The Universal Program of Nonlinear Hyperelasticity*

Arash Yavari[†1,2] and Alain Goriely[3]

[1] *School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*
[2] *The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*
[3] *Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK.*

June 6, 2022

Dedicated to Professor Roger L. Fosdick on the occasion of his 85th birthday.

Abstract

For a given class of materials, universal deformations are those that can be maintained in the absence of body forces by applying only boundary tractions. Universal deformations play a crucial role in nonlinear elasticity. To date, their classification has been accomplished for homogeneous isotropic solids following Ericksen's seminal work, and homogeneous anisotropic solids and inhomogeneous isotropic solids in our recent works. In this paper we study universal deformations for inhomogeneous anisotropic solids defined as materials whose energy function depends on position. We consider both compressible and incompressible transversely isotropic, orthotropic, and monoclinic solids. We show that the *universality constraints*—the constraints that are dictated by the equilibrium equations and the arbitrariness of the energy function—for inhomogeneous anisotropic solids include those of inhomogeneous isotropic and homogeneous anisotropic solids. For compressible solids, universal deformations are homogeneous and the material preferred directions are uniform. For each of the three classes of anisotropic solids we find the corresponding *universal inhomogeneities*—those inhomogeneities that are consistent with the universality constraints. For incompressible anisotropic solids we find the universal inhomogeneities for each of the six known families of universal deformations. This work provides a systematic approach to study analytically functionally-graded fiber-reinforced elastic solids.

Keywords: Universal deformations, nonlinear elasticity, anisotropic elasticity, inhomogeneity, functionally-graded materials.

Contents

*To appear in the *Journal of Elasticity*.

†Corresponding author, e-mail: arash.yavari@ce.gatech.edu

1 Introduction

In elasticity, for a given class of materials, universal deformations are those deformations that can be maintained in the absence of body forces by applying only boundary tractions for an arbitrary energy function in that class.[1] They are particularly important in nonlinear elasticity since they exist independently of a particular choice of energy function. Therefore, they can be used experimentally to study material properties and analytically as a basis for more complicated deformations or to gain insight into basic properties of materials. The history of a theory of universal deformations goes back to the seminal work of Ericksen who showed that for homogeneous compressible isotropic solids, universal deformations are homogeneous [Ericksen, 1955]. From that original seed, grew a large body of work addressing the same problems for materials that have constraints such as incompressibility, may be anisotropic, may be inhomogenous, may be anelastic, or linear as shown in Fig. 1. The problem of finding universal deformations in the presence of internal constraints is more difficult [Saccomandi, 2001]. For homogeneous incompressible isotropic solids, in a second seminal paper that was motivated by the earlier works of Rivlin [Rivlin, 1948, 1949a,b], Ericksen [1954] found four families of universal deformations. He conjectured that a deformation with constant principal invariants has to be homogeneous. This conjecture turned out to be incorrect [Fosdick, 1966], and motivated the discovery of a fifth family of universal deformations [Singh and Pipkin, 1965, Klingbeil and Shield, 1966]. The six known families of universal deformations are:

- Family 0: Homogeneous deformations
- Family 1: Bending, stretching, and shearing of a rectangular block
- Family 2: Straightening, stretching, and shearing of a sector of a cylindrical shell
- Family 3: Inflation, bending, torsion, extension, and shearing of a sector of an annular wedge
- Family 4: Inflation/inversion of a sector of a spherical shell
- Family 5: Inflation, bending, extension, and azimuthal shearing of an annular wedge

We should emphasize that for incompressible isotropic solids Ericksen's problem has not been solved completely to this day; the case of deformations with constant principal invariants is still an open problem. However, the conjecture is that there are no other possible families of universal deformations. In related works, there have been several studies of universal deformations and universal steady-state temperature fields

[1]See Pucci et al. [2015] for definitions of controllable, general, universal, and partial solutions in nonlinear elasticity.

in nonlinear thermoelasticity (see [Petroski and Carlson, 1968, Saccomandi, 1999, Dunwoody, 2005a,b], and references therein).

Based on Ericksen's seminal work, we embarked a few years ago into what we now refer to as the *universal program*: to generalize Ericken's results to anisotropic and inhomogeneous materials for all hyperelastic materials, anelastic materials, and linear materials (see Fig.1). Indeed, the analogue of universal deformations in linear elasticity are universal displacements [Truesdell, 1966, Gurtin, 1972, Yavari et al., 2020]. In [Yavari et al., 2020], it was shown that universal displacements explicitly depend on the symmetry class of the material; the larger the symmetry group is the larger the corresponding space of universal displacements is. More recently, we studied universal inhomogeneities in anisotropic linear elasticity [Yavari and Goriely, 2022]. There have been recent extensions of Ericksen's analysis to anelasticity. Yavari and Goriely [2016] proved that in compressible anelasticity universal deformations must be covariantly homogeneous. In the case of incompressible anelasticity, Goodbrake et al. [2020] observed that a key feature of the analysis is that the extra fields entering the analysis should follow the same symmetry as the universal deformations. They also showed that the six known families of universal deformations are invariant under certain Lie subgroups of the special Euclidean group.

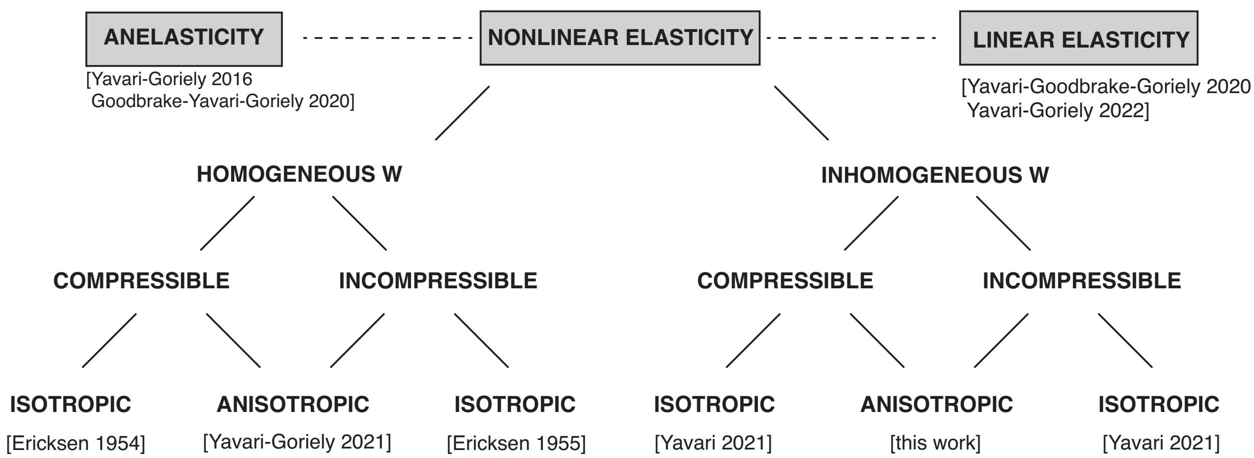

Figure 1: *The universal program: Finding all the universal deformations and displacements, together with the associated universal material preferred directions, and universal inhomogeneities, for both compressible and incompressible solids. These are the different cases considered so far with partial or complete solutions. Here, nonlinear elasticity refers to hyperleasticity and the existence of a strain-energy density W is assumed that can either be homogeneous or non-homogenous, isotropic or anisotropic.*

Until recently, there was no systematic study of universal deformations in anisotropic solids. There were early studies restricted to a subset of Family 1 deformations for two cases of homogeneous anisotropy, and Family 3 deformations for an example of homogeneous anisotropy [Ericksen and Rivlin, 1954] (see also [Adkins, 1955a,b]). However, many examples of universal deformations for anisotropic fiber-reinforced systems were known and widely used [Spencer, 1982, Qiu and Pence, 1997, Melnik and Goriely, 2013, Holzapfel et al., 2000, Demirkoparan and Pence, 2007, Goriely and Tabor, 2013, Demirkoparan and Pence, 2015, Goriely, 2017]. Recently, we studied universal deformations and universal material preferred directions in homogeneous compressible and incompressible anisotropic solids [Yavari and Goriely, 2021]. More specifically, we considered compressible and incompressible transversely isotropic, orthotropic, and monoclinic solids. We assumed that the material preferred directions can vary from point to point. In the case of compressible solids we showed that universal deformations are homogeneous and universal material preferred directions for the three classes of anisotropic solids must be uniform. In the case of homogeneous incompressible transversely isotropic, orthotropic, and monoclinic solids, we showed that in addition to the nine universality constraints for isotropic solids [Ericksen, 1954], there are extra 25, 74, and 152, respectively, extra universality constraints that must be satisfied. For each known family of universal deformations we obtained the universal material preferred directions assuming that they have the symmetry of the corresponding universal deformations (that are encoded in the symmetries of the right Cauchy-Green strain).[2]

Motivated by a result in [Golgoon and Yavari, 2021], Yavari [2021] extended the analysis of universal deformations to inhomogeneous isotropic solids (with position-dependent strain-energy density), and showed that in addition to those of homogeneous isotropic solids there are some extra universality constraints. It was shown that inhomogeneous compressible isotropic solids do not admit universal deformations. In the case of inhomogeneous incompressible solids the following results were obtained for each of the six known families of universal deformations.

- For inhomogeneous incompressible isotropic solids it was incorrectly concluded that Family 0 deformations are not universal. This is discussed in §4.1, and the corrected statement is given in Footnote 4.
- Family 1 deformations are universal for any energy function of the form $W = W(X, I_1, I_2)$, where (X, Y, Z) is a Cartesian coordinate system with coordinate lines normal to the faces of an undeformed rectangular block. Note that with respect to cylindrical coordinates (r, θ, z) in the deformed configuration, Family 1 deformations have the form: $(r, \theta, z) = \left(\sqrt{C_1(2X + C_4)}, C_2(Y + C_5), \frac{Z}{C_1 C_2} - C_2 C_3 Y + C_6\right)$, where $C_1, ..., C_6$ are constants.
- Families 2, 3, and 4 deformations are universal for any energy function of the form $W = W(R, I_1, I_2)$, where R is the radial coordinate in the undeformed configuration of a cylindrical shell, an annular wedge, and a spherical shell, for Families 2, 3, and 4, respectively.
- For inhomogeneous incompressible isotropic solids, Family 5 deformations are not universal.

The remaining problem to be solved to complete Ericksen's program is to study elastic materials that are inhomogenous, and anisotropic. Therefore, we study universal deformations for inhomogeneous anisotropic solids and generalize the results of [Yavari and Goriely, 2021, Yavari, 2021]. We consider both compressible and incompressible transversely isotropic, orthotropic, and monoclinic solids. It is shown that the *universality constraints*—the constraints that are dictated by the equilibrium equations and the arbitrariness of the energy function— for inhomogeneous anisotropic solids include those of inhomogeneous isotropic and homogeneous anisotropic solids as special cases. For compressible solids, universal deformations are homogeneous and the material preferred directions are uniform. For each of the three classes of anisotropic solids we find the corresponding *universal inhomogeneities*—those inhomogeneities (position dependence of the energy function) that are compatible with the universality constraints. For incompressible anisotropic solids we find the universal inhomogeneities for each of the six known families of universal deformations.

This paper is organized as follows. In §2 we tersely review nonlinear anisotropic elasticity. In §3, we consider inhomogeneous compressible transversely isotropic, orthotropic, and monoclinic solids. The universal deformations, universal material preferred directions, and universal inhomogeneities of inhomogeneous incompressible transversely isotropic solids are analyzed for each of the six known families in §4. Similar analyses for inhomogeneous incompressible orthotropic and inhomogeneous incompressible monoclinic solids are given in §5 and §6, respectively. Conclusions are given in §7.

2 Nonlinear Anisotropic Elasticity

Kinematics. Consider an elastic body $\mathcal{B}$. In nonlinear anelasticity the body is identified with a Riemannian manifold $(\mathcal{B}, \mathbf{G})$ whose metric $\mathbf{G}$ is used in calculating the natural distances between material points in the body. In nonlinear elasticity $(\mathcal{B}, \mathbf{G})$ is flat, and is a submanifold of the Euclidean 3-space. A deformation of the body is a map $\varphi : \mathcal{B} \to \mathcal{S}$, where $\mathcal{S}$ is the Euclidean ambient space, and $\mathbf{g}$ is the Euclidean metric. The material velocity is defined as

$$\mathbf{V}_t : \mathcal{B} \to T_{\varphi_t(\mathbf{X})}\mathcal{S}\,, \quad \mathbf{V}_t(\mathbf{X}) = \mathbf{V}(\mathbf{X}, t) = \frac{\partial \varphi(\mathbf{X}, t)}{\partial t}\,. \tag{2.1}$$

The spatial velocity is defined as $\mathbf{v} = \mathbf{V} \circ \varphi_t^{-1}$. The deformation gradient—the tangent map (or derivative) of φ—is denoted by $\mathbf{F} = T\varphi$. With respect to local coordinate charts $\{x^a\}$ and $\{X^A\}$ on $\mathcal{S}$ and $\mathcal{B}$, respectively,

[2]Unfortunately, there was a small mistake in calculating the universal material preferred directions for Family 5 deformations. The correct universal material preferred directions are given in (4.83), (5.17), and (6.31), for transversely isotropic, orthotropic, and monoclinic solids, respectively.

deformation gradient is defined as

$$\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}\,, \quad F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X})\,. \tag{2.2}$$

The deformation gradient is a linear map that maps vectors in the tangent space at a material point in the reference configuration to vectors in the tangent space of the same material point in the current configuration. The transpose of deformation gradient is defined as

$$\mathbf{F}^{\mathsf{T}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}, \quad \langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v} \rangle\!\rangle_{\mathbf{g}} = \langle\!\langle \mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v} \rangle\!\rangle_{\mathbf{G}}, \quad \forall \mathbf{V} \in T_{\mathbf{X}}\mathcal{B},\ \mathbf{v} \in T_{\mathbf{x}}\mathcal{S}\,, \tag{2.3}$$

where $\langle\!\langle , \rangle\!\rangle_{\mathbf{G}}$ and $\langle\!\langle , \rangle\!\rangle_{\mathbf{g}}$ are the inner products induced by the material and ambient space metrics, respectively. $\mathbf{F}^{\mathsf{T}}$ has the following components

$$(F^{\mathsf{T}}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X})\,. \tag{2.4}$$

The right Cauchy-Green deformation tensor is defined as

$$\mathbf{C}(X) = \mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}\,, \quad C^A_B = (F^{\mathsf{T}})^A{}_a F^a{}_B\,. \tag{2.5}$$

The pulled-back metric is denoted by $\mathbf{C}^\flat = \varphi^*\mathbf{g}$, and is defined as

$$\langle\!\langle \mathbf{U}, \mathbf{W} \rangle\!\rangle_{\varphi^*\mathbf{g}} = \langle\!\langle \mathbf{F}\mathbf{U}, \mathbf{F}\mathbf{W} \rangle\!\rangle_{\mathbf{g}}, \quad \forall \mathbf{U}, \mathbf{W} \in T_{\mathbf{X}}\mathcal{B}\,, \tag{2.6}$$

where $\flat$ is the flat operator induced by the metric $\mathbf{g}$. $\mathbf{C}^\flat$ has components $C_{AB} = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B$. The left Cauchy-Green deformation tensor is defined as

$$\mathbf{B}^\sharp = \varphi^*(\mathbf{g}^\sharp)\,, \quad B^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b\, g^{ab}\,. \tag{2.7}$$

The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are denoted by $\mathbf{c}^\flat$ and $\mathbf{b}^\sharp$ (the Finger deformation tensor), respectively, and are defined as

$$\begin{aligned} \mathbf{c}^\flat &= \varphi_*(\mathbf{G}), \qquad & c_{ab} &= \left(F^{-1}\right)^A{}_a \left(F^{-1}\right)^B{}_b\, G_{AB}, \\ \mathbf{b}^\sharp &= \varphi_*(\mathbf{G}^\sharp), \qquad & b^{ab} &= F^a{}_A\, F^b{}_B\, G^{AB}. \end{aligned} \tag{2.8}$$

The second-order tensors $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants I_1, I_2, and I_3 that are defined as [Ogden, 1984]

$$\begin{aligned} I_1 &= \operatorname{tr}\mathbf{b} = b^a{}_a = b^{ab}\, g_{ab}, \\ I_2 &= \frac{1}{2}\left(I_1^2 - \operatorname{tr}\mathbf{b}^2\right) = \frac{1}{2}\left(I_1^2 - b^a{}_b\, b^b{}_a\right) = \frac{1}{2}\left(I_1^2 - b^{ab} b^{cd}\, g_{ac}\, g_{bd}\right), \\ I_3 &= \det \mathbf{b}. \end{aligned} \tag{2.9}$$

Balance laws. The referential forms of the mass conservation and the balance of linear and angular momenta read

$$\frac{\partial \rho_0}{\partial t} = 0\,, \quad \operatorname{Div}\mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}\,, \quad \mathbf{P}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{P}^{\mathsf{T}}\,, \tag{2.10}$$

where ρ_0 is the material mass density, $\mathbf{B}$ is body force per unit referential volume, $\mathbf{A}$ is the material acceleration, and $\mathbf{P}$ is the first Piola-Kirchhoff stress. The spatial forms of conservation of mass and balance of linear and angular momenta read

$$\mathbf{L}_{\mathbf{v}}\rho = 0\,, \quad \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}\,, \quad \boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}\,, \tag{2.11}$$

where ρ is the spatial mass density, $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{b} = \mathbf{B} \circ \varphi_t^{-1}$, $\mathbf{a}$ is the spatial acceleration, and $\mathbf{L}_{\mathbf{v}}\rho$ is the Lie derivative of the spatial mass density with respect to the spatial velocity. $\mathbf{P}$ and $\boldsymbol{\sigma}$ are related as $J\sigma^{ab} = P^{aA}F^b{}_A$. The Jacobian of deformation $J = \sqrt{I_3}$ relates the material (dV) and spatial (dv) Riemannian volume forms as $dv = JdV$, and is given by

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\, \det \mathbf{F}\,. \tag{2.12}$$

Constitutive equations. For an inhomogeneous anisotropic hyperelastic solid the energy function (per unit undeformed volume) has the following functional form

$$W = \hat{W}(\mathbf{X}, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)\,, \tag{2.13}$$

where W explicitly depends on $\mathbf{X}$ (inhomogeneity), and the *structural tensors* $\boldsymbol{\zeta}_i, i = 1, \ldots, n$ characterize the material symmetry group of the solid. Using structural tensors the energy function becomes an isotropic function of its arguments. Instead of (2.13) one can write the energy as a function of an integrity basis for the set of tensors $\{\mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n\}$. Denoting the integrity basis by $I_j, j = 1, \ldots, m$, one can write $W = W(\mathbf{X}, I_1, ..., I_m)$. The second Piola-Kirchhoff stress tensor has the following representation [Doyle and Ericksen, 1956, Marsden and Hughes, 1994, Yavari et al., 2006]

$$\mathbf{S} = 2\frac{\partial \hat{W}}{\partial \mathbf{C}^\flat} = \sum_{j=1}^{m} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \qquad W_j = W_j(\mathbf{X}, I_1, ..., I_m) := \frac{\partial W}{\partial I_j}\,, \quad j = 1, \ldots, m\,. \tag{2.14}$$

The relations between the second Piola-Kirchhoff stress, and the first Piola-Kirchhoff and Cauchy stresses are: $S^{AB} = (F^{-1})^A{}_a P^{aB} = J(F^{-1})^A{}_a (F^{-1})^B{}_b\, \sigma^{ab}$.

Isotropic solids. For an inhomogeneous isotropic solid, $W = W(\mathbf{X}, I_1, I_2, I_3)$, where I_1, I_2, and I_3 were defined in (2.9). From (2.14) one writes

$$\mathbf{S} = 2W_1\, \mathbf{G}^\sharp + 2W_2\, (I_2\, \mathbf{C}^{-1} - I_3\, \mathbf{C}^{-2}) + 2W_3\, I_3\, \mathbf{C}^{-1}\,. \tag{2.15}$$

The Cauchy stress has the following representation

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}} \left[W_1\, b^{ab} + (I_2\, W_2 + I_3\, W_3) g^{ab} - I_3\, W_2\, c^{ab}\right]\,, \tag{2.16}$$

where $c^{ab} = (F^{-1})^M{}_m (F^{-1})^N{}_n G_{MN}\, g^{am}\, g^{bn}$. For an incompressible isotropic solid $I_3 = 1$, and hence

$$\begin{aligned} \mathbf{S} &= -p\, \mathbf{C}^{-1} + 2W_1\, \mathbf{G}^\sharp - 2W_2\, \mathbf{C}^{-2}\,, \\ \boldsymbol{\sigma} &= -p\, \mathbf{g}^\sharp + 2W_1\, \mathbf{b}^\sharp - 2W_2\, \mathbf{c}^{-1}\,, \end{aligned} \tag{2.17}$$

where p is the Lagrange multiplier associated with the incompressibility constraint $J = \sqrt{I_3} = 1$. Eq. $(2.17)_2$ in components reads $\sigma^{ab} = -p\, g^{ab} + 2W_1\, b^{ab} - 2W_2\, c^{ab}$.

Transversely isotropic solids. In a transversely isotropic solid at every point there is a single material preferred direction, which is normal to the plane of isotropy at that point. We assume that a unit vector $\mathbf{N}(\mathbf{X})$ identifies the material preferred direction at $\mathbf{X} \in \mathcal{B}$. The energy function for an inhomogeneous transversely isotropic solid has the form $W = W(\mathbf{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$, where $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$ is a structural tensor [Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000]. The energy function W depends on five independent invariants that are defined as

$$\begin{aligned} &I_1 = \operatorname{tr} \mathbf{C} = C^A{}_A\,, \quad I_2 = \det \mathbf{C} \operatorname{tr} \mathbf{C}^{-1} = \det(C^A{}_B)(C^{-1})^D{}_D\,, \quad I_3 = \det \mathbf{C} = \det(C^A{}_B) \\ &I_4 = \mathbf{N} \cdot \mathbf{C} \cdot \mathbf{N} = N^A N^B\, C_{AB}\,, \quad I_5 = \mathbf{N} \cdot \mathbf{C}^2 \cdot \mathbf{N} = N^A N^B\, C_{BM}\, C^M{}_A\,. \end{aligned} \tag{2.18}$$

The second Piola-Kirchhoff stress tensor has the following representation

$$\mathbf{S} = \sum_{j=1}^{5} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \qquad W_j = W_j(\mathbf{X}, I_1, ..., I_5) := \frac{\partial W}{\partial I_j}\,, \, j = 1, \ldots, 5\,, \tag{2.19}$$

where

$$\begin{aligned} &\frac{\partial I_1}{\partial \mathbf{C}^\flat} = \mathbf{G}^\sharp\,, \quad \frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2 \mathbf{C}^{-1} - I_3 \mathbf{C}^{-2}\,, \quad \frac{\partial I_3}{\partial \mathbf{C}^\flat} = I_3 \mathbf{C}^{-1}\,, \\ &\frac{\partial I_4}{\partial \mathbf{C}^\flat} = \mathbf{N} \otimes \mathbf{N}\,, \quad \frac{\partial I_5}{\partial \mathbf{C}^\flat} = \mathbf{N} \otimes (\mathbf{C} \cdot \mathbf{N}) + (\mathbf{C} \cdot \mathbf{N}) \otimes \mathbf{N}\,. \end{aligned} \tag{2.20}$$

Thus

$$\begin{aligned}\mathbf{S} = 2W_1\,\mathbf{G}^\sharp + 2W_2\left(I_2\,\mathbf{C}^{-1} - I_3\,\mathbf{C}^{-2}\right) + 2W_3\,I_3\,\mathbf{C}^{-1}\\ + 2W_4\,(\mathbf{N}\otimes\mathbf{N}) + 2W_5\,[\mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}]\,.\end{aligned} \tag{2.21}$$

The Cauchy stress has the representation [Ericksen and Rivlin, 1954, Golgoon and Yavari, 2018a,b]

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}\left[W_1 b^{ab} + (I_2W_2 + I_3W_3)\,g^{ab} - I_3W_2\,c^{ab} + W_4\,n^a n^b + W_5\,\ell^{ab}\right]\,, \tag{2.22}$$

where $n^a = F^a{}_A N^A$, and

$$\ell^{ab} = n^a\,b^{bc}\,n_c + n^b\,b^{ac}\,n_c\,. \tag{2.23}$$

In the case of an incompressible transversely isotropic solid ($I_3 = 1$), $W = W(\mathbf{X}, I_1, I_2, I_4, I_5)$, and hence

$$\mathbf{S} = -p\,\mathbf{C}^{-1} + 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\,\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + 2W_4\,(\mathbf{N}\otimes\mathbf{N}) + 2W_5\,[\mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}]\,. \tag{2.24}$$

Similarly, the Cauchy stress has the following representation [Ericksen and Rivlin, 1954, Spencer, 1986, Golgoon and Yavari, 2018a,b]

$$\sigma^{ab} = -p\,g^{ab} + 2W_1\,b^{ab} - 2W_2\,c^{ab} + 2W_4\,n^a\,n^b + 2W_5\left(n^a\,b^{bc}\,n^d\,g_{cd} + n^b\,b^{ac}\,n^d\,g_{cd}\right)\,. \tag{2.25}$$

Orthotropic solids. An orthotropic solid has reflection symmetry with respect to three mutually perpendicular planes at every point. Let three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1(\mathbf{X})$, $\mathbf{N}_2(\mathbf{X})$, and $\mathbf{N}_3(\mathbf{X})$ specify the orthotropic axes at a point $\mathbf{X}$ in the reference configuration. The three tensors $\mathbf{A}_1 = \mathbf{N}_1\otimes\mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2\otimes\mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3\otimes\mathbf{N}_3$ are structural tensors. However, because $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$, only two of them are independent. The energy function of an inhomogeneous orthotropic solid has the functional form $W = W(\mathbf{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)$ [Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000]. It can be rewritten as a function of the following seven independent invariants:

$$\begin{aligned}&I_1 = \operatorname{tr}\mathbf{C}\,,\quad I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}\,,\quad I_3 = \det\mathbf{C}\,,\\ &I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1\,,\quad I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1\,,\\ &I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2\,,\quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2\,.\end{aligned} \tag{2.26}$$

Thus

$$\mathbf{S} = \sum_{j=1}^{7} 2W_j\,\frac{\partial I_j}{\partial\mathbf{C}^\flat}\,,\qquad W_j = W_j(\mathbf{X}, I_1, ..., I_7) := \frac{\partial W}{\partial I_j}\,,\quad j = 1,\ldots,7\,. \tag{2.27}$$

The second Piola-Kirchhoff stress tensor has the following representation

$$\begin{aligned}\mathbf{S} =& 2W_1\,\mathbf{G}^\sharp + 2W_2\left(I_2\,\mathbf{C}^{-1} - I_3\,\mathbf{C}^{-2}\right) + 2W_3\,I_3\,\mathbf{C}^{-1}\\ &+ 2W_4\,(\mathbf{N}_1\otimes\mathbf{N}_1) + 2W_5\,[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1]\\ &+ 2W_6\,(\mathbf{N}_2\otimes\mathbf{N}_2) + 2W_7\,[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2]\,.\end{aligned} \tag{2.28}$$

Similarly, the Cauchy stress is written as [Smith and Rivlin, 1958, Spencer, 1986, Golgoon and Yavari, 2018a,b]

$$\begin{aligned}\sigma^{ab} =& \frac{2}{\sqrt{I_3}}\Big[W_1\,b^{ab} + (I_2\,W_2 + I_3\,W_3)g^{ab} - I_3\,W_2\,c^{ab}\\ &+ W_4\,n_1^a\,n_1^b + W_5\left(n_1^a\,b^{bc}\,n_1^d\,g_{cd} + n_1^b\,b^{ac}\,n_1^d\,g_{cd}\right)\\ &+ W_6\,n_2^a\,n_2^b + W_7\left(n_2^a\,b^{bc}\,n_2^d\,g_{cd} + n_2^b\,b^{ac}\,n_2^d\,g_{cd}\right)\Big]\,,\end{aligned} \tag{2.29}$$

where $n_1^a = F^a{}_A N_1^A$, and $n_2^a = F^a{}_A N_2^A$. In the case of an incompressible orthotropic solid ($I_3 = 1$), $W = W(\mathbf{X}, I_1, I_2, I_4, I_5, I_6, I_7)$. Thus, using (2.28), one has

$$\begin{aligned}\mathbf{S} =& -p\mathbf{C}^{-1} + 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right)\\ &+ 2W_4\,(\mathbf{N}_1\otimes\mathbf{N}_1) + 2W_5\,[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1]\\ &+ 2W_6\,(\mathbf{N}_2\otimes\mathbf{N}_2) + 2W_7\,[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2]\,.\end{aligned} \tag{2.30}$$

Similarly, the Cauchy stress tensor is written as

$$\sigma^{ab} = -pg^{ab} + 2W_1\, b^{ab} - 2W_2\, c^{ab} + 2W_4\, n_1^a\, n_1^b + 2W_5\, \ell_1^{ab} + 2W_6\, n_2^a\, n_2^b + 2W_7\, \ell_2^{ab}\,, \tag{2.31}$$

where $\ell_1^{ab} = n_1^a\, b^{bc}\, n_1^d\, g_{cd} + n_1^b\, b^{ac}\, n_1^d\, g_{cd}$, and $\ell_2^{ab} = n_2^a\, b^{bc}\, n_2^d\, g_{cd} + n_2^b\, b^{ac}\, n_2^d\, g_{cd}$.

Monoclinic solids. A monoclinic solid has three material preferred directions that are specified by three unit vectors $\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ such that $\mathbf{N}_1 \cdot \mathbf{N}_2 \neq 0$ and $\mathbf{N}_3$ is normal to the plane of $\mathbf{N}_1$ and $\mathbf{N}_2$ [Merodio and Ogden, 2020]. The energy function of a monoclinic solid depends on nine invariants [Spencer, 1986], seven of which are identical to those of orthotropic solids (2.26). The two extra invariants are

$$I_8 = g\, \mathbf{N}_1 \cdot \mathbf{C} \cdot \mathbf{N}_2, \quad I_9 = g^2\,, \tag{2.32}$$

where $g = \mathbf{N}_1 \cdot \mathbf{N}_2$. Note that

$$\frac{\partial I_8}{\partial \mathbf{C}^\flat} = \frac{g}{2}\left(\mathbf{N}_1 \otimes \mathbf{N}_2 + \mathbf{N}_2 \otimes \mathbf{N}_1\right)\,, \quad \frac{\partial I_9}{\partial \mathbf{C}^\flat} = \mathbf{0}\,. \tag{2.33}$$

For orthotropic solids the second Piola-Kirchhoff stress has the following representation

$$\begin{aligned}\mathbf{S} =& 2W_1\, \mathbf{G}^\sharp + 2W_2\left(I_2\, \mathbf{C}^{-1} - I_3\, \mathbf{C}^{-2}\right) + 2W_3\, I_3 \mathbf{C}^{-1} \\ &+ 2W_4\left(\mathbf{N}_1 \otimes \mathbf{N}_1\right) + 2W_5\left[\mathbf{N}_1 \otimes (\mathbf{C} \cdot \mathbf{N}_1) + (\mathbf{C} \cdot \mathbf{N}_1) \otimes \mathbf{N}_1\right] \\ &+ 2W_6\left(\mathbf{N}_2 \otimes \mathbf{N}_2\right) + 2W_7\left[\mathbf{N}_2 \otimes (\mathbf{C} \cdot \mathbf{N}_2) + (\mathbf{C} \cdot \mathbf{N}_2) \otimes \mathbf{N}_2\right] \\ &+ gW_8\left(\mathbf{N}_1 \otimes \mathbf{N}_2 + \mathbf{N}_2 \otimes \mathbf{N}_1\right)\,,\end{aligned} \tag{2.34}$$

where $W_i = W_i\left(\mathbf{X}, I_1, ..., I_9\right)$, $i = 1, ..., 8$. Similarly, the Cauchy stress can be written as

$$\begin{aligned}\sigma^{ab} =& \frac{2}{\sqrt{I_3}}\Big[W_1\, b^{ab} + (I_2\, W_2 + I_3\, W_3) g^{ab} - I_3\, W_2\, c^{ab} \\ &+ W_4\, n_1^a\, n_1^b + W_5\left(n_1^a\, b^{bc}\, n_1^d\, g_{cd} + n_1^b\, b^{ac}\, n_1^d\, g_{cd}\right) \\ &+ W_6\, n_2^a\, n_2^b + W_7\left(n_2^a\, b^{bc}\, n_2^d\, g_{cd} + n_2^b\, b^{ac}\, n_2^d\, g_{cd}\right) \\ &+ gW_8\left(n_1^a\, n_2^b + n_1^b\, n_2^a\right)\Big]\,.\end{aligned} \tag{2.35}$$

In the case of incompressible monoclinic solids ($I_3 = 1$), $W = W\left(\mathbf{X}, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9\right)$. Thus

$$\begin{aligned}\mathbf{S} =& -p\, \mathbf{C}^{-1} + 2W_1\, \mathbf{G}^\sharp + 2W_2\left(I_2\, \mathbf{C}^{-1} - \mathbf{C}^{-2}\right) \\ &+ 2W_4\left(\mathbf{N}_1 \otimes \mathbf{N}_1\right) + 2W_5\left[\mathbf{N}_1 \otimes (\mathbf{C} \cdot \mathbf{N}_1) + (\mathbf{C} \cdot \mathbf{N}_1) \otimes \mathbf{N}_1\right] \\ &+ 2W_6\left(\mathbf{N}_2 \otimes \mathbf{N}_2\right) + 2W_7\left[\mathbf{N}_2 \otimes (\mathbf{C} \cdot \mathbf{N}_2) + (\mathbf{C} \cdot \mathbf{N}_2) \otimes \mathbf{N}_2\right] \\ &+ gW_8\left(\mathbf{N}_1 \otimes \mathbf{N}_2 + \mathbf{N}_2 \otimes \mathbf{N}_1\right)\,.\end{aligned} \tag{2.36}$$

Similarly, the Cauchy stress tensor is written as

$$\sigma^{ab} = -p\, g^{ab} + 2W_1\, b^{ab} - 2I_3\, W_2\, c^{ab} + 2W_4\, n_1^a\, n_1^b + 2W_5\, \ell_1^{ab} + 2W_6\, n_2^a\, n_2^b + 2W_7\, \ell_2^{ab} + W_8\, \ell_3^{ab}\,, \tag{2.37}$$

where $\ell_3^{ab} = g\left(n_1^a\, n_2^b + n_1^b\, n_2^a\right)$.

3 Compressible Inhomogeneous Anisotropic Solids

3.1 Transversely isotropic solids

We first consider an inhomogeneous body made of compressible transversely isotropic solids. We do not specify the material preferred direction $\mathbf{N}(\mathbf{X})$ a priori. In the absence of body forces, the equilibrium equations in Cartesian coordinates read ${\sigma^{ab}}_{,b} = 0$. Substituting (2.22) into the equilibrium equations one obtains [Yavari and Goriely, 2021]

$$
\begin{aligned}
&- I_3^{-\frac{3}{2}} I_{3,b} \left[W_1\, b^{ab} + (I_2\, W_2 + I_3\, W_3)\delta^{ab} - I_3\, W_2\, c^{ab} + W_4\, n^a\, n^b + W_5\, \ell^{ab}\right] \\
&+ 2 I_3^{-\frac{1}{2}} \Big[(I_{2,b}\, W_2 + I_2\, W_{2,b} + I_{3,b}\, W_3 + I_3\, W_{3,b})\delta^{ab} + W_1\, {b^{ab}}_{,b} + W_{1,b}\, b^{ab} \\
&- I_{3,b}\, W_2\, c^{ab} - I_3\, W_{2,b}\, c^{ab} - I_3\, W_2\, {c^{ab}}_{,b} \\
&+ W_{4,b}\, n^a\, n^b + W_4\, {n^a}_{,b}\, n^b + W_4\, n^a\, {n^b}_{,b} + W_{5,b}\, \ell^{ab} + W_5\, {\ell^{ab}}_{,b}\Big] = 0\,.
\end{aligned}
\tag{3.1}
$$

For universal deformations the equilibrium equations hold for an arbitrary energy function W. Knowing that W is an arbitrary function of its arguments, the coefficient of W_1, W_2, W_3, W_3, and W_5 must vanish separately. Thus [Yavari and Goriely, 2021]

$$
\begin{aligned}
W_1 :&\quad {b^{ab}}_{,b} = 0\,, \\
W_2 :&\quad I_{2,b}\, \delta^{ab} - I_3\, {c^{ab}}_{,b} = 0\,, \\
W_3 :&\quad I_{3,b} = 0\,, \\
W_4 :&\quad (n^a\, n^b)_{,b} = 0\,, \\
W_5 :&\quad {\ell^{ab}}_{,b} = 0\,.
\end{aligned}
\tag{3.2}
$$

The above constraints simplify (3.1) to read

$$
b^{ab}\, W_{1,b} + (I_2\, \delta^{ab} - I_3\, c^{ab}) W_{2,b} + I_3 \delta^{ab}\, W_{3,b} + n^a\, n^b\, W_{4,b} + \ell^{ab}\, W_{5,b} = 0\,. \tag{3.3}
$$

Note that $I_{3,b} = 0$ from $(3.2)_3$ and

$$
\begin{aligned}
W_{1,b} &= (F^{-1})^A{}_b\, W_{1,A} + W_{11}\, I_{1,b} + W_{12}\, I_{2,b} + W_{14}\, I_{4,b} + W_{15}\, I_{5,b}\,, \\
W_{2,b} &= (F^{-1})^A{}_b\, W_{2,A} + W_{12}\, I_{1,b} + W_{22}\, I_{2,b} + W_{24}\, I_{4,b} + W_{25}\, I_{5,b}\,, \\
W_{3,b} &= (F^{-1})^A{}_b\, W_{3,A} + W_{13}\, I_{1,b} + W_{23}\, I_{2,b} + W_{34}\, I_{4,b} + W_{35}\, I_{5,b}\,, \\
W_{4,b} &= (F^{-1})^A{}_b\, W_{4,A} + W_{14}\, I_{1,b} + W_{24}\, I_{2,b} + W_{44}\, I_{4,b} + W_{45}\, I_{5,b}\,, \\
W_{5,b} &= (F^{-1})^A{}_b\, W_{5,A} + W_{15}\, I_{1,b} + W_{25}\, I_{2,b} + W_{45}\, I_{4,b} + W_{55}\, I_{5,b}\,,
\end{aligned}
\tag{3.4}
$$

where

$$
W_{i,A} = \frac{\partial W_i}{\partial X^A}\,, \qquad W_{ij} = \frac{\partial^2 W}{\partial I_i \partial I_j},\ i \leq j\,. \tag{3.5}
$$

Notice that the first term on the right-hand side of each equation in (3.4) vanishes for homogeneous solids [Yavari and Goriely, 2021]. Substituting the above relations into (3.3) the coefficients of W_{13} and W_{23} read

$$
\begin{aligned}
W_{13} :&\quad I_3\, I_{1,b}\, \delta^{ab} = 0\,, \\
W_{23} :&\quad I_3\, I_{2,b}\, \delta^{ab} = 0\,.
\end{aligned}
\tag{3.6}
$$

Thus, $I_{1,b} = I_{2,b} = 0$. Substituting these into (3.4) and using (3.3) the coefficients of W_{34} and W_{35} read

$$
\begin{aligned}
W_{34} :&\quad I_3\, I_{4,b}\, \delta^{ab} = 0\,, \\
W_{35} :&\quad I_3\, I_{5,b}\, \delta^{ab} = 0\,.
\end{aligned}
\tag{3.7}
$$

Hence, $I_{4,b} = I_{5,b} = 0$. Therefore, we have the following universality constraints

$$
I_1,\ I_2,\ \text{and}\ I_3\ \text{are constant}, \tag{3.8}
$$

$$
{b^{ab}}_{,b} = {c^{ab}}_{,b} = 0\,, \tag{3.9}
$$

$$
I_4,\ \text{and}\ I_5\ \text{are constant}, \tag{3.10}
$$

$$
(n^a\, n^b)_{,b} = {\ell^{ab}}_{,b} = 0\,. \tag{3.11}
$$

Note that (3.8) and (3.9) are the universality constraints for isotropic solids [Ericksen, 1955, Yavari and Goriely, 2016] and imply that ${F^a}_{A|B} = 0$, i.e., universal deformations are homogeneous. In addition, since $I_{4,b} = I_{4,A}(F^{-1})^A{}_b = 0$, we have $I_{4,A} = 0$. Similarly, $I_{5,A} = 0$. The constraints (3.10) and (3.11) imply that $\mathbf{N}$ is a constant unit vector [Yavari and Goriely, 2021].

For inhomogeneous solids one has the following extra five sets of universality constraints:

$$
\begin{aligned}
b^{ab}\,(F^{-1})^A{}_b\,W_{1,A} &= 0\,,\\
\left(I_2\,\delta^{ab} - I_3\,c^{ab}\right)(F^{-1})^A{}_b\,W_{2,A} &= 0\,,\\
I_3\,\delta^{ab}\,(F^{-1})^A{}_b\,W_{3,A} &= 0\,,\\
n^a\,n^b(F^{-1})^A{}_b\,W_{4,A} &= 0\,,\\
\ell^{ab}(F^{-1})^A{}_b\,W_{5,A} &= 0\,.
\end{aligned}
\tag{3.12}
$$

The first three constraints in (3.12) are identical to those of isotropic solids [Yavari, 2021], and imply that

$$
W_{1,A} = W_{2,A} = W_{3,A} = 0\,, \quad A = 1,2,3\,. \tag{3.13}
$$

The constraint $(3.12)_4$ implies that $n^b(F^{-1})^A{}_b\,W_{4,A} = W_{4,A}\,N^A = 0$. As $\mathbf{N}$ is a constant unit vector we can choose the Cartesian coordinates (X^1, X^2, X^3) in the reference configuration such that

$$
\mathbf{N} = \frac{\partial}{\partial X^1}\,, \tag{3.14}
$$

i.e., $N^A = \delta^A_1$. Here we have used the notation ∂_{X^i} to denote the unit (tangent) vector along the ith Cartesian direction as is customary in differential geometry. With this choice of coordinates the constraint $W_{4,A}\,N^A = 0$ reads

$$
\frac{\partial W_4}{\partial X^1} = 0\,. \tag{3.15}
$$

Note that $n^a = F^a{}_A\,N^A = F^a{}_A\,\delta^A_1 = F^a{}_1$.

Eq.$(3.12)_5$ is equivalent to

$$
(F^{-1})^B{}_a\,\ell^{ab}\,(F^{-1})^A{}_b\,W_{5,A} = 0, \quad B = 1,2,3\,. \tag{3.16}
$$

Using (2.23) the above constraints can be rewritten as

$$
\left(N^A\,C^B{}_D\,N^D + N^B\,C^A{}_D\,N^D\right)W_{5,A} = 0, \quad B = 1,2,3\,. \tag{3.17}
$$

Knowing that $N^A = \delta^A_1$, this last expression can be rewritten as

$$
C^B{}_1\,W_{5,1} + \delta^B_1\,C^A{}_1\,W_{5,A} = 0, \quad B = 1,2,3\,. \tag{3.18}
$$

For $B = 2$, it implies that $C^2{}_1 W_{5,1} = 0$, which must hold for arbitrary homogeneous deformations, i.e., for arbitrary constant $C^2{}_1$. Thus, $W_{5,1} = 0$. Now the constraint for $B = 3$ is trivially satisfied. For $B = 1$, $C^2{}_1\,W_{5,2} + C^3{}_1\,W_{5,3} = 0$, which must be satisfied for arbitrary constants $C^2{}_1$, and $C^3{}_1$. Therefore, $W_{5,2} = W_{5,3} = 0$. Thus, the constraint $(3.12)_5$ implies that $W_{5,A} = 0$. In summary, we have the following constraints

$$
W_{1,A} = W_{2,A} = W_{3,A} = W_{5,A} = 0,\; A = 1,2,3, \quad \& \quad W_{4,1} = 0\,. \tag{3.19}
$$

This implies that

$$
\frac{\partial W}{\partial X^1} = f_1(\mathbf{X})\,, \quad \frac{\partial W}{\partial X^2} = f_2(\mathbf{X}, I_4)\,, \quad \frac{\partial W}{\partial X^3} = f_3(\mathbf{X}, I_4)\,, \tag{3.20}
$$

for some scalar functions f_A. Note that $\frac{\partial f_1}{\partial X^2} = \frac{\partial f_2}{\partial X^1}$. Since f_1 does not depend on I_4, one has

$$
f_2(\mathbf{X}, I_4) = \bar{f}_2(X^2, X^3, I_4) + \bar{\bar{f}}_2(\mathbf{X})\,. \tag{3.21}
$$

Similarly, $\frac{\partial f_1}{\partial X^3} = \frac{\partial f_3}{\partial X^1}$ implies that

$$f_3(\mathbf{X}, I_4) = \bar{f}_3(X^2, X^3, I_4) + \bar{\bar{f}}_3(\mathbf{X})\,. \tag{3.22}$$

From (3.20)$_1$, one writes

$$W(\mathbf{X}, I_i) = \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3)\, dX^1 + h(X^2, X^3, I_i)\,, \tag{3.23}$$

where X_0^1 is some fixed value of X^1, h is some scalar function, and $W(\mathbf{X}, I_i)$ and $h(X^2, X^3, I_i)$ are short for $W(\mathbf{X}, I_1, I_2, I_3, I_4, I_5)$ and $h(X^2, X^3, I_1, I_2, I_3, I_4, I_5)$, respectively. Taking partial derivative with respect to X^2 of both sides one obtains

$$\begin{aligned}
\frac{\partial W}{\partial X^2} &= \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3)}{\partial X^2}\, dX^1 + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,, \\
&= \int_{X_0^1}^{X^1} \frac{\partial f_2(X^1, X^2, X^3, I_4)}{\partial X^1}\, dX^1 + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,, \\
&= f_2(X^1, X^2, X^3, I_4) - f_2(X_0^1, X^2, X^3, I_4) + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,.
\end{aligned} \tag{3.24}$$

From (3.24) and (3.20)$_2$ one concludes that

$$\frac{\partial h(X^2, X^3, I_i)}{\partial X^2} = f_2(X_0^1, X^2, X^3, I_4)\,. \tag{3.25}$$

Thus

$$\int_{X_0^2}^{X^2} \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\, dX^2 = \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2\,, \tag{3.26}$$

where X_0^2 is some fixed value of X^2. Hence

$$h(X^2, X^3, I_i) = h(X_0^2, X^3, I_i) + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2\,. \tag{3.27}$$

Using the above relation in (3.23), one writes

$$W(\mathbf{X}, I_i) = h(X_0^2, X^3, I_i) + \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3)\, dX^1 + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2\,. \tag{3.28}$$

Taking partial derivative with respect to X^3 of the above relation one obtains

$$\begin{aligned}
\frac{\partial W}{\partial X^3} &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3)}{\partial X^3}\, dX^1 + \int_{X_0^2}^{X^2} \frac{\partial f_2(X_0^1, X^2, X^3, I_4)}{\partial X^3}\, dX^2\,, \\
&= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + \int_{X_0^1}^{X^1} \frac{\partial f_3(X^1, X^2, X^3, I_4)}{\partial X^1}\, dX^1 + \int_{X_0^2}^{X^2} \frac{\partial f_3(X_0^1, X^2, X^3, I_4)}{\partial X^2}\, dX^2\,, \\
&= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + f_3(X^1, X^2, X^3, I_4) - f_3(X_0^1, X_0^2, X^3, I_4)\,.
\end{aligned} \tag{3.29}$$

Thus using (3.20)$_3$ one concludes that

$$\frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} = f_3(X_0^1, X_0^2, X^3, I_4)\,. \tag{3.30}$$

Hence

$$\int_{X_0^3}^{X^3} \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3}\, dX^3 = \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4)\, dX^3\,, \tag{3.31}$$

where X_0^3 is some fixed value of X^3. Thus

$$h(X_0^2, X^3, I_i) = h(X_0^2, X_0^3, I_i) + \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4)\, dX^3\,. \tag{3.32}$$

Substituting the above relation into (3.28) one obtains

$$\begin{aligned} W(\mathbf{X}, I_i) = h(X_0^2, X_0^3, I_i) &+ \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3)\, dX^1 + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2 \\ &+ \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4)\, dX^3\,. \end{aligned} \tag{3.33}$$

Substituting (3.21) and (3.22) into the above relation one finds that $W(\mathbf{X}, I_i) = \hat{W}(\mathbf{X}) + \overline{W}(I_i) + \widetilde{W}(X^2, X^3, I_4)$. Note that the term $\hat{W}(\mathbf{X})$ is mechanically inconsequential, and hence we have proved that the only universal deformations are homogeneous and the only possible dependence on the position is through I_4 and in directions normal to a constant vector $\mathbf{N}$:

Proposition 3.1. *For compressible nonlinear transversely isotropic solids, universal deformations are homogeneous, the universal material preferred direction is at all points a constant unit vector* $\mathbf{N}$*, and the universal inhomogeneity has the following form*

$$W(\mathbf{X}, I_1, I_2, I_3, I_4, I_5) = \overline{W}(I_1, I_2, I_3, I_4, I_5) + \widetilde{W}(X^2, X^3, I_4)\,, \tag{3.34}$$

where the Cartesian X^1*-coordinate line is parallel to* $\mathbf{N}$.

3.2 Orthotropic solids

For inhomogeneous compressible orthotropic solids there are two sets of universality constraints. The first set of constraints are identical to those of homogeneous compressible orthotropic solids and read [Yavari and Goriely, 2021]:

$$I_1,\ I_2,\ \text{and } I_3 \text{ are constant}\,, \tag{3.35}$$

$$b^{ab}{}_{,b} = c^{ab}{}_{,b} = 0\,, \tag{3.36}$$

$$I_4,\ \text{and } I_5 \text{ are constant}\,, \tag{3.37}$$

$$(n_1^a\, n_1^b)_{,b} = \ell_1^{ab}{}_{,b} = 0\,, \tag{3.38}$$

$$I_6,\ \text{and } I_7 \text{ are constant}\,, \tag{3.39}$$

$$(n_2^a\, n_2^b)_{,b} = \ell_2^{ab}{}_{,b} = 0\,. \tag{3.40}$$

These constraints imply again that universal deformations are homogeneous and the material preferred directions are uniform. In the reference configuration we choose the Cartesian coordinates (X^1, X^2, X^3) such that

$$\mathbf{N}_1 = \frac{\partial}{\partial X^1}\,, \quad \mathbf{N}_2 = \frac{\partial}{\partial X^2}\,, \quad \mathbf{N}_3 = \frac{\partial}{\partial X^3}\,. \tag{3.41}$$

The second set of universality constraints are:

$$
\begin{aligned}
b^{ab}\,(F^{-1})^A{}_b\,W_{1,A} &= 0\,,\\
\left(I_2\,\delta^{ab} - I_3\,c^{ab}\right)(F^{-1})^A{}_b\,W_{2,A} &= 0\,,\\
I_3\,\delta^{ab}\,(F^{-1})^A{}_b\,W_{3,A} &= 0\,,\\
n_1^a\,n_1^b(F^{-1})^A{}_b\,W_{4,A} &= 0\,,\\
\ell_1^{ab}(F^{-1})^A{}_b\,W_{5,A} &= 0\,,\\
n_2^a\,n_2^b(F^{-1})^A{}_b\,W_{6,A} &= 0\,,\\
\ell_2^{ab}(F^{-1})^A{}_b\,W_{7,A} &= 0\,.
\end{aligned} \tag{3.42}
$$

The first three constraints are identical to those of isotropic solids [Yavari, 2021], and imply that $W_{1,A} = W_{2,A} = W_{3,A} = 0$. Similarly to the universality constraints of transversely isotropic solids, $(3.42)_4$ and $(3.42)_6$ imply that

$$
\frac{\partial W_4}{\partial X^A}N_1^A = \frac{\partial W_4}{\partial X^1} = 0, \qquad \frac{\partial W_6}{\partial X^A}N_2^A = \frac{\partial W_6}{\partial X^2} = 0\,. \tag{3.43}
$$

The universality constraints $(3.42)_5$ and $(3.42)_7$ imply that

$$
W_{5,A} = W_{7,A} = 0, \quad A = 1,2,3\,. \tag{3.44}
$$

This means that

$$
\frac{\partial W}{\partial X^1} = f_1(\mathbf{X}, I_6)\,, \quad \frac{\partial W}{\partial X^2} = f_2(\mathbf{X}, I_4)\,, \quad \frac{\partial W}{\partial X^3} = f_3(\mathbf{X}, I_4, I_6)\,. \tag{3.45}
$$

Note that

$$
\frac{\partial f_1(\mathbf{X}, I_6)}{\partial X^2} = \frac{\partial f_2(\mathbf{X}, I_4)}{\partial X^1}\,, \quad \frac{\partial f_1(\mathbf{X}, I_6)}{\partial X^3} = \frac{\partial f_3(\mathbf{X}, I_4, I_6)}{\partial X^1}\,, \quad \frac{\partial f_2(\mathbf{X}, I_4)}{\partial X^3} = \frac{\partial f_3(\mathbf{X}, I_4, I_6)}{\partial X^2}\,. \tag{3.46}
$$

Thus

$$
\begin{aligned}
f_1(\mathbf{X}, I_6) &= \bar{f}_1(X^1, X^3, I_6) + \bar{\bar{f}}_1(\mathbf{X})\,,\\
f_2(\mathbf{X}, I_4) &= \bar{f}_2(X^2, X^3, I_4) + \bar{\bar{f}}_2(\mathbf{X})\,.
\end{aligned} \tag{3.47}
$$

Using $(3.45)_1$, one writes

$$
W(\mathbf{X}, I_i) = \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3, I_6)\,dX^1 + h(X^2, X^3, I_i)\,, \tag{3.48}
$$

where h is some scalar function, and X_0^1 is some fixed value of X^1. Taking partial derivative with respect to X^2 of both sides one obtains

$$
\begin{aligned}
\frac{\partial W}{\partial X^2} &= \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3, I_6)}{\partial X^2}\,dX^1 + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,,\\
&= \int_{X_0^1}^{X^1} \frac{\partial f_2(X^1, X^2, X^3, I_4)}{\partial X^1}\,dX^1 + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,,\\
&= f_2(X^1, X^2, X^3, I_4) - f_2(X_0^1, X^2, X^3, I_4) + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,.
\end{aligned} \tag{3.49}
$$

From (3.49) and $(3.45)_2$ one concludes that

$$
\frac{\partial h(X^2, X^3, I_i)}{\partial X^2} = f_2(X_0^1, X^2, X^3, I_4)\,. \tag{3.50}
$$

Thus

$$
\int_{X_0^2}^{X^2} \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,dX^2 = \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\,dX^2\,, \tag{3.51}
$$

where X_0^2 is some fixed value of X^2. Hence

$$h(X^2, X^3, I_i) = h(X_0^2, X^3, I_i) + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2 \,. \tag{3.52}$$

Using the above relation in (3.48), one has

$$W(\mathbf{X}, I_i) = h(X_0^2, X^3, I_i) + \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3, I_6)\, dX^1 + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2 \,. \tag{3.53}$$

Taking partial derivative with respect to X^3 of the above relation one obtains

$$\begin{aligned} \frac{\partial W}{\partial X^3} &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3, I_6)}{\partial X^3}\, dX^1 + \int_{X_0^2}^{X^2} \frac{\partial f_2(X_0^1, X^2, X^3, I_4)}{\partial X^3}\, dX^2 \,, \\ &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + \int_{X_0^1}^{X^1} \frac{\partial f_3(X^1, X^2, X^3, I_4, I_6)}{\partial X^1}\, dX^1 + \int_{X_0^2}^{X^2} \frac{\partial f_3(X_0^1, X^2, X^3, I_4, I_6)}{\partial X^2}\, dX^2 \,, \\ &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + f_3(X^1, X^2, X^3, I_4, I_6) - f_3(X_0^1, X_0^2, X^3, I_4, I_6) \,. \end{aligned} \tag{3.54}$$

Thus using $(3.45)_3$ one concludes that

$$\frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} = f_3(X_0^1, X_0^2, X^3, I_4, I_6) \,. \tag{3.55}$$

Hence

$$\int_{X_0^3}^{X^3} \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3}\, dX^3 = \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4, I_6)\, dX^3 \,, \tag{3.56}$$

where X_0^3 is some fixed value of X^3. Thus

$$h(X_0^2, X^3, I_i) = h(X_0^2, X_0^3, I_i) + \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4, I_6)\, dX^3 \,. \tag{3.57}$$

Using the above relation in (3.53), one obtains

$$\begin{aligned} W(\mathbf{X}, I_i) = h(X_0^2, X_0^3, I_i) &+ \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3, I_6)\, dX^1 + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4)\, dX^2 \\ &+ \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4, I_6)\, dX^3 \,. \end{aligned} \tag{3.58}$$

Substituting (3.47) into (3.58) one finds

$$\begin{aligned} W(\mathbf{X}, I_i) = \hat{W}(\mathbf{X}) + \overline{W}(I_i) &+ \int_{X_0^1}^{X^1} \bar{\bar{f}}_1(X^1, X^3, I_6)\, dX^1 + \int_{X_0^2}^{X^2} \bar{\bar{f}}_2(X^2, X^3, I_4)\, dX^2 \\ &+ \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4, I_6)\, dX^3 \,. \end{aligned} \tag{3.59}$$

Noting that the term $\hat{W}(\mathbf{X})$ is mechanically inconsequential, we have proved the following result.

Proposition 3.2. *For compressible nonlinear orthotropic solids universal deformations are homogeneous, the universal material preferred directions are everywhere the same three mutually orthogonal constant unit vectors* $\mathbf{N}_1, \mathbf{N}_2$*, and* $\mathbf{N}_3$*, and the universal inhomogeneity has the following form*

$$\begin{aligned} W(\mathbf{X}, I_1, I_2, I_3, I_4, I_5, I_6, I_7) &= \overline{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7) \\ &\quad + \widetilde{W}(X^3, I_4, I_6) + \widehat{W}(X^2, X^3, I_4) + \widehat{W}(X^1, X^3, I_6) \,, \end{aligned} \tag{3.60}$$

where the Cartesian coordinate lines are the orthotropy directions.

While the form of this strain-energy density seems involved, it can be written explicitly in terms of the Cartesian components of $\mathbf{C}$ as

$$W(\mathbf{X},\mathbf{C}) = \overline{W}(\mathbf{C}) + \widetilde{W}(X^3, C_{11}, C_{22}) + \widehat{W}(X^2, X^3, C_{11}) + \widehat{W}(X^1, X^3, C_{22})\,. \tag{3.61}$$

3.3 Monoclinic solids

Note that orthogonality of the material preferred directions was not assumed when deriving the constraints (3.35)-(3.40), i.e., these universality constraints hold for monoclinic solids as well. However, there are the following extra universality constraints [Yavari and Goriely, 2021]:

$$I_8,\ \text{and}\ I_9\ \text{are constant}\,, \tag{3.62}$$

$$(n_3^a\, n_3^b)_{,b} = \ell_3^{ab}{}_{,b} = 0\,. \tag{3.63}$$

For compressible monoclinic solids the universality constraints (3.35)-(3.40) imply that universal deformations are homogeneous, and the three unit vectors $\mathbf{N}_1, \mathbf{N}_2$, and $\mathbf{N}_3$ are constant. This means that (3.62), (3.63) are trivially satisfied. Let us assume that the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$ is θ ($0 < \theta < \frac{\pi}{2}$). In the reference configuration we choose a Cartesian coordinate system (X^1, X^2, X^3) such that

$$\mathbf{N}_3 = \frac{\partial}{\partial X^3}\,. \tag{3.64}$$

In general, $\mathbf{N}_1$ makes and angle α with the X_1-axis, and thus

$$\mathbf{N}_1 = \cos\alpha\,\frac{\partial}{\partial X^1} + \sin\alpha\,\frac{\partial}{\partial X^2}\,,\quad \mathbf{N}_2 = \cos(\alpha+\theta)\,\frac{\partial}{\partial X^1} + \sin(\alpha+\theta)\,\frac{\partial}{\partial X^2}\,. \tag{3.65}$$

The second set of universality constraints for inhomogeneous monoclinic solids include those of orthotropic solids, i.e., Eqs.(3.42). There is one extra universality constrain that reads:

$$\ell_3^{ab}(F^{-1})^A{}_b\, W_{8,A} = 0\,. \tag{3.66}$$

This is equivalent to $\left(N_1^B N_2^A + N_1^A N_2^B\right) W_{8,A} = 0$, and is trivially satisfied for $B=3$. For $B=1,2$ it gives us

$$\begin{aligned} 2\cos\alpha\,\cos(\alpha+\theta)\,W_{8,1} + \sin(2\alpha+\theta)\,W_{8,2} &= 0\,,\\ \sin(2\alpha+\theta)W_{8,1} + 2\sin\alpha\,\sin(\alpha+\theta)\,W_{8,2} &= 0\,. \end{aligned} \tag{3.67}$$

These need to be satisfied for arbitrary α, and θ, and hence

$$W_{8,1} = W_{8,2} = 0\,. \tag{3.68}$$

The first three universality constraints in (3.42), and $(3.42)_5$ and $(3.42)_7$ imply that

$$W_{1,A} = W_{2,A} = W_{3,A} = W_{5,A} = W_{7,A} = 0,\quad A = 1,2,3\,. \tag{3.69}$$

The constraint $(3.42)_4$ implies that

$$\frac{\partial W_4}{\partial X^A} N_1^A = \cos\alpha\,\frac{\partial W_6}{\partial X^1} + \sin\alpha\,\frac{\partial W_6}{\partial X^2} = 0\,, \tag{3.70}$$

which must hold for any α, and hence

$$\frac{\partial W_4}{\partial X^1} = \frac{\partial W_4}{\partial X^2} = 0\,. \tag{3.71}$$

The constraint $(3.42)_6$ implies that

$$\frac{\partial W_6}{\partial X^A} N_2^A = \cos(\alpha+\theta)\,\frac{\partial W_6}{\partial X^1} + \sin(\alpha+\theta)\,\frac{\partial W_6}{\partial X^2} = 0\,. \tag{3.72}$$

This needs to hold for any $0 < \theta < \frac{\pi}{2}$, and hence

$$\frac{\partial W_6}{\partial X^1} = \frac{\partial W_6}{\partial X^2} = 0\,. \tag{3.73}$$

From Eqs. (3.68), (3.69), (3.71), and (3.73) one has

$$\frac{\partial W}{\partial X^1} = f_1(\mathbf{X}, I_9)\,, \quad \frac{\partial W}{\partial X^2} = f_2(\mathbf{X}, I_9)\,, \quad \frac{\partial W}{\partial X^3} = f_3(\mathbf{X}, I_4, I_6, I_8, I_9)\,. \tag{3.74}$$

Using $(3.74)_1$, one can write

$$W(\mathbf{X}, I_i) = \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3, I_9)\, dX^1 + h(X^2, X^3, I_i)\,. \tag{3.75}$$

Taking partial derivative with respect to X^2 of both sides one obtains

$$\begin{aligned} \frac{\partial W}{\partial X^2} &= \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3, I_9)}{\partial X^2}\, dX^1 + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,, \\ &= \int_{X_0^1}^{X^1} \frac{\partial f_2(X^1, X^2, X^3, I_9)}{\partial X^1}\, dX^1 + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,, \\ &= f_2(X^1, X^2, X^3, I_9) - f_2(X_0^1, X^2, X^3, I_9) + \frac{\partial h(X^2, X^3, I_i)}{\partial X^2}\,. \end{aligned} \tag{3.76}$$

From (3.76) and $(3.74)_2$ one concludes that

$$\frac{\partial h(X^2, X^3, I_i)}{\partial X^2} = f_2(X_0^1, X^2, X^3, I_9)\,. \tag{3.77}$$

Thus

$$h(X^2, X^3, I_i) = h(X_0^2, X^3, I_i) + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_4, I_9)\, dX^2\,. \tag{3.78}$$

Using the above relation in (3.75), one writes

$$W(\mathbf{X}, I_i) = h(X_0^2, X^3, I_i) + \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3, I_9)\, dX^1 + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_9)\, dX^2\,. \tag{3.79}$$

Taking partial derivative with respect to X^3 of the above relation one obtains

$$\begin{aligned} \frac{\partial W}{\partial X^3} &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3, I_9)}{\partial X^3}\, dX^1 + \int_{X_0^2}^{X^2} \frac{\partial f_2(X_0^1, X^2, X^3, I_9)}{\partial X^3}\, dX^2\,, \\ &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + \int_{X_0^1}^{X^1} \frac{\partial f_3(X^1, X^2, X^3, I_4, I_6, I_8, I_9)}{\partial X^1}\, dX^1 \\ &\quad + \int_{X_0^2}^{X^2} \frac{\partial f_3(X_0^1, X^2, X^3, I_4, I_6, I_8, I_9)}{\partial X^2}\, dX^2\,, \\ &= \frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} + f_3(X^1, X^2, X^3, I_4, I_6, I_8, I_9) - f_3(X_0^1, X_0^2, X^3, I_4, I_6, I_8, I_9)\,. \end{aligned} \tag{3.80}$$

Thus using $(3.74)_3$ one concludes that

$$\frac{\partial h(X_0^2, X^3, I_i)}{\partial X^3} = f_3(X_0^1, X_0^2, X^3, I_4, I_6, I_8, I_9)\,. \tag{3.81}$$

Hence

$$h(X_0^2, X^3, I_i) = h(X_0^2, X_0^3, I_i) + \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4, I_6, I_8, I_9)\, dX^3 \,. \tag{3.82}$$

Using the above relation in (3.79), one obtains

$$\begin{aligned} W(\mathbf{X}, I_i) = h(X_0^2, X_0^3, I_i) &+ \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3, I_9)\, dX^1 + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3, I_9)\, dX^2 \\ &+ \int_{X_0^3}^{X^3} f_3(X_0^1, X_0^2, X^3, I_4, I_6, I_8, I_9)\, dX^3 \,. \end{aligned} \tag{3.83}$$

Note that the second and third terms on the right-hand side are mechanically inconsequential, and hence, we have proved the following result.

Proposition 3.3. *For compressible nonlinear monoclinic solids universal deformations are homogeneous, the universal material preferred directions are everywhere the same three constant unit vectors $\mathbf{N}_1, \mathbf{N}_2$, and $\mathbf{N}_3$, such that $\mathbf{N}_3$ is perpendicular to the plane of $\mathbf{N}_1$ and $\mathbf{N}_2$, and the universal inhomogeneity has the following form*

$$W(\mathbf{X}, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9) = \overline{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9) + \widetilde{W}(X^3, I_4, I_6, I_8, I_9)\,, \tag{3.84}$$

where the Cartesian X^3-coordinate line is along $\mathbf{N}_3$.

Table 1 summarizes our results for inhomogeneous compressible anisotropic solids.

Symmetry Class	Energy Function	Universal Energy Function
Transversely Isotropic	$W(X^1, X^2, X^3, I_1, I_2, I_3, I_4, I_5)$	$\overline{W}(I_1, I_2, I_3, I_4, I_5) + \widetilde{W}(X^2, X^3, I_4)$
Orthotropic	$W(X^1, X^2, X^3, I_1, I_2, I_3, I_4, I_5, I_6, I_7)$	$\overline{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7) + \widetilde{W}(X^3, I_4, I_6)$ $+\widehat{W}(X^2, X^3, I_4) + \widehat{W}(X^1, X^3, I_6)$
Monoclinic	$W(X^1, X^2, X^3, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9)$	$\overline{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9) + \widetilde{W}(X^3, I_4, I_6, I_8, I_9)$

Table 1: *Universal inhomogeneities for compressible transversely isotropic, orthotropic, and monoclinic solids.*

4 Incompressible Inhomogeneous Transversely Isotropic Elastic Solids

For a body made of an incompressible transversely isotropic solid, the equilibrium equations in the absence of body forces read:

$$p_{,b}\, g^{ab} = 2\left[W_1\, b^{ab} - W_2\, c^{ab} + W_4\, n^a\, n^b + W_5\, \ell^{ab}\right]_{|b} \,. \tag{4.1}$$

This is equivalent to exactness of the 1-form

$$\boldsymbol{\xi} = g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n^m n^n + W_5\, \ell^{mn}\right]_{|n} dx^a = \xi_a dx^a, \tag{4.2}$$

where

$$\xi_a = [W_1\, b^n_a - W_2\, c^n_a + W_4\, n_a\, n^n + W_5\, \ell^n_a]_{|n}$$
$$= W_{1,n}\, b^n_a - W_{2,n}\, c^n_a + W_{4,n}\, n_a\, n^n + W_{5,n}\, \ell^n_a + W_1\, b^n_{a\,|n} - W_2\, c^n_{a\,|n} + W_4\, (n_a n^n)_{|n} + W_5\, \ell^n_{a\,|n}\,. \tag{4.3}$$

The exactness of $\boldsymbol{\xi}$ implies that $\boldsymbol{d\xi}=\mathbf{0}$ [Yavari, 2013], or equivalently, $\xi_{a|b}=\xi_{b|a}$, where

$$\begin{aligned}\xi_{a|b} &= W_1 b^n_{a\,|nb} - W_2\, c^n_{a\,|nb} + W_4\, (n_a n^n)_{|nb} + W_5\, \ell^n_{a\,|nb}\\ &\quad + W_{1,n} b^n_{a\,|b} - W_{2,n}\, c^n_{a\,|b} + W_{4,n}\, (n_a n^n)_{|b} + W_{5,n}\, \ell^n_{a\,|b}\\ &\quad + W_{1,b} b^n_{a\,|n} - W_{2,b}\, c^n_{a\,|n} + W_{4,b}\, (n_a n^n)_{|n} + W_{5,b}\, \ell^n_{a\,|n}\\ &\quad + W_{1|nb} b^n_a - W_{2|nb}\, c^n_a + W_{4|nb}\, n_a n^n + W_{5|nb}\, \ell^n_a\,.\end{aligned} \tag{4.4}$$

Note that $W_i = W_i(\mathbf{X}, I_1, I_2, I_4, I_5)$, $i = 1,2,4,5$, and thus

$$\begin{aligned}W_{1,b} &= (F^{-1})^A{}_n\, W_{1,A} + W_{11}\, I_{1,b} + W_{12} I_{2,b} + W_{14}\, I_{4,b} + W_{15}\, I_{5,b}\,,\\ W_{2,b} &= (F^{-1})^A{}_n\, W_{2,A} + W_{12}\, I_{1,b} + W_{22} I_{2,b} + W_{24}\, I_{4,b} + W_{25}\, I_{5,b}\,,\\ W_{4,b} &= (F^{-1})^A{}_n\, W_{4,A} + W_{14}\, I_{1,b} + W_{24} I_{2,b} + W_{44}\, I_{4,b} + W_{45}\, I_{5,b}\,,\\ W_{5,b} &= (F^{-1})^A{}_n\, W_{5,A} + W_{15}\, I_{1,b} + W_{25} I_{2,b} + W_{45}\, I_{4,b} + W_{55}\, I_{5,b}\,.\end{aligned} \tag{4.5}$$

Note also that

$$\begin{aligned}W_{1|bn} = (W_{1,b})_{|n} &= W_{11}\, I_{1|bn} + W_{12}\, I_{2|bn} + W_{14}\, I_{4|bn} + W_{15}\, I_{5|bn} + W_{11,n}\, I_{1,b} + W_{12,n}\, I_{2,b}\\ &\quad + W_{14,n}\, I_{4,b} + W_{15,n}\, I_{5,b} + \left[(F^{-1})^A{}_b\, W_{1,A}\right]_{|n}\,.\end{aligned} \tag{4.6}$$

The last term on the right hand-side is simplified as

$$\begin{aligned}\left[(F^{-1})^A{}_b\, W_{1,A}\right]_{|n} &= \frac{\partial}{\partial x^n}\left[(F^{-1})^A{}_b\, W_{1,A}\right] - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\, W_{1,A}\\ &= (F^{-1})^B{}_n (F^{-1})^A{}_{b,B}\, W_{1,A} + (F^{-1})^A{}_b\, \frac{\partial}{\partial x^n} W_{1,A} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\, W_{1,A}\,.\end{aligned} \tag{4.7}$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial x^n} W_{1,A} &= (F^{-1})^B{}_n W_{1,AB} + \frac{\partial}{\partial X^A}\left[W_{11}\, I_{1,n} + W_{12}\, I_{2,n} + W_{14}\, I_{4,n} + W_{15}\, I_{5,n}\right]\\ &= (F^{-1})^B{}_n W_{1,AB} + W_{11,A}\, I_{1,n} + W_{12,A}\, I_{2,n} + W_{14,A}\, I_{4,n} + W_{15,A}\, I_{5,n}\,.\end{aligned} \tag{4.8}$$

Thus

$$\begin{aligned}\left[(F^{-1})^A{}_b\, W_{1,A}\right]_{|n} &= \left[(F^{-1})^B{}_n (F^{-1})^A{}_{b,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right] W_{1,A} + (F^{-1})^A{}_b\, (F^{-1})^B{}_n W_{1,AB}\\ &\quad + (F^{-1})^A{}_b \left[W_{11}\, I_{1,n} + W_{12}\, I_{2,n} + W_{14}\, I_{4,n} + W_{15}\, I_{5,n}\right]\\ &= \left[(F^{-1})^B{}_n (F^{-1})^A{}_{b,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right] W_{1,A}\\ &\quad + \frac{1}{2}\left[(F^{-1})^A{}_b\, (F^{-1})^B{}_n + (F^{-1})^B{}_b\, (F^{-1})^A{}_n\right] W_{1,AB}\\ &\quad + (F^{-1})^A{}_b \left[W_{11}\, I_{1,n} + W_{12}\, I_{2,n} + W_{14}\, I_{4,n} + W_{15}\, I_{5,n}\right]\,.\end{aligned} \tag{4.9}$$

Let us denote the independent third order derivatives of the energy function by $W_{ijk} = \frac{\partial^3 W}{\partial I_i \partial I_j \partial I_k}$, $(\, i \leq j \leq k)$. Thus

$$\begin{aligned}W_{11,n} &= (F^{-1})^A{}_n\, W_{11,A} + W_{111}\, I_{1,n} + W_{112}\, I_{2,n} + W_{114}\, I_{4,n} + W_{115}\, I_{5,n}\,,\\ W_{12,n} &= (F^{-1})^A{}_n\, W_{12,A} + W_{112}\, I_{1,n} + W_{122}\, I_{2,n} + W_{124}\, I_{4,n} + W_{125}\, I_{5,n}\,,\\ W_{14,n} &= (F^{-1})^A{}_n\, W_{14,A} + W_{114}\, I_{1,n} + W_{124}\, I_{2,n} + W_{144}\, I_{4,n} + W_{145}\, I_{5,n}\,,\\ W_{15,n} &= (F^{-1})^A{}_n\, W_{15,A} + W_{115}\, I_{1,n} + W_{125}\, I_{2,n} + W_{145}\, I_{4,n} + W_{155}\, I_{5,n}\,.\end{aligned} \tag{4.10}$$

Hence[3]

$$
\begin{aligned}
W_{1|bn} = {} & W_{11}\, I_{1|bn} + W_{12}\, I_{2|bn} + W_{14}\, I_{4|bn} + W_{15}\, I_{5|bn} \\
& + W_{111}\, I_{1,n} I_{1,b} + W_{112}(I_{2,n}\, I_{1,b} + I_{1,n}\, I_{2,b}) + W_{114}(I_{4,n}\, I_{1,b} + I_{1,n}\, I_{4,b}) \\
& + W_{115}(I_{5,n}\, I_{1,b} + I_{1,n}\, I_{5,b}) + W_{122} I_{2,n}\, I_{2,b} + W_{124}(I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n}) \\
& + W_{125}(I_{5,n}\, I_{2,b} + I_{2,n}\, I_{5,b}) + W_{144} I_{4,n}\, I_{4,b} + W_{145}(I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n} + W_{155} I_{5,n}\, I_{5,b} \\
& + \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right] W_{1,A} \\
& + \frac{1}{2}\left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right] W_{1,AB} \\
& + \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right] W_{11,A} + \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right] W_{12,A} \\
& + \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n}\right] W_{14,A} + \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n}\right] W_{15,A}\,.
\end{aligned}
\tag{4.11}
$$

Similarly,

$$
\begin{aligned}
W_{2|bn} = {} & W_{12}\, I_{1|bn} + W_{22}\, I_{2|bn} + W_{24}\, I_{4|bn} + W_{25}\, I_{5|bn} \\
& + W_{112}\, I_{1,n}\, I_{1,b} + W_{122}(I_{2,n}\, I_{1,b} + I_{1,n}\, I_{2,b}) + W_{222}\, I_{2,n}\, I_{2,b} \\
& + W_{244} I_{4,n}\, I_{4,b} + W_{255} I_{5,n}\, I_{5,b} + W_{124}(I_{4,n}\, I_{1,b} + I_{4,b}\, I_{1,n}) \\
& + W_{125}(I_{5,n}\, I_{1,b} + I_{1,n}\, I_{5,b}) + W_{224}(I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n}) \\
& + W_{225}(I_{5,n}\, I_{2,b} + I_{5,b}\, I_{2,n} + W_{245}(I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n}) \\
& + \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right] W_{2,A} \\
& + \frac{1}{2}\left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right] W_{2,AB} \\
& + \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right] W_{12,A} + \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right] W_{22,A} \\
& + \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n}\right] W_{24,A} + \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n}\right] W_{25,A}\,,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
W_{4|bn} = {} & W_{14}\, I_{1|bn} + W_{24}\, I_{2|bn} + W_{44}\, I_{4|bn} + W_{45}\, I_{5|bn} \\
& + W_{114}\, I_{1,n}\, I_{1,b} + W_{224}\, I_{2,n}\, I_{2,b} + W_{444}\, I_{4,n}\, I_{4,b} + W_{455}\, I_{5,n}\, I_{5,b} \\
& + W_{124}(I_{2,n}\, I_{1,b} + I_{1,n}\, I_{2,b}) + W_{144}(I_{4,n}\, I_{1,b} + I_{4,b}\, I_{1,n}) \\
& + W_{244}(I_{4,n}\, I_{2,b} + I_{2,n}\, I_{4,b}) + W_{145}(I_{5,n}\, I_{1,b} + I_{5,b}\, I_{1,n}) \\
& + W_{245}(I_{5,n}\, I_{2,b} + I_{2,n}\, I_{5,b}) + W_{445}(I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n}) \\
& + \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right] W_{4,A} \\
& + \frac{1}{2}\left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right] W_{4,AB} \\
& + \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right] W_{14,A} + \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right] W_{24,A} \\
& + \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n}\right] W_{44,A} + \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n}\right] W_{45,A}\,,
\end{aligned}
\tag{4.13}
$$

[3]The factor "$\frac{1}{2}$" on the sixth line is missing in Eqs. (4.7)-(4.9) in [Yavari, 2021]. However, this typo did not affect any of the results of that work.

and

$$
\begin{aligned}
W_{5|bn} = {} & W_{15}\, I_{1|bn} + W_{25}\, I_{2|bn} + W_{45}\, I_{4|bn} + W_{55}\, I_{5|bn} \\
& + W_{115}\, I_{1,n}\, I_{1,b} + W_{225}\, I_{2,n}\, I_{2,b} + W_{445}\, I_{4,n}\, I_{4,b} + W_{555}\, I_{5,n}\, I_{5,b} \\
& + W_{125}(I_{2,n}\, I_{1,b} + I_{1,n}\, I_{2,b}) + W_{145}(I_{4,n}\, I_{1,b} + I_{4,b}\, I_{2,1}) \\
& + W_{155}(I_{5,n}\, I_{1,b} + I_{1,n}\, I_{5,b}) + W_{245}(I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n}) \\
& + W_{255}(I_{5,n}\, I_{2,b} + I_{2,n}\, I_{5,b}) + W_{455}(I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n}) \\
& + \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right] W_{5,A} \\
& + \frac{1}{2}\left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right] W_{5,AB} \\
& + \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right] W_{15,A} + \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right] W_{25,A} \\
& + \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n}\right] W_{45,A} + \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n}\right] W_{55,A}\,.
\end{aligned}
\tag{4.14}
$$

The symmetry $\xi_{a|b} = \xi_{b|a}$ forces the coefficient of each partial derivative of the energy function to be symmetric. Following the notation introduced in [Yavari and Goriely, 2021], we define $\mathcal{A}^{\kappa}_{ab}$ to be the matrix of the coefficient of W_κ, where κ is a multi-index. The first nine terms are identical to those of homogeneous isotropic solids: $\kappa \in \mathcal{K}_{\text{iso}} = \{1, 2, 11, 22, 12, 111, 222, 112, 122\}$. They read

$$
\begin{aligned}
\mathcal{A}^{1}_{ab} &= b^n_{a\,|bn}\,, \\
\mathcal{A}^{2}_{ab} &= -c^n_{a\,|bn}\,, \\
\mathcal{A}^{11}_{ab} &= b^n_{a\,|n}\, I_{1,b} + (b^n_a\, I_{1,n})_{|b}\,, \\
\mathcal{A}^{22}_{ab} &= -c^n_{a\,|n}\, I_{2,b} - (c^n_a\, I_{2,n})_{|b}\,, \\
\mathcal{A}^{12}_{ab} &= (b^n_a\, I_{2,n})_{|b} + b^n_{a\,|n}\, I_{2,b} - \left[(c^n_a\, I_{1,n})_{|b} + c^n_{a\,|n}\, I_{1,b}\right], \\
\mathcal{A}^{111}_{ab} &= b^n_a\, I_{1,n}\, I_{1,b}\,, \\
\mathcal{A}^{222}_{ab} &= -c^n_a\, I_{2,n}\, I_{2,b}\,, \\
\mathcal{A}^{112}_{ab} &= b^n_a\, (I_{1,b}\, I_{2,n} + I_{1,n}\, I_{2,b}) - c^n_a\, I_{1,n}\, I_{1,b}\,, \\
\mathcal{A}^{122}_{ab} &= b^n_a\, I_{2,b}\, I_{2,n} - c^n_a\, (I_{1,b}\, I_{2,n} + I_{1,n}\, I_{2,b})\,,
\end{aligned}
\tag{4.15}
$$

where $b^n_a = b^{mn} g_{ma}$, and $c^n_a = c^{mn} g_{ma}$. It is well known that the symmetry of the above nine terms admits six families of deformations [Ericksen, 1954, Singh and Pipkin, 1965, Klingbeil and Shield, 1966]. For both homogeneous and inhomogeneous transversely isotropic solids, we have 25 extra terms:

$$
\begin{aligned}
\mathcal{K} = \{&4, 5, 44, 55, 14, 15, 24, 25, 45, 444, 555, 114, 115, 124, 125, \\
&144, 145, 155, 224, 225, 244, 245, 255, 445, 455\}\,.
\end{aligned}
\tag{4.16}
$$

These terms read [Yavari and Goriely, 2021]:

$$
\begin{aligned}
\mathcal{A}^{4}_{ab} &= (n_a\, n^n)_{|nb}\,, \\
\mathcal{A}^{5}_{ab} &= \ell^n_{a\,|nb}\,, \\
\mathcal{A}^{44}_{ab} &= (n_a\, n^n)_{|n}\, I_{4,b} + (n_a\, n^n\, I_{4,n})_{|b}\,, \\
\mathcal{A}^{55}_{ab} &= \ell^n_{a\,|n}\, I_{5,b} + (\ell^n_a\, I_{5,n})_{|b}\,, \\
\mathcal{A}^{14}_{ab} &= b^n_{a\,|n}\, I_{4,b} + (b^n_a\, I_{4,n})_{|b} + (n_a\, n^n)_{|n}\, I_{1,b} + (n_a\, n^n I_{1,n})_{|b}\,, \\
\mathcal{A}^{15}_{ab} &= b^n_{a\,|n}\, I_{5,b} + (b^n_a\, I_{5,n})_{|b} + \ell^n_{a\,|n}\, I_{1,b} + (\ell^n_a\, I_{1,n})_{|b}\,, \\
\mathcal{A}^{24}_{ab} &= (n_a\, n^n)_{|n}\, I_{2,b} + (n_a\, n^n\, I_{2,n})_{|b} - \left[c^n_{a\,|n}\, I_{4,b} + (c^n_a\, I_{4,n})_{|b}\right], \\
\mathcal{A}^{25}_{ab} &= \ell^n_{a\,|n}\, I_{2,b} + (\ell^n_a\, I_{2,n})_{|b} - \left[c^n_{a\,|n}\, I_{5,b} + (c^n_a\, I_{5,n})_{|b}\right], \\
\mathcal{A}^{45}_{ab} &= (n_a\, n^n)_{|n}\, I_{5,b} + (n_a\, n^n\, I_{5,n})_{|b} + \ell^n_{a\,|n}\, I_{4,b} + (\ell^n_a\, I_{4,n})_{|b}\,,
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
\mathcal{A}^{444}_{ab} &= n_a\, n^n\, I_{4,n}\, I_{4,b}\,, \\
\mathcal{A}^{555}_{ab} &= \ell^n_a\, I_{5,n}\, I_{5,b}\,, \\
\mathcal{A}^{114}_{ab} &= b^n_a\, (I_{4,n}\, I_{1,b} + I_{4,b}\, I_{1,n}) + n_a\, n^n I_{1,n}\, I_{1,b}\,, \\
\mathcal{A}^{115}_{ab} &= b^n_a\, (I_{5,n}\, I_{1,b} + I_{5,b}\, I_{1,n}) + \ell^n_a\, I_{1,n}\, I_{1,b}\,, \\
\mathcal{A}^{124}_{ab} &= b^n_a\, (I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n}) - c^n_a\, (I_{4,n}\, I_{1,b} + I_{4,b}\, I_{1,n}) + n_a\, n^n\, (I_{2,n}\, I_{1,b} + I_{2,b}\, I_{1,n})\ , \\
\mathcal{A}^{125}_{ab} &= b^n_a\, (I_{5,n}\, I_{2,b} + I_{5,b}\, I_{2,n}) - c^n_a\, (I_{5,n}\, I_{1,b} + I_{5,b}\, I_{1,n}) + \ell^n_a\, (I_{2,n}\, I_{1,b} + I_{2,b}\, I_{1,n})\ , \\
\mathcal{A}^{144}_{ab} &= b^n_a\, I_{4,n}\, I_{4,b} + n_a\, n^n\, (I_{4,n}\, I_{1,b} + I_{4,b}\, I_{1,n})\ , \\
\mathcal{A}^{145}_{ab} &= b^n_a\, (I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n}) + n_a\, n^n\, (I_{5,n}\, I_{1,b} + I_{5,b}\, I_{1,n}) + \ell^n_a\, (I_{4,n}\, I_{1,b} + I_{4,b}\, I_{1,n})\ , \\
\mathcal{A}^{155}_{ab} &= b^n_a\, I_{5,n} I_{5,b} + \ell^n_a\, (I_{5,n}\, I_{1,b} + I_{5,b}\, I_{1,n})\ , \\
\mathcal{A}^{224}_{ab} &= n_a\, n^n\, I_{2,n}\, I_{2,b} - c^n_a\, (I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n})\ , \\
\mathcal{A}^{225}_{ab} &= \ell^n_a\, I_{2,n}\, I_{2,b} - c^n_a\, (I_{5,n}\, I_{2,b} + I_{5,b}\, I_{2,n})\ , \\
\mathcal{A}^{244}_{ab} &= -c^n_a\, I_{4,n}\, I_{4,b} + n_a\, n^n\, (I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n})\ , \\
\mathcal{A}^{245}_{ab} &= n_a\, n^n\, (I_{5,n}\, I_{2,b} + I_{5,b}\, I_{2,n}) + \ell^n_a\, (I_{4,n}\, I_{2,b} + I_{4,b}\, I_{2,n}) - c^n_a\, (I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n})\ , \\
\mathcal{A}^{255}_{ab} &= \ell^n_a\, (I_{5,n}\, I_{2,b} + I_{5,b}\, I_{2,n}) - c^n_a I_{5,n}\, I_{5,b}\,, \\
\mathcal{A}^{445}_{ab} &= n_a\, n^n\, (I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n}) + \ell^n_a\, I_{4,n}\, I_{4,b}\,, \\
\mathcal{A}^{455}_{ab} &= n_a\, n^n\, I_{5,n}\, I_{5,b} + \ell^n_a\, (I_{5,n}\, I_{4,b} + I_{5,b}\, I_{4,n})\ .
\end{aligned}
\tag{4.18}
$$

It turns out that the known universal deformations are invariant with respect to certain Lie subgroups of the special Euclidean group [Goodbrake et al., 2020]. In [Yavari and Goriely, 2021] we conjectured that for each family of universal deformations the corresponding universal material preferred direction vector $\mathbf{N}$ is invariant under the same Lie subgroup. For each of the six families of universal deformations we found the corresponding universal material preferred directions.

For inhomogeneous incompressible transversely isotropic solids, in addition to the universality constraints (4.15), (4.17), and (4.18), there are the following eighteen extra sets of universality constraints (each term must be symmetric in (ab) for $A = 1, 2, 3$, and $B \geq A$):

$$
\begin{aligned}
\mathcal{C}^{1A}_{ab} &= (F^{-1})^A{}_n\, b^n_{a|b} + (F^{-1})^A{}_b\, b^n_{a|n} + b^n_a \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right]\ , \\
\mathcal{C}^{2A}_{ab} &= (F^{-1})^A{}_n\, c^n_{a|b} + (F^{-1})^A{}_b\, c^n_{a|n} + c^n_a \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right]\ , \\
\mathcal{C}^{11A}_{ab} &= b^n_a \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right]\ , \\
\mathcal{C}^{22A}_{ab} &= c^n_a \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right]\ , \\
\mathcal{C}^{12A}_{ab} &= b^n_a \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right] - c^n_a \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right]\ , \\
\mathcal{C}^{1AB}_{ab} &= b^n_a \left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right]\ , \\
\mathcal{C}^{2AB}_{ab} &= c^n_a \left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right]\ ,
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
C^{4A}_{ab} &= (F^{-1})^A{}_n\,(n_a n^n)_{|b} + (F^{-1})^A{}_b\,(n_a n^n)_{|n} + n_a\,n^n\left[(F^{-1})^B{}_b\,(F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\,(F^{-1})^A{}_m\right],\\
C^{5A}_{ab} &= (F^{-1})^A{}_n\,\ell^n_{a|b} + (F^{-1})^A{}_b\,\ell^n_{a|n} + \ell^n_a\left[(F^{-1})^B{}_b\,(F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\,(F^{-1})^A{}_m\right],\\
C^{14A}_{ab} &= b^n_a\left[(F^{-1})^A{}_n\,I_{4,b} + (F^{-1})^A{}_b\,I_{4,n}\right] + n_a\,n^n\left[(F^{-1})^A{}_n\,I_{1,b} + (F^{-1})^A{}_b\,I_{1,n}\right],\\
C^{15A}_{ab} &= b^n_a\left[(F^{-1})^A{}_n\,I_{5,b} + (F^{-1})^A{}_b\,I_{5,n}\right] + \ell^n_a\left[(F^{-1})^A{}_n\,I_{1,b} + (F^{-1})^A{}_b\,I_{1,n}\right],\\
C^{24A}_{ab} &= -c^n_a\left[(F^{-1})^A{}_n\,I_{4,b} + (F^{-1})^A{}_b\,I_{4,n}\right] + n_a\,n^n\left[(F^{-1})^A{}_n\,I_{2,b} + (F^{-1})^A{}_b\,I_{2,n}\right],\\
C^{25A}_{ab} &= -c^n_a\left[(F^{-1})^A{}_n\,I_{5,b} + (F^{-1})^A{}_b\,I_{5,n}\right] + \ell^n_a\left[(F^{-1})^A{}_n\,I_{2,b} + (F^{-1})^A{}_b\,I_{2,n}\right],\\
C^{44A}_{ab} &= n_a\,n^n\left[(F^{-1})^A{}_n\,I_{4,b} + (F^{-1})^A{}_b\,I_{4,n}\right],\\
C^{45A}_{ab} &= n_a\,n^n\left[(F^{-1})^A{}_n\,I_{5,b} + (F^{-1})^A{}_b\,I_{5,n}\right] + \ell^n_a\left[(F^{-1})^A{}_n\,I_{4,b} + (F^{-1})^A{}_b\,I_{4,n}\right],\\
C^{55A}_{ab} &= \ell^n_a\left[(F^{-1})^A{}_n\,I_{5,b} + (F^{-1})^A{}_b\,I_{5,n}\right],\\
C^{4AB}_{ab} &= n_a\,n^n\left[(F^{-1})^A{}_n\,(F^{-1})^B{}_b + (F^{-1})^B{}_n\,(F^{-1})^A{}_b\right],\\
C^{5AB}_{ab} &= \ell^n_a\left[(F^{-1})^A{}_n\,(F^{-1})^B{}_b + (F^{-1})^B{}_n\,(F^{-1})^A{}_b\right].
\end{aligned}
\tag{4.20}
$$

The set of universality constraints (4.19) are identical to those of inhomogeneous isotropic solids [Yavari, 2021]. For a given family of deformations and material preferred directions that are consistent with (4.15), (4.17), and (4.18), the corresponding inhomogeneities that respect (4.19) and (4.20) are celled the *universal inhomogeneities*. In the following subsections, for each of the six families of universal deformations the corresponding universal inhomogeneities will be determined. This will be done by looking at each term in (4.19) and (4.20) and examining its symmetries. If a particular term cannot be symmetric the corresponding derivative of W has to vanish, giving us a constraint on the form of W.

4.1 Family 0: Homogeneous deformations

With respect to the Cartesian coordinates $\{X^A\}$ and $\{x^a\}$ in the reference and current configurations, respectively, a homogeneous deformation has the representation $x^a(\mathbf{X}) = F^a{}_A X^A + c^a$, where $[F^a{}_A]$ is a constant matrix and $[c^a]$ is a constant vector. The incompressibility constraint is then $\det[F^a{}_A] = 1$. For a homogeneous deformation the right Cauchy-Green tensor has the constant components $C_{AB} = F^a{}_A F^a{}_A\,\delta_{ab}$, which implies that $\mathbf{C}^\flat$ is invariant under the action of $T(3) \subset SE(3)$—the group of translations. In [Yavari and Goriely, 2021] it was assumed that $\mathbf{N}(\mathbf{X})$ is invariant under $T(3)$ as well, i.e., $\mathbf{N}$ is a constant unit vector. We choose the Cartesian coordinates (X^1, X^2, X^2) such that

$$
\mathbf{N} = \frac{\partial}{\partial X^1}\,,
\tag{4.21}
$$

i.e., $N^A = \delta^A_1$. With this assumption the universality constraints (4.17) and (4.18) are satisfied. For homogeneous deformations, the first five sets of universality constraints (4.19) are trivially satisfied. The last two sets force the deformation to be the identity [Yavari, 2021]. This implies that

$$
W_{1,AB} = W_{2,AB} = 0\,,\quad A,B = 1,2.3\,.
\tag{4.22}
$$

For isotropic solids, the relations $W_{1,AB} = (W_{1,A})_{,B} = 0$, and $W_{2,AB} = (W_{2,A})_{,B} = 0$ imply that

$$
\begin{aligned}
&\frac{\partial W_1}{\partial X^1} = f_1(I_1,I_2), \quad \frac{\partial W_1}{\partial X^2} = f_2(I_1,I_2), \quad \frac{\partial W_1}{\partial X^3} = f_3(I_1,I_2)\,,\\
&\frac{\partial W_2}{\partial X^1} = g_1(I_1,I_2), \quad \frac{\partial W_2}{\partial X^2} = g_2(I_1,I_2), \quad \frac{\partial W_2}{\partial X^3} = g_3(I_1,I_2)\,.
\end{aligned}
\tag{4.23}
$$

Note that

$$
\frac{\partial f_1(I_1,I_2)}{\partial I_2} = \frac{\partial g_1(I_1,I_2)}{\partial I_1}\,,\quad \frac{\partial f_2(I_1,I_2)}{\partial I_2} = \frac{\partial g_2(I_1,I_2)}{\partial I_1}\,,\quad \frac{\partial f_3(I_1,I_2)}{\partial I_2} = \frac{\partial g_3(I_1,I_2)}{\partial I_1}\,.
\tag{4.24}
$$

From $(4.23)_1$, one concludes that

$$W_1(\mathbf{X}, I_1, I_2) = f_0(I_1, I_2) + f_1(I_1, I_2) X^1 + f_2(I_1, I_2) X^2 + f_3(I_1, I_2) X^3 \,. \tag{4.25}$$

Thus

$$\begin{aligned} W(\mathbf{X}, I_1, I_2) = \int f_0(I_1, I_2)\, dI_1 + X^1 \int f_1(I_1, I_2)\, dI_1 + X^2 \int f_2(I_1, I_2)\, dI_1 \\ + X^3 \int f_3(I_1, I_2)\, dI_1 + R(\mathbf{X}, I_2) \,, \end{aligned} \tag{4.26}$$

for some function $R(\mathbf{X}, I_2)$. Hence

$$\begin{aligned} W_2(\mathbf{X}, I_1, I_2) &= \int \frac{\partial f_0(I_1, I_2)}{\partial I_2}\, dI_1 + X^1 \int \frac{\partial f_1(I_1, I_2)}{\partial I_2}\, dI_1 + X^2 \int \frac{\partial f_2(I_1, I_2)}{\partial I_2}\, dI_1 \\ &\quad + X^3 \int \frac{\partial f_3(I_1, I_2)}{\partial I_2}\, dI_1 + \frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} \\ &= \int \frac{\partial f_0(I_1, I_2)}{\partial I_2}\, dI_1 + X^1 \int \frac{\partial g_1(I_1, I_2)}{\partial I_1}\, dI_1 + X^2 \int \frac{\partial g_2(I_1, I_2)}{\partial I_1}\, dI_1 \\ &\quad + X^3 \int \frac{\partial g_3(I_1, I_2)}{\partial I_1}\, dI_1 + \frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} \\ &= \int \frac{\partial f_0(I_1, I_2)}{\partial I_2}\, dI_1 + g_1(I_1, I_2) X^1 + g_2(I_1, I_2) X^2 + g_3(I_1, I_2) X^3 + \frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} \,. \end{aligned} \tag{4.27}$$

Substituting the above identity into $(4.23)_2$ one concludes that

$$\frac{\partial}{\partial X^1} \frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} = \frac{\partial}{\partial X^2} \frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} = \frac{\partial}{\partial X^3} \frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} = 0 \,. \tag{4.28}$$

This implies that

$$\frac{\partial R(\mathbf{X}, I_2)}{\partial I_2} = r(I_2) \,, \tag{4.29}$$

and hence $R(\mathbf{X}, I_2) = R_1(\mathbf{X}) + R_2(I_2)$. Using this in (4.26), up to a mechanically inconsequential $\mathbf{X}$ dependent term one concludes that for an incompressible isotropic solid the energy function is a linear function of the Cartesian coordinates, i.e.,

$$W(\mathbf{X}, I_1, I_2) = \overline{W}(I_1, I_2) + \mathbf{H}(I_1, I_2) \cdot \mathbf{X} \,, \tag{4.30}$$

for some vector $\mathbf{H}(I_1, I_2)$.[4]

In the case of inhomogeneous transversely isotropic solids, one still has the constraints (4.22). The first nine sets of universality constraints (4.20) are trivially satisfied for homogeneous deformations and constant $\mathbf{N}$. The last two sets of constraints in (4.20) are nontrivial. The universality constraints corresponding to Eq.$(4.20)_{10}$ read

$$n_a \left[N^A \, (F^{-1})^B{}_b + N^B \, (F^{-1})^A{}_b \right] W_{4,AB} = n_b \left[N^A \, (F^{-1})^B{}_a + N^B \, (F^{-1})^A{}_a \right] W_{4,AB} \,. \tag{4.31}$$

Knowing that $N^A = \delta^A_1$, the above constraints are rewritten as

$$\left[n_a (F^{-1})^A{}_b - n_b (F^{-1})^A{}_a \right] W_{4,1A} = 0 \,, \quad a, b = 1, 2, 3 \,. \tag{4.32}$$

This is equivalent to

$$F^a{}_M F^b{}_N \left[n_a (F^{-1})^A{}_b - n_b (F^{-1})^A{}_a \right] W_{4,1A} = 0 \,, \quad M, N = 1, 2, 3 \,, \tag{4.33}$$

[4] In [Yavari, 2021] from (4.22) it was incorrectly concluded that $W(\mathbf{X}, I_1, I_2) = \overline{W}(I_1, I_2)$. Proposition 4.1 in [Yavari, 2021] should be corrected to read: "For inhomogeneous incompressible nonlinear isotropic solids, Family 0 deformations are universal for any energy function of the form $W(\mathbf{X}, I_1, I_2) = \overline{W}(I_1, I_2) + \mathbf{H}(I_1, I_2) \cdot \mathbf{X}$."

which is simplified to read

$$C_{M1}\, W_{4,1N} - C_{N1}\, W_{4,1M} = 0\,, \quad M,N = 1,2,3\,. \tag{4.34}$$

These are three constraints corresponding to $(M,N) = (1,2),\ (1,3)$, and $(2,3)$, and read

$$\begin{aligned} C_{11}\, W_{4,12} - C_{21}\, W_{4,11} &= 0\,,\\ C_{11}\, W_{4,13} - C_{31}\, W_{4,11} &= 0\,,\\ C_{21}\, W_{4,13} - C_{31}\, W_{4,12} &= 0\,. \end{aligned} \tag{4.35}$$

Notice that these need to be satisfied for an arbitrary matrix $[C_{AB}]$ with unit determinant. This means that $W_{4,11} = W_{4,12} = W_{4,13} = 0$.

The universality constraints corresponding to Eq.$(4.20)_{11}$ read

$$\begin{aligned} &\ell^n_a \left[(F^{-1})^A{}_n\,(F^{-1})^B{}_b + (F^{-1})^B{}_n\,(F^{-1})^A{}_b\right] W_{5,AB}\\ &\quad = \ell^n_b \left[(F^{-1})^A{}_n\,(F^{-1})^B{}_a + (F^{-1})^B{}_n\,(F^{-1})^A{}_a\right] W_{5,AB}\,, \quad a,b = 1,2,3\,. \end{aligned} \tag{4.36}$$

This can be simplified to read

$$C^K{}_1 \left[C_{M1}\, W_{5,KN} + C_{MK}\, W_{5,1N} - C_{N1}\, W_{5,KM} - C_{NK}\, W_{5,1M}\right] = 0\,, \quad M,N = 1,2,3\,. \tag{4.37}$$

These are three constraints corresponding to $(M,N) = (1,2),\ (1,3)$, and $(2,3)$, and read

$$\begin{aligned} &\left(2C^1{}_1C_{21} + C^2{}_1C_{22} + C^3{}_1C_{23}\right) W_{5,11} - \left(2C^1{}_1C_{11} + C^3{}_1C_{13}\right) W_{5,12} + C^3{}_1C_{21}W_{5,13}\\ &\quad - C^2{}_1C_{11}W_{5,22} - C^3{}_1C_{11}W_{5,23} = 0\,,\\ &\left(2C^1{}_1C_{31} + C^2{}_1C_{32} + C^3{}_1C_{33}\right) W_{5,11} + C^2{}_1C_{31}W_{5,12} - \left(2C^1{}_1C_{11} + C^2{}_1C_{12}\right) W_{5,13}\\ &\quad - C^2{}_1C_{11}W_{5,23} - C^3{}_1C_{11}W_{5,33} = 0\,,\\ &\left(2C^1{}_1C_{31} + C^2{}_1C_{32} + C^3{}_1C_{33}\right) W_{5,12} - \left(2C^1{}_1C_{21} + C^2{}_1C_{22} + C^3{}_1C_{23}\right) W_{5,13} + C^2{}_1C_{31}W_{5,22}\\ &\quad + \left(C^3{}_1C_{31} - C^2{}_1C_{21}\right) W_{5,23} - C^3{}_1C_{21}W_{5,33} = 0\,. \end{aligned} \tag{4.38}$$

These must be satisfied for an arbitrary matrix $[C_{AB}]$ with unit determinant. If $[C_{AB}]$ is diagonal, one concludes that $W_{5,12} = W_{5,13} = 0$. Considering simple shear in the X^1X^2-plane ($C_{13} = C_{23} = 0$), one concludes that $W_{5,11} = W_{5,22} = W_{5,23} = 0$. Substituting these in the above equations, one concludes that $W_{5,33} = 0$. Therefore, $W_{4,AB} = 0$.

In summary, for the universality constraints to hold one must have

$$(W_{1,A})_{,B} = (W_{2,A})_{,B} = (W_{4,1})_{,B} = (W_{5,A})_{,B} = 0\,, \quad A,B = 1,2,3\,. \tag{4.39}$$

Using arguments similar to those used in deriving (4.30), one can show that the above constraints imply the following proposition.

Proposition 4.1. *For inhomogeneous incompressible nonlinear transversely isotropic solids with material preferred direction parallel to the* X^1*-axis in a Cartesian coordinate system* (X^1, X^2, X^3)*, Family 0 deformations are universal for any energy function of the following form*

$$W(\mathbf{X}, I_1, I_2, I_4, I_5) = \overline{W}(I_1, I_2, I_4, I_5) + \mathbf{H}(I_1, I_2, I_4, I_5)\cdot\mathbf{X} + \widetilde{W}(X^2, X^3, I_4)\,. \tag{4.40}$$

Remark 4.2. Note that the last term of the energy function in (4.40) has a form identical to that of compressible orthotropic solids (3.34).

4.2 Family 1: Bending, stretching, and shearing of a rectangular block

Consider a rectangular block and a Cartesian coordinate system (X, Y, Z) with coordinate planes parallel to the faces of the block. In the current configuration cylindrical coordinates (r, θ, z) are used. With respect to these coordinates, the deformations given by Family 1 have the following representation

$$r(X,Y,Z) = \sqrt{C_1(2X+C_4)}\,, \quad \theta(X,Y,Z) = C_2(Y+C_5)\,, \quad z(X,Y,Z) = \frac{Z}{C_1\,C_2} - C_2\,C_3 Y + C_6\,, \tag{4.41}$$

where $C_1, ..., C_6$ are constants. The right Cauchy-Green strain reads

$$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2\left[C_1(2X+C_4)+C_3^2\right] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2\,C_2^2} \end{bmatrix}, \tag{4.42}$$

and is independent of Y and Z, i.e., $\mathbf{C}^\flat$ is invariant under the action of $T(2) \subset SE(3)$. Yavari and Goriely [2021] assumed that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(X,Y,Z) = \begin{bmatrix} N^1(X) \\ N^2(X) \\ N^3(X) \end{bmatrix}, \tag{4.43}$$

where $(N^1(X))^2 + (N^2(X))^2 + (N^3(X))^2 = 1$. It was shown that the universal material preferred direction has the following possible forms

$$\mathbf{N} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 \\ \cos\psi(X) \\ \pm\sin\psi(X) \end{bmatrix}, \tag{4.44}$$

where $\psi(X)$ is an arbitrary function. Notice that $(4.44)_1$ corresponds to a uniform distribution of fibers parallel to the X-axis. In the other universal material preferred direction distribution $(4.44)_2$, for fixed X fibers make an angle $\psi(X)$ with the Y-axis, and are distributed uniformly in the YZ-plane.

In [Yavari, 2021] it was shown that the constraints (4.19) imply that

$$\frac{\partial W_1}{\partial Y} = \frac{\partial W_1}{\partial Z} = \frac{\partial W_2}{\partial Y} = \frac{\partial W_2}{\partial Z} = 0\,. \tag{4.45}$$

The above relations hold for inhomogeneous transversely isotropic solids as well. For the universal material preferred direction $(4.44)_1$, one can show that[5]

$$\begin{aligned} &C^{4A}_{[ab]}=0,\ \text{for}\ (A,a,b)=(2,1,2)\Rightarrow\ C_1^2=0\,,\\ &C^{4A}_{[ab]}=0,\ \text{for}\ (A,a,b)=(3,1,2)\Rightarrow\ C_1^3\,C_2\,C_3=0\,,\\ &C^{5A}_{[ab]}=0,\ \text{for}\ (A,a,b)=(2,1,2)\Rightarrow\ \left[C_1(C_4+2X)\right]^{\frac{3}{2}}C_4=0\,,\\ &C^{5A}_{[ab]}=0,\ \text{for}\ (A,a,b)=(3,1,2)\Rightarrow\ C_2\,C_3\left[C_1(C_4+2X)\right]^{\frac{5}{2}}=0\,. \end{aligned} \tag{4.46}$$

These constraints cannot be satisfied, and hence

$$\frac{\partial W_4}{\partial Y}=\frac{\partial W_4}{\partial Z}=\frac{\partial W_5}{\partial Y}=\frac{\partial W_5}{\partial Z}=0\,. \tag{4.47}$$

Similarly, for the universal material preferred direction $(4.44)_2$ one has the following constraints:

- $C^{4A}_{[ab]}=0$, for $(A,a,b)=(2,1,2)$ requires that

$$C_2\sqrt{C_1(C_4+2X)}\cos\psi\left[2(C_4+2X)\psi'\sin\psi-3\cos\psi\right]=0\,. \tag{4.48}$$

- $C^{4A}_{[ab]}=0$, for $(A,a,b)=(3,1,2)$ implies that

$$C_2\sqrt{C_1(C_4+2X)}\left[C_1C_2^2C_3\cos^2\psi+(C_4+2X)\psi'\cos2\psi+\sin2\psi\right]=0\,. \tag{4.49}$$

- $C^{5A}_{[ab]}=0$, for $(A,a,b)=(2,1,2)$ requires that

$$\begin{aligned} C_2\sqrt{C_1(C_4+2X)}\Big\{&2(C_4+2X)\psi'\left[C_1C_2^2\sin2\psi\left(C_1C_4+2C_1X+C_3^2\right)+C_3\cos2\psi\right]\\ &-2C_1\,C_2^2\cos^2\psi\left[5C_1(C_4+2X)+3C_3^2\right]+3C_3\sin2\psi\Big\}=0\,. \end{aligned} \tag{4.50}$$

- $C^{5A}_{[ab]}=0$, for $(A,a,b)=(3,1,2)$ requires that

$$\begin{aligned} &(C_4+2X)^2\psi'\cos2\psi\left[C_1^2\,C_2^4\left(C_1(C_4+2X)+C_3^2\right)+1\right]\\ &\quad+(C_4+2X)\Big\{C_1^3C_2^6C_3\cos2\psi\left(C_1\,C_4+2C_1X+C_3^2\right)+\sin2\psi\left[2C_1^3\,C_2^4(C_4+2X)+1\right]\\ &\qquad+C_1\,C_2^2\,C_3\left[C_1^2\,C_2^4\left(C_1\,C_4+2C_1X+C_3^2\right)-2\right]\Big\}=0\,. \end{aligned} \tag{4.51}$$

None of the above constraints can be satisfied, and hence, (4.47) holds for this case as well. From (4.45) and (4.47) one concludes that up to a mechanically inconsequential function of (X,Y,Z), the energy function must have the form $W=W(X,I_1,I_2,I_4,I_5)$. For energy functions of this form, in (4.19) and (4.20) one only needs to check the symmetry of the terms with $A=1$, and $A=B=1$. All those terms are symmetric.

Proposition 4.3. *For inhomogeneous incompressible nonlinear transversely isotropic solids with any of the universal material preferred directions given in* (4.44)*, Family 1 deformations are universal for any energy function of the form* $W=W(X,I_1,I_2,I_4,I_5)$.

[5]All the symbolic computations in this paper were performed using Mathematica Version 12.3.0.0, Wolfram Research, Champaign, IL.

4.3 Family 2: Straightening, stretching, and shearing of a sector of a cylindrical shell

Consider a sector of a cylindrical shell that is parametrized by cylindrical coordinates (R, Θ, Z). In the deformed configuration Cartesian coordinates (x, y, z) are used. Family 2 deformations have the following representation

$$x(R,\Theta,Z) = \frac{1}{2}C_1\, C_2^2\, R^2 + C_4\,, \quad y(R,\Theta,Z) = \frac{\Theta}{C_1\, C_2} + C_5\,, \quad z(R,\Theta,Z) = \frac{C_3}{C_1\, C_2}\Theta + \frac{1}{C_2}Z + C_6\,, \tag{4.52}$$

and hence

$$[C_{AB}] = \begin{bmatrix} C_1^2\, C_2^4 R^2 & 0 & 0 \\ 0 & \frac{C_3^2+1}{C_1^2\, C_2^2} & \frac{C_3}{C_1\, C_2^2} \\ 0 & \frac{C_3}{C_1\, C_2^2} & \frac{1}{C_2^2} \end{bmatrix}. \tag{4.53}$$

It is seen that the right Cauchy-Green strain is independent of Θ and Z. In [Yavari and Goriely, 2021] it was assumed that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} N^1(R) \\ N^2(R) \\ N^3(R) \end{bmatrix}, \tag{4.54}$$

such that $(N^1(R))^2 + R^2(N^2(R))^2 + (N^3(R))^2 = 1$. It was shown that there are two solutions for the universal material preferred direction:

$$\mathbf{N} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}, \tag{4.55}$$

where $\chi(R)$ is an arbitrary function. In the case of $(4.55)_1$ fibers are distributed radially. In the solution $(4.55)_2$, if $\cos\psi(R) \neq 0, \pm 1$ fibers are distributed helically, if $\cos\psi(R) = 0$ they are distributed parallel to the axis of the shell, and if $\cos\psi(R) = \pm 1$ they are concentric circles in the (R, Θ)-plane.

In [Yavari, 2021] it was shown that the constraints (4.19) imply that

$$\frac{\partial W_1}{\partial \Theta} = \frac{\partial W_1}{\partial Z} = \frac{\partial W_2}{\partial \Theta} = \frac{\partial W_2}{\partial Z} = 0\,. \tag{4.56}$$

The above relations hold for inhomogeneous transversely isotropic solids as well. For the universal material preferred direction $(4.55)_1$, one can show that

$$\begin{aligned}
&C^{4A}_{[ab]} = 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow \; C_1^2\, C_2^3 = 0\,,\\
&C^{4A}_{[ab]} = 0, \text{ for } (A,a,b) = (3,1,2) \Rightarrow \; C_1\, C_2^3\, C_3 = 0\,,\\
&C^{5A}_{[ab]} = 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow \; C_1^4\, C_2^7\, R^2 = 0\,,\\
&C^{5A}_{[ab]} = 0, \text{ for } (A,a,b) = (3,1,2) \Rightarrow \; C_1^3\, C_2^7\, C_3\, R^2 = 0\,.
\end{aligned} \tag{4.57}$$

These constraints cannot be satisfied, and thus

$$\frac{\partial W_4}{\partial \Theta} = \frac{\partial W_4}{\partial Z} = \frac{\partial W_5}{\partial \Theta} = \frac{\partial W_5}{\partial Z} = 0\,. \tag{4.58}$$

Similarly, for the universal material preferred direction $(4.55)_2$ one has the following constraints:

- $C^{4A}_{[ab]} = 0$, for $(A,a,b) = (2,1,2)$ requires that

$$\cos\chi(R)\,[R\,\chi'(R)\sin\chi(R) + \cos\chi(R)] = 0\,. \tag{4.59}$$

- $C^{4A}_{[ab]} = 0$, for $(A,a,b) = (3,1,2)$ implies that

$$\sin 2\chi(R) - 2R\,\chi'(R)\cos 2\chi(R) = 0\,. \tag{4.60}$$

- $C^{5A}_{[ab]} = 0$, for $(A,a,b) = (2,1,2)$ requires that

$$\begin{aligned}
&R\,\chi'(R)\left[\left(1 + C_3^2\right)\sin 2\chi(R) - C_1 C_3 R\cos 2\chi(R)\right]\\
&+ \cos\chi(R)\left[3C_1C_3R\sin\chi(R) + 4\left(1 + C_3^2\right)\cos\chi(R)\right] = 0\,.
\end{aligned} \tag{4.61}$$

- $C^{5A}_{[ab]} = 0$, for $(A,a,b) = (3,1,2)$ requires that

$$-R\left(1 + C_1^2R^2 + C_3^2\right)\chi'(R)\cos 2\chi(R) + \left(3 + C_1^2R^2 + 3C_3^2\right)\sin 2\chi(R) + 2C_1\,C_3R = 0\,. \tag{4.62}$$

None of the above constraints can be satisfied,[6] and thus, (4.58) holds for this case as well. From (4.56) and (4.58) one concludes that up to a mechanically inconsequential function of (R,Θ,Z), the energy function must have the form $W = W(R, I_1, I_2, I_4, I_5)$. For energy functions of this form, in (4.19) and (4.20) one only needs to check the symmetry of the terms with $A = 1$, and $A = B = 1$. All those terms are symmetric.

Proposition 4.4. *For inhomogeneous incompressible nonlinear transversely isotropic solids with any of the universal material preferred directions given in* (4.55)*, Family 2 deformations are universal for any energy function of the form* $W = W(R, I_1, I_2, I_4, I_5)$.

4.4 Family 3: Inflation, bending, torsion, extension, and shearing of a sector of an annular wedge

Family 3 deformations, with respect to the cylindrical coordinates (R,Θ,Z) and (r,θ,z) in the reference and current configurations, respectively, have the following representation

$$r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1\,C_4 - C_2\,C_3} + C_5}\,, \quad \theta(R,\Theta,Z) = C_1\Theta + C_2 Z + C_6\,, \quad z(R,\Theta,Z) = C_3\Theta + C_4 Z + C_7\,, \tag{4.63}$$

[6]Note that we are finding the universal inhomogeneities for an arbitrary universal material preferred direction in $(4.55)_2$, and hence, $\cos\chi(R) \neq 0$, in general, i.e., (4.59) cannot be satisfied.

and hence

$$[C_{AB}] = \begin{bmatrix} \frac{R^2}{K(KC_5+R^2)} & 0 & 0 \\ 0 & C_3^2 + C_1^2\left[\frac{R^2}{K}+C_5\right] & C_1C_2\left[\frac{R^2}{K}+C_5\right]+C_3C_4 \\ 0 & C_1C_2\left[\frac{R^2}{K}+C_5\right]+C_3C_4 & C_4^2 + C_2^2\left[\frac{R^2}{K}+C_5\right] \end{bmatrix}, \tag{4.64}$$

where $K = C_1C_4 - C_2C_3$. Notice that $\mathbf{C}^\flat$ is independent of Θ and Z. In [Yavari and Goriely, 2021] it was assumed that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} N^1(R) \\ N^2(R) \\ N^3(R) \end{bmatrix}, \tag{4.65}$$

where $(N^1(R))^2 + R^2(N^2(R))^2 + (N^3(R))^2 = 1$. It was shown that there are two solutions for the universal material preferred direction:

$$\mathbf{N} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\psi(R) \\ \pm\sin\psi(R) \end{bmatrix}, \tag{4.66}$$

where $\psi(R)$ is an arbitrary function.

In [Yavari, 2021] it was shown that for this family of deformations constraints (4.19) imply that

$$\frac{\partial W_1}{\partial \Theta} = \frac{\partial W_1}{\partial Z} = \frac{\partial W_2}{\partial \Theta} = \frac{\partial W_2}{\partial Z} = 0\,. \tag{4.67}$$

The above relations hold for inhomogeneous transversely isotropic solids as well. For the universal material preferred direction $(4.66)_1$, one can show that

$$\begin{aligned} C^{4A}_{[ab]} &= 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow C_4\left(-2C_2\,C_3\,C_5 + 2C_1\,C_4\,C_5 + R^2\right) = 0\,, \\ C^{4A}_{[ab]} &= 0, \text{ for } (A,a,b) = (3,1,2) \Rightarrow C_3\left(-2C_2\,C_3\,C_5 + 2C_1\,C_4\,C_5 + R^2\right) = 0\,, \\ C^{5A}_{[ab]} &= 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow C_4\left(-4C_2\,C_3\,C_5 + 4C_1\,C_4\,C_5 + R^2\right) = 0\,, \\ C^{5A}_{[ab]} &= 0, \text{ for } (A,a,b) = (3,1,2) \Rightarrow C_3\left(-4C_2\,C_3\,C_5 + 4C_1\,C_4\,C_5 + R^2\right) = 0\,. \end{aligned} \tag{4.68}$$

These constraints cannot be satisfied, and hence

$$\frac{\partial W_4}{\partial \Theta} = \frac{\partial W_4}{\partial Z} = \frac{\partial W_5}{\partial \Theta} = \frac{\partial W_5}{\partial Z} = 0\,. \tag{4.69}$$

Similarly, for the universal material preferred direction $(4.66)_2$ $C^{4A}_{[ab]}=0$, for $(A,a,b)=(2,1,2)$ requires that:

$$
\begin{split}
&\left(C_1\,C_4\,C_5-C_2\,C_3\,C_5+R^2\right)\Big\{2C_1^3\,C_4^2\,C_5-4C_1^2\,C_2\,C_3\,C_4\,C_5+C_1^2\,C_2\,C_4^2\,C_5R\sin 2\psi(R)\\
&-C_1^2\,C_4R^2+\cos 2\psi(R)\left[2C_1^3\,C_4^2\,C_5-C_1^2\,C_4\left(4C_2\,C_3\,C_5+R^2\right)+2C_1C_2^2C_3^2C_5+C_2^2C_4R^4\right]\\
&+2C_1\,C_2^2\,C_3^2\,C_5-2C_1\,C_2^2\,C_3\,C_4\,C_5\,R\sin 2\psi(R)\\
&-2R\psi'(R)(C_2\,C_3-C_1\,C_4)\left(-C_1\,C_4\,C_5+C_2\,C_3\,C_5-R^2\right)\left[C_2\,R\cos 2\psi(R)-C_1\,\sin 2\psi(R)\right]\\
&-3C_1\,C_2\,C_4\,R^3\sin 2\psi(R)+C_2^3\,C_3^2\,C_5\,R\sin 2\psi(R)+C_2^2\,C_3\,R^3\sin 2\psi(R)-C_2^2\,C_4\,R^4\Big\}=0\,.
\end{split}
\tag{4.70}
$$

The constraints $C^{4A}_{[ab]}=0$, for $(A,a,b)=(3,1,2)$, $C^{5A}_{[ab]}=0$, for $(A,a,b)=(2,1,2)$, and $C^{5A}_{[ab]}=0$, for $(A,a,b)=(3,1,2)$ require vanishing of some lengthy expressions that we do not report here. None of these four constraints can be satisfied, and thus, (4.69) holds for this case as well. Similar to Family 2 deformations, from (4.67) and (4.69) one concludes that up to a mechanically inconsequential function of (R,Θ,Z), the energy function must have the form $W=W(R,I_1,I_2,I_4,I_5)$. For energy functions of this form, in (4.19) and (4.20) one only needs to check the symmetry of the terms with $A=1$, and $A=B=1$. All those terms are symmetric.

Proposition 4.5. *For inhomogeneous incompressible nonlinear transversely isotropic solids with any of the universal material preferred directions given in* (4.66)*, Family 3 deformations are universal for any energy function of the form* $W=W(R,I_1,I_2,I_4,I_5)$.

Physically, this universal inhomogeneity and directions can be understood as follows: A particular case consists of a single homogeneous cylindrical tube with helical preferred directions. Now, consider a series of encased homogeneous cylindrical tubes in the reference configuration, each with its own helical material preferred directions as describe in [Goriely, 2017]. The solution from Proposition 4.5 is a continuous version of this problem where the variation in helical fibers and material properties only depends on R.

4.5 Family 4: Inflation/inversion of a sector of a spherical shell

Family 4 deformations with respect to the spherical coordinates (R,Θ,Φ) and (r,θ,ϕ) in the reference and current configurations, respectively, have the following representation

$$
r(R,\Theta,\Phi)=(\pm R^3+C_1^3)\,,\quad \theta(R,\Theta,\Phi)=\pm\Theta\,,\quad \phi(R,\Theta,\Phi)=\Phi\,. \tag{4.71}
$$

Thus

$$
[C_{AB}]=\begin{bmatrix}\frac{R^4}{\left(C_1^3\pm R^3\right)^{4/3}} & 0 & 0\\ 0 & \left(C_1^3\pm R^3\right)^{2/3} & 0\\ 0 & 0 & \left(C_1^3\pm R^3\right)^{2/3}\sin^2\Theta\end{bmatrix}, \tag{4.72}
$$

which can be written as [Goodbrake et al., 2020]

$$
\mathbf{C}^\flat(\mathbf{X})=\frac{R^4}{\left(C_1^3\pm R^3\right)^{4/3}}\hat{\mathbf{R}}\otimes\hat{\mathbf{R}}+\frac{\left(C_1^3\pm R^3\right)^{2/3}}{R^2}(\mathbf{1}-\hat{\mathbf{R}}\otimes\hat{\mathbf{R}})\,, \tag{4.73}
$$

where $\mathbf{1}$ is the identity tensor, and $\hat{\mathbf{R}}=\frac{\mathbf{X}}{|\mathbf{X}|}$. This implies that at a given point $\mathbf{X}$, $\mathbf{C}^\flat$ is invariant under all those rotations that fix $\mathbf{X}$. Yavari and Goriely [2021] assumed that $\mathbf{N}(\mathbf{X})$ has the same symmetry, i.e., it is invariant under all those rotations that fix $\mathbf{X}$. Thus, $\mathbf{N}(\mathbf{X})$ is parallel to $\mathbf{X}$, and knowing that it is a unit vector one concludes that

$$\mathbf{N}(\mathbf{X}) = \pm\frac{\mathbf{X}}{|\mathbf{X}|} = \pm\hat{\mathbf{R}}\,. \tag{4.74}$$

This means that the universal material preferred direction is radial, i.e., with respect to the spherical coordinates

$$\mathbf{N}(\mathbf{X}) = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.75}$$

In [Yavari, 2021] it was shown that for this family of deformations constraints (4.19) imply that

$$\frac{\partial W_1}{\partial \Theta} = \frac{\partial W_1}{\partial \Phi} = \frac{\partial W_2}{\partial \Theta} = \frac{\partial W_2}{\partial \Phi} = 0\,. \tag{4.76}$$

The above relations hold for inhomogeneous transversely isotropic solids as well. For the universal material preferred direction (4.75), one can show that

$$\begin{aligned}
&\mathcal{C}^{4A}_{[ab]} = 0,\ \text{for } (A,a,b) = (2,1,2) \Rightarrow\ 4C_1^3\,R - R^4 = 0\,,\\
&\mathcal{C}^{4A}_{[ab]} = 0,\ \text{for } (A,a,b) = (3,1,3) \Rightarrow\ 4C_1^3\,R - R^4 = 0\,,\\
&\mathcal{C}^{5A}_{[ab]} = 0,\ \text{for } (A,a,b) = (2,1,2) \Rightarrow\ -8C_1^3\,R^5 + R^8 = 0\,,\\
&\mathcal{C}^{5A}_{[ab]} = 0,\ \text{for } (A,a,b) = (3,1,3) \Rightarrow\ -8C_1^3\,R^5 + R^8 = 0\,.
\end{aligned} \tag{4.77}$$

These constraints cannot be satisfied, and hence

$$\frac{\partial W_4}{\partial \Theta} = \frac{\partial W_4}{\partial \Phi} = \frac{\partial W_5}{\partial \Theta} = \frac{\partial W_5}{\partial \Phi} = 0\,. \tag{4.78}$$

From (4.76) and (4.78) one concludes that up to a mechanically inconsequential function of (R,Θ,Φ), the energy function must have the form $W = W(R, I_1, I_2, I_4, I_5)$. For any energy function of this form, in (4.19) and (4.20) one only needs to check the symmetry of the terms with $A = 1$, and $A = B = 1$. All those terms are symmetric.

Proposition 4.6. *For inhomogeneous incompressible nonlinear transversely isotropic solids with the universal material preferred directions given in (4.75), Family 4 deformations are universal for any energy function of the form $W = W(R, I_1, I_2, I_4, I_5)$.*

Again, this result can be understood physically as the continuous limit of a finite number of encased homogeneous spherical shells with different material properties.

4.6 Family 5: Inflation, bending, extension, and azimuthal shearing of an annular wedge

Family 5 deformations with respect to the cylindrical coordinates (R,Θ,Z) and (r,θ,z) in the reference and current configurations, respectively, have the following representation

$$r(R,\Theta,Z) = C_1 R\,,\quad \theta(R,\Theta,Z) = C_2\log R + C_3\Theta + C_4\,,\quad z(R,\Theta,Z) = \frac{1}{C_1^2\,C_3}Z + C_5\,. \tag{4.79}$$

Thus

$$
[C_{AB}] = \begin{bmatrix} C_1^2\left(C_2^2+1\right) & C_1^2\,C_2\,C_3R & 0 \\ C_1^2\,C_2\,C_3R & C_1^2\,C_3^2R^2 & 0 \\ 0 & 0 & \frac{1}{C_1^4\,C_3^2} \end{bmatrix}, \tag{4.80}
$$

which only depends on R. Yavari and Goriely [2021] assumed that $\mathbf{N}$ has the same symmetry, i.e.,

$$
\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} N^1(R) \\ N^2(R) \\ N^3(R) \end{bmatrix}, \tag{4.81}
$$

where $(N^1(R))^2+(N^2(R))^2+(N^3(R))^2 = 1$. They obtained the following two solutions for universal material preferred directions

$$
\mathbf{N} = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \cos\xi \\ \pm\frac{1}{R}\sin\xi \\ 0 \end{bmatrix}, \tag{4.82}
$$

for arbitrary constants η, and ξ. Unfortunately, there was a mistake in checking the universality constraints for solution $(4.82)_1$: This solution satisfies all the universality constraints other than the symmetry of the coefficient of W_4 for $(a,b)=(1,3)$, which gives $C_2\cos\eta\sin\eta=0$. Note that $\sin\eta=0$ in $(4.82)_1$ corresponds to $\cos\xi=0$ in $(4.82)_2$. This means that the correct set of universal material preferred directions for Family 5 are:

$$
\mathbf{N} = \begin{bmatrix} 0 \\ 0 \\ \pm1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \cos\xi \\ \pm\frac{1}{R}\sin\xi \\ 0 \end{bmatrix}, \tag{4.83}
$$

for an arbitrary constant ξ.

In [Yavari, 2021] it was shown that the for Family 5 deformations constraints (4.19) imply that

$$
\frac{\partial W_1}{\partial R} = \frac{\partial W_1}{\partial \Theta} = \frac{\partial W_1}{\partial Z} = \frac{\partial W_2}{\partial R} = \frac{\partial W_2}{\partial \Theta} = \frac{\partial W_2}{\partial Z} = 0\,. \tag{4.84}
$$

The above relations hold for inhomogeneous transversely isotropic solids as well.

For the universal material preferred direction $(4.83)_1$, all the terms in $(4.20)_{(1-9)}$ are symmetric. In the last two sets of equations the following four terms are not symmetric:

$$\begin{aligned} &C^{4AB}_{[ab]} \neq 0, \text{ for } (A,B,a,b) = (1,3,1,3)\,, \\ &C^{4AB}_{[ab]} \neq 0, \text{ for } (A,B,a,b) = (2,3,1,3)\,, \\ &C^{5AB}_{[ab]} \neq 0, \text{ for } (A,B,a,b) = (1,3,1,3)\,, \\ &C^{5AB}_{[ab]} \neq 0, \text{ for } (A,B,a,b) = (2,3,1,3)\,. \end{aligned} \tag{4.85}$$

This implies that

$$\frac{\partial^2 W_4}{\partial R\,\partial Z} = \frac{\partial^2 W_4}{\partial \Theta\,\partial Z} = \frac{\partial^2 W_5}{\partial R\,\partial Z} = \frac{\partial W_5}{\partial \Theta\,\partial Z} = 0\,. \tag{4.86}$$

From (4.84) and (4.86) one concludes that $W(\mathbf{X}, I_1, I_2, I_4, I_5) = \overline{W}(I_1, I_2, I_4, I_5) + \widetilde{W}(R, \Theta, I_4, I_5) + \widehat{W}(Z, I_4, I_5)$. For an energy function of this form, in (4.19) and (4.20) one only needs to check the symmetry of the terms with $A = 1$, and $A = B = 1$. It turns out that all those terms are symmetric.

For the universal material preferred direction $(4.83)_2$, one can show that

$$\begin{aligned} &C^{4A}_{[ab]} = 0, \text{ for } (A,a,b) = (1,1,2) \Rightarrow C_1 \cos\xi\,[C_2 \cos\xi + C_3 \sin\xi] = 0\,, \\ &C^{4A}_{[ab]} = 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow C_1 \left[\left(1 + C_2^2\right)\cos^2\xi - C_3^2 \sin^2\xi\right] = 0\,, \\ &C^{4A}_{[ab]} = 0, \text{ for } (A,a,b) = (3,2,3) \Rightarrow C_1^3\, C_3 \left[\left(-1 + C_2^2\right)\cos^2\xi + C_3 \sin\xi\,(2C_2 \cos\xi + C_3 \sin\xi)\right] = 0\,, \\ &C^{5A}_{[ab]} = 0, \text{ for } (A,a,b) = (1,1,2) \Rightarrow \\ &\qquad C_1^3 \Big\{2C_2\left[1 + C_2^2 + C_3^2 + \left(1 + C_2^2\right)\cos 2\xi\right] + C_3\left(1 + 3C_2^2 + C_3^2\right)\sin 2\xi\Big\} = 0\,, \\ &C^{5A}_{[ab]} = 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow \\ &\qquad C_1^3 \Big\{\left[\left(1 + C_2^2\right)^2 + C_3^4\right]\cos 2\xi + \left(1 + C_2^2 - C_3^2\right)\left(1 + C_2^2 + C_3^2 + C_2 C_3 \sin 2\xi\right)\Big\} = 0\,, \\ &C^{5A}_{[ab]} = 0, \text{ for } (A,a,b) = (3,2,3) \Rightarrow \\ &\qquad C_1^6\, C_3 \Big\{2C_2\left[1 + C_2^2 + C_3^2 + \left(1 + C_2^2\right)\cos 2\xi\right] + C_3\left(1 + 3C_2^2 + C_3^2\right)\sin 2\xi\Big\} = 0\,. \end{aligned} \tag{4.87}$$

None of the above constraints can be satisfied,[7] and hence

$$\frac{\partial W_4}{\partial R} = \frac{\partial W_4}{\partial \Theta} = \frac{\partial W_4}{\partial Z} = \frac{\partial W_5}{\partial R} = \frac{\partial W_5}{\partial \Theta} = \frac{\partial W_5}{\partial Z} = 0\,. \tag{4.88}$$

From (4.84) and (4.88) one concludes that the energy function must be homogeneous. This means that Family 5 deformations are not universal for inhomogeneous incompressible transversely isotropic solids with the universal material preferred directions $(4.83)_2$.

Proposition 4.7. *For inhomogeneous incompressible nonlinear transversely isotropic solids with the universal material preferred directions given in* $(4.83)_1$*, Family 5 deformations are universal for any energy function of the form* $W(\mathbf{X}, I_1, I_2, I_4, I_5) = \overline{W}(I_1, I_2, I_4, I_5) + \widetilde{W}(R, \Theta, I_4, I_5) + \widehat{W}(Z, I_4, I_5)$*. Family 5 deformations are not universal for inhomogeneous incompressible transversely isotropic solids with the universal material preferred directions* $(4.83)_2$*.*

Table 2 summarizes our results for inhomogeneous incompressible transversely isotropic solids.

[7]Note that we are finding the universal inhomogeneities of the energy function for an arbitrary member of this class. That means that $\cos\xi \neq 0$, in general, i.e., $(4.87)_1$ cannot be satisfied.

Family	Universal Deformations	Universal material preferred directions	Universal inhomogeneities
0	$x^a(X) = F^a{}_A X^A + c^a$	Any constant unit vector $\mathbf{N}$	$W = \overline{W}(I_1, I_2, I_4, I_5) + \mathbf{H}(I_1, I_2, I_4, I_5) \cdot \mathbf{X} + \widetilde{W}(X^2, X^3, I_4)$
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X + C_4)} \\ \theta(X,Y,Z) = C_2(Y + C_5) \\ z(X,Y,Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6 \end{cases}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\psi(X) \\ \pm\sin\psi(X) \end{bmatrix}$	$W = W(X, I_1, I_2, I_4, I_5)$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2}C_1C_2^2R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6 \end{cases}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}$	$W = W(R, I_1, I_2, I_4, I_5)$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1C_4 - C_2C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4Z + C_7 \end{cases}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}$	$W = W(R, I_1, I_2, I_4, I_5)$
4	$\begin{cases} r(R,\Theta,\Phi) = (\pm R^3 + C_1^3) \\ \theta(R,\Theta,\Phi) = \pm\Theta \\ \phi(R,\Theta,\Phi) = \Phi \end{cases}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}$	$W = W(R, I_1, I_2, I_4, I_5)$
5	$\begin{cases} r(R,\Theta,Z) = C_1R \\ \theta(R,\Theta,Z) = C_2\log R + C_3\Theta + C_4 \\ z(R,\Theta,Z) = \frac{1}{C_1^2C_3}Z + C_5 \end{cases}$	$\hat{\mathbf{N}} = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}$	$W = \overline{W}(I_1, I_2, I_4, I_5) + \widetilde{W}(R, \Theta, I_4, I_5) + \widehat{W}(Z, I_4, I_5)$

Table 2: *Universal deformations, universal material preferred directions, and universal inhomogeneities for incompressible transversely isotropic solids for the six known families of universal deformations.*

5 Incompressible Orthotropic Elastic Solids

For inhomogeneous orthotropic solids

$$\xi_a = g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n_1^m n_1^n + W_5\, \ell_1^{mn} + W_6\, n_2^m n_2^n + W_7\, \ell_2^{mn}\right]_{|n} . \tag{5.1}$$

In order to satisfy the symmetry $\xi_{a|b} = \xi_{b|a}$ for an arbitrary energy function the coefficient of each partial derivative of W must be symmetric. There are five groups of terms. The first four were derived in [Yavari and Goriely, 2021]. In order for this work to be self contained, all the five groups are reported below. The first four groups of terms that must be symmetric for both incompressible and compressible orthotropic solids are:

i) Nine terms that need to be symmetric for isotropic solids as well:

$$\mathscr{K}_{\text{iso}} = \{1, 2, 11, 22, 12, 111, 222, 112, 122\} . \tag{5.2}$$

ii) 25 terms associated to $\mathbf{N}_1$:

$$\begin{aligned}\mathscr{K}_i = \{&4,5,44,55,14,15,24,25,45,444,555,114,115,124,125,\\ &144,145,155,224,225,244,245,255,445,455\}\,.\end{aligned} \tag{5.3}$$

iii) 25 terms associated to $\mathbf{N}_2$:

$$\begin{aligned}\mathscr{K}_{ii} = \{&6,7,66,77,16,17,26,27,67,666,777,116,117,126,127,\\ &166,167,177,226,227,266,267,277,667,677\}\,.\end{aligned} \tag{5.4}$$

iv) 24 terms corresponding to coupling of $\mathbf{N}_1$ and $\mathbf{N}_2$:

$$\begin{aligned}\mathscr{K}_{iii} = \{&46,47,56,57,146,147,156,157,246,247,256,257,446,447,\\ &456,457,556,557,466,467,566,567,477,577\}\,.\end{aligned} \tag{5.5}$$

v) 33 terms that correspond to the inhomogeneity of the energy function. 18 of these are identical to those of isotropic (4.19) and transversely isotropic solids (4.20).

In [Yavari and Goriely, 2021] it was noticed that $\mathscr{K}_i$ and $\mathscr{K}_{ii}$ universality constraints have forms identical to those of $\mathscr{K}$ universality constraints (4.16). This implies that $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ is universal for orthotropic solids if i) $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$ are universal for transversely isotropic solids, and ii) the three pairs $(\mathbf{N}_1, \mathbf{N}_2)$, $(\mathbf{N}_2, \mathbf{N}_3)$, and $(\mathbf{N}_3, \mathbf{N}_1)$ satisfy the $\mathscr{K}_{iii}$ universality constraints. We follow the notation introduced in [Yavari and Goriely, 2021], and let $(\boldsymbol{n}, \boldsymbol{m}) = (\mathbf{n}_1, \mathbf{n}_2)$, and $(\ell^{ab}, \hbar^{ab}) = (\ell_1^{ab}, \ell_2^{ab})$. The coefficients of the derivatives of the energy function associated to the set $\mathscr{K}_{iii}$ are:

$$\begin{aligned}\mathscr{A}_{ab}^{46} &= [n_a\, I_{6,n}\, n^n]_{|b} + I_{6,b}[n_a\, n^n]_{|n} + [m_a\, I_{4,n}\, m^n]_{|b} + I_{4,b}[m_a\, m^n]_{|n}\,,\\ \mathscr{A}_{ab}^{47} &= [n_a\, I_{7,n}\, n^n]_{|b} + I_{7,b}[n_a\, n^n]_{|n} + (\hbar_a^{\ n} I_{4,n})_{|b} + \hbar_a^{\ n}{}_{|n} I_{4,b}\,,\\ \mathscr{A}_{ab}^{56} &= (\ell_a^{\ n}\, I_{6,n})_{|b} + \ell_a^{\ n}{}_{|n}\, I_{6,b} + (m_a\, I_{5,n}\, m^n)_{|b} + (m_a\, m^n)_{|n} I_{5,b}\,,\\ \mathscr{A}_{ab}^{57} &= (\ell_a^{\ n}\, I_{7,n})_{|b} + \ell_a^{\ n}{}_{|n}\, I_{7,b} + (\hbar_a^{\ n}\, I_{5,n})_{|b} + \hbar_a^{\ n}{}_{|n}\, I_{5,b}\,,\end{aligned} \tag{5.6}$$

and

$$
\begin{aligned}
\mathcal{A}^{146}_{ab} &= b^n_a \left(I_{4,b}\, I_{6,n} + I_{4,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{147}_{ab} &= b^n_a \left(I_{4,b}\, I_{7,n} + I_{4,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{156}_{ab} &= b^n_a \left(I_{5,b}\, I_{6,n} + I_{5,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{157}_{ab} &= b^n_a \left(I_{5,b}\, I_{7,n} + I_{5,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{246}_{ab} &= c^n_a \left(I_{4,b}\, I_{6,n} + I_{4,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{247}_{ab} &= c^n_a \left(I_{4,b}\, I_{7,n} + I_{4,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{256}_{ab} &= c^n_a \left(I_{5,b}\, I_{6,n} + I_{5,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{257}_{ab} &= c^n_a \left(I_{5,b}\, I_{7,n} + I_{5,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{446}_{ab} &= n_a\, n^n \left(I_{4,b}\, I_{6,n} + I_{4,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{447}_{ab} &= n_a\, n^n \left(I_{4,b}\, I_{7,n} + I_{4,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{456}_{ab} &= n_a\, n^n \left(I_{5,b}\, I_{6,n} + I_{5,n}\, I_{6,b} \right) + \ell^n_a \left(I_{4,b}\, I_{6,n} + I_{4,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{457}_{ab} &= n_a\, n^n \left(I_{5,b}\, I_{7,n} + I_{5,n}\, I_{7,b} \right) + \ell^n_a \left(I_{4,b}\, I_{7,n} + I_{4,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{466}_{ab} &= m_a\, m^n \left(I_{4,b}\, I_{6,n} + I_{4,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{467}_{ab} &= m_a\, m^n \left(I_{4,b}\, I_{7,n} + I_{4,n}\, I_{7,b} \right) + k^n_a \left(I_{4,b}\, I_{6,n} + I_{4,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{477}_{ab} &= k^n_a \left(I_{4,b}\, I_{7,n} + I_{4,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{556}_{ab} &= \ell^n_a \left(I_{5,b}\, I_{6,n} + I_{5,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{557}_{ab} &= \ell^n_a \left(I_{5,b}\, I_{7,n} + I_{5,n}\, I_{7,b} \right) \,, \\
\mathcal{A}^{566}_{ab} &= m_a\, m^n \left(I_{5,b}\, I_{6,n} + I_{5,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{567}_{ab} &= m_a\, m^n \left(I_{5,b}\, I_{7,n} + I_{5,n}\, I_{7,b} \right) + k^n_a \left(I_{5,b}\, I_{6,n} + I_{5,n}\, I_{6,b} \right) \,, \\
\mathcal{A}^{577}_{ab} &= k^n_a \left(I_{5,b}\, I_{7,n} + I_{5,n}\, I_{7,b} \right) \,.
\end{aligned}
\tag{5.7}
$$

For inhomogeneous incompressible orthotropic solids, in addition to the universality constraints (4.19), and (4.20) there are the following 15 extra sets of universality constraints (each term must be symmetric in (ab) for $A = 1, 2, 3$, and $B \geq A$):

$$
\begin{aligned}
C^{6A}_{ab} &= (F^{-1})^A{}_n\, (m_a\, m^n)_{|b} + (F^{-1})^A{}_b\, (m_a\, m^n)_{|n} + m_a\, m^n \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m \right] \,, \\
C^{7A}_{ab} &= (F^{-1})^A{}_n\, k^n_{a|b} + (F^{-1})^A{}_b\, k^n_{a|n} + k^n_a \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m \right] \,, \\
C^{16A}_{ab} &= b^n_a \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n} \right] + m_a\, m^n \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n} \right] \,, \\
C^{17A}_{ab} &= b^n_a \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n} \right] + k^n_a \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n} \right] \,, \\
C^{26A}_{ab} &= -c^n_a \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n} \right] + m_a\, m^n \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n} \right] \,, \\
C^{27A}_{ab} &= -c^n_a \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n} \right] + k^n_a \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n} \right] \,, \\
C^{46A}_{ab} &= n_a\, n^n \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n} \right] + m_a\, m^n \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n} \right] \,, \\
C^{47A}_{ab} &= n_a\, n^n \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n} \right] + k^n_a \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n} \right] \,, \\
C^{56A}_{ab} &= \ell^n_a \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n} \right] + m_a\, m^n \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n} \right] \,, \\
C^{57A}_{ab} &= \ell^n_a \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n} \right] + k^n_a \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n} \right] \,, \\
C^{66A}_{ab} &= m_a\, m^n \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n} \right] \,, \\
C^{67A}_{ab} &= k^n_a \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n} \right] + m_a\, m^n \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n} \right] \,, \\
C^{77A}_{ab} &= k^n_a \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n} \right] \,, \\
C^{6AB}_{ab} &= m_a\, m^n \left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b \right] \,, \\
C^{7AB}_{ab} &= k^n_a \left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b \right] \,.
\end{aligned}
\tag{5.8}
$$

5.1 Family 0

In [Yavari and Goriely, 2021] it was shown that for homogeneous orthotropic solids homogeneous deformations are universal for any three constant unit vectors $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ that are mutually orthogonal. In the reference configuration we choose the Cartesian coordinates (X^1, X^2, X^3) such that

$$\mathbf{N}_1 = \frac{\partial}{\partial X^1}\,, \quad \mathbf{N}_2 = \frac{\partial}{\partial X^2}\,, \quad \mathbf{N}_3 = \frac{\partial}{\partial X^3}\,. \tag{5.9}$$

The universality constraints still imply (4.39). For homogeneous deformations and constant $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$, only the last two sets of universality constraints in (5.8) are nontrivial, and imply that

$$(W_{6,2})_{,B} = (W_{7,A})_{,B} = 0\,, \quad A, B = 1, 2, 3\,. \tag{5.10}$$

Using a fairly lengthy but standard argument (similar to those of §3.2) one can show that the constraints (4.39) and (5.10) imply the following result.

Proposition 5.1. *For inhomogeneous incompressible nonlinear orthotropic solids, Family 0 deformations are universal for any energy function of the following form*

$$\begin{aligned} W(\mathbf{X}, I_1, I_2, I_4, I_5, I_6, I_7) &= \overline{W}(I_1, I_2, I_4, I_5, I_6, I_7) + \mathbf{H}(I_1, I_2, I_4, I_5, I_6, I_7) \cdot \mathbf{X} \\ &\quad + \widetilde{W}(X^3, I_4, I_6) + \widehat{W}(X^2, X^3, I_4) + \widehat{W}(X^1, X^3, I_6)\,, \end{aligned} \tag{5.11}$$

where $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ *are constant unit vectors given in* (5.9).

Remark 5.2. Note that the last three terms of the energy function in (5.11) have identical forms to that of compressible orthotropic solids (3.60).

5.2 Family 1

In [Yavari and Goriely, 2021] it was shown that for Family 1 universal deformations the universal material preferred directions are

$$\mathbf{N}_1 = \begin{bmatrix} \pm 1 \\ \\ 0 \\ \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 \\ \\ \cos\psi(X) \\ \\ \pm\sin\psi(X) \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ \\ \sin\psi(X) \\ \\ \mp\cos\psi(X) \end{bmatrix}, \tag{5.12}$$

where $\psi(X)$ is an arbitrary function. The constraints (4.45) and (4.47) hold for orthotropic solids as well. Similarly, from $(5.8)_{1-2}$ one concludes that

$$\frac{\partial W_6}{\partial Y} = \frac{\partial W_6}{\partial Z} = \frac{\partial W_7}{\partial Y} = \frac{\partial W_7}{\partial Z} = 0\,. \tag{5.13}$$

All the other universality constraints are satisfied. Therefore, we have the following result.

Proposition 5.3. *For inhomogeneous incompressible nonlinear orthotropic solids with any of the universal material preferred directions given in* (5.12)*, Family 1 deformations are universal for any energy function of the form* $W = W(X, I_1, I_2, I_4, I_5, I_6, I_7)$.

5.3 Families 2 and 3

In [Yavari and Goriely, 2021] it was shown that for Families 2 and 3 the following family of material preferred directions are universal.

$$\mathbf{N}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 \\ \frac{\cos\chi(R)}{R} \\ \pm\sin\chi(R) \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ \frac{\sin\chi(R)}{R} \\ \mp\cos\chi(R) \end{bmatrix}, \tag{5.14}$$

where $\chi(R)$ is an arbitrary function. The constraints (4.56) and (4.58) still hold. Similarly, from $(5.8)_{1-2}$ one concludes that

$$\frac{\partial W_6}{\partial \Theta} = \frac{\partial W_6}{\partial Z} = \frac{\partial W_7}{\partial \Theta} = \frac{\partial W_7}{\partial Z} = 0\,. \tag{5.15}$$

All the other universality constraints are satisfied. Thus, we have the following result.

Proposition 5.4. *For inhomogeneous incompressible nonlinear orthotropic solids with any of the universal material preferred directions given in (5.14), Family 2 and 3 deformations are universal for any energy function of the form* $W = W(R, I_1, I_2, I_4, I_5, I_6, I_7)$.

Yavari and Goriely [2021] showed that for homogeneous incompressible orthotropic solids Family 4 deformations are not universal. This is the case for inhomogeneous incompressible orthotropic solids as well.

5.4 Family 5

In [Yavari and Goriely, 2021] the following universal material preferred directions were reported.

$$\begin{cases} \mathbf{N}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} \cos\xi \\ \pm\frac{1}{R}\sin\xi \\ 0 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} \sin\xi \\ \mp\frac{1}{R}\cos\xi \\ 0 \end{bmatrix}, \\[2ex] \mathbf{N}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ \frac{1}{R}\sin\eta \\ \mp\cos\eta \end{bmatrix}. \end{cases} \tag{5.16}$$

As was mentioned in §4.6, there was a mistake in one of the families of universal material preferred directions. In $(5.16)_2$ either $\cos\eta = 0$, or $\sin\eta = 0$, which are already included in $(5.16)_1$. Therefore, the correct families of universal material preferred directions are (we have relabeled them so that $\mathbf{N}_3$ is parallel to the Z-axis):

$$\mathbf{N}_1 = \begin{bmatrix} \cos\xi \\ \pm\frac{1}{R}\sin\xi \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} \sin\xi \\ \mp\frac{1}{R}\cos\xi \\ 0 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \tag{5.17}$$

where ξ is an arbitrary constant.

In [Yavari, 2021] it was shown that the for Family 5 deformations constraints (4.19) imply that

$$\frac{\partial W_1}{\partial R} = \frac{\partial W_1}{\partial \Theta} = \frac{\partial W_1}{\partial Z} = \frac{\partial W_2}{\partial R} = \frac{\partial W_2}{\partial \Theta} = \frac{\partial W_2}{\partial Z} = 0\,. \tag{5.18}$$

The above relations hold for inhomogeneous orthotropic isotropic solids as well. If we check the universality constraints for the pair $(\mathbf{N}_1, \mathbf{N}_2)$ given in (5.17), from §4.6 we know that $W_{4,A} = W_{5,A} = W_{6,A} = W_{7,A} = 0$, and hence the energy function must be uniform:

Proposition 5.5. *For inhomogeneous incompressible nonlinear orthotropic solids Family 5 deformations are not universal.*

Table 3 summarizes our results for inhomogeneous incompressible orthotropic solids.

Family	Universal Deformations	Universal material preferred directions	Universal inhomogeneities
0	$x^a(X) = F^a{}_A X^A + c^a$	Any three mutually orthogonal constant unit vectors $(\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2, \hat{\mathbf{N}}_3)$	$W = \overline{W}(I_1, I_2, I_4, I_5, I_6, I_7) + \mathbf{H}(I_1, I_2, I_4, I_5, I_6, I_7) \cdot \mathbf{X}$ $+\widetilde{W}(X^3, I_4, I_6) + \widehat{W}(X^2, X^3, I_4) + \widehat{W}(X^1, X^3, I_6)$
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X + C_4)} \\ \theta(X,Y,Z) = C_2(Y + C_5) \\ z(X,Y,Z) = \frac{Z}{C_1 C_2} - C_2 C_3 Y + C_6 \end{cases}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\psi(X) \\ \pm\sin\psi(X) \end{bmatrix}, \quad \hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\psi(X) \\ \mp\cos\psi(X) \end{bmatrix}$	$W = W(X, I_1, I_2, I_4, I_5, I_6, I_7)$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2} C_1 C_2^2 R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1 C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1 C_2}\Theta + \frac{1}{C_2} Z + C_6 \end{cases}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}, \quad \hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\chi(R) \\ \mp\cos\chi(R) \end{bmatrix}$	$W = W(R, I_1, I_2, I_4, I_5, I_6, I_7)$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1 C_4 - C_2 C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2 Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4 Z + C_7 \end{cases}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}, \quad \hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\chi(R) \\ \mp\cos\chi(R) \end{bmatrix}$	$W = W(R, I_1, I_2, I_4, I_5, I_6, I_7)$

Table 3: *Universal deformations, universal material preferred directions, and universal inhomogeneities for incompressible orthotropic solids for the six known families of universal deformations.*

6 Incompressible Monoclinic Elastic Solids

In the case of monoclinic solids

$$\xi_a = g_{am} \left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n_1^m n_1^n + W_5\, \ell_1^{mn} + W_6\, n_2^m n_2^n + W_7\, \ell_2^{mn} + \frac{1}{2} W_8\, \ell_3^{mn} \right]_{|n} . \tag{6.1}$$

The universality constraint $\xi_{a|b} = \xi_{b|a}$ forces the coefficient of each partial derivative of W to be symmetric. Yavari and Goriely [2021] showed that for monoclinic solids there are an extra 78 terms corresponding to the following set:

$$\mathscr{K}_{iv} = \{8, 18, 19, 28, 29, 48, 49, 58, 59, 68, 69, 78, 79, 88, 89,
118, 119, 128, 129, 148, 149, 158, 159, 168, 169, 178, 179, 188, 189, 199, 228, 229,
248, 249, 258, 259, 268, 269, 278, 279, 288, 289, 299, 448, 449, 458, 459, 468, 469,
478, 479, 488, 489, 499, 558, 559, 568, 569, 578, 579, 588, 589, 599, 668, 669,
678, 679, 688, 689, 699, 778, 779, 788, 789, 799, 888, 889, 999\}\,. \tag{6.2}$$

We follow the notation in [Yavari and Goriely, 2021] and write $(\boldsymbol{n}, \boldsymbol{m}) = (\mathbf{n}_1, \mathbf{n}_2)$, and $(\ell^{ab}, \hbar^{ab}, q^{ab}) = (\ell_1^{ab}, \ell_2^{ab}, \ell_3^{ab})$. The terms corresponding to the set $\mathscr{K}_{iv}$ are:

$$\begin{aligned}
\mathscr{A}^{8}_{ab} &= q^n_{a\,|nb}\,,\\
\mathscr{A}^{18}_{ab} &= q^n_{a\,|n}\, I_{1,b} + (q^n_a\, I_{1,n})_{|b} + (b^n_a\, I_{8,n})_{|b} + b^n_{a\,|n}\, I_{8,b}\,,\\
\mathscr{A}^{19}_{ab} &= (b^n_a I_{9,n})_{|b} + b^n_{a\,|n}\, I_{9,b}\,,\\
\mathscr{A}^{28}_{ab} &= q^n_{a\,|n}\, I_{2,b} + (q^n_a\, I_{2,n})_{|b} - (c^n_a I_{8,n})_{|b} - c^n_{a\,|n}\, I_{8,b}\,,\\
\mathscr{A}^{29}_{ab} &= -(c^n_a\, I_{9,n})_{|b} - c^n_{a\,|n}\, I_{9,b}\,,\\
\mathscr{A}^{48}_{ab} &= q^n_{a\,|n}\, I_{4,b} + (q^n_a\, I_{4,n})_{|b} + (n_a\, n^n\, I_{8,n})_{|b} + (n_a\, n^n)_{|n}\, I_{8,b}\,,\\
\mathscr{A}^{49}_{ab} &= (n_a\, n^n\, I_{9,n})_{|b} + (n_a\, n^n)_{|n}\, I_{9,b}\,,\\
\mathscr{A}^{58}_{ab} &= q^n_{a\,|n}\, I_{5,b} + (q^n_a\, I_{5,n})_{|b} + (\ell^n_a\, I_{8,n})_{|b} + \ell^n_{a\,|n}\, I_{8,b}\,,\\
\mathscr{A}^{59}_{ab} &= (\ell^n_a\, I_{9,n})_{|b} + \ell^n_{a\,|n}\, I_{9,b}\,,\\
\mathscr{A}^{68}_{ab} &= q^n_{a\,|n}\, I_{6,b} + (q^n_a\, I_{6,n})_{|b} + (m_a\, m^n\, I_{8,n})_{|b} + (m_a\, m^n)_{|n}\, I_{8,b}\,,\\
\mathscr{A}^{69}_{ab} &= (m_a\, m^n\, I_{9,n})_{|b} + (m_a\, m^n)_{|n}\, I_{9,b}\,,\\
\mathscr{A}^{78}_{ab} &= q^n_{a\,|n}\, I_{7,b} + (q^n_a\, I_{7,n})_{|b} + (\hbar^n_a\, I_{8,n})_{|b} + \hbar^n_{a\,|n}\, I_{8,b}\,,\\
\mathscr{A}^{79}_{ab} &= (\hbar^n_a\, I_{9,n})_{|b} + \hbar^n_{a\,|n}\, I_{9,b}\,,\\
\mathscr{A}^{88}_{ab} &= q^n_{a\,|n}\, I_{8,b} + (q^n_a\, I_{8,n})_{|b}\,,\\
\mathscr{A}^{89}_{ab} &= q^n_{a\,|n}\, I_{9,b} + (q^n_a\, I_{9,n})_{|b}\,,
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
\mathscr{A}^{118}_{ab} &= b^n_a\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{119}_{ab} &= b^n_a\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{128}_{ab} &= b^n_a\,(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}) - c^n_a\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{129}_{ab} &= b^n_a\,(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9,b}) - c^n_a\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{148}_{ab} &= b^n_a\,(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}) + n_a\, n^n\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{149}_{ab} &= b^n_a\,(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}) + n_a\, n^n\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{158}_{ab} &= b^n_a\,(I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b}) + \ell^n_a\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{159}_{ab} &= b^n_a\,(I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b}) + \ell^n_a\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{168}_{ab} &= b^n_a\,(I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b}) + m_a\, m^n\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{169}_{ab} &= b^n_a\,(I_{6,b}\, I_{9,n} + I_{6,n}\, I_{9,b}) + m_a\, m^n\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{178}_{ab} &= b^n_a\,(I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b}) + \hbar^n_a\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{179}_{ab} &= b^n_a\,(I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b}) + \hbar^n_a\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{188}_{ab} &= b^n_a\, I_{8,b} I_{8,n} + q^n_a\,(I_{1,b}\, I_{8,n} + I_{1,n}\, I_{8,b})\,,\\
\mathscr{A}^{189}_{ab} &= b^n_a\,(I_{8,b}\, I_{9,n} + I_{8,n}\, I_{9,b}) + q^n_a\,(I_{1,b}\, I_{9,n} + I_{1,n}\, I_{9,b})\,,\\
\mathscr{A}^{199}_{ab} &= b^n_a\, I_{9,b}\, I_{9,n}\,,
\end{aligned} \tag{6.4}$$

$$
\begin{aligned}
\mathcal{A}_{ab}^{228} &= -c_a^n \left(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{229} &= -c_a^n \left(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{248} &= -c_a^n \left(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}\right) + n_a\, n^n \left(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{249} &= -c_a^n \left(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}\right) + n_a\, n^n \left(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9b}\right) \,, \\
\mathcal{A}_{ab}^{258} &= -c_a^n \left(I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b}\right) + \ell_a^n \left(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{259} &= -c_a^n \left(I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b}\right) + \ell_a^n \left(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{268} &= -c_a^n \left(I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b}\right) + m_a\, m^n \left(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{269} &= -c_a^n \left(I_{6,b}\, I_{9,n} + I_{6,n}\, I_{9,b}\right) + m_a\, m^n \left(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{278} &= -c_a^n \left(I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b}\right) + k_a^n \left(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{279} &= -c_a^n \left(I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b}\right) + k_a^n \left(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{288} &= -c_a^n\, I_{8,b} I_{8,n} + q_a^n \left(I_{2,b}\, I_{8,n} + I_{2,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{289} &= -c_a^n \left(I_{8,b}\, I_{9,n} + I_{8,n}\, I_{9,b}\right) + q_a^n \left(I_{2,b}\, I_{9,n} + I_{2,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{299} &= -c_a^n\, I_{9,b}\, I_{9,n}, \\
\mathcal{A}_{ab}^{448} &= n_a\, n^n \left(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{449} &= n_a\, n^n \left(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}\right) \,,
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
\mathcal{A}_{ab}^{458} &= n_a\, n^n \left(I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b}\right) + \ell_a^n \left(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{459} &= n_a\, n^n \left(I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b}\right) + \ell_a^n \left(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{468} &= n_a n^n \left(I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b}\right) + m_a\, m^n \left(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{469} &= n_a\, n^n \left(I_{6,b}\, I_{9,n} + I_{6,n}\, I_{9,b}\right) + m_a\, m^n \left(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{478} &= n_a\, n^n \left(I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b}\right) + k_a^n \left(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{479} &= n_a\, n^n \left(I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b}\right) + k_a^n \left(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{488} &= n_a\, n^n\, I_{8,b}\, I_{8,n} + q_a^n \left(I_{4,b}\, I_{8,n} + I_{4,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{489} &= n_a\, n^n \left(I_{8,b}\, I_{9,n} + I_{8,n}\, I_{9,b}\right) + q_a^n \left(I_{4,b}\, I_{9,n} + I_{4,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{499} &= n_a\, n^n\, I_{9,b}\, I_{9,n} \,, \\
\mathcal{A}_{ab}^{558} &= \ell_a^n \left(I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{559} &= \ell_a^n \left(I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{568} &= \ell_a^n \left(I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b}\right) + m_a\, m^n \left(I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{569} &= \ell_a^n \left(I_{6,b} I_{9,n} + I_{6,n} I_{9,b}\right) + m_a\, m^n \left(I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b}\right) \,, \\
\mathcal{A}_{ab}^{578} &= \ell_a^n \left(I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b}\right) + k_a^n \left(I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b}\right) \,, \\
\mathcal{A}_{ab}^{579} &= \ell_a^n \left(I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b}\right) + k_a^n \left(I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b}\right) \,,
\end{aligned}
\tag{6.6}
$$

and

$$
\begin{aligned}
\mathcal{A}^{588}_{ab} &= \ell^n_a\, I_{8,b}\, I_{8,n} + q^n_a\, (I_{5,b}\, I_{8,n} + I_{5,n}\, I_{8,b})\ ,\\
\mathcal{A}^{589}_{ab} &= \ell^n_a\, (I_{8,b}\, I_{9,n} + I_{8,n}\, I_{9,b}) + q^n_a\, (I_{5,b}\, I_{9,n} + I_{5,n}\, I_{9,b})\ ,\\
\mathcal{A}^{599}_{ab} &= \ell^n_a\, I_{9,b}\, I_{9,n}\ ,\\
\mathcal{A}^{668}_{ab} &= m_a\, m^n\, (I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b})\ ,\\
\mathcal{A}^{669}_{ab} &= m_a\, m^n\, (I_{6,b}\, I_{9,n} + I_{6,n}\, I_{9,b})\ ,\\
\mathcal{A}^{678}_{ab} &= m_a\, m^n\, (I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b}) + k^n_a\, (I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b})\ ,\\
\mathcal{A}^{679}_{ab} &= m_a\, m^n\, (I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b}) + k^n_a\, (I_{6,b}\, I_{9,n} + I_{6,n}\, I_{9,b})\ ,\\
\mathcal{A}^{688}_{ab} &= m_a\, m^n\, I_{8,b}\, I_{8,n} + q^n_a\, (I_{6,b}\, I_{8,n} + I_{6,n}\, I_{8,b})\ ,\\
\mathcal{A}^{689}_{ab} &= m_a m^n\, (I_{8,b} I_{9,n} + I_{8,n} I_{9,b}) + q^n_a\, (I_{6,b}\, I_{9,n} + I_{6,n}\, I_{9,b})\ ,\\
\mathcal{A}^{699}_{ab} &= m_a\, m^n\, I_{9,b}\, I_{9,n}\ ,\\
\mathcal{A}^{778}_{ab} &= k^n_a\, (I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b})\ ,\\
\mathcal{A}^{779}_{ab} &= k^n_a\, (I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b})\ ,\\
\mathcal{A}^{788}_{ab} &= k^n_a\, I_{8,b}\, I_{8,n} + q^n_a\, (I_{7,b}\, I_{8,n} + I_{7,n}\, I_{8,b})\ ,\\
\mathcal{A}^{789}_{ab} &= k^n_a\, (I_{8,b}\, I_{9,n} + I_{8,n}\, I_{9,b}) + q^n_a\, (I_{7,b}\, I_{9,n} + I_{7,n}\, I_{9,b})\ ,\\
\mathcal{A}^{799}_{ab} &= k^n_a\, I_{9,b}\, I_{9,n}\ ,\\
\mathcal{A}^{888}_{ab} &= q^n_a\, I_{8,b}\, I_{8,n}\ ,\\
\mathcal{A}^{889}_{ab} &= q^n_a\, (I_{8,b}\, I_{9,n} + I_{8,n}\, I_{9,b})\ ,\\
\mathcal{A}^{999}_{ab} &= q^n_a\, I_{9,b}\, I_{9,n}\ .
\end{aligned}
\tag{6.7}
$$

For inhomogeneous incompressible monoclinic solids, in addition to the universality constraints (4.19), and (4.20) there are the following 16 extra sets of universality constraints (each term must be symmetric in (ab) for $A = 1, 2, 3$, and $B \geq A$):

$$
\begin{aligned}
\mathcal{C}^{8A}_{ab} &= (F^{-1})^A{}_n\, q^n_{a|b} + (F^{-1})^A{}_b\, q^n_{a|b} + q^n_a\, \left[(F^{-1})^B{}_b\, (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb}\, (F^{-1})^A{}_m\right]\ ,\\
\mathcal{C}^{18A}_{ab} &= b^n_a\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right] + q^n_a\, \left[(F^{-1})^A{}_n\, I_{1,b} + (F^{-1})^A{}_b\, I_{1,n}\right]\ ,\\
\mathcal{C}^{19A}_{ab} &= b^n_a\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{28A}_{ab} &= -c^n_a\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right] + q^n_a\, \left[(F^{-1})^A{}_n\, I_{2,b} + (F^{-1})^A{}_b\, I_{2,n}\right]\ ,\\
\mathcal{C}^{29A}_{ab} &= -c^n_a\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{48A}_{ab} &= n_a\, n^n\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right] + q^n_a\, \left[(F^{-1})^A{}_n\, I_{4,b} + (F^{-1})^A{}_b\, I_{4,n}\right]\ ,\\
\mathcal{C}^{49A}_{ab} &= n_a\, n^n\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{58A}_{ab} &= q^n_a\, \left[(F^{-1})^A{}_n\, I_{5,b} + (F^{-1})^A{}_b\, I_{5,n}\right] + \ell^n_a\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right]\ ,\\
\mathcal{C}^{59A}_{ab} &= \ell^n_a\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{68A}_{ab} &= m_a\, m^n\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right] + q^n_a\, \left[(F^{-1})^A{}_n\, I_{6,b} + (F^{-1})^A{}_b\, I_{6,n}\right]\ ,\\
\mathcal{C}^{69A}_{ab} &= m_a\, m^n\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{78A}_{ab} &= k^n_a\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right] + q^n_a\, \left[(F^{-1})^A{}_n\, I_{7,b} + (F^{-1})^A{}_b\, I_{7,n}\right]\ ,\\
\mathcal{C}^{79A}_{ab} &= k^n_a\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{88A}_{ab} &= q^n_a\, \left[(F^{-1})^A{}_n\, I_{8,b} + (F^{-1})^A{}_b\, I_{8,n}\right]\ ,\\
\mathcal{C}^{89A}_{ab} &= q^n_a\, \left[(F^{-1})^A{}_n\, I_{9,b} + (F^{-1})^A{}_b\, I_{9,n}\right]\ ,\\
\mathcal{C}^{8AB}_{ab} &= q^n_a\, \left[(F^{-1})^A{}_n\, (F^{-1})^B{}_b + (F^{-1})^B{}_n\, (F^{-1})^A{}_b\right]\ .
\end{aligned}
\tag{6.8}
$$

6.1 Family 0

In [Yavari and Goriely, 2021] it was shown that for homogeneous incompressible monoclinic solids homogeneous deformations are universal for any three constant unit vectors $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ such that $\mathbf{N}_1$ and $\mathbf{N}_2$ are non-parallel, and $\mathbf{N}_3$ is normal to the plane of $\mathbf{N}_1$ and $\mathbf{N}_2$. We assume that the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$ is θ $(0 < \theta < \frac{\pi}{2})$, and hence, $g = \mathbf{N}_1 \cdot \mathbf{N}_2 = \cos\theta$. In the reference configuration let us choose the Cartesian coordinates (X^1, X^2, X^3) such that[8]

$$\mathbf{N}_1 = \frac{\partial}{\partial X^1}\,, \quad \mathbf{N}_2 = \cos\theta\,\frac{\partial}{\partial X^1} + \sin\theta\,\frac{\partial}{\partial X^2}\,, \quad \mathbf{N}_3 = \frac{\partial}{\partial X^3}\,, \tag{6.9}$$

i.e.,

$$N_1^A = \delta_1^A, \quad N_2^A = \begin{cases} \cos\theta, & A = 1 \\ \sin\theta, & A = 2 \\ 0, & A = 3 \end{cases}. \tag{6.10}$$

For monoclinic solids the constraints (4.39) still hold. Notice that only the last two sets of constraints in (5.8) are nontrivial. The universality constraint $(5.8)_{14}$ implies that

$$C_{MK} N_2^K\, N_2^A\, W_{6,AN} = C_{NK} N_2^K\, N_2^A\, W_{6,AM}\,, \quad M, N = 1,2,3\,. \tag{6.11}$$

Explicitly, we have

$$\begin{aligned} &\left(C_{M1}\cos^2\theta + C_{M2}\cos\theta\sin\theta\right) W_{6,1N} + \left(C_{M1}\cos\theta\sin\theta + C_{M2}\sin^2\theta\right) W_{6,2N} \\ &= \left(C_{N1}\cos^2\theta + C_{N2}\cos\theta\sin\theta\right) W_{6,1M} + \left(C_{N1}\cos\theta\sin\theta + C_{N2}\sin^2\theta\right) W_{6,2M}\,. \end{aligned} \tag{6.12}$$

These are three constraints corresponding to $(M, N) = (1,2), (1,3), (2,3)$, and read

$$\begin{aligned} &\left(C_{12}\cos^2\theta + C_{22}\cos\theta\sin\theta\right) W_{6,11} - \left(C_{11}\cos\theta\sin\theta + C_{12}\sin^2\theta\right) W_{6,22} \\ &\qquad + \left(-C_{11}\cos^2\theta + C_{22}\sin^2\theta\right) W_{6,12} = 0\,, \\ &\left(C_{13}\cos^2\theta + C_{23}\cos\theta\sin\theta\right) W_{6,11} + \left(C_{13}\cos\theta\sin\theta + C_{23}\sin^2\theta\right) W_{6,12} \\ &\qquad - \left(C_{11}\cos^2\theta + C_{12}\cos\theta\sin\theta\right) W_{6,13} - \left(C_{11}\cos\theta\sin\theta + C_{12}\sin^2\theta\right) W_{6,23} = 0\,, \\ &\left(C_{13}\cos^2\theta + C_{23}\cos\theta\sin\theta\right) W_{6,12} + \left(C_{13}\cos\theta\sin\theta + C_{23}\sin^2\theta\right) W_{6,22} \\ &\qquad - \left(C_{12}\cos^2\theta + C_{22}\cos\theta\sin\theta\right) W_{6,13} - \left(C_{12}\cos\theta\sin\theta + C_{22}\sin^2\theta\right) W_{6,23} = 0\,. \end{aligned} \tag{6.13}$$

Suppose $[C_{AB}]$ is diagonal. From $(6.13)_2$, one concludes that $\cos\theta\, W_{6,13} + \sin\theta\, W_{6,23} = 0$, which must hold for any $\theta \in (0, \frac{\pi}{2})$. This implies that $W_{6,13} = W_{6,23} = 0$. Substituting this back into $(6.13)_2$ one concludes that $(C_{13}\cos\theta + C_{23}\sin\theta)(\cos\theta\, W_{6,11} + \sin\theta\, W_{6,12}) = 0$, which implies that $W_{6,11} = W_{6,12} = 0$. Substituting these into $(6.13)_3$ one obtains $\left(C_{13}\cos\theta\sin\theta + C_{23}\sin^2\theta\right) W_{6,22} = 0$, which implies that $W_{6,22} = 0$. Therefore, we have shown that

$$(W_{6,1})_{,A} = (W_{6,2})_{,A} = 0, \quad A = 1,2,3\,. \tag{6.14}$$

The universality constraint $(5.8)_{15}$ implies that

$$\begin{aligned} &\left(C_{M1}\cos\theta + C_{M2}\sin\theta\right)\left(C_1{}^K\cos\theta\, W_{7,KN} + C_2{}^K\sin\theta\, W_{7,KN}\right) \\ &+ \left(C_{MK}C_1{}^K\cos\theta + C_{MK}C_2{}^K\sin\theta\right)\left(\cos\theta\, W_{7,1N} + \sin\theta\, W_{7,2N}\right)\,, \end{aligned} \tag{6.15}$$

is symmetric in (MN). For $M = 1, N = 3$, and diagonal $[C_{AB}]$, the universality constraint is simplified to read

$$C_{11}\left(2\cos\theta\, W_{7,13} + \sin\theta\, W_{7,23}\right) + C_{22}\sin\theta\, W_{7,23} = 0\,. \tag{6.16}$$

[8] In order to make the calculations simpler we have chosen $\alpha = 0$ in (3.65).

This must hold for arbitrary C_{11} and C_{22}, and hence, $W_{7,13} = W_{7,23} = 0$. Substituting this back into the universality constraint and considering simple shear deformations for which $C_{12} = C_{13} = 0$, one concludes that $\cos\theta\, W_{7,11} + 2\sin\theta\, W_{7,12} = 0$, which must hold for an arbitrary θ. Thus, $W_{7,11} = W_{7,12} = 0$. Substituting these back into the constraint for simple shear, one concludes that $W_{7,33} = 0$. For $M = 2, N = 3$, and simple shear deformations for which $C_{12} = C_{13} = 0$, the universality constraint is simplified to read: $C_{23}\,(2C_{22} + C_{33})\sin^2\theta\, W_{7,22} = 0$, which implies that $W_{7,22} = 0$. Therefore, we have concluded that $W_{7,AB} = (W_{7,A})_{,B} = 0$, $A, B = 1, 2, 3$.

For homogeneous deformations and uniform material preferred directions only the last set of constraints in (6.8) are non-trivial and are rewritten in terms of the referential quantities as (for $K, N = 1, 2, 3$)

$$\begin{aligned} & C_{MN}\left[N_1^M\, N_2^A\, \delta_K^B + N_2^M\, N_1^B\, \delta_K^A + N_1^M\, N_2^B\, \delta_K^A + N_2^M\, N_1^A\, \delta_K^B\right] W_{8,AB} \\ & = C_{MK}\left[N_1^M\, N_2^A\, \delta_N^B + N_2^M\, N_1^B\, \delta_N^A + N_1^M\, N_2^B\, \delta_N^A + N_2^M\, N_1^A\, \delta_N^B\right] W_{8,AB}\,. \end{aligned} \tag{6.17}$$

Thus, we have

$$\begin{aligned} & C_{1N}\left(\cos\theta\, W_{8,1K} + \sin\theta\, W_{8,2K}\right) + \left(C_{N1}\cos\theta + C_{N2}\sin\theta\right) W_{8,1K} \\ & = C_{1K}\left(\cos\theta\, W_{8,1N} + \sin\theta\, W_{8,2N}\right) + \left(C_{K1}\cos\theta + C_{K2}\sin\theta\right) W_{8,1N}\,. \end{aligned} \tag{6.18}$$

Eq. (6.18) are three constraints corresponding to $(K, N) = (1, 2)$, $(1, 3)$, and $(2, 3)$, and read

$$\begin{aligned} & (2C_{12}\cos\theta + C_{22}\sin\theta)\, W_{8,11} - 2C_{11}\cos\theta\, W_{8,12} - C_{11}\sin\theta\, W_{8,22} = 0\,, \\ & (2C_{13}\cos\theta + C_{23}\sin\theta)\, W_{8,11} + C_{13}\sin\theta\, W_{8,12} - (2C_{11}\cos\theta + C_{12}\sin\theta)\, W_{8,13} \\ & \qquad - C_{11}\sin\theta\, W_{8,23} = 0\,, \\ & (2C_{13}\cos\theta + C_{23}\sin\theta)\, W_{8,12} - (2C_{12}\cos\theta + C_{22}\sin\theta)\, W_{8,13} + C_{13}\sin\theta\, W_{8,22} \\ & \qquad - C_{12}\sin\theta\, W_{8,23} = 0\,. \end{aligned} \tag{6.19}$$

The above constraints need to be satisfied for an arbitrary matrix $[C_{AB}]$ with unit determinant. For simple shear in the X^2X^3-plane ($C_{12} = C_{13} = 0$), $(6.19)_3$ gives $C_{23}\sin\theta\, W_{8,12} - C_{22}\sin\theta\, W_{8,13} = 0$, which must hold for arbitrary C_{23}, and hence $W_{8,12} = W_{8,13} = 0$. Thus, (6.19) is simplified to read

$$\begin{aligned} & (2C_{12}\cos\theta + C_{22}\sin\theta)\, W_{8,11} - C_{11}\sin\theta\, W_{8,22} = 0\,, \\ & (2C_{13}\cos\theta + C_{23}\sin\theta)\, W_{8,11} - C_{11}\sin\theta\, W_{8,23} = 0\,, \\ & C_{13}\sin\theta\, W_{8,22} - C_{12}\sin\theta\, W_{8,23} = 0\,. \end{aligned} \tag{6.20}$$

For simple shear in the X^1X^2-plane ($C_{13} = C_{23} = 0$), $(6.20)_2$ gives $C_{11}\sin\theta\, W_{8,23} = 0$, which implies that $W_{8,23} = 0$. Thus

$$\begin{aligned} & (2C_{12}\cos\theta + C_{22}\sin\theta)\, W_{8,11} - C_{11}\sin\theta\, W_{8,22} = 0\,, \\ & (2C_{13}\cos\theta + C_{23}\sin\theta)\, W_{8,11} = 0\,, \\ & C_{13}\sin\theta\, W_{8,22} = 0\,. \end{aligned} \tag{6.21}$$

The last two equations imply that $W_{8,11} = W_{8,22} = 0$. Thus, (6.19) implies that $(W_{8,1})_{,A} = (W_{8,2})_{,A} = 0$, $A = 1, 2, 3$.

In summary, the universality constraints give us the following

$$\begin{aligned} & (W_{1,A})_{,B} = (W_{2,A})_{,B} = (W_{5,A})_{,B} = (W_{7,A})_{,B} = 0\,, \quad A, B = 1, 2, 3\,, \\ & (W_{4,1})_{,A} = (W_{6,1})_{,A} = (W_{6,2})_{,A} = (W_{8,1})_{,A} = (W_{8,2})_{,A} = 0\,, \quad A = 1, 2, 3\,. \end{aligned} \tag{6.22}$$

Using a lengthy but standard argument (similar to those of §3.2) one can show that the constraints (6.22) imply the following result.

Proposition 6.1. *For inhomogeneous incompressible nonlinear monoclinic solids, Family 0 deformations are universal for any energy function of the following form*

$$\begin{aligned} W(\mathbf{X}, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) = & \overline{W}(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) + \mathbf{H}(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) \cdot \mathbf{X} \\ & + \widetilde{W}(X^3, I_4, I_6, I_8, I_9)\,, \end{aligned} \tag{6.23}$$

where $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ *are constant unit vectors such that* $\mathbf{N}_3$ *is parallel to the Cartesian* X^3*-axis.*

Remark 6.2. Note that the last term of the energy function in (6.23) has a form identical to that of compressible monoclinic solids (3.84).

6.2 Family 1

In [Yavari and Goriely, 2021] it was shown that for Family 1 deformations of homogeneous incompressible monoclinic solids the universal material preferred directions are

$$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\psi_1(X) \\ \pm\sin\psi_1(X) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\psi_2(X) \\ \pm\sin\psi_2(X) \end{bmatrix}, \tag{6.24}$$

where $\psi_1(X)$ and $\psi_2(X)$ are arbitrary functions such that $\psi_1(X) \neq \psi_2(X)$. The constraints (4.45), (4.47), and (5.13) hold for monoclinic solids as well. The constraints $C^{8A}_{[ab]} = 0$, for $(A, a, b) = (2, 1, 2)$ and $(A, a, b) = (3, 1, 2)$, require vanishing of some lengthy expressions that we do not report here. Neither of these two constraints can be satisfied, and hence

$$\frac{\partial W_8}{\partial Y} = \frac{\partial W_8}{\partial Z} = 0\,. \tag{6.25}$$

All the other universality constraints are satisfied. Therefore, we conclude that $W = \overline{W}(X, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) + \widetilde{W}(X, Y, Z, I_9)$. Noting that the term $\widetilde{W}(X, Y, Z, I_9)$ is mechanically inconsequential, we have proved the following result.

Proposition 6.3. *For inhomogeneous incompressible nonlinear monoclinic solids with any of the universal material preferred directions given in (6.24), Family 1 deformations are universal for any energy function of the form $W = W(X, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9)$.*

6.3 Families 2 and 3

In [Yavari and Goriely, 2021] it was shown that for Family 2 and 3 deformations of homogeneous incompressible monoclinic solids the universal material preferred directions are

$$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\chi_1(R) \\ \pm\sin\chi_1(R) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi_2(R) \\ \pm\sin\chi_2(R) \end{bmatrix}, \tag{6.26}$$

where $\chi_1(R) \neq \chi_2(R)$ are arbitrary functions.

For monoclinic solids, the constraints (4.56), (4.58), and (5.15) still hold. The constraints $C^{8A}_{[ab]} = 0$, for $(A, a, b) = (2, 1, 2)$ and $(A, a, b) = (3, 1, 2)$, require vanishing of some lengthy expressions that we do not report here. Neither of these two constraints can be satisfied, and hence

$$\frac{\partial W_8}{\partial \Theta} = \frac{\partial W_8}{\partial Z} = 0\,. \tag{6.27}$$

All the other universality constraints are satisfied. Therefore we conclude that $W = \overline{W}(R, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) + \widetilde{W}(R, \Theta, Z, I_9)$. Noting that the term $\widetilde{W}(R, \Theta, Z, I_9)$ is mechanically inconsequential, we have proved the following result.

Proposition 6.4. *For inhomogeneous incompressible nonlinear monoclinic solids with any of the universal material preferred directions given in (6.26), Family 2 and 3 deformations are universal for any energy function of the form $W = W(R, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9)$.*

Yavari and Goriely [2021] showed that for homogeneous incompressible monoclinic solids Family 4 deformations are not universal. This is the case for inhomogeneous incompressible monoclinic solids as well.

6.4 Family 5

In [Yavari and Goriely, 2021] the following universal material preferred directions were reported:

$$\text{Class (i)}: \quad \hat{\mathbf{N}}_1 = \begin{bmatrix} \cos\xi_1 \\ \pm\sin\xi_1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} \cos\xi_2 \\ \pm\sin\xi_2 \\ 0 \end{bmatrix}, \quad \xi_1 \neq \xi_2 \,, \tag{6.28}$$

$$\text{Class (ii)}: \quad \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \sin\eta \neq 0 \,, \tag{6.29}$$

$$\text{Class (iii)}: \quad \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \cos\eta \neq 0 \,. \tag{6.30}$$

Noting that $\sin\eta\cos\eta = 0$, Classes (ii) and (iii) become unacceptable ($\hat{\mathbf{N}}_1 \cdot \hat{\mathbf{N}}_2 = 0$), and hence the correct universal material preferred directions are:

$$\hat{\mathbf{N}}_1 = \begin{bmatrix} \cos\xi_1 \\ \pm\sin\xi_1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} \cos\xi_2 \\ \pm\sin\xi_2 \\ 0 \end{bmatrix}, \quad \xi_1 \neq \xi_2 \,. \tag{6.31}$$

This means that the material preferred directions are two families of fibers that are parallel to the (R, Θ) plane and are distributed uniformly in two distinct fixed directions.

In [Yavari, 2021] it was shown that the for Family 5 deformations constraints (4.19) imply that

$$\frac{\partial W_1}{\partial R} = \frac{\partial W_1}{\partial \Theta} = \frac{\partial W_1}{\partial Z} = \frac{\partial W_2}{\partial R} = \frac{\partial W_2}{\partial \Theta} = \frac{\partial W_2}{\partial Z} = 0\,. \tag{6.32}$$

The above relations hold for inhomogeneous monoclinic solids as well. As was shown in §4.6 the universality constraints (4.20) imply that $W_{4,A} = W_{5,A} = W_{6,A} = W_{7,A} = 0$. For the universal material preferred direction (6.31), one can show that

$$\begin{aligned}
& C^{8A}_{[ab]} = 0, \text{ for } (A,a,b) = (1,1,2) \Rightarrow \\
& \qquad C_1^3 \Big\{ C_3 \sin\xi_1 \left[\left(1 + C_2^2\right) \cos\xi_2 + C_2\, C_3 \sin\xi_2 \right] \\
& \qquad + \cos\xi_1 \left[C_2 \left(2 + 2C_2^2 + C_3^2\right) \cos\xi_2 + C_3 \left(2C_2^2 + C_3^2\right) \sin\xi_2 \right] \Big\} = 0\,, \\
& C^{8A}_{[ab]} = 0, \text{ for } (A,a,b) = (2,1,2) \Rightarrow \\
& \qquad C1^3 \Big\{ \cos\xi_1 \left[\left(-2 + C2^2(-6 - 4C2^2 + C3^2)\right) \cos\xi_2 + C_2\, C_3 (-2 - 4C_2^2 + C_3^2) \sin\xi_2 \right] \\
& \qquad + C_3 \sin\xi_1 \left[C_2 \left(1 + C2^2 + 6C3^2\right) \cos\xi_2 + C_3 \left(C_2^2 + 6C_3^2\right) \sin\xi_2 \right] \Big\} = 0\,, \\
& C^{8A}_{[ab]} = 0, \text{ for } (A,a,b) = (3,2,3) \Rightarrow \\
& \qquad C_1^6\, C_3 \Big\{ C_3 \sin\xi_1 \left[\left(1 + C_2^2\right) \cos\xi_2 + C_2\, C_3 \sin\xi_2 \right] \\
& \qquad + \cos\xi_1 \left[C_2 \left(2 + 2C_2^2 + C_3^2\right) \cos\xi_2 + C_3 \left(2C_2^2 + C_3^2\right) \sin\xi_2 \right] \Big\} = 0\,.
\end{aligned} \tag{6.33}$$

None of the above constraints can be satisfied, and hence

$$\frac{\partial W_8}{\partial R} = \frac{\partial W_8}{\partial \Theta} = \frac{\partial W_8}{\partial Z}\,. \tag{6.34}$$

In summary, we have proved the following result.

Proposition 6.5. *For inhomogeneous incompressible nonlinear monoclinic solids Family 5 deformations are not universal.*

Table 4 summarizes our results for inhomogeneous incompressible monoclinic solids.

7 Concluding Remarks

In this paper we studied universal deformations in inhomogeneous anisotropic bodies. Equilibrium equations in the absence of body forces, and arbitrariness of energy functions in a given class of materials impose certain constraints that we call *universality constraints*. We observed that the universality constraints of inhomogeneous solids include those of the corresponding homogeneous solids. In other words, for a given class of materials universal deformations and universal material preferred directions are determined by the universality constraints of the corresponding homogeneous solids. Universal inhomogeneities (position dependence of the energy function) are those inhomogeneities that are consistent with the universality constraints. We characterized the universal inhomogeneities for inhomogeneous compressible transversely isotropic, orthotropic, and monoclinic solids. In the case of inhomogeneous incompressible solids, for each of the six known families of universal deformations, and material preferred directions we characterized the corresponding universal inhomogeneities for inhomogeneous incompressible transversely isotropic, orthotropic, and monoclinic solids. Table 1 summarizes our results for inhomogeneous compressible transversely isotropic, orthotropic, and

Family	Universal Deformations	Universal material preferred directions	Universal inhomogeneities
0	$x^a(X) = F^a{}_A X^A + c^a$	Any two non-parallel constant unit vectors $\hat{\mathbf{N}}_1$, and $\hat{\mathbf{N}}_2$	$W = \overline{W}(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) + \mathbf{H}(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9) \cdot \mathbf{X} + \widetilde{W}(X^3, I_4, I_6, I_8, I_9)$
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X + C_4)} \\ \theta(X,Y,Z) = C_2(Y + C_5) \\ z(X,Y,Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6 \end{cases}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\psi_1(X) \\ \pm\sin\psi_1(X) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\psi_2(X) \\ \pm\sin\psi_2(X) \end{bmatrix}$	$W = W(X, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9)$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2}C_1C_2^2R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6 \end{cases}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\chi_1(R) \\ \pm\sin\chi_1(R) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi_2(R) \\ \pm\sin\chi_2(R) \end{bmatrix}, \quad \chi_1(R) \neq \chi_2(R)$	$W = W(R, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9)$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1C_4 - C_2C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4Z + C_7 \end{cases}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\chi_1(R) \\ \pm\sin\chi_1(R) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi_2(R) \\ \pm\sin\chi_2(R) \end{bmatrix}, \quad \chi_1(R) \neq \chi_2(R)$	$W = W(R, I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9)$

Table 4: *Universal deformations, universal material preferred directions, and universal inhomogeneities for incompressible monoclinic solids for the six known families of universal deformations. For inhomogeneous monoclinic solids Family 4 and Family 5 deformations are not universal. Also, note that $\hat{\mathbf{N}}_3$ is normal to the plane of $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$.*

monoclinic solids. Tables 2, 3, and 4 summarize our results for inhomogeneous incompressible transversely isotropic, orthotropic, and monoclinic solids.

This classification of universal solutions concludes our universal program for hyperelastic materials. It provides a complete collection of solutions that can be used for applications and can be systematically analyzed by stability methods to look for the existence of nearby solutions. In our construction we have assumed that the choice of material preferred directions is consistent with the underlying symmetries of the deformation (e.g. radial fibers for radial deformations). Therefore, our results do not preclude the existence of other universal solutions that would not preserve the underlying symmetry of the deformations. However, we believe that these solutions are unlikely to exist and we conjecture that this classification, like the cases of isotropic incompressible solids, and isotropic anelastic solids is complete.

Acknowledgments

Arash Yavari was supported by ARO W911NF-18-1-0003 and NSF – Grant No. CMMI 1939901. This work was also supported by the Engineering and Physical Sciences Research Council grant EP/R020205/1 to Alain Goriely.

References

J. E. Adkins. Finite deformation of materials exhibiting curvilinear aeolotropy. *Proceedings of the Royal Society of London A*, 229(1176):119–134, 1955a.

J. E. Adkins. Some general results in the theory of large elastic deformations. *Proceedings of the Royal Society of London A*, 231(1184):75–90, 1955b.

H. Demirkoparan and T. J. Pence. Swelling of an internally pressurized nonlinearly elastic tube with fiber reinforcing. *International Journal of Solids and Structures*, 44(11-12):4009–4029, 2007.

H. Demirkoparan and T. J. Pence. Magic angles for fiber reinforcement in rubber-elastic tubes subject to pressure and swelling. *International Journal of Non-Linear Mechanics*, 68:87–95, 2015.

T. C. Doyle and J. L. Ericksen. Nonlinear elasticity. *Advances in Applied Mechanics*, 4:53–115, 1956.

J Dunwoody. On a class of controllable deformations of isotropic incompressible elastic solids with simple material inhomogeneity. *IMA Journal of Applied Mathematics*, 70(1):64–79, 2005a.

J Dunwoody. On a class of controllable deformations of isotropic incompressible elastic solids with simple material inhomogeneity: II. *IMA Journal of Applied Mathematics*, 70(6):937–953, 2005b.

J. L. Ericksen. Deformations possible in every isotropic, incompressible, perfectly elastic body. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 5(6):466–489, 1954.

J. L. Ericksen. Deformations possible in every compressible, isotropic, perfectly elastic material. *Studies in Applied Mathematics*, 34(1-4):126–128, 1955.

J. L. Ericksen and R. S. Rivlin. Large elastic deformations of homogeneous anisotropic materials. *Journal of Rational Mechanics and Analysis*, 3:281–301, 1954.

R. L. Fosdick. Remarks on Compatibility. *Modern Developments in the Mechanics of Continua*, pages 109–127, 1966.

A. Golgoon and A. Yavari. Nonlinear elastic inclusions in anisotropic solids. *Journal of Elasticity*, 130(2): 239–269, 2018a.

A. Golgoon and A. Yavari. Line and point defects in nonlinear anisotropic solids. *Zeitschrift für angewandte Mathematik und Physik*, 69(3):1–28, 2018b.

A Golgoon and A Yavari. On Hashin's hollow cylinder and sphere assemblages in anisotropic nonlinear elasticity. *Journal of Elasticity*, 146(1):65–82, 2021.

C. Goodbrake, A. Yavari, and A. Goriely. The anelastic Ericksen problem: Universal deformations and universal eigenstrains in incompressible nonlinear anelasticity. *Journal of Elasticity*, 142(2):291–381, 2020.

A. Goriely. *The Mathematics and Mechanics of Biological Growth.* Springer Verlag, New York, 2017.

A. Goriely and M. Tabor. Rotation, inversion and perversion in anisotropic elastic cylindrical tubes and membranes. *Proceedings of the Royal Society of London A*, 469(2153), 2013.

M. E. Gurtin. The linear theory of elasticity. In *Handbuch der Physik, Band VIa/2.* Springer-Verlag, Berlin, 1972.

G. A. Holzapfel, T. C. Gasser, and R. W. Ogden. A new constitutive framework for arterial wall mechanics and a comparative study of material models. *J. Elasticity*, 61:1–48, 2000.

W. W. Klingbeil and R. T. Shield. On a class of solutions in plane finite elasticity. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 17(4):489–511, 1966.

J. Lu and P. Papadopoulos. A covariant constitutive description of anisotropic non-linear elasticity. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 51(2):204–217, 2000.

J. E. Marsden and T. J. R. Hughes. *Mathematical Foundations of Elasticity.* Dover Publications, New York, 1994.

A. V. Melnik and A. Goriely. Dynamic fiber reorientation in a fiber-reinforced hyperelastic material. *Mathematics and Mechanics of Solids*, 18(6):634–648, 2013.

J. Merodio and R. W. Ogden. Finite deformation elasticity theory. In *Constitutive Modelling of Solid Continua*, pages 17–52. Springer, 2020.

R. W. Ogden. *Non-Linear Elastic Deformations*. Dover, 1984.

Henry J. Petroski and Donald E. Carlson. Controllable states of elastic heat conductors. *Archive for Rational Mechanics and Analysis*, 31:127–150, 1968.

Edvige Pucci, KR Rajagopal, and Giuseppe Saccomandi. On the determination of semi-inverse solutions of nonlinear Cauchy elasticity: The not so simple case of anti-plane shear. *International Journal of Engineering Science*, 88:3–14, 2015.

G. Y. Qiu and T. J. Pence. Remarks on the behavior of simple directionally reinforced incompressible nonlinearly elastic solids. *Journal of Elasticity*, 49(1):1–30, 1997.

R. S. Rivlin. Large elastic deformations of isotropic materials IV. Further developments of the general theory. *Philosophical Transactions of the Royal Society of London A*, 241(835):379–397, 1948.

R. S. Rivlin. Large elastic deformations of isotropic materials. V. the problem of flexure. *Proceedings of the Royal Society of London A*, 195(1043):463–473, 1949a.

R. S. Rivlin. A note on the torsion of an incompressible highly elastic cylinder. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 45, pages 485–487. Cambridge University Press, 1949b.

G. Saccomandi. Universal solutions and relations in finite elasticity. In *Topics in finite elasticity*, pages 95–130. Springer, 2001.

Giuseppe Saccomandi. On inhomogeneous deformations in finite thermoelasticity. *IMA journal of applied mathematics*, 63(2):131–148, 1999.

M. Singh and A. C. Pipkin. Note on Ericksen's problem. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 16(5):706–709, 1965.

G. F. Smith and R. S. Rivlin. The strain-energy function for anisotropic elastic materials. *Transactions of the American Mathematical Society*, 88(835):175–193, 1958.

A. J. M. Spencer. The formulation of constitutive equation for anisotropic solids. In *Mechanical Behavior of Anisotropic Solids/Comportment Méchanique des Solides Anisotropes*, pages 3–26. Springer, 1982.

A. J. M. Spencer. Modelling of finite deformations of anisotropic materials. In *Large Deformations of Solids: Physical Basis and Mathematical Modelling*, pages 41–52. Springer, 1986.

C Truesdell. *The Elements of Continuum Mechanics*. Springer-Verlag, 1966.

A Yavari. Compatibility equations of nonlinear elasticity for non-simply-connected bodies. *Archive for Rational Mechanics and Analysis*, 209(1):237–253, 2013.

A. Yavari. Universal deformations in inhomogeneous isotropic nonlinear elastic solids. *Proceedings of the Royal Society A*, 477(2253):20210547, 2021.

A. Yavari and A. Goriely. The anelastic Ericksen problem: Universal eigenstrains and deformations in compressible isotropic elastic solids. *Proceedings of the Royal Society A*, 472(2196):20160690, 2016.

A. Yavari and A. Goriely. Universal deformations in anisotropic nonlinear elastic solids. *Journal of the Mechanics and Physics of Solids*, 156:104598, 2021.

A. Yavari and A. Goriely. The universal program of linear elasticity. *Mathematics and Mechanics of Solids*, 2022.

A. Yavari, J. E. Marsden, and M. Ortiz. On the spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics*, 47:85–112, 2006.

A. Yavari, C. Goodbrake, and A. Goriely. Universal displacements in linear elasticity. *Journal of the Mechanics and Physics of Solids*, 135:103782, 2020.