the Eulerian strain-rate Σ_{ij}, which is the symmetric part of the Eulerian deformation-rate $\Gamma_{ij} = \partial\dot{x}_i/\partial x_j = \dot{A}_{i\alpha}B_{j\alpha}$, which reduces to $\dot{A}_{i\alpha}\delta_{j\alpha}$ when $\mathbf{A}$ is the identity, $A_{i\alpha} = \delta_{i\alpha}$. The time derivative of λ_r, and the eigenvector $\mathbf{u}^{(r)}$, satisfy

$$\Sigma_{ij}\delta_{j\alpha}u^{(r)}_\alpha = \dot{\lambda}_r\delta_{i\alpha}u^{(r)}_\alpha. \tag{6.28}$$

Thus,

$$\Sigma_{ij} = \sum_{r=1}^{3}\dot{\lambda}_r\delta_{i\alpha}\delta_{j\beta}u^{(r)}_\alpha u^{(r)}_\beta. \tag{6.29}$$

This demonstrates explicitly, by comparison with (6.24) when $\lambda_r = 1$, that *all* stress-rates coincide with the Eulerian strain-rate when the current configuration is chosen as reference. Expressed differently, when $\mathbf{A} = \mathbf{I}$,

$$\dot{e}^f_{\beta\gamma} = \frac{\partial e^f_{\beta\gamma}}{\partial A_{i\alpha}}\dot{A}_{i\alpha} = \delta_{j\beta}\delta_{k\gamma}\Sigma_{jk} = \tfrac{1}{2}(\delta_{j\beta}\dot{A}_{j\gamma} + \delta_{j\gamma}\dot{A}_{j\beta}). \tag{6.30}$$

It follows that

$$\frac{\partial e^f_{\beta\gamma}}{\partial A_{i\alpha}} = \tfrac{1}{2}(\delta_{i\beta}\delta_{\alpha\gamma} + \delta_{i\gamma}\delta_{\alpha\beta}), \tag{6.31}$$

and hence that

$$S_{i\alpha} = \delta_{i\beta}T^f_{\beta\alpha}. \tag{6.32}$$

Thus, relative to the current configuration, all measures of stress coincide, and may be identified with a single stress $T_{\alpha\beta}$ which in fact is also coincident with the Cauchy stress.

Relations between different stress-rates are now investigated by differentiating (6.26) to give

$$\dot{S}_{\alpha i} = \dot{T}^f_{\beta\gamma}\frac{\partial e^f_{\beta\gamma}}{\partial A_{i\alpha}} + T^f_{\beta\gamma}\frac{\partial^2 e^f_{\beta\gamma}}{\partial A_{i\alpha}\partial A_{j\nu}}\dot{A}_{j\nu}. \tag{6.33}$$

It is shown in Appendix 2A that, when $\mathbf{A} = \mathbf{I}$,

$$\begin{aligned}\frac{\partial^2 e^f_{\beta\gamma}}{\partial A_{i\alpha}\partial A_{j\nu}} &= \tfrac{1}{2}\Big\{\tfrac{1}{4}(f''(1)-1)(\delta_{\alpha\beta}\delta_{ij}\delta_{\gamma\nu} + \delta_{\alpha\gamma}\delta_{ij}\delta_{\beta\nu} + \delta_{\alpha\beta}\delta_{i\nu}\delta_{j\gamma} + \delta_{j\alpha}\delta_{\beta\nu}\delta_{i\gamma} \\ &\qquad + \delta_{j\alpha}\delta_{i\beta}\delta_{\gamma\nu} + \delta_{j\beta}\delta_{i\nu}\delta_{\alpha\gamma} + \delta_{i\beta}\delta_{j\gamma}\delta_{\alpha\nu} + \delta_{j\beta}\delta_{\alpha\nu}\delta_{i\gamma}) \\ &\quad + \sum_r u^{(r)}_\beta u^{(r)}_\gamma u^{(r)}_\lambda u^{(r)}_\mu \delta_{ij}(\delta_{\lambda\nu}\delta_{\alpha\mu} + \delta_{\alpha\lambda}\delta_{\mu\nu}) \\ &\quad + \tfrac{1}{2}\sum_r\sum_{s\neq r}(u^{(r)}_\beta u^{(s)}_\gamma + u^{(s)}_\beta u^{(r)}_\gamma)\delta_{ij}(\delta_{\lambda\nu}\delta_{\alpha\mu} + \delta_{\alpha\lambda}\delta_{\mu\nu})\Big\}. \end{aligned} \tag{6.34}$$

The desired relation between $\dot{S}_{\alpha i}$ and $\dot{T}^f_{\beta\gamma}$ now follows by substituting (6.31) and (6.34) into (6.33). This is not written out explicitly, but it is noted that the rate associated with $\mathbf{T}^f$ depends on the function f only through the term multiplying the coefficient $(f''(1)-1)$. It is now not difficult to conclude, for $\hat{\mathbf{T}}^f$ and $\hat{\mathbf{T}}^g$ (the latter being defined with the function g replacing f), that

$$\frac{\delta \hat{T}^f_{ij}}{\delta t} = \frac{\delta \hat{T}^g_{ij}}{\delta t} - \tfrac{1}{2}[f''(1) - g''(1)](\Sigma_{ik}\hat{T}_{kj} + \hat{T}_{ik}\Sigma_{kj}), \tag{6.35}$$

where $\hat{\mathbf{T}}$ denotes either of $\hat{\mathbf{T}}^f$ or $\hat{\mathbf{T}}^g$, since these coincide when the current configuration is taken as reference.

All that remains now is to evaluate one particular stress-rate. This is easy to do for the tensor $\hat{\mathbf{T}}^{(2)}$. Differentiating (6.21) and then setting $\mathbf{A} = \mathbf{I}$ gives

$$\frac{d\hat{T}^{(2)}_{ij}}{dt} = \delta_{i\alpha}\delta_{j\beta}\dot{T}^{(2)}_{\alpha\beta} + (\dot{A}_{i\alpha}\delta_{j\beta} + \delta_{i\alpha}\dot{A}_{j\beta})T^{(2)}_{\alpha\beta}, \tag{6.36}$$

whence it follows that

$$\frac{\delta \hat{T}^{(2)}_{ij}}{\delta t} = \frac{d\hat{T}_{ij}}{dt} - \Gamma_{ik}\hat{T}_{kj} - \Gamma_{jk}\hat{T}_{ki}. \tag{6.37}$$

It is also of interest, for later use, to derive an expression for the nominal stress-rate. The simplest course is to differentiate the relation (6.18). This gives (when $\mathbf{A} = \mathbf{I}$)

$$\dot{T}^{(2)}_{\alpha\beta} = \dot{S}_{\alpha i}\delta_{i\beta} - S_{\alpha i}\delta_{i\gamma}\dot{A}_{k\gamma}\delta_{k\beta}.$$

This delivers the nominal stress-rate

$$\delta_{j\alpha}\dot{S}_{\alpha i} = \frac{\delta \hat{T}^{(2)}_{ji}}{\delta t} + \Gamma_{ik}\hat{T}_{kj} = \frac{d\hat{T}_{ji}}{dt} - \Gamma_{jk}\hat{T}_{ki}. \tag{6.38}$$

It is remarked finally that the Cauchy stress tensor, previously introduced as $\mathbf{T}$ with components T_{ij}, is *not* conjugate to any strain measure. However, the relation $\hat{\mathbf{T}}^{(2)} = \det(\mathbf{A})\mathbf{T}$ between Kirchhoff stress and Cauchy stress implies, when $\mathbf{A} = \mathbf{I}$, that

$$\frac{d\hat{T}_{ij}}{dt} = \frac{dT_{ij}}{dt} + \Gamma_{kk}T_{ij}. \tag{6.39}$$

6.2 Elastic constitutive equations

This section records the constitutive relations of finite-deformation elasticity and then their specialization, first to ‘incremental’ deformation and then, further, to classical linear elasticity (which may be viewed as an increment of deformation from an unstressed reference configuration).

The general elastic constitutive relation

An elastic body is taken as one which stores any energy that is put into it. If thermal effects are disregarded, any mechanical work done on the body must either generate kinetic energy or else be stored within the body as strain energy. The energy that is stored, per unit reference volume, is W, which is a function, $W(\mathbf{A})$, of the local deformation gradient. If the deformation is sufficiently slow for isothermal conditions to prevail, then W is the free energy function, with the temperature fixed at its ambient value. More generally, energy accounts must be performed, making allowance for temperature (and entropy) variations and the input of heat, but this is not discussed further.

The total rate of input of mechanical work has already been given, as $\dot{w}$, in equation (6.14). The statement just made translates into the equation

$$\dot{w} = \left(\frac{d}{dt}\right)\int_{\mathcal{B}_0}\left[\tfrac{1}{2}\rho_0 v_i v_i + W(\mathbf{A})\right]\mathbf{dX}. \tag{6.40}$$

Comparison of (6.14) and (6.40) (both of which also apply when $\mathcal{B}_0$ is replaced by any part of $\mathcal{B}_0$) shows that

$$\begin{aligned} S_{\alpha i}\dot{A}_{i\alpha} &= \frac{dW(\mathbf{A})}{dt} \\ &= \frac{\partial W}{\partial A_{i\alpha}}\dot{A}_{i\alpha}. \end{aligned}$$

Since this equation remains true for any motion[15], it follows that

$$S_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}}. \tag{6.41}$$

Equation (6.41) is the constitutive equation for finite-deformation elasticity.

There is, however, a restriction on the form of the function W: it has to be 'objective'. It is common experience that the stored energy of an elastic body undergoes no change if it is subjected to a rigid motion. Hence, W can depend on $\mathbf{A}$ only in some combination that recognises only length and angle changes – that is, $\mathbf{A}$ must appear only through some measure of strain. All are equivalent, since all can be constructed from (6.8) or from $\mathbf{U}$, but not all are equally convenient. If, however, W is expressed as a function of

[15] It is a simple exercise to construct deformations which are independent of $\mathbf{X}$, for which $\mathbf{A}$ and $\dot{\mathbf{A}}$ take chosen values at some specific time.

some particular strain $\mathbf{e}^f$, it follows immediately from the definition of work-conjugacy that the associated stress, $\mathbf{T}^f$, is given by

$$T^f_{\alpha\beta} = \frac{\partial W}{\partial e^f_{\alpha\beta}}. \tag{6.42}$$

Incremental deformations

This discussion is restricted to the case of a small, possibly dynamic, perturbation of a static finite deformation, defined by a deformation gradient $\mathbf{A}^0$, say, and corresponding nominal stress $\mathbf{S}^0$, so that

$$S^0_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}}(\mathbf{A}^0).$$

This initial deformation is maintained by body force $\mathbf{b}^0$, so that, from the equation of motion (6.12) in the case of no time-dependence,

$$S^0_{\alpha i,\alpha} + \rho_0 b^0_i = 0. \tag{6.43}$$

Some boundary condition must also be specified; for simplicity we assume that $\mathbf{x} = \chi^0(\mathbf{X})$ is prescribed for $\mathbf{X} \in \partial\mathcal{B}_0$.

Now change the body force to $\mathbf{b}^0 + \mathbf{f}$ and displace the boundary to $\chi^0 + \mathbf{w}$, where $\mathbf{f}$ and $\mathbf{w}$ are small, but possibly depend on time t. The body undergoes an increment of displacement $\mathbf{u}$ and the total deformation gradient and nominal stress now have components

$$A_{i\alpha} = A^0_{i\alpha} + u_{i,\alpha} \quad \text{and} \quad S_{\alpha i} = S^0_{\alpha i} + s_{\alpha i}, \quad \text{say.}$$

The equation of motion (6.12) now implies, on taking account of (6.43),

$$s_{\alpha i,\alpha} + \rho_0 f_i = \rho_0 \frac{d^2 u_i}{dt^2} \tag{6.44}$$

and the boundary condition gives

$$\mathbf{u} = \mathbf{w}, \quad \mathbf{X} \in \partial\mathcal{B}_0. \tag{6.45}$$

The system (6.44), (6.45) is completed by appending an incremental version of the constitutive equation (6.41). This is

$$s_{\alpha i} = c_{\alpha i \beta j} u_{j,\beta}, \tag{6.46}$$

where

$$c_{\alpha i\beta j} = \frac{\partial^2 W}{\partial A_{i\alpha}\partial A_{j\beta}}(\mathbf{A}^0). \tag{6.47}$$

It should be noted that, although the tensor $\mathbf{c}$ does not have all of the usual symmetries assigned to elastic moduli, it does have the crucial symmetry

$$c_{\alpha i\beta j} = c_{\beta j\alpha i}, \tag{6.48}$$

which renders the equations self-adjoint. These equations form the basis for analyses of stability and, when their solution is unique, most of the standard techniques of linear elasticity can be deployed for their solution. Computational schemes for static finite-deformation problems are also usually approached by an incremental formulation, equivalent to the one given here.

Linear elasticity

Classical linear elasticity is a special case of small deformations superposed on a finite deformation: it is necessary only to choose the special values $\mathbf{b}^0 = 0$, $\mathbf{A}^0 = \mathbf{I}$, $\mathbf{S}^0 = 0$. The coordinate bases $\{\mathbf{E}_\alpha\}$ and $\{\mathbf{e}_i\}$ can be taken to coincide and it is usual to take the coordinates to be $\{x_i\}$ and avoid the use of Greek suffixes. All stress tensors coincide – and increments too, in the absence of pre-strain – and can be regarded as Cauchy stress, with components T_{ij}. All strain tensors likewise reduce to the infinitesimal strain tensor – called $\mathbf{e}$ – with components

$$e_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

and, since the strain energy function W must depend upon $\mathbf{e}$, the elastic moduli are given by

$$c_{ijkl} = \frac{\partial^2 W}{\partial e_{ij}\partial e_{kl}}$$

and have the symmetries

$$c_{ijkl} = c_{jikl} = c_{klij}.$$

(This is consistent with the symmetry (6.13) of the stress tensor $\mathbf{S}$, which follows when $\mathbf{A} = \mathbf{I}$.)

The energy function at zero strain is taken to be zero, and zero strain is taken to correspond to zero stress (unless pre-stress was a feature to be modelled). Hence, for small strains, the energy function becomes

$$W(\mathbf{e}) = \tfrac{1}{2}c_{ijkl}e_{ij}e_{kl}. \tag{6.49}$$

Since the energy at any non-zero level of strain should be positive, a restriction on the quadratic form (6.49) is that it should be positive-definite.

In the special case of isotropy, the tensor of elastic moduli **c** takes the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{6.50}$$

The energy function (6.49) can be written

$$\begin{aligned} W(\mathbf{e}) &= \tfrac{1}{2}\lambda e_{ii}e_{kk} + \mu e_{ij}e_{ij} \\ &= \tfrac{1}{2}\kappa e_{ii}e_{kk} + \mu e'_{ij}e'_{ij}, \end{aligned}$$

where

$$\kappa = \lambda + \tfrac{2}{3}\mu$$

and

$$e'_{ij} = e_{ij} - \tfrac{1}{3}e_{kk}.$$

Positive-definiteness of (6.49) is then equivalent to the inequalities

$$\kappa > 0, \qquad \mu > 0.$$

6.3 Some Energy Considerations

This section deals with the classical energy principles. Also, just to finish off, a discussion of energy flux is included; this can equally well be done for a general continuum and specialized to elasticity afterwards, so this sequence is followed. The topics have some fundamental significance. In particular, they have significant bearing on the theory of forces on defects.

The minimum energy principle

It is a reasonable physical postulate that an elastic body, when in equilibrium, adopts the configuration that minimizes its total energy, allowing for the constraints to which it is subjected[16]. This statement can be put into mathematical form, only once these constraints have been made explicit. Although there are other possibilities, it will be assumed here that the body is subjected to body force $\mathbf{b}(\mathbf{X})$ per unit mass and that, at each point of its boundary $\partial\mathcal{B}_0$, one of the pair $\{x_i, t_i\}$ is prescribed, for each i, where t_i

[16] Of course this can apply only to constraints which can be associated with a potential energy.

denotes a prescribed value for $N_\alpha S_{\alpha i}$. Thus, any loads that are prescribed are of 'dead loading' type; configuration-dependent loads (that depend on $\mathbf{x}$) are more complicated. The energy *of the system* comprising the body and its loading mechanism is now

$$E(\mathbf{x}(\mathbf{X})) = \int_{\mathcal{B}_0} [W(\mathbf{A}) - \rho_0 b_i x_i] \, \mathbf{dX} - \int_{\partial\mathcal{B}_0} \{t_i x_i\} \, dS_0, \tag{6.51}$$

where the curly bracket implies evaluating the sum at any point $\mathbf{X}$ only over those values of i for which t_i is prescribed. The physical postulate that the body selects for itself the function $\mathbf{x}(\mathbf{X})$ that minimizes E, subject to its components x_i taking any values that may be prescribed for $\mathbf{X} \in \partial\mathcal{B}_0$, is hard to verify mathematically (and, when W has several minima corresponding to phase transformations, is a subject of active research) but it is easy to verify that the equations of equilibrium are satisfied when E is stationary. To see this, let $\mathbf{x}(\mathbf{X})$ be the solution and let $\mathbf{u}$ be any admissible variation. The statement that, to first order,

$$E(\mathbf{x} + \mathbf{u}) = E(\mathbf{x})$$

implies

$$\int_{\mathcal{B}_0} \left[\frac{\partial W}{\partial A_{i\alpha}} u_{i,\alpha} - \rho_0 b_i u_i \right] \mathbf{dX} - \int_{\partial\mathcal{B}_0} \{t_i u_i\} dS_0 = 0. \tag{6.52}$$

An application of the divergence theorem gives

$$\int_{\partial\mathcal{B}_0} [N_\alpha S_{\alpha i} u_i - \{t_i u_i\}] \, dS_0 - \int_{\mathcal{B}_0} [S_{\alpha i,\alpha} + \rho_0 b_i] \, u_i \mathbf{dX} = 0. \tag{6.53}$$

The requirement that (6.53) should hold for any $\mathbf{u}(\mathbf{X})$ for which $u_i(\mathbf{X}) = 0$ whenever x_i is prescribed, generates the equilibrium equation (the time-independent version of (6.12)) and the traction boundary conditions.

In the case of linear elasticity, W is a convex function of $\mathbf{e}$ and it is easy to prove that the stationary point is a minimum. This follows from the calculation

$$E(\hat{\mathbf{x}}) - E(\mathbf{x}) = \int_{\mathcal{B}_0} [W(\hat{\mathbf{e}}) - W(\mathbf{e}) - b_i(\hat{x}_i - x_i)] \, \mathbf{dX} - \int_{\partial\mathcal{B}_0} \{t_i(\hat{x}_i - x_i)\} dS_0,$$

where $\hat{\mathbf{x}}$ is any admissible field. It follows since W is convex that

$$\begin{aligned} W(\hat{\mathbf{e}}) - W(\mathbf{e}) &\geq (\hat{e}_{ij} - e_{ij}) \frac{\partial W}{\partial e_{ij}}(\mathbf{e}) \\ &= (\hat{e}_{ij} - e_{ij}) T_{ij}. \end{aligned}$$

This inequality, coupled with an application of the divergence theorem, gives the desired result, that

$$E(\hat{\mathbf{x}}) \geq E(\mathbf{x}).$$

Notice that the proof, in this form, requires W to be convex but not necessarily quadratic, and so applies also to ‘physically nonlinear’ problems, under the assumption of small deformations (the deformation theory of plasticity falls within this class).

The complementary energy principle

Although there is a stationary principle of complementary energy for finite deformations, a minimum principle has only been established in the case of small deformations. Then, W is a convex function of $\mathbf{e}$ and there is no difficulty in defining a complementary energy density function

$$W^*(\mathbf{T}) = \sup_{\mathbf{e}} \left[T_{ij}e_{ij} - W(\mathbf{e})\right]. \tag{6.54}$$

The supremum is attained when

$$T_{ij} = \frac{\partial W}{\partial e_{ij}};$$

the equality

$$e_{ij} = \frac{\partial W^*}{\partial T_{ij}}$$

is satisfied simultaneously. The complementary energy principle states that

$$F(\mathbf{T}) = \int_{\mathcal{B}_0} W^*(\mathbf{T})\mathbf{dX} - \int_{\partial\mathcal{B}_0} \{\{N_j T_{ji} x_i\}\} dS_0$$

is minimized by the actual stress field $\mathbf{T}$, amongst stress fields that satisfy the equilibrium equations and any prescribed traction boundary conditions ($N_j T_{ji} = t_i$). The double curly bracket in (6.54) implies summation only over those values of i for which x_i is prescribed. The function W^* is convex and the proof follows that outlined for the minimum energy principle.

Energy flux considerations

This section is valid for any continuum and even thermal effects are admitted; the formulae can easily be specialized to elasticity. Suppose that (part of) a body occupies a domain $\mathcal{B}_0$ in the reference configuration, as discussed earlier. It is subjected to surface forces $t_i = N_\alpha S_{\alpha i}$. In addition, there is a flux of heat out of $\mathcal{B}_0$ across the surface

$\partial\mathcal{B}_0$, which is expressed as $N_\alpha q_\alpha$ per unit area in the reference configuration, so that $\mathbf{q}$ represents a 'nominal heat flux vector'. For simplicity, it is assumed that there is no body force or direct input of heat from an external source, except through the boundary. Since energy is conserved, the rate of energy input into $\mathcal{B}_0$ must equal the rate of increase of energy within $\mathcal{B}_0$. Thus,

$$\int_{\partial\mathcal{B}_0} [N_\alpha S_{\alpha i} v_i - N_\alpha q_\alpha] dS_0 = \int_{\mathcal{B}_0} \frac{d}{dt} \left[U + \tfrac{1}{2}\rho_0 v_i v_i\right] \mathbf{dX}, \tag{6.55}$$

where U denotes the internal energy per unit reference volume (this will, in general, be a function of the current state, as specified by the deformation gradient $\mathbf{A}$, the entropy and some set of internal variables). Application of the divergence theorem to (6.55), coupled with the equation of motion (6.12) with $\mathbf{b} = 0$, now implies the local energy balance equation

$$S_{\alpha i} v_{i,\alpha} - q_{\alpha,\alpha} = \dot{U}, \tag{6.56}$$

since (6.55) must apply for any domain $\mathcal{B}_0$.

So far, the domain $\mathcal{B}_0$, once chosen, is fixed. It transforms, during the motion of the body, to a domain $\mathcal{B}_1$, which depends upon t but always consists of the same set of material points. Now consider, however, a more general case, where a domain $\mathcal{B}_1(t)$ is chosen *a priori*, restricted only so that it varies smoothly with t. Such a domain maps back onto the reference configuration, through the inverse of the mapping (6.2), to a domain $\mathcal{B}_0(t)$, which now depends on t. Of course, $\mathcal{B}_0(t)$ could be chosen first, to induce a corresponding $\mathcal{B}_1(t)$. The rate of change of energy within $\mathcal{B}_1(t)$ becomes

$$\begin{aligned} &\left(\frac{d}{dt}\right) \int_{\mathcal{B}_0(t)} \left[U + \tfrac{1}{2}\rho_0 v_i v_i\right] \mathbf{dX} \\ &\quad = \int_{\mathcal{B}_0(t)} \frac{d}{dt} \left[U + \tfrac{1}{2}\rho_0 v_i v_i\right] \mathbf{dX} + \int_{\partial\mathcal{B}_0(t)} \left[U + \tfrac{1}{2}\rho_0 v_i v_i\right] (N_\alpha C_\alpha) dS_0, \end{aligned} \tag{6.57}$$

where the components of the velocity of $\partial\mathcal{B}_0$ are C_α (these may depend on position $\mathbf{X} \in \partial\mathcal{B}_0$). Use of the divergence theorem, the local energy balance (6.56) and the equation of motion shows that the volume integral on the right side of (6.57) satisfies (6.56), even though $\mathcal{B}_0$ depends on t, and hence

$$\left(\frac{d}{dt}\right) \int_{\mathcal{B}_0(t)} \left[U + \tfrac{1}{2}\rho_0 v_i v_i\right] \mathbf{dX} = \int_{\partial\mathcal{B}_0(t)} N_\alpha \left[S_{\alpha i} v_i - q_\alpha + C_\alpha \left(U + \tfrac{1}{2}\rho_0 v_i v_i\right)\right] dS_0. \tag{6.58}$$

The right side of (6.58) may thus be interpreted as the flux of energy across the moving surface $\partial\mathcal{B}_1(t)$, whose image in the reference configuration is $\partial\mathcal{B}_0(t)$.

6.4 Inelastic constitutive equations

The study of the response of solids which are not elastic of course is extremely broad. This section just presents the briefest outline, sufficient for the purposes of these notes.

A rather general framework for discussing the response of solids is to postulate that stress at some material point $\mathbf{X}$ and some time t depends on the entire history of the strain at the point $\mathbf{X}$, at all times before t, and including the time t. ("Non-local" continua, in which stress depends on the strain in a *neighbourhood* of $\mathbf{X}$ are excluded for the present.) Conversely, the strain at $\mathbf{X}$ at time t depends on the history of the stress. We shall assume here that "stress" and "strain" are some chosen work-conjugate pair, as discussed earlier.

No progress can be made unless some further structure is assumed. Here, we suppose that the rate of strain at time t is expressible in the form

$$\dot{\mathbf{e}}^f = \mathbf{F}(\mathbf{T}^f, \dot{\mathbf{T}}^f, H), \quad \text{or} \quad \dot{e}^f_{\alpha\beta} = F_{\alpha\beta}(\mathbf{T}^f, \dot{\mathbf{T}}^f, H), \tag{6.59}$$

where $\mathbf{F}$ is a (tensor-valued) function and H symbolically denotes the entire history of the deformation at times prior to t, perhaps encapsulated in some set of parameters, called "state variables". A possible response to an imposed stress is displayed in Fig. 6.3. This depicts just a uniaxial stress S and the corresponding component e of strain[17]. The stress increases from zero at O, up to a maximum at B, and then decreases down to C. The noteworthy feature is that the response is linear from O to A. Then, it deviates significantly from linear behaviour. Of course the entire curve that is shown will depend not only upon the stress but also on its rate of change. That is, the figure actually shows the curve obtained by eliminating the time t between the stress and the strain, both obtained as functions of t.

It is common experience that the linear portion OA that is depicted really does exist for the majority of solid materials, and that this portion is independent of the rate of change of the stress. Thus, at low stress or strain, the material behaves according to linear elasticity. The point A is not so easy to define rigorously. It is true that a rapid deviation from linear elastic behaviour occurs in the vicinity of A. The phenomenon is called "plastic yielding". Exactly where it is considered to occur depends on the sensitivity of the experiment. It is probably true to assert that there is *some* contamination of the linear elastic response, even at very small stress or strain, but this is insignificant until the point A is reached. Thus, the point A will be defined in practice as that point beyond which the linear elastic approximation is no longer acceptable.

[17] S can be regarded as the 11 component of nominal stress and e as the proportional elongation, $\lambda_1 - 1$; these are natural observables in a simple tension test.

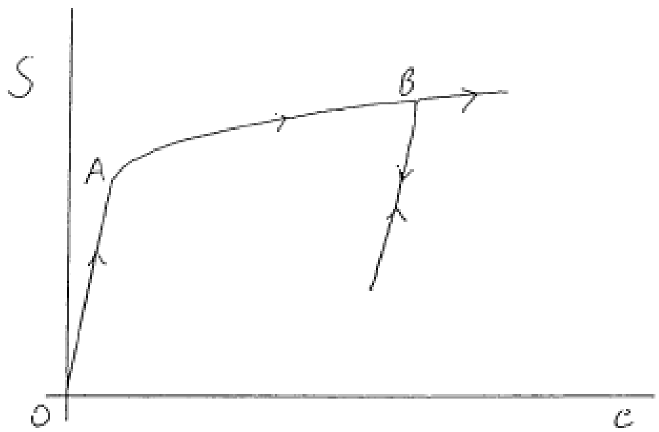

Fig. 6.3. Representation of elastoplastic response.

Now consider the point B, at which the stress is reduced: the incremental response again becomes linear, but if the stress were reduced to zero there would be some residual strain. Again, exactly what happens near B will depend on the rate of change of the stress, and some deviation from linearity will occur, though this can almost invariably be ignored in practice.

Unless the rate of stress (or, equivalently, the strain-rate) is very large, it is found for metals at moderate temperatures (less than one-third the melting point, for instance) that the dependence on stress- or strain-rate is negligible. An example stress-strain relation, in one dimension, that illustrates this is

$$\dot{e} = \dot{S}/E + \dot{e}^p, \tag{6.60}$$

where the "plastic strain-rate" $\dot{e}^p$ is given by

$$\dot{e}^p = A(|S|/S_0)^N S/|S|, \tag{6.61}$$

where E is an elastic modulus, S_0 depends on the history H in some rate-independent fashion (such as $H = \int_0^t |\dot{e}^p| dt$) and $N \gg 1$. Since N is large, the stress will remain close to S_0 so long as $S \geq S_0$ and is increasing with time, since otherwise the strain-rate would be large. Conversely, if $|S| < S_0$, the plastic strain-rate is negligible. Thus, to lowest approximation, the response (6.60) during continued loading can be expressed in the form

$$\dot{e} = \left(\frac{1}{E} + \frac{1}{h}\right)\dot{S}, \tag{6.62}$$

where $h = S_0'(H)$ if $H = \int_0^t |\dot{e}^p|\,dt$. This approximation will not be uniform near the point A, which requires the more complete relation (6.60, 2.61) for resolution.

However, for many practical purposes, the rate-independent idealisation is sufficient. This is now developed more systematically. Some generic instant, at which the stress and strain tensors have the values $\mathbf{T}^f$, $\mathbf{e}^f$, is considered. There is an elastic domain in stress space (and a corresponding domain in strain space). The stress lies either within the elastic domain or on its boundary. The boundary is called the "yield surface". If the stress lies within the elastic domain, then increments of stress and strain are related elastically: say

$$\dot{e}^f_{\alpha\beta} = \mathcal{M}_{\alpha\beta\gamma\delta}\dot{T}^f_{\gamma\delta}. \tag{6.63}$$

The compliance tensor $\mathcal{M}_{\alpha\beta\gamma\delta}$ may depend on whatever inelastic deformation has occurred previously — that is, upon H. Its inverse, $\mathcal{L}_{\alpha\beta\gamma\delta}$, is the tensor of elastic moduli. If the stress is currently at the boundary of the elastic domain — that is, on the yield surface — the relation (6.63) still applies if the stress-rate is directed into the interior of the elastic domain. This is assumed to be convex, so the appropriate restriction when the stress is on the yield surface is that the stress-rate lies within a cone $\mathcal{K}$ in stress-rate space, which locally defines the interior of the elastic domain. In the simplest case, the yield surface is smooth in a neighbourhood of $\mathbf{T}^f$; then, $\mathcal{K}$ will have the form

$$\mathcal{K} = \{\dot{\mathbf{T}}^f : Q_{\alpha\beta}\dot{T}^f_{\alpha\beta} \leq 0\}, \tag{6.64}$$

where $\mathbf{Q}$ is symmetric and depends upon the local state of stress and strain (and history). However, it is also possible that the yield surface might have a vertex. If $\mathbf{T}^f$ is on the yield surface and its rate $\dot{\mathbf{T}}^f$ lies outside the cone $\mathcal{K}$, then $\mathbf{T}^f$ remains on the yield surface which therefore, in general, moves as the stress changes. The strain-rate has a plastic contribution as well, so that

$$\dot{e}^f_{\alpha\beta} = \mathcal{M}_{\alpha\beta\gamma\delta}\dot{T}^f_{\gamma\delta} + F_{\alpha\beta}(\dot{\mathbf{T}}^f, H), \tag{6.65}$$

where $F_{\alpha\beta}$ are homogeneous functions of degree one in $\dot{\mathbf{T}}^f$, and H represents history, as before. Most models of plasticity, in fact, can be expressed, when the current stress is on the yield surface,

$$\begin{aligned} \dot{e}^f_{\alpha\beta} &= \mathcal{M}_{\alpha\beta\gamma\delta}\dot{T}^f_{\gamma\delta} \text{ if } \dot{\mathbf{T}}^f \in \mathcal{K} \\ &= M_{\alpha\beta\gamma\delta}\dot{T}^f_{\gamma\delta} \text{ otherwise,} \end{aligned} \tag{6.66}$$

where the tensor $\mathbf{M}$ is a homogeneous function of degree zero in $\dot{\mathbf{T}}^f$. The case described as "otherwise" is referred to as "during plastic loading", or "during plastic deformation". The elastic relation also applies when the stress is within the elastic domain.

In the case that the cone takes the form (6.64) — at a smooth point on the yield surface — the relation (6.66) is often given in the form

$$\begin{aligned} \dot{e}^f_{\alpha\beta} &= \mathcal{M}_{\alpha\beta\gamma\delta}\dot{T}^f_{\gamma\delta} \ \text{ if } \ Q_{\alpha\beta}\dot{T}^f_{\alpha\beta} \leq 0 \\ &= \mathcal{M}_{\alpha\beta\gamma\delta}\dot{T}^f_{\gamma\delta} + P_{\alpha\beta}(Q_{\gamma\delta}\dot{T}^f_{\gamma\delta})/h \ \text{ otherwise.} \end{aligned} \tag{6.67}$$

Thus, in this case,

$$M_{\alpha\beta\gamma\delta} = \mathcal{M}_{\alpha\beta\gamma\delta} + P_{\alpha\beta}Q_{\gamma\delta}/h. \tag{6.68}$$

It is usual to normalise $\mathbf{P}$ and $\mathbf{Q}$ so that $P_{\alpha\beta}P_{\alpha\beta} = 1$ (and similarly for $\mathbf{Q}$).

Another very common assumption (good for metals but not applicable to granular media, for instance) is that of "normality": during loading, the increment of plastic strain must lie within the cone of normals to the yield surface. In the simple case represented by (6.68), this means simply that $\mathbf{P} = \mathbf{Q}$.

In summary, therefore, our theory of plasticity is as follows: the constitutive equation will be taken to be (6.66). Usually, the instantaneous compliances $M_{\alpha\beta\gamma\delta}$ will have the symmetry $M_{\alpha\beta\gamma\delta} = M_{\gamma\delta\alpha\beta}$. The stress-rate versus strain-rate relation will be assumed to be invertible. Thus, when the stress is on the yield surface,

$$\begin{aligned} \dot{T}^f_{\alpha\beta} &= \mathcal{L}_{\alpha\beta\gamma\delta}\dot{e}^f_{\gamma\delta} \ \text{ if } \ \dot{\mathbf{T}} \in \mathcal{K}, \\ &= L_{\alpha\beta\gamma\delta}\dot{e}^f_{\gamma\delta} \ \text{ if } \ \dot{\mathbf{T}} \notin \mathcal{K}. \end{aligned} \tag{6.69}$$

The tensor $\mathbf{L}$ is homogeneous of degree zero in $\dot{\mathbf{T}}^f$. Equivalently, $\mathbf{L}$ may be considered to be a homogeneous function of degree zero of $\dot{\mathbf{e}}^f$.

Three remarks will be made, in conclusion. The first is that nonlinear elastic response can be described in the form $\dot{T}^f_{\alpha\beta} = L_{\alpha\beta\gamma\delta}\dot{e}^f_{\gamma\delta}$, for any increment of stress. It is necessary only to take

$$L_{\alpha\beta\gamma\delta} = \frac{\partial^2 W}{\partial e^f_{\alpha\beta}\partial e^f_{\gamma\delta}}(\mathbf{e}^f),$$

which automatically has the symmetry given above. The second is that materials other than those normally considered as elastic-plastic may have local constitutive response with the form (6.66). The last is that, if the relation (6.66) is known for any conjugate stress-strain pair, then the corresponding relation can be deduced for any other. Furthermore, a similar relation can be deduced for the relation between nominal stress and deformation gradient, by use of (6.32), (6.33) and (6.34). In the particular case that $\mathbf{T}^f$ is the second Piola-Kirchhoff stress, $S_{\alpha i} = A_{i\beta}T^{(2)}_{\alpha\beta}$ from (6.18), and $\dot{e}^{(2)}_{\alpha\beta} = \frac{1}{2}(\dot{A}_{k\alpha}A_{k\beta} + A_{k\alpha}\dot{A}_{k\beta})$. Therefore,

$$\dot{S}_{\alpha i} = c_{\alpha i\beta j}\dot{A}_{j\beta}, \tag{6.70}$$

where

$$\begin{aligned} c_{\alpha i\beta j} &= A_{i\gamma}\mathcal{L}_{\alpha\gamma\beta\delta}A_{j\delta} + \delta_{ij}S_{\alpha k}B_{k\beta} \text{ if } \dot{\mathbf{S}} \in \mathcal{K}', \\ &= A_{i\gamma}L_{\alpha\gamma\beta\delta}A_{j\delta} + \delta_{ij}S_{\alpha k}B_{k\beta} \text{ if } \dot{\mathbf{S}} \notin \mathcal{K}', \end{aligned} \tag{6.71}$$

where $\mathcal{K}'$ is defined so that $\dot{\mathbf{S}} \in\mathcal{K}'$ implies $\dot{\mathbf{T}}^f\in\mathcal{K}$, and conversely.

6.5 Virtual work; relation to finite element computation

It has been seen in Section 2.3 that an *elastic* body subjected to loads of a certain fairly general type is in equilibrium when it attains a minimum-energy configuration. In fact, equation (6.53) was shown to generate the field equations and the boundary conditions.

Now consider the opposite reasoning, for a somewhat more general problem. Assume the equations of motion (6.12), together with initial conditions

$$\mathbf{x} = \mathbf{x}_0(\mathbf{X}), \;\; \dot{\mathbf{x}} = \dot{\mathbf{x}}_0(\mathbf{X}), \;\; \mathbf{X} \in \mathcal{B}_0, \;\; t = 0 \tag{6.72}$$

and boundary conditions of the type considered in Section 2.3: one of the pair $(x_i, N_\alpha S_{\alpha i})$ is given at each point of the boundary, for each i, *except* that the traction component $N_\alpha S_{\alpha i}$, where given, may be configuration-dependent,

$$N_\alpha S_{\alpha i} = \psi_i(\mathbf{x}, \mathbf{A}, \mathbf{X}, t). \tag{6.73}$$

The functions ψ_i are restricted so that they only involve derivatives of $\mathbf{x}$ in directions tangent to the surface, so that they can be expressed in terms of surface values of $\mathbf{x}$ alone.

Now multiply the equation of motion (6.12) by a "virtual displacement" $w_i(\mathbf{X}, t)$, sum over i and integrate over $\mathcal{B}_0$. Use of the divergence theorem then gives

$$\int_{\partial\mathcal{B}_0} N_\alpha S_{\alpha i} w_i dS_0 = \int_{\mathcal{B}_0} [w_{i,\alpha}S_{\alpha i} - \rho_0 b_i w_i + \rho_0 \dot{v}_i w_i]\mathbf{dX}. \tag{6.74}$$

This is true for any field w_i. Now restrict w_i so that $w_i = 0$ wherever x_i is prescribed. Then, the left side of (6.74) reduces to

$$\int_{\partial\mathcal{B}_0} \{\psi_i w_i\} dS_0 = \int_{\mathcal{B}_0} [w_{i,\alpha}S_{\alpha i} - \rho_0 b_i w_i + \rho_0 \dot{v}_i w_i]\mathbf{dX}, \tag{6.75}$$

where the curly bracket implies summation over only those values of i for which ψ_i is given.

The statement (6.75) permits the construction of finite-element approximations. The field x_i is approximated as

$$x_i = \sum_K U_i^K(t)\phi_K(\mathbf{X}), \tag{6.76}$$

where the functions $\{\phi_K\}$ take the value 1 at the node labelled K and are zero at all other nodes. Thus, U_i^K provides an approximation for u_i, evaluated at the node K. Now w_i is given similarly:

$$w_i = \sum_K W_i^K(t)\phi_K(\mathbf{X}). \tag{6.77}$$

The requirement that (6.75) should be satisfied for all w_i of the form (6.77) generates a system of nonlinear equations for the nodal values $\{U_i^K\}$. Formally, the system has the structure

$$F_{Ki}(\mathbf{U}) = \sum_L M_{KL}\ddot{U}_i^L, \quad \text{or} \quad \mathbf{F}(\mathbf{U}) = \mathbf{M}\ddot{\mathbf{U}}, \tag{6.78}$$

together with initial conditions. The function (or functional) $\mathbf{F}$ cannot easily be given explicitly in the general case. It requires that the nominal stress be expressed, through the constitutive equation, in terms of the displacement field, in the approximation represented by (6.76). The boundary term ψ_i has to be similarly approximated, in terms of the values of the displacements at the boundary nodes. The "mass matrix" has the components

$$M_{KL} = \int_{\mathcal{B}_0} \rho_0\phi_K\phi_L\mathbf{dX}. \tag{6.79}$$

The practical implementation of the finite element method is beyond the scope of this set of notes.

Appendix 6A: Calculations relating to stress- and strain-rates

The problem that is addressed is to calculate the rates of change of the principal stretches λ_r and the corresponding principal directions $\mathbf{u}^{(r)}$ $(r = 1, 2, 3)$. They satisfy the equations

$$\mathbf{A}^T\mathbf{A}\mathbf{u}^{(r)} = \lambda_r^2\mathbf{u}^{(r)}, \quad \text{or} \quad A_{k\beta}A_{k\gamma}u_\gamma^{(r)} = \lambda_r^2 u_\beta^{(r)}. \tag{6A.1}$$

Differentiating with respect to time gives

$$(\dot{A}_{k\beta}A_{k\gamma} + A_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)} + A_{k\beta}A_{k\gamma}\dot{u}_\gamma^{(r)} = 2\lambda_r\dot{\lambda}_r u_\beta^{(r)} + \lambda_r^2\dot{u}_\beta^{(r)}. \tag{6A.2}$$

Now multiply (6A.2) by $u_\beta^{(r)}$ and sum over β. This gives

$$2\lambda_r\dot{\lambda}_r = u_\beta^{(r)}(\dot{A}_{k\beta}A_{k\gamma} + A_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)}, \tag{6A.3}$$

since (6A.1) holds, and $\mathbf{u}^{(r)}$ is a unit vector. Substituting the now-known value of $\dot{\lambda}_r$ back into (6A.2) gives

$$(A_{k\beta}A_{k\gamma} - \lambda_r^2\delta_{\beta\gamma})\dot{u}_\gamma^{(r)} = u_\beta^{(r)}[u_\nu^{(r)}(\dot{A}_{k\nu}A_{k\gamma} + A_{k\nu}\dot{A}_{k\gamma})u_\gamma^{(r)}] - (\dot{A}_{k\beta}A_{k\gamma} + A_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)}. \qquad (6A.4)$$

The matrix on the left side of this equation is singular, but a solution exists because the right side is orthogonal to the eigenvector $\mathbf{u}^{(r)}$. It is rendered unique by the requirement that $\dot{\mathbf{u}}^{(r)}$ must be orthogonal to $\mathbf{u}^{(r)}$, since the latter must remain a unit vector. The explicit solution is obtained by writing (6A.4) in the spectral representation

$$\sum_{s\neq r}(\lambda_s^2 - \lambda_r^2)u_\beta^{(s)}u_\gamma^{(s)})\dot{u}_\gamma^{(r)} = u_\beta^{(r)}[u_\nu^{(r)}(\dot{A}_{k\nu}A_{k\gamma} + A_{k\nu}\dot{A}_{k\gamma})u_\gamma^{(r)}] - (\dot{A}_{k\beta}A_{k\gamma} + A_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)}. \qquad (6A.5)$$

Inversion is immediate. The matrix on the left, regarded as an operator on the subspace spanned by the eigenvectors $\{\mathbf{u}^{(s)}\}$ with $s \neq r$, has inverse with $\beta\gamma$ component

$$\sum_{s\neq r}\frac{1}{\lambda_s^2 - \lambda_r^2}u_\beta^{(s)}u_\gamma^{(s)}.$$

Thus,

$$\dot{u}_\alpha^{(r)} = \sum_{s\neq r}\frac{1}{\lambda_r^2 - \lambda_s^2}u_\alpha^{(s)}[u_\beta^{(s)}(\dot{A}_{k\beta}A_{k\gamma} + A_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)}]. \qquad (6A.6)$$

These values may now be substituted into (6.24) to give

$$\dot{e}_{\beta\gamma}^f = \sum_r\left\{\frac{f'(\lambda_r)}{2\lambda_r}u_\beta^{(r)}u_\gamma^{(r)}u_\lambda^{(r)}(\dot{A}_{k\lambda}A_{k\mu} + A_{k\lambda}\dot{A}_{k\mu})u_\mu^{(r)}\right.$$
$$\left. + \sum_{s\neq r}\frac{f(\lambda_r)}{\lambda_r^2 - \lambda_s^2}(u_\beta^{(r)}u_\gamma^{(s)} + u_\beta^{(s)}u_\gamma^{(r)})[u_\lambda^{(r)}(\dot{A}_{k\lambda}A_{k\mu} + A_{k\lambda}\dot{A}_{k\mu})u_\mu^{(s)}]\right\}. \qquad (6A.7)$$

Since

$$\dot{e}_{\beta\gamma}^f = \frac{\partial e_{\beta\gamma}^f}{\partial A_{i\alpha}}\dot{A}_{i\alpha},$$

a formal differentiation of (6A.7) with respect to $\dot{A}_{i\alpha}$ yields the result (6.26).

The case $\mathbf{A} = \mathbf{I}$ is degenerate, and it is easier to start again than to take the necessary limits. Equation (6A.1) is satisfied identically. The eigenvectors $\mathbf{u}^{(r)}$ are arbitrary at the precise instant that $\mathbf{A} = \mathbf{I}$, but it is sensible to require them to be continuous functions of time. The relation (6A.2) reduces to

$$\tfrac{1}{2}(\dot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\dot{A}_{k_\gamma})u_\gamma^{(r)} = \dot{\lambda}_r u_\beta^{(r)}. \qquad (6A.8)$$

Equivalently,

$$\Gamma_{ij}\delta_{j\alpha}u_\alpha^{(r)} = \dot{\lambda}_r\delta_{i\alpha}u_\alpha^{(r)}.$$

Thus, $\{\dot{\lambda}_r\}$ $(r = 1, 2, 3)$ are the eigenvalues of the Eulerian strain-rate tensor, and $\{\mathbf{u}^{(r)}\}$ are the corresponding eigenvectors, resulting in the spectral representation (6.29).

Now differentiate (6A.2) with respect to time, and afterwards set $\mathbf{A} = \mathbf{I}$ (and $\lambda_r = 1$). This gives

$$(\ddot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\ddot{A}_{k\gamma} + 2\dot{A}_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)} + (\dot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\dot{A}_{k_\gamma})\dot{u}_\gamma^{(r)} = 2(\ddot{\lambda}_r + \dot{\lambda}_r^2)u_\beta^{(r)} + 4\dot{\lambda}_r\dot{u}_\beta^{(r)}. \qquad (6A.9)$$

Multiplying by $u_\beta^{(r)}$ and summing over β gives

$$u_\beta^{(r)}(\ddot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\ddot{A}_{k\gamma} + 2\dot{A}_{k\beta}\dot{A}_{k\gamma})u_\gamma^{(r)} = 2(\ddot{\lambda}_r + \dot{\lambda}_r^2). \qquad (6A.10)$$

Then, substituting back into (6A.9) gives

$$\begin{aligned}(\dot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\dot{A}_{k_\gamma} - 2\dot{\lambda}_r\delta_{\beta\gamma})\dot{u}_\gamma^{(r)} = \{[u_\lambda^{(r)}(\ddot{A}_{k\lambda}\delta_{k\mu} + \delta_{k\lambda}\ddot{A}_{k\mu} + 2\dot{A}_{k\lambda}\dot{A}_{k\mu})u_\mu^{(r)}]\delta_{k\gamma} \\ - (\ddot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\ddot{A}_{k\gamma} + 2\dot{A}_{k\beta}\dot{A}_{k\gamma})\}u_\gamma^{(r)} \qquad (6A.11)\end{aligned}$$

This equation can be treated in the same way that (6A.5) was. It has solution

$$\dot{u}_\alpha^{(r)} = \tfrac{1}{2}\sum_{s\neq r}\frac{1}{\dot{\lambda}_r - \dot{\lambda}_s}u_\alpha^{(s)}u_\beta^{(s)}[\tfrac{1}{2}(\ddot{A}_{k\beta}\delta_{k\gamma} + \delta_{k\beta}\ddot{A}_{k\gamma}) + \dot{A}_{k\beta}\dot{A}_{k\gamma}]u_\gamma^{(r)}. \qquad (6A.12)$$

Now differentiating (6.24) with respect to time, and then setting $\mathbf{A} = \mathbf{I}$, gives

$$\ddot{e}_{\beta\gamma}^f = \sum_r (f''(1)\dot{\lambda}_r^2 + \ddot{\lambda}_r)u_\beta^{(r)}u_\gamma^{(r)} + 2\dot{\lambda}_r(\dot{u}_\beta^{(r)}u_\gamma^{(r)} + u_\beta^{(r)}\dot{u}_\gamma^{(r)}). \qquad (6A.13)$$

Also,

$$\ddot{e}_{\beta\gamma}^f = \frac{\partial e_{\beta\gamma}^f}{\partial A_{i\alpha}}\ddot{A}_{i\alpha} + \frac{\partial^2 e_{\beta\gamma}^f}{\partial A_{i\alpha}\partial A_{j\nu}}\dot{A}_{i\alpha}\dot{A}_{j\nu}. \qquad (6A.14)$$

Therefore, the expressions (6A.12) for $\dot{u}_\alpha^{(r)}$ and (6A.10) for $\ddot{\lambda}_r$ can be substituted into (6A.13), and comparison of the resulting expression with (6A.14) will give $\partial^2 e_{\beta\gamma}^f/\partial A_{i\alpha}\partial A_{j\nu}$. It suffices for this purpose to consider the special case $\ddot{\mathbf{A}} = 0$. It is noted, too, that $\sum_r \dot{\lambda}_r^2 u_\beta^{(r)}u_\gamma^{(r)}$ can be given explicitly, by exploiting the spectral representation of the operator in (6A.8): the sum is just that operator, squared.

Implementing this plan gives the result

$$\begin{aligned}\frac{\partial^2 e_{\beta\gamma}^f}{\partial A_{i\alpha}\partial A_{j\nu}}\dot{A}_{i\alpha}\dot{A}_{j\nu} = \Big\{\tfrac{1}{4}(f''(1) - 1)(\dot{A}_{k\beta}\delta_{k\delta} + \dot{A}_{k\delta}\delta_{k\beta})(\dot{A}_{l\gamma}\delta_{l\delta} + \dot{A}_{l\delta}\delta_{k\gamma}) \\ + \sum_r u_\beta^{(r)}u_\gamma^{(r)}u_\lambda^{(r)}\dot{A}_{k\lambda}\dot{A}_{k\mu}u_\mu^{(r)} + \tfrac{1}{2}\sum_r\sum_{s\neq r}(u_\lambda^{(s)}\dot{A}_{k\lambda}\dot{A}_{k\mu}u_\mu^{(r)})(u_\beta^{(s)}u_\gamma^{(r)} + u_\beta^{(r)}u_\gamma^{(s)})\Big\}. \qquad (6A.15)\end{aligned}$$

Formal differentiation of this result with respect to $\dot{A}_{i\alpha}$ and $\dot{A}_{j\nu}$ gives *twice* $\partial^2 e_{\beta\gamma}^f/\partial A_{i\alpha}\partial A_{j\nu}$. The result (6.34) follows directly.