Geometric nonlinear thermoelasticity and the time evolution of thermal stresses*†

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Abstract

In this paper we formulate a geometric theory of nonlinear thermoelasticity that can be used to calculate the time evolution of the temperature and thermal stress fields in a nonlinear elastic body. In particular, this formulation can be used to calculate residual thermal stresses. In this theory the material manifold (natural stress-free configuration of the body) is a Riemannian manifold with a temperature-dependent metric. Evolution of the geometry of the material manifold is governed by a generalized heat equation. As examples, we consider an infinitely long circular cylindrical bar with a cylindrically-symmetric temperature distribution and a spherical ball with a spherically-symmetric temperature distribution. In both cases we assume that the body is made of an arbitrary incompressible isotropic solid. We numerically solve for the evolution of thermal stress fields induced by thermal inclusions in both a cylindrical bar and a spherical ball and compare the linear and nonlinear solutions.

Keywords: Geometric mechanics; nonlinear elasticity; nonlinear thermoelasticity; thermal stresses; coupled heat equation; referential evolution; evolving metric.

Contents

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1 Introduction

Following the seminal work of Fourier [1822] on thermal conduction, Duhamel [1836, 1837] was the first to study the thermo-mechanical behavior of solids. He considered the superposition of the uncoupled elasticity and thermal conduction problems and in this framework solved for radial temperature fields in the spherical and cylindrical geometries. The linear heat conduction problem in solids—uncoupled from elasticity—has since been studied extensively and several exact solutions have been found, in particular for the spherical and cylindrical geometries (see for example [Carslaw and Jaeger, 1986]). Biot [1956] derived the governing equations for coupled linear thermoelasticity by combining the elasticity governing equations with the first and the second laws of thermodynamics. He also proved a variational principle based on his generalization of the definition of the free energy to nonuniform systems [Biot, 1955]. Boley and Weiner [1960] presented an extensive account on the theory of linear thermoelasticity and its applications. They introduced the coupling between thermal conduction and elasticity by considering the temperature dependence of the specific free energy and using a power series expansion of it in terms of strain invariants and temperature. Limiting the expansion to quadratic terms, they derived the coupled linearized thermoelasticity equations. By extending this expansion to the cubic order, Dillon Jr [1962] derived a nonlinear version of the theory including the deviatoric components of strain and solved for the temperature field resulting from the torsion of a bar through the thermoelastic coupling. Jaeger [1945] numerically solved for the thermal stresses in circular cylinders.

Based on the contributions of Green et al. [1970] on thermo-mechanical constraints, Trapp [1971] studied an incompressible elastic solid reinforced by inextensible cords and obtained several exact controllable solutions for stress and displacement fields. Later, Erbe [1974] introduced a temperature-dependent incompressibility condition to study the thermoelastic behavior of rubber. In this paper we will discuss this modified incompressibility condition in our geometric formulation.

Petroski and Carlson [1968, 1970] explored universal/controllable states of elastic heat conductors and provided a few exact homogeneous solutions to the equations of nonlinear thermoelasticity under steady-state conditions in the absence of body force and heat supply. For this class of solutions, Petroski [1975a,b] studied the deformation/heating of spherical sectors and the torsion/radial heating of a cylinder. Rajagopal [1995] and Rajagopal et al. [1996] investigated inhomogeneous deformations in non-linear thermoelasticity and found, for a generalized neo-Hookean material with elastic properties that depend on stretch and temperature, exact solutions that exhibit a boundary-layer structure (i.e., the deformation in the core is homogeneous (or inhomogeneous) while in a layer adjacent to the boundary it is inhomogeneous (or homogeneous)). Tanigawa and Takeuti [1982a,b] and Tanigawa et al. [1983] studied the three-dimensional linear coupled thermoelasticity and carried out numerical computations for hollow spheres and cylinders. Jabbari et al. [2010, 2011] studied coupled linear thermoelasticity and gave exact solutions in the case of a radially-symmetric problem in both spherical and cylindrical geometries. Shahani and Nabavi [2007] and Shahani and Bashusqeh [2014, 2013] analytically solved the steady-state and the dynamical problem of uncoupled linear thermoelasticity in a thick-walled cylinder and a sphere.

Under non-uniform temperature fields, thermal stresses may arise. Because a body cannot leave the three-dimensional Euclidean ambient space, it is therefore constrained to deform in its Euclidean geometry and this can lead to thermal stresses. In the context of nonlinear thermoelasticity, the study of thermal stresses goes back to Stojanović et al. [1964]; Stojanović [1969]. They suggested a multiplicative decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_T$ based on a conceptual stress release of the material configuration from its current state under a non-uniform temperature field to a stress-free state. $\boldsymbol{F}$ is decomposed into a thermal relaxation $\boldsymbol{F}_T$ followed by an elastic deformation $\boldsymbol{F}_e$. For an orthotropic material, they gave a formula for $\boldsymbol{F}_T$ that then allows for the calculation of thermal stresses for a static temperature field [Stojanović, 1972; Lu and Pister, 1975; Vujošević and Lubarda, 2002; Lubarda, 2004]. However, to the best of our knowledge, there has not been any previous study in the literature on the evolution of thermal stresses in nonlinear thermoelasticity. We should also mention that the decomposition of deformation gradient is not uniquely determined and makes use of a mathematically ambiguous imaginary relaxed state of the material. Ozakin and Yavari [2010] formulated a geometric theory of thermal stresses by considering a temperature-dependent material configuration through a metric that explicitly depends on temperature. This geometric framework interpreted the imaginary intermediate relaxed configuration for non-uniform temperature fields as a Riemannian manifold not necessarily embedded in the Euclidean ambient space and allowed for a mathematically meaningful formulation of the multiplicative decomposition of deformation gradient. It also provided a systematic approach to finding stress-free temperature fields in non-linear thermoelasticity.

In this paper, following the idea of Ozakin and Yavari [2010], we formulate a geometric theory of nonlinear thermoelasticity by considering an evolving material metric that explicitly depends on temperature and the thermal expansion properties of the medium to study the coupling of heat conduction with elasticity (see [Yavari, 2010; Yavari and Goriely, 2013] for other examples of an evolving material metric). In Section 2, we establish the connection between the evolution of the geometry of the material manifold on the one hand and the temperature field and the thermal expansion properties of the medium on the other hand. We explicitly write the temperature-dependent material metric and generalize it to the thermally inhomogeneous anisotropic case. Then we discuss a systematic approach following from Riemannian geometry to find the stress-free temperature fields in nonlinear elasticity. In Section 3, we establish the theoretical framework for geometric nonlinear thermoelasticity. In doing so, we review the kinematics of nonlinear elasticity and derive the governing equations of motion in the scope of our theory. We also derive the response functions for the hyperelastic constitutive model and find the generalized nonlinear thermoelastic coupling equation based on the laws of thermodynamics. In Section 4 we illustrate the capability of the theory by solving for the static and time-dependent thermal stresses induced by thermal inclusions in both a circular cylindrical bar and a spherical ball in the case of arbitrary incompressible isotropic elastic solids with a radially-symmetric temperature distribution.

2 The material manifold in nonlinear thermoelasticity

2.1 The material metric

Let B be a three-dimensional body identified with a three-dimensional Riemannian manifold $(\mathcal{B}, \boldsymbol{G})$—the material manifold. Also, let $(\mathcal{S}, \boldsymbol{g})$ be a three-dimensional Riemannian manifold—the ambient space manifold. We adopt the standard convention to denote objects and indices by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$) and by lowercase characters in the spatial manifold $\mathcal{S}$ (e.g., $x \in \mathcal{S}$). Let $\{X^A\}$ and $\{x^a\}$ be local coordinate charts on $\mathcal{B}$ and $\mathcal{S}$, respectively. Also, let $\partial_A = \frac{\partial}{\partial X^A}$ and $\partial_a = \frac{\partial}{\partial x^a}$ denote the local coordinate bases corresponding to $\{X^A\}$ and $\{x^a\}$, respectively, and let $\{dX^A\}$ and $\{dx^a\}$ denote the corresponding dual bases. We also adopt Einstein's repeated index summation convention.

In this paper, for the purpose of formulating a geometric theory of nonlinear thermoelasticity and coupling heat conduction with elasticity, we define, following Ozakin and Yavari [2010], a material metric that explicitly depends on the thermal expansion properties of the material and the absolute temperature T, i.e., $\boldsymbol{G} = \boldsymbol{G}(X, T)$. Let $\boldsymbol{G}_0$ be a metric for the configuration of $\mathcal{B}$ corresponding to a given stress-free temperature field $T_0 = T_0(X)$ (cf. Section 2.2). The manifold $(\mathcal{B}, \boldsymbol{G}_0)$ is flat and by Riemann's theorem there exists a local coordinate chart $\{Y^A\}$ in which the metric is Euclidean, i.e.

$$\boldsymbol{G}_0(X) = \delta_{AB} dY^A \otimes dY^B . \tag{2.1}$$

In order to represent the thermal expansion properties, we introduce three real-valued functions of temperature and position $\{\omega_A(X, T)\}$, $A = 1, 2, 3$, to describe the thermal expansion properties of the material in the directions $\{\frac{\partial}{\partial Y^A}\}$ at every material point X. Following Ozakin and Yavari [2010], we define the temperature-dependent material metric as

$$\boldsymbol{G}(X, T) = \sum_K e^{2\omega_K(X,T)} dY^K \otimes dY^K . \tag{2.2}$$

Let $\zeta_{(K)} : I \to \mathcal{B}$ (where $I \subset \mathbb{R}$ is an interval and $K = 1, 2, 3$) be a curve in $(\mathcal{B}, \boldsymbol{G})$ such that, in the coordinate chart $\{Y^A\}$, we have $(\zeta_{(K)})^A(s) = (\delta^A{}_K)s$, for $s \in I$. At a point $X = \zeta_{(K)}(s)$, the arc length of the curve $\zeta_{(K)}$ measures the length in the direction $\frac{\partial}{\partial Y^K}$. It is given by (no summation on K)

$$
\begin{aligned}
dL[\zeta_{(K)}](s,T) =& \sqrt{\boldsymbol{G}(\zeta_{(K)}(s),T)\left(\dot{\zeta}_{(K)}(s), \dot{\zeta}_{(K)}(s)\right)}ds \\
=& \sqrt{G_{KK}(\zeta_{(K)}(s),T)}ds \\
=& e^{\omega_K(X,T)}ds\,.
\end{aligned}
\tag{2.3}
$$

Therefore

$$
\frac{\partial(dL[\zeta_{(K)}])}{\partial T} = \frac{\partial \omega_K}{\partial T} dL[\zeta_{(K)}]\,,
\tag{2.4}
$$

and one can read the linear thermal expansion coefficient of the material in the direction $\frac{\partial}{\partial Y^K}$ as

$$
\alpha_K(X,T) = \frac{\partial \omega_K}{\partial T}(X,T)\,.
\tag{2.5}
$$

Note that for the stress-free temperature field T_0, there is no stretching of the material and hence $\omega_K(X, T_0) = 0$ for $K = 1, 2, 3$. Therefore, $\boldsymbol{G}(X, T_0) = \boldsymbol{G}_0(X)$. Following (2.2), one can equivalently represent $\boldsymbol{G}$ in $\{Y^A\}$ as

$$
\boldsymbol{G}(X,T) = \sum_{K,L} e^{2\omega_K} \delta^K{}_L \frac{\partial}{\partial Y^K} \otimes dY^L = \sum_K e^{2\omega_K} \frac{\partial}{\partial Y^K} \otimes dY^K\,.
\tag{2.6}
$$

Let the change of basis between $\{X^A\}$ and $\{Y^A\}$ be written as

$$
dY^K = A^K{}_J dX^J \quad \text{and} \quad \frac{\partial}{\partial Y^K} = (A^{-1})^I{}_K \frac{\partial}{\partial X^I}\,.
\tag{2.7}
$$

Then

$$
\boldsymbol{G}(X,T) = \left(\sum_K (A^{-1})^I{}_K e^{2\omega_K} A^K{}_J\right) \frac{\partial}{\partial X^I} \otimes dX^J\,.
\tag{2.8}
$$

Let $\boldsymbol{\omega}$ be the $\binom{1}{1}$-tensor with the following matrix representation in $\{Y^A\}$

$$
\hat{\boldsymbol{\omega}} = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}.
\tag{2.9}
$$

Now its representation in $\{X^A\}$ reads off as

$$
\omega^I{}_J = \sum_K (A^{-1})^I{}_K \omega_K A^K{}_J\,,
\tag{2.10}
$$

and $e^{\boldsymbol{\omega}}$ has the following representation in $\{X^A\}$

$$
(e^{2\boldsymbol{\omega}})^I{}_J = \sum_K (A^{-1})^I{}_K e^{2\omega_K} A^K{}_J\,.
\tag{2.11}
$$

Therefore

$$
\boldsymbol{G}(X,T) = \left(e^{2\boldsymbol{\omega}}\right)^I{}_J \frac{\partial}{\partial X^I} \otimes dX^J\,.
\tag{2.12}
$$

Hence

$$
\boldsymbol{G}(X,T) = (G_0)_{IK} \left(e^{2\boldsymbol{\omega}}\right)^K{}_J dX^I \otimes dX^J\,,
\tag{2.13}
$$

where $(G_0)_{IK}$ are the components of $\boldsymbol{G}_0$ in $\{X^A\}$. It finally follows that

$$
\boldsymbol{G}(X,T) = \boldsymbol{G}_0(X) e^{2\boldsymbol{\omega}(X,T)}\,.
\tag{2.14}
$$

As noted earlier, for the stress-free temperature field $T_0(X)$ we have $\boldsymbol{G}(X,T_0)=\boldsymbol{G}_0(X)$, which corresponds to $\boldsymbol{\omega}(X,T_0(X))=\mathbf{0}$. The Riemannian volume form associated with this metric is

$$dV(X,\boldsymbol{G})=\sqrt{\det\boldsymbol{G}}\,dX^1\wedge dX^2\wedge dX^3=\sqrt{\det\boldsymbol{G}_0}\,e^{\mathrm{tr}(\boldsymbol{\omega}(X,T))}dX^1\wedge dX^2\wedge dX^3=e^{\mathrm{tr}(\boldsymbol{\omega}(X,T))}dV_0(X)\,, \tag{2.15}$$

where dV_0 is the Riemannian volume form associated with the metric $\boldsymbol{G}_0$. Thus

$$\frac{d}{dT}dV(X,\boldsymbol{G})=\frac{\partial(\mathrm{tr}(\boldsymbol{\omega}(X,T)))}{\partial T}e^{\mathrm{tr}(\boldsymbol{\omega}(X,T))}dV_0(X)=\frac{\partial(\mathrm{tr}(\boldsymbol{\omega}(X,T)))}{\partial T}dV(X,T)\,. \tag{2.16}$$

Therefore, the volumetric thermal expansion coefficient $\beta(X,T)$ of the material reads

$$\beta(X,T)=\frac{\partial}{\partial T}\left[\mathrm{tr}(\boldsymbol{\omega}(X,T))\right]\,. \tag{2.17}$$

If one further assumes that the material is thermally isotropic, the matrix $\boldsymbol{\omega}$ reduces to a scalar function ω times the identity matrix, and one recovers the metric introduced for the isotropic case by Ozakin and Yavari [2010], i.e., $\boldsymbol{G}(X,T)=\boldsymbol{G}_0(X)e^{2\omega(X,T)}$. The Riemannian volume form associated with this metric is

$$dV(X,\boldsymbol{G})=e^{3\omega(X,T)}dV_0(X)\,, \tag{2.18}$$

and the thermal expansion coefficient $\alpha(X,T)$ reads

$$\alpha(X,T)=\frac{1}{3}\beta(X,T)=\frac{\partial\omega(X,T)}{\partial T}\,. \tag{2.19}$$

Therefore

$$\omega(X,T)=\int_{T_0}^{T}\alpha(X,\tau)d\tau\,. \tag{2.20}$$

Remark 2.1. The material metric $\boldsymbol{G}$ is defined in such a way to include the thermal expansion properties of the material in order to capture any change of shape due to the temperature field. In other words, the geometry of the material manifold explicitly depends on the material thermal expansion properties and the temperature field; it is not purely kinematic. This is in contrast with the material manifold of solids with distributed defects, which is purely kinematic and only depends on the density of defects [Yavari and Goriely, 2012a,b, 2013, 2014a,b].

2.2 Riemannian geometry and the stress-free temperature fields

In the following we tersely review some elements from Riemannian geometry about linear connections and curvature on a manifold. For more details see [Hicks, 1965; do Carmo, 1992; Lee, 1997]. A linear connection on a manifold $\mathcal{B}$ is a map $\nabla:\mathcal{X}(\mathcal{B})\times\mathcal{X}(\mathcal{B})\to\mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\mathcal{X}(\mathcal{B}),\forall f,g\in C^\infty(\mathcal{B})$, we have

$$\nabla_{\boldsymbol{X}}(\boldsymbol{Y}+\boldsymbol{Z})=\nabla_{\boldsymbol{X}}\boldsymbol{Y}+\nabla_{\boldsymbol{X}}\boldsymbol{Z}, \tag{2.21a}$$

$$\nabla_{f\boldsymbol{X}+g\boldsymbol{Y}}\boldsymbol{Z}=f\nabla_{\boldsymbol{X}}\boldsymbol{Z}+g\nabla_{\boldsymbol{Y}}\boldsymbol{Z}, \tag{2.21b}$$

$$\nabla_{\boldsymbol{X}}(f\boldsymbol{Y})=f\nabla_{\boldsymbol{X}}\boldsymbol{Y}+(\boldsymbol{X}f)\boldsymbol{Y}\,. \tag{2.21c}$$

The vector field $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ is called the covariant derivative of $\boldsymbol{Y}$ along $\boldsymbol{X}$. In a local chart $\{X^A\}$, we have $\nabla_{\partial_A}\partial_B\in\mathcal{X}(\mathcal{B})$, and hence there exist scalars $\Gamma^C{}_{AB}$, called the Christoffel symbol of the connection, such that $\nabla_{\partial_A}\partial_B=\Gamma^C{}_{AB}\partial_C$. We define the Lie bracket of two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ as the vector denoted by $[\boldsymbol{X},\boldsymbol{Y}]$ such that $\forall f\in C^\infty(\mathcal{B})$, we have

$$[\boldsymbol{X},\boldsymbol{Y}]f=\boldsymbol{X}\left(\boldsymbol{Y}f\right)-\boldsymbol{Y}\left(\boldsymbol{X}f\right)\,. \tag{2.22}$$

A linear connection is said to be compatible with a metric $\boldsymbol{G}$ on the manifold if

$$\boldsymbol{X}\left(\boldsymbol{G}\left(\boldsymbol{Y},\boldsymbol{Z}\right)\right)=\boldsymbol{G}\left(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\right)+\boldsymbol{G}\left(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}\right)\,. \tag{2.23}$$

It can be shown that ∇ is compatible with $\boldsymbol{G}$ if and only if $\nabla\boldsymbol{G}=\boldsymbol{0}$, which in components reads

$$G_{AB|C}=\frac{\partial G_{AB}}{\partial X^C}-\Gamma^K{}_{CA}G_{KB}-\Gamma^K{}_{CB}G_{AK}=0. \tag{2.24}$$

The torsion of a connection is defined as

$$\boldsymbol{T}(\boldsymbol{X},\boldsymbol{Y})=\nabla_{\boldsymbol{X}}\boldsymbol{Y}-\nabla_{\boldsymbol{Y}}\boldsymbol{X}-[\boldsymbol{X},\boldsymbol{Y}]. \tag{2.25}$$

In components $T^A{}_{BC}=\Gamma^A{}_{BC}-\Gamma^A{}_{CB}$. ∇ is said to be symmetric if it is torsion-free, i.e., $\nabla_{\boldsymbol{X}}\boldsymbol{Y}-\nabla_{\boldsymbol{Y}}\boldsymbol{X}=[\boldsymbol{X},\boldsymbol{Y}]$. It can be shown that on any Riemannian manifold $(\mathcal{B},\boldsymbol{G})$ there is a unique linear connection ∇ that is both compatible with $\boldsymbol{G}$ and is torsion-free. This result is the fundamental theorem of Riemannian geometry and such a connection is called the Levi-Civita connection. It can be shown that Christoffel symbol of the Levi-Civita connection associated with the metric $\boldsymbol{G}$ reads

$$\Gamma^A{}_{BC}=\frac{1}{2}\sum_K G^{AK}\left(\partial_C G_{KB}+\partial_B G_{KC}-\partial_K G_{BC}\right). \tag{2.26}$$

We denote in the remainder of the paper the Levi-Civita connections of the material manifold $(\mathcal{B},\boldsymbol{G})$ and the ambient space $(\mathcal{S},\boldsymbol{g})$ by ∇ and $\bar{\nabla}$, respectively. The curvature tensor $\boldsymbol{\mathcal{R}}$ of a Riemannian manifold $(\mathcal{B},\boldsymbol{G})$ is given in terms of the Levi-Civita connection ∇ by

$$\boldsymbol{\mathcal{R}}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}=\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z}-\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z}-\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}, \tag{2.27}$$

for $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\mathcal{X}(\mathcal{B})$. In components

$$\mathcal{R}^A{}_{BCD}=dX^A(\boldsymbol{\mathcal{R}}(\partial_C,\partial_D)\partial_B)=\frac{\partial\Gamma^A{}_{DB}}{\partial X^C}-\frac{\partial\Gamma^A{}_{CB}}{\partial X^D}+\Gamma^A{}_{CK}\Gamma^K{}_{DB}-\Gamma^A{}_{DK}\Gamma^K{}_{CB}. \tag{2.28}$$

In order to study the stress-free temperature fields of a body, let the material manifold $\mathcal{B}$ be given an arbitrary temperature field $T=T(X)$ and assume that there exists an embedding φ of the material manifold into the ambient space. The embedded configuration lies in the spatial manifold and hence distances are measured by the metric $\boldsymbol{g}$. As noted earlier, any change in the temperature field affects the geometry of the material manifold and a configuration of the body is stress-free if no stretch occurs as a consequence of such a temperature field. A measure of stretch is provided by the right Cauchy-Green tensor, which is the pull-back of the spatial metric by the embedding, i.e., $\boldsymbol{C}^\flat=\varphi^*\boldsymbol{g}$. Therefore, a temperature field is stress free when the material metric agrees with the pullback of the spatial metric by the embedding of the material manifold in the ambient space, i.e., the material manifold is isometrically embedded in the ambient space. If the ambient space is flat, a temperature field is stress free if and only if the metric is flat. A classical result from Riemannian geometry addresses this issue following from the fact that a Riemannian manifold is locally flat if and only if its curvature tensor vanishes identically (see for example [Lee, 1997]). Therefore, a stress-free temperature field exists if and only if the curvature tensor $\boldsymbol{\mathcal{R}}$ of the metric $\boldsymbol{G}(X,T)$ is identically zero (assuming that the body is simply connected). We can therefore find all the stress-free temperature fields of a simply-connected body by solving for T the tensorial equation $\boldsymbol{\mathcal{R}}(T(X))=\boldsymbol{0}$. See [Ozakin and Yavari, 2010] for a more detailed discussion on stress-free temperature fields.

3 Geometric nonlinear thermoelasticity

3.1 Kinematics of nonlinear elasticity

We review in the following some elements of the geometric formulation of the kinematics of nonlinear elasticity. For more details, see [Marsden and Hughes, 1983]. A configuration of $\mathcal{B}$ is a smooth embedding $\varphi:\mathcal{B}\to\mathcal{S}$. We denote the set of all configurations of $\mathcal{B}$ by $\mathcal{C}$. A motion of $\mathcal{B}$ is a smooth curve $t\in\mathbb{R}^+\to\varphi_t\in\mathcal{C}$ that assigns a spatial point $x=\varphi(X,t)=\varphi_t(X)$ at any time t to every material point X. For a fixed $X\in\mathcal{B}$ we write $\varphi_X(t)=\varphi(X,t)$. The material velocity of the motion is defined as the mapping

$$\boldsymbol{V}:\mathcal{B}\times\mathbb{R}^+\to T\mathcal{S}\ \text{ such that }\ \boldsymbol{V}(X,t)=d_t\varphi_X\left[\frac{\partial}{\partial t}\right]\in T_{\varphi_X(t)}\mathcal{S}. \tag{3.1}$$

The spatial velocity is defined as the mapping

$$\boldsymbol{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S} \;\; \text{such that} \;\; \boldsymbol{v}(x,t) = \boldsymbol{V}(\varphi_t^{-1}(x),t) \in T_x\mathcal{S} \,. \tag{3.2}$$

The material acceleration is defined as the mapping

$$\boldsymbol{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S} \;\; \text{such that} \;\; \boldsymbol{A}(X,t) = \bar{\nabla}_{\boldsymbol{V}(X,t)}\boldsymbol{V}(X,t) \in T_{\varphi(X)}\mathcal{S} \,. \tag{3.3}$$

In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc}V^bV^c$, where $\gamma^a{}_{bc}$ denote the Christoffel symbols of the connection $\bar{\nabla}$ in the local coordinate chart $\{x^a\}$, i.e., $\bar{\nabla}_{\partial_b}\partial_c = \gamma^a{}_{bc}\partial_a$. The spatial acceleration is defined as the mapping

$$\boldsymbol{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S} \;\; \text{such that} \;\; \boldsymbol{a}(x,t) = \boldsymbol{A}(\varphi_t^{-1}(x),t) \in T_x\mathcal{S} \,. \tag{3.4}$$

The deformation gradient $\boldsymbol{F}$ is defined as the tangent map of $\varphi_t : \mathcal{B} \to \mathcal{S}$, i.e.

$$\boldsymbol{F}(X,t) = d\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{S} \,. \tag{3.5}$$

The adjoint $\boldsymbol{F}^{\mathsf{T}}$ of $\boldsymbol{F}$ is defined by

$$\begin{gathered} \boldsymbol{F}^{\mathsf{T}}(X,t) : T_{\varphi(X)}\mathcal{S} \to T_X\mathcal{B} \,, \\ \forall (\boldsymbol{W},\boldsymbol{w}) \in (T_X\mathcal{B} \times T_{\varphi_t(X)}\mathcal{S}) : \boldsymbol{g}(\boldsymbol{F}\boldsymbol{W},\boldsymbol{w}) = \boldsymbol{G}(\boldsymbol{W},\boldsymbol{F}^{\mathsf{T}}\boldsymbol{w}) \,. \end{gathered} \tag{3.6}$$

In components, $(F^{\mathsf{T}})^A{}_a = g_{ab}F^b{}_BG^{AB}$. The Jacobian of the motion J relates the material and spatial Riemannian volume elements $dV(X,\boldsymbol{G})$ and $dv(x,\boldsymbol{g})$ by

$$dv(\varphi_t(X),\boldsymbol{g}) = J(X,\varphi_t,\boldsymbol{G},\boldsymbol{g})dV(X,\boldsymbol{G}) \,. \tag{3.7}$$

It can be shown that [Marsden and Hughes, 1983]

$$J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F} = e^{-\operatorname{tr}(\boldsymbol{\omega}(X,T))}\sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}_0}} \det \boldsymbol{F} \,. \tag{3.8}$$

Remark 3.1. In our geometric framework, the deformation gradient $\boldsymbol{F}$ is purely elastic and any temperature evolution of the body is reflected in the geometry of the material manifold through the temperature dependence of $\boldsymbol{G}$. As a matter of fact, in the framework of the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}_e\boldsymbol{F}_T$ introduced by Stojanović et al. [1964], $\boldsymbol{F}_e = \boldsymbol{F}\boldsymbol{F}_T^{-1}$ corresponds to our purely elastic deformation gradient. For thermally isotropic materials, Lu and Pister [1975] suggested $\boldsymbol{F}_T = e^{\int_{T_0}^{T}\alpha(X,\tau)d\tau}\boldsymbol{I}$, which gives an equation identical to (3.8), i.e., $\det \boldsymbol{F}_e = e^{-3\int_{T_0}^{T}\alpha(X,\tau)d\tau}\det \boldsymbol{F}$. Given the purely elastic character of the deformation gradient in our formulation, the incompressibility condition is simply written as $J=1$. In terms of the flat metric $\boldsymbol{G}_0$, the incompressibility condition reads $\sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}_0}}\det \boldsymbol{F} = e^{\operatorname{tr}(\boldsymbol{\omega}(X,T))}$, which is similar to the condition for incompressible thermoelasticity introduced by Erbe [1974] written as $\det \boldsymbol{F} = \frac{1}{g(T)}$, where $g(T) \neq 0$ and is identical to the incompressibility condition suggested by Lu and Pister [1975] in the case of thermally isotropic materials, i.e., $\det \boldsymbol{F} = e^{-\int_{T_0}^{T}3\alpha(X,\tau)d\tau}$.

The right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C}(X,t) = \boldsymbol{F}^{\mathsf{T}}(X,t)\boldsymbol{F}(X,t) : T_X\mathcal{B} \to T_X\mathcal{B} \,. \tag{3.9}$$

In components, $C^A{}_B = G^{AK}F^a{}_KF^b{}_Bg_{ab}$. We note that $\boldsymbol{C}^\flat$ agrees with the pull-back of the spatial metric $\boldsymbol{g}$ by φ_t , i.e., $\boldsymbol{C}^\flat = \varphi_t{}^*\boldsymbol{g}$, where $^\flat$ denotes the flat operator. The material strain tensor is defined as the difference between the pull back of the spatial metric and the material metric, i.e.

$$\boldsymbol{E} = \frac{1}{2}\left(\varphi_t{}^*\boldsymbol{g} - \boldsymbol{G}\right) = \frac{1}{2}(\boldsymbol{C}^\flat - \boldsymbol{G}). \tag{3.10}$$

In components, $E_{AB} = \frac{1}{2}\left(C_{AB} - G_{AB}\right)$.

3.2 The governing equations of motion

Conservation of mass. We start by revisiting the transport theorem for the case of the temperature-dependent metric (2.14) considered in this paper.

Lemma 3.1 (Transport theorem)**.** *If f is a real-valued smooth function of position $x \in \varphi_t(\mathcal{B})$, temperature T, and time,i.e., $f = f(x, T, t)$, and $\mathcal{U}$ is an arbitrary open set in $\mathcal{B}$, then*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} f dv = \int_{\varphi_t(\mathcal{U})} \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial t} + div(f\boldsymbol{v})\right] dv\,, \tag{3.11}$$

where div denotes the spatial divergence operator.

Proof. We have by a change of variable $x = \varphi_t(X)$

$$\begin{aligned}
\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} f(\varphi_t(X), T, t) dv(\varphi_t(X), \boldsymbol{g}) &= \frac{d}{dt}\int_{\mathcal{U}} f(\varphi_t(X), T, t) J(X, \varphi_t, \boldsymbol{G}, \boldsymbol{g}) dV(X, \boldsymbol{G}) \\
&= \frac{d}{dt}\int_{\mathcal{U}} f(\varphi_t(X), T, t)\sqrt{\det \boldsymbol{C}^\flat}(X) dX^1 \wedge dX^2 \wedge dX^3 \\
&= \int_{\varphi_t(\mathcal{U})} \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial t} + \langle \boldsymbol{d}f, \boldsymbol{v}\rangle + \frac{1}{2} f \operatorname{tr}_{\boldsymbol{C}^\flat}\left(\frac{d\boldsymbol{C}^\flat}{dt}\right)\right] dv \\
&= \int_{\varphi_t(\mathcal{U})} \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial t} + \langle \boldsymbol{d}f, \boldsymbol{v}\rangle + f \operatorname{div}\boldsymbol{v}\right] dv\,,
\end{aligned}$$

where we have used the identity

$$\frac{d}{dt}\left[\det \boldsymbol{A}(t)\right] = \det \boldsymbol{A}(t) \operatorname{tr}\left[\boldsymbol{A}^{-1}(t)\frac{d}{dt}\boldsymbol{A}(t)\right],$$

and where $\boldsymbol{d}f(x) = \frac{\partial f}{\partial x^a} dx^a$ is the differential of a function, $\operatorname{tr}_{\boldsymbol{C}^\flat}$ is the trace taken with respect to the metric $\boldsymbol{C}^\flat$, and $\langle\,.\,,\,.\,\rangle$ is the natural pairing of a one-form and a vector. $\square$

Let ρ and ϱ, respectively, denote the material and spatial mass densities. For any open set $\mathcal{U}$ in $\mathcal{B}$, conservation of mass can be written as

$$\int_{\varphi_t(\mathcal{U})} \varrho dv = \int_{\mathcal{U}} \rho dV\,. \tag{3.12}$$

By applying the change of variable $X = \varphi_t^{-1}(x)$ to the right hand-side of (3.12) and by the arbitrariness of $\mathcal{U}$, we find that conservation of mass is equivalent to

$$\rho = J\varrho. \tag{3.13}$$

Note that even though both ρ and dV can be time dependent, the material mass form $dM = \rho\, dV$ is time-independent, and since $dV = dV(X, \boldsymbol{G})$, it follows that the material mass density should depend on the position and the material metric as well, i.e.

$$\rho = \rho(X, \boldsymbol{G})\,. \tag{3.14}$$

Note also that since $J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F} = J(X, \varphi_t, \boldsymbol{G}, \boldsymbol{g})$ it follows from (3.13) that[1]

$$\varrho = \varrho(X, \varphi_t, \boldsymbol{G}, \boldsymbol{g}). \tag{3.15}$$

Equation (3.12) along with the fact that dM is constant yield

$$\frac{d}{dt}\left(\int_{\varphi_t(\mathcal{U})} \varrho dv\right) = 0\,. \tag{3.16}$$

[1]Note that both ρ and ϱ depend implicitly on temperature via the material metric $\boldsymbol{G}$. Therefore, we write $\frac{d\rho}{dT} = \frac{\partial \rho}{\partial \boldsymbol{G}} : \frac{\partial \boldsymbol{G}}{\partial T}$ and $\frac{d\varrho}{dT} = \frac{\partial \varrho}{\partial \boldsymbol{G}} : \frac{\partial \boldsymbol{G}}{\partial T}$.

Using (3.11) and arbitrariness of $\mathcal{U}$, (3.16) is equivalent to

$$\frac{d\varrho}{dT}\frac{\partial T}{\partial t} + \operatorname{div}(f\boldsymbol{v}) = 0\,. \tag{3.17}$$

Combining (3.13) and (3.17), the conservation of mass in the material manifold reads[2]

$$\frac{d\rho}{dT} + \beta\rho = 0\,. \tag{3.18}$$

It follows that

$$\rho(X,\boldsymbol{G}) = \rho_0(X)e^{-\operatorname{tr}(\omega)(X,T)} = \rho_0(X)e^{-\frac{1}{2}\operatorname{tr}(\boldsymbol{G})(X,T)}\,, \tag{3.19}$$

where for the stress-free material metric $\boldsymbol{G}_0(X)$ we have

$$\rho_0(X) = \rho(X,\boldsymbol{G}_0)\,. \tag{3.20}$$

Balance laws. The balance of linear and angular momenta read[3]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho\boldsymbol{V}dV = \int_{\mathcal{U}} \rho\boldsymbol{B}dV + \int_{\partial\mathcal{U}} \boldsymbol{T}dA\,, \tag{3.21a}$$

$$\frac{d}{dt}\int_{\mathcal{U}} \rho\boldsymbol{X}\times\boldsymbol{V}dV = \int_{\mathcal{U}} \rho\boldsymbol{X}\times\boldsymbol{B}dV + \int_{\partial\mathcal{U}} \boldsymbol{X}\times\boldsymbol{T}dA\,, \tag{3.21b}$$

where $\boldsymbol{B}$ is the body force per unit undeformed mass and $\boldsymbol{T}$ is the traction at the boundary. Assuming the conservation of mass (3.18), the governing equations of motion follow by localization of (3.21a) and (3.21b) and read [Marsden and Hughes, 1983]

$$\operatorname{Div}\boldsymbol{P} + \rho\boldsymbol{B} = \rho\boldsymbol{A}\,, \tag{3.22a}$$

$$\boldsymbol{P}\boldsymbol{F}^{\mathsf{T}} = \boldsymbol{F}\boldsymbol{P}^{\mathsf{T}}\,, \tag{3.22b}$$

where $\boldsymbol{P}(X,t)$ denotes the first Piola-Kirchhoff stress tensor $P^{aA} = J(F^{-1})^A{}_b\sigma^{ab}$, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and Div denotes the material divergence operator. In local coordinates

$$\operatorname{Div}\boldsymbol{P} = P^{aA}{}_{|A}\partial_a = \left(\frac{\partial P^{aA}}{\partial X^A} + \Gamma^A{}_{AB}P^{aB} + \gamma^a{}_{bc}F^b{}_AP^{cA}\right)\partial_a\,. \tag{3.23}$$

Writing (3.22a) and (3.22b) in components, we find

$$\rho A^a = \rho B^a + P^{aA}{}_{|A}\,, \tag{3.24a}$$

$$P^{aA}F^b{}_A = P^{bA}F^a{}_A\,. \tag{3.24b}$$

[2] Note that from (3.12) and (3.16), we have

$$\frac{d}{dt}\int_{\mathcal{U}} \rho(X,\boldsymbol{G})dV(X,\boldsymbol{G}) = 0\,.$$

Therefore

$$\frac{d\rho}{dT} + \frac{1}{2}\rho\operatorname{tr}\left(\frac{\partial\boldsymbol{G}}{\partial T}\right) = 0\,.$$

But since

$$\frac{1}{2}\operatorname{tr}\left(\frac{\partial\boldsymbol{G}}{\partial T}\right) = \frac{\partial}{\partial T}\left[\operatorname{tr}(\boldsymbol{\omega}(X,T))\right] = \beta\,,$$

the material conservation of mass (3.18) follows.

[3] Note that the balance laws in the form of equations (3.21a) and (3.21b) make sense only if we consider a Euclidean ambient space. One should note that integrating a vector field is meaningless in a general manifold, but we can make sense of it in a Euclidean space by using its linear structure. Alternative approaches for deriving the governing equations of motion (3.22a) and (3.22b) for a general manifold are found in the literature using the Lagrangian field theory or the covariance of the balance of energy [Green and Rivlin, 1964; Marsden and Hughes, 1983; Yavari et al., 2006; Yavari and Marsden, 2012]. In Appendix B, we derive the equations of motion for nonlinear thermoelasticity by using the Lagrangian field theory.

3.3 The first law of thermodynamics

The first law of thermodynamics postulates the existence of a state function, namely the internal energy that satisfies, in the case of a static material metric, the following balance of energy [Truesdell, 1952; Gurtin and Williams, 1967; Gurtin, 1974; Marsden and Hughes, 1983; Yavari et al., 2006]

$$\frac{d}{dt}\int_{\mathcal{U}}\rho\left(\mathcal{E}+\frac{1}{2}\boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})\right)dV=\int_{\mathcal{U}}\rho\left(\boldsymbol{g}(\boldsymbol{B},\boldsymbol{V})+R\right)dV+\int_{\partial\mathcal{U}}\left(\boldsymbol{g}(\boldsymbol{T},\boldsymbol{V})+H\right)dA\,, \tag{3.25}$$

where $\mathcal{E}$ is the material specific internal energy, $R(X,t)$ is the heat supply per unit undeformed mass and $H(X,t,\boldsymbol{N})$ is the heat flux across a surface with unit normal $\boldsymbol{N}$. Now, because we have a time-dependent metric (implicit time dependence through the temperature dependence), the energy balance must be modified to include the rate of change of the metric as a variable that contributes to the rate of change of internal energy in order to capture possible changes of shape due to the temperature field and its energy contribution. The energy balance is therefore modified to read (see [Epstein and Maugin, 2000; Lubarda and Hoger, 2002; Yavari, 2010] for the modified energy balance in the case of growing bodies).

$$\frac{d}{dt}\int_{\mathcal{U}}\rho\left(\mathcal{E}+\frac{1}{2}\boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})\right)dV=\int_{\mathcal{U}}\rho\left(\boldsymbol{g}(\boldsymbol{B},\boldsymbol{V})+R+\frac{\partial\mathcal{E}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}\right)dV+\int_{\partial\mathcal{U}}\left(\boldsymbol{g}(\boldsymbol{T},\boldsymbol{V})+H\right)dA\,, \tag{3.26}$$

and in localized form, the material balance of energy reads

$$\rho\dot{\mathcal{E}}=\boldsymbol{S}\!:\!\boldsymbol{D}-\operatorname{Div}\boldsymbol{Q}+\rho R+\rho\frac{\partial\mathcal{E}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}\,, \tag{3.27}$$

where a doted quantity denotes its total time derivative, e.g., $\dot{\mathcal{E}}=\frac{d\mathcal{E}}{dt}$, $\boldsymbol{S}$ is the second Piola-Kirchhoff stress tensor, in components $S^{AB}=J(F^{-1})^A{}_a(F^{-1})^B{}_b\sigma^{ab}$, $\boldsymbol{D}=\frac{1}{2}\dot{\boldsymbol{C}}^\flat(X,t)$ is the material rate of deformation tensor, $\boldsymbol{Q}=\boldsymbol{Q}(X,t)$ is the external heat flux vector per unit area, where $\boldsymbol{D}T=\frac{\partial T}{\partial X^A}dX^A$. In local coordinates, we write the divergence of $\boldsymbol{Q}$ as

$$\operatorname{Div}\boldsymbol{Q}=Q^A{}_{|A}=\frac{\partial Q^A}{\partial X^A}+\Gamma^A_{AB}Q^B\,. \tag{3.28}$$

3.4 The second law of thermodynamics

The second law of thermodynamics postulates the existence of a state function, namely the entropy that satisfies, in the case of a static material metric, the material Clausius-Duhem inequality [Truesdell, 1952; Gurtin and Williams, 1967; Gurtin, 1974; Marsden and Hughes, 1983]

$$\frac{d}{dt}\int_{\mathcal{U}}\rho\mathcal{N}dV\geq\int_{\mathcal{U}}\rho\frac{R}{T}dV+\int_{\partial\mathcal{U}}\frac{H}{T}dA\,, \tag{3.29}$$

where $\mathcal{N}=\mathcal{N}(X,T,\boldsymbol{C}^\flat,\boldsymbol{G})$ is the specific entropy. In the case of a time-dependent material metric, the Clausius-Duhem inequality needs to be modified to include the rate of change of the material metric. The Clausius-Duhem inequality in our geometric formulation reads (see [Epstein and Maugin, 2000; Lubarda and Hoger, 2002; Yavari, 2010] for the modified Clausius-Duhem inequality in the case of growing bodies[4]).

$$\frac{d}{dt}\int_{\mathcal{U}}\rho\mathcal{N}dV\geq\int_{\mathcal{U}}\rho\frac{R}{T}dV+\int_{\partial\mathcal{U}}\frac{H}{T}dA+\int_{\mathcal{U}}\rho\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}dV\,, \tag{3.30}$$

[4]Note that in [Yavari, 2010], there was a typo in the modified Clausius-Duhem inequality (Eq. (2.37)), which should read

$$\frac{d}{dt}\int_{\mathcal{U}}\rho_0\mathcal{N}dV\geq\int_{\mathcal{U}}\rho_0\frac{R}{T}dV+\int_{\partial\mathcal{U}}\frac{H}{T}dA+\int_{\mathcal{U}}\mathcal{N}\left[\frac{\partial\rho_0}{\partial t}+\frac{1}{2}\rho_0\operatorname{tr}\left(\frac{\partial\boldsymbol{G}}{\partial t}\right)\right]dV+\int_{\mathcal{U}}\rho\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}dV\,,$$

where in the last term $\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}$ was mistakenly written as $\frac{\partial\mathcal{E}}{\partial\boldsymbol{G}}$. This correction leads to a few changes in Section 2.4 of [Yavari, 2010] without affecting other sections of the paper. Eq. (2.38) should read

$$\rho_0\frac{d\mathcal{N}}{dt}\geq\rho_0\frac{R}{T}-\operatorname{Div}\left(\frac{\boldsymbol{Q}}{T}\right)+\rho_0\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}\,.$$

Consequently, Eqs. (2.40),(2.41), and (2.43) should, respectively, read

$$\frac{\rho_0}{T}\frac{d\mathcal{E}}{dt}-\frac{\rho_0}{T}\frac{d\Psi}{dt}-\rho_0\frac{\dot{T}}{T^2}\left(\mathcal{E}-\Psi\right)\geq\rho_0\frac{R}{T}-\operatorname{Div}\left(\frac{\boldsymbol{Q}}{T}\right)+\rho_0\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}:\frac{\partial\boldsymbol{G}}{\partial t}\,,$$

and in localized form, the material Clausius-Duhem inequality reads

$$\rho\dot{\mathcal{N}} \geq \rho\frac{R}{T} - \operatorname{Div}\left(\frac{\boldsymbol{Q}}{T}\right) + \rho\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}\,. \tag{3.31}$$

3.5 The response functions

Free energy. By considering the Clausius-Duhem inequality as a restriction on the constitutive behavior of the material, Coleman and Noll [1959, 1963] derived the response functions for entropy and stress. We prove in the following a version of this result in nonlinear thermoelasticity with a temperature-dependent metric inspired by the proof given in [Coleman and Noll, 1963; Gurtin, 1974]. In thermoelasticity, the (hyperelastic) constitutive model is given by the specific free energy function

$$\Psi = \Psi(X, T, \boldsymbol{C}^\flat, \boldsymbol{G})\,, \tag{3.32}$$

such that the specific internal energy $\mathcal{E}$ is the Legendre transform of $-\Psi$ with respect to the conjugate variables T and $\mathcal{N}$, i.e.

$$\mathcal{E} = T\mathcal{N} + \Psi\,, \tag{3.33a}$$

$$\mathcal{N} = -\frac{\partial\Psi}{\partial T}\,. \tag{3.33b}$$

It follows that the specific internal energy $\mathcal{E}$ is such that

$$\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \boldsymbol{C}^\flat, \boldsymbol{G})\,, \quad \frac{\partial\mathcal{E}}{\partial\mathcal{N}} = T\,, \quad \frac{\partial\mathcal{E}}{\partial\boldsymbol{G}} = \frac{\partial\Psi}{\partial\boldsymbol{G}}\,, \quad \frac{\partial\mathcal{E}}{\partial\boldsymbol{C}^\flat} = \frac{\partial\Psi}{\partial\boldsymbol{C}^\flat}\,. \tag{3.34}$$

In the following, we investigate the restrictions imposed by the Clausius-Duhem inequality (3.31) on the constitutive equations. By using (3.27) and (3.34) in (3.31), we find

$$\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^\flat}\!:\!\dot{\boldsymbol{C}}^\flat + \rho T\dot{N} - \rho T\frac{\partial\mathcal{N}}{\partial\boldsymbol{C}^\flat}\!:\!\dot{\boldsymbol{C}}^\flat - \rho T\frac{\partial\mathcal{N}}{\partial T}\dot{T} - \boldsymbol{S}\!:\!\boldsymbol{D} + \frac{1}{T}\left\langle \boldsymbol{D}T, \boldsymbol{Q}\right\rangle \leq 0\,. \tag{3.35}$$

Therefore, recalling that $\boldsymbol{D} = \frac{1}{2}\dot{\boldsymbol{C}}^\flat$, we can rewrite (3.35) as

$$\left(2\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^\flat} - \boldsymbol{S}\right)\!:\!\boldsymbol{D} + \rho T\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}} + \frac{1}{T}\left\langle \boldsymbol{D}T, \boldsymbol{Q}\right\rangle \leq 0\,. \tag{3.36}$$

The above inequality holds for all deformations φ and metrics $\boldsymbol{G}$. In particular, if we choose φ to be time-independent, (3.36) reads

$$\rho T\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}} + \frac{1}{T}\left\langle \boldsymbol{D}T, \boldsymbol{Q}\right\rangle \leq 0\,. \tag{3.37}$$

Note that $\boldsymbol{D}T$ and $\dot{\boldsymbol{G}} = \frac{d\boldsymbol{G}}{dT}\dot{T}$ can be chosen arbitrarily and independently. Let T be homogeneous, i.e., $\boldsymbol{D}T = \boldsymbol{0}$, then $\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}} \leq 0$ must hold for every $\dot{\boldsymbol{G}}$. Therefore, we must have

$$\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}} = \boldsymbol{0}\,. \tag{3.38}$$

$$\boldsymbol{P}\!:\!\nabla_0\boldsymbol{V} - \rho_0\frac{d\Psi}{dt} + \rho_0\frac{\partial\Psi}{\partial\boldsymbol{G}}\!:\!\frac{\partial\boldsymbol{G}}{\partial t} - \rho_0\mathcal{N}\dot{T} \geq \frac{1}{T}\boldsymbol{D}T.\boldsymbol{Q}\,,$$

$$\rho_0\left(\frac{\partial\Psi}{\partial T} + \mathcal{N}\right)\dot{T} + \left(\rho_0\frac{\partial\Psi}{\partial\boldsymbol{F}} - \boldsymbol{P}\right)\!:\!\nabla_0\boldsymbol{V} + \frac{1}{T}\left\langle \boldsymbol{D}T, \boldsymbol{Q}\right\rangle \leq 0\,.$$

Finally, the entropy production inequality (after Eq. (2.44)) is reduced to read

$$\frac{1}{T}\boldsymbol{D}T.\boldsymbol{Q} \leq 0\,.$$

Similarly, in Equation (2.60), $\frac{\partial E'}{\partial\boldsymbol{G}} : \frac{\partial\boldsymbol{G}}{\partial t}$ should be substituted by $\frac{\partial N'}{\partial\boldsymbol{G}} : \frac{\partial\boldsymbol{G}}{\partial t}$, and in Equation (2.62) and what follows, $\frac{\partial E}{\partial\boldsymbol{G}} : \frac{\partial\boldsymbol{G}}{\partial t}$ should be substituted by $\frac{\partial N}{\partial\boldsymbol{G}} : \frac{\partial\boldsymbol{G}}{\partial t}$.

Now we assume that temperature is homogeneous and time-independent. Thus, the inequality (3.36) reads

$$\left(2\rho\frac{\partial\Psi}{\partial \boldsymbol{C}^\flat}-\boldsymbol{S}\right)\!:\!\boldsymbol{D}\leq 0\,, \tag{3.39}$$

and must hold for every $\boldsymbol{D}$. It follows that

$$\boldsymbol{S}=2\rho\frac{\partial\Psi}{\partial \boldsymbol{C}^\flat}\,. \tag{3.40}$$

Consequently, the energy balance (3.27) reduces to

$$\rho T\dot{\mathcal{N}}=\rho R-\mathrm{Div}\boldsymbol{Q}\,. \tag{3.41}$$

Remark 3.2. Note that for a material with an internal constraint of the form $\kappa(\boldsymbol{C},T)=0$, we have

$$\mathcal{N}=-\left.\frac{\partial\Psi}{\partial T}\right|_{\kappa(\boldsymbol{C},T)=0}-\left.\frac{q}{\rho}\frac{\partial\kappa}{\partial T}\right|_{\kappa(\boldsymbol{C},T)=0}, \tag{3.42a}$$

$$\boldsymbol{S}=2\rho\left.\frac{\partial\Psi}{\partial \boldsymbol{C}^\flat}\right|_{\kappa(\boldsymbol{C},T)=0}+2q\left.\frac{\partial\kappa}{\partial \boldsymbol{C}^\flat}\right|_{\kappa(\boldsymbol{C},T)=0}, \tag{3.42b}$$

where q is the Lagrange multiplier corresponding to the constraint κ. In the case of incompressibility we have $\kappa(\boldsymbol{C},T)=J-1$ and $q=-p$, where p is the pressure field due to the constraint of incompressibility.[5]

Heat conduction. It follows from (3.41) that the entropy production inequality (3.35) reduces to

$$\langle \boldsymbol{D}T,\boldsymbol{Q}\rangle\leq 0\,. \tag{3.43}$$

For an arbitrary isotropic solid the heat flux response function has the following representation [Truesdell and Noll, 2004]

$$\boldsymbol{Q}=\left(\phi_0\boldsymbol{C}^{-1}+\phi_1\boldsymbol{G}+\phi_2\boldsymbol{C}\right)\mathbf{D}T\,, \tag{3.44}$$

where $\phi_k=\phi_k(X,T,\mathbf{D}T,\boldsymbol{C},\boldsymbol{G})$, $k=-1,0,1$, are scalar functions.[6] If we set $K^{AB}=-(\phi_0(C^{-1})^{AB}+\phi_1G^{AB}+\phi_2C^{AB})$, then by (3.43), $\boldsymbol{K}$ is a positive semi-definite symmetric material $\binom{2}{0}$-tensor and we can write a generalized version of the Fourier's law of thermal conduction $\boldsymbol{Q}=-\boldsymbol{K}\mathbf{D}T$. In components, $\boldsymbol{Q}=Q^A\partial_A=-K^{AB}\frac{\partial T}{\partial X^B}\partial_A$.

3.6 Coupled nonlinear thermoelasticity

We substitute (3.33b) and (3.40) into (3.41) and obtain

$$\mathrm{Div}\,\boldsymbol{Q}=\rho\frac{\partial^2\Psi}{\partial T^2}T\dot{T}+\frac{1}{2}T\frac{\partial\boldsymbol{S}}{\partial T}\!:\!\dot{\boldsymbol{C}}^\flat+\rho R\,. \tag{3.45}$$

The specific heat capacity at constant strain c_E is defined as the quantity of heat required to produce a unit temperature increase in a unit mass of material at constant strain ($\dot{\boldsymbol{C}}^\flat=\mathbf{0}$), i.e.

$$\mathrm{Div}\,\boldsymbol{Q}=-\rho c_E\dot{T}\,. \tag{3.46}$$

Comparing (3.45) and (3.46) at constant strain and without external heat supply ($R=0$), we find

$$c_E=-T\frac{\partial^2\Psi}{\partial T^2}\,. \tag{3.47}$$

[5] Note that the pressure field $-p$ is the Lagrange multiplier corresponding to the incompressibility constraint and is, in general, different from the mean stress $\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$.

[6] Truesdell and Noll [2004] gave the spatial version of the heat flux response function $\boldsymbol{q}=\left(\psi_0\boldsymbol{G}+\psi_1\boldsymbol{B}+\psi_2\boldsymbol{B}^2\right)\mathbf{d}T$, where $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^\mathsf{T}$ is the left Cauchy-Green tensor and $\mathbf{d}T$ denotes the derivative of $T(x,t)$ with respect to the spatial variable x. One can easily find (3.44) by using the Piola transform $\boldsymbol{Q}=J\varphi_t{}^*\boldsymbol{q}$ and setting $\phi_k=J\psi_k$, $k=0,1,2$.

Remark 3.3. Note that the partial derivative with respect to temperature in (3.45) and (3.47) is a partial derivative with respect to temperature with $\boldsymbol{C}$ and $\boldsymbol{G}$ being fixed, i.e., $\frac{\partial}{\partial T} = \frac{\partial}{\partial T}|_{\boldsymbol{C},\boldsymbol{G}}$.

We can now rewrite (3.45) as

$$\operatorname{Div}\boldsymbol{Q} = -\rho c_E \dot{T} + \frac{1}{2} T \frac{\partial \boldsymbol{S}}{\partial T} \!:\! \dot{\boldsymbol{C}}^\flat + \rho R \,. \tag{3.48}$$

Along with the boundary conditions for φ on $\partial\mathcal{B}$ (prescribed in terms of displacement or traction), the boundary conditions for T on $\partial\mathcal{B}$ (prescribed in terms of temperature or flux), and the initial temperature field $T(X, t = 0) = T_{\text{init}}(X)$, the equations (3.18), (3.22a), (3.22b), and (3.48) constitute the governing equations for the general nonlinear thermoelastic problem.

By using the generalized Fourier's law of thermal conduction for an arbitrary isotropic solid (cf. (3.44)), (3.48) takes the form of a second-order nonlinear partial differential equation that is a generalization of the classical heat equation with the following extra forcing terms arising from the thermoelastic coupling:

$$\operatorname{Div}(\boldsymbol{K}\boldsymbol{D}T) = \rho c_E \dot{T} - \frac{1}{2} T \frac{\partial \boldsymbol{S}}{\partial T} \!:\! \dot{\boldsymbol{C}}^\flat - \rho R \,. \tag{3.49}$$

Remark 3.4. Note that the left hand-side of (3.49) can be written as[7]

$$\operatorname{Div}(\boldsymbol{K}\boldsymbol{D}T) = \boldsymbol{K} : \boldsymbol{\Delta} T + \langle \operatorname{Div}\boldsymbol{K}, \boldsymbol{D}T \rangle \,, \tag{3.50}$$

where $\operatorname{Div}\boldsymbol{K} = K^{AB}{}_{|B}\partial_A$, $K^{AB}{}_{|B} = K^{AB}{}_{,B} + \Gamma^A{}_{BC}K^{CB} + \Gamma^B{}_{BC}K^{AC}$, and $\boldsymbol{\Delta}$ denotes the Hessian operator, i.e.

$$\boldsymbol{\Delta} T = \left[\frac{\partial^2 T}{\partial X^A \partial X^B} - \Gamma^C{}_{AB} \frac{\partial T}{\partial X^C} \right] dX^A \otimes dX^B \,. \tag{3.51}$$

In components, (3.49) reads

$$K^{AB} \left[\frac{\partial^2 T}{\partial X^A \partial X^B} - \Gamma^C{}_{AB} \frac{\partial T}{\partial X^C} \right] + K^{AB}{}_{|B} \frac{\partial T}{\partial X^A} = \rho c_E \dot{T} - \frac{1}{2} T \frac{\partial S^{AB}}{\partial T} \!:\! \dot{C}_{AB} - \rho R \,. \tag{3.52}$$

Linearization of the generalized heat equation. In the following we linearize the generalized heat equation for a thermally isotropic material without external heat supply. The generalized heat equation in this case reads[8]

$$\left(k G^{AB} \frac{\partial(\delta T)}{\partial X^B} \right)_{|A} = \rho c_E \dot{T} - \frac{1}{2} T \frac{\partial S^{AB}}{\partial T} \dot{C}_{AB} \,. \tag{3.53}$$

Let $T_\epsilon(X, t)$ be a 1-parameter family of temperature fields such that $T_{\epsilon=0}(X, t) = T_0$ is a uniform temperature field and let the temperature variation be defined as

$$\delta T(X, t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} T_\epsilon(X, t) \,. \tag{3.54}$$

The corresponding 1-parameter family of metrics is $\boldsymbol{G}_\epsilon(X) = \boldsymbol{G}(X, T_\epsilon) = \boldsymbol{G}_0(X) e^{2\omega(X, T_\epsilon)}$. Variation of the material metric is calculated as

$$\delta\boldsymbol{G}(X) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \boldsymbol{G}_\epsilon(X) = 2\alpha(X, T_0)\delta T(X, t)\boldsymbol{G}(X) \,. \tag{3.55}$$

Let φ_ϵ be the corresponding 1-parameter family of equilibrium configurations. The variation of the equilibrium configuration is a spatial tangent vector [Yavari and Ozakin, 2008]

$$\boldsymbol{U}(X, t) = \delta\varphi(X, t) = d_\epsilon \varphi_{t,X} \left[\partial_\epsilon\right]\big|_{\epsilon=0} \tag{3.56}$$

[7]Recall that $\boldsymbol{K}$ is symmetric.

[8]Note that $\left(kG^{AB}\frac{\delta T}{\partial X^B}\right)_{|A} = kG^{AB}\left(\frac{\partial^2 T}{\partial X^A \partial X^B} - \Gamma^C{}_{AB}\frac{\partial T}{\partial X^C}\right) + (kG^{AB})_{|A}\frac{\partial T}{\partial X^B}$.

$\boldsymbol{u} = \boldsymbol{U} \circ \varphi^{-1}$ can be thought of as the geometric analogue of the displacement field in the classical theory of linear elasticity. Note that

$$U^a{}_{|A} = F^b{}_A u^a{}_{|b} \,. \tag{3.57}$$

The linearization of the generalized heat equation (3.53) is defined as [Marsden and Hughes, 1983; Yavari and Ozakin, 2008]

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[k_\epsilon G_\epsilon{}^{AB} \left(\frac{\partial^2 T_\epsilon}{\partial X^A \partial X^B} - \Gamma_\epsilon{}^C{}_{AB} \frac{\partial T_\epsilon}{\partial X^C} \right) + (k_\epsilon G_\epsilon{}^{AB})_{|A} \frac{\partial T_\epsilon}{\partial X^B} \right] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\rho_\epsilon c_{E\epsilon} \dot{T}_\epsilon - \frac{1}{2} T_\epsilon \frac{\partial S_\epsilon^{AB}}{\partial T} \dot{C}_{\epsilon AB} \right]. \tag{3.58}$$

We assume that the ambient space is Euclidean. We also assume that, for a given uniform temperature field T_0, the material manifold is Euclidean. Thus, we have in Cartesian coordinates $G_{0AB} = \delta_{AB}$, and $\Gamma_0{}^C{}_{AB} = 0$. Therefore, the linearized heat equation (3.58) is simplified to read

$$\begin{aligned}
&\frac{dk_\epsilon}{d\epsilon}\Big|_{\epsilon=0} \delta^{AB} \frac{\partial^2 T_\epsilon}{\partial X^A \partial X^B}\Big|_{\epsilon=0} - 2k_0 \alpha_0 \delta T \delta^{AB} \frac{\partial^2 T_\epsilon}{\partial X^A \partial X^B}\Big|_{\epsilon=0} \\
&+ k_0 \delta^{AB} \left(\frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\frac{\partial^2 T_\epsilon}{\partial X^A \partial X^B} \right] - \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\Gamma_\epsilon{}^C{}_{AB} \right] \frac{\partial T_\epsilon}{\partial X^C}\Big|_{\epsilon=0} \right) \\
&+ \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[(k_\epsilon G_\epsilon{}^{AB})_{|A} \right] \frac{\partial T_\epsilon}{\partial X^B}\Big|_{\epsilon=0} + (k_\epsilon G_\epsilon{}^{AB})_{|A}\Big|_{\epsilon=0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\frac{\partial T_\epsilon}{\partial X^B} \right] \\
&= \frac{d(\rho_\epsilon c_{E\epsilon})}{d\epsilon}\Big|_{\epsilon=0} \dot{T}_\epsilon\Big|_{\epsilon=0} + \rho_0 c_{E0} \frac{d\dot{T}_\epsilon}{d\epsilon}\Big|_{\epsilon=0} - \frac{1}{2} \delta T \frac{\partial S_\epsilon{}^{AB}}{\partial T}\Big|_{\epsilon=0} \dot{C}_{\epsilon AB}\Big|_{\epsilon=0} \\
&- \frac{1}{2} T_0 \left[\frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{\partial S_\epsilon{}^{AB}}{\partial T} \right] \dot{C}_{\epsilon AB}\Big|_{\epsilon=0} - \frac{1}{2} T_0 \frac{\partial S_\epsilon{}^{AB}}{\partial T}\Big|_{\epsilon=0} \frac{d\dot{C}_{\epsilon AB}}{d\epsilon}\Big|_{\epsilon=0} .
\end{aligned} \tag{3.59}$$

Note that the parameter ϵ is independent of time t and position X, and hence

$$\begin{aligned}
\frac{\partial T_\epsilon}{\partial X^C}\Big|_{\epsilon=0} &= \frac{\partial T_0}{\partial X^C} = 0 \,, \\
\frac{\partial^2 T_\epsilon}{\partial X^A \partial X^B}\Big|_{\epsilon=0} &= \frac{\partial^2 T_0}{\partial X^A \partial X^B} = 0 \,, \\
\frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\frac{\partial^2 T_\epsilon}{\partial X^A \partial X^B} \right] &= \frac{\partial^2 \delta T}{\partial X^A \partial X^B} \,, \\
(k_\epsilon G_\epsilon{}^{AB})_{|A}\Big|_{\epsilon=0} &= \frac{\partial k_0}{\partial X^A} \delta^{AB} \,, \\
\dot{T}_\epsilon\Big|_{\epsilon=0} &= \dot{T}_0 = 0 \,, \\
\frac{d\dot{T}_\epsilon}{d\epsilon}\Big|_{\epsilon=0} &= \frac{\partial(\delta T)}{\partial t} \,, \\
\frac{\partial S_\epsilon{}^{AB}}{\partial T}\Big|_{\epsilon=0} &= \frac{\partial S_0{}^{AB}}{\partial T} \,, \\
\frac{d\dot{C}_{\epsilon AB}}{d\epsilon}\Big|_{\epsilon=0} &= \frac{d}{dt} \left(\frac{dC_{\epsilon AB}}{d\epsilon}\Big|_{\epsilon=0} \right) .
\end{aligned} \tag{3.60}$$

Therefore, the linearized heat equation simplifies to read

$$\left(k_0 \delta^{AB} \frac{\partial(\delta T)}{\partial X^B} \right)_{|A} = \rho_0 c_{E0} \frac{\partial \delta T}{\partial t} - \frac{1}{2} T_0 \frac{\partial S_0{}^{AB}}{\partial T} \frac{d}{dt} \left(\frac{dC_{\epsilon AB}}{d\epsilon}\Big|_{\epsilon=0} \right) . \tag{3.61}$$

Since $C_{AB} = F^a{}_A F^b{}_B \delta_{ab}$, we find [Yavari, 2010]

$$\begin{aligned} \frac{dC_{\epsilon AB}}{d\epsilon}\Big|_{\epsilon=0} &= \frac{dF_\epsilon{}^a{}_A}{d\epsilon}\Big|_{\epsilon=0} F_0{}^b{}_B \delta_{ab} + F_0{}^a{}_B \frac{dF_\epsilon{}^b{}_B}{d\epsilon}\Big|_{\epsilon=0} \delta_{ab} \\ &= U^a{}_{|A} F_0{}^b{}_B \delta_{ab} + F_0{}^a{}_A U^b{}_{|B} \delta_{ab} \\ &= F_0{}^c{}_A F_0{}^b{}_B \delta_{ab} u^a{}_{|c} + F_0{}^a{}_A F_0{}^c{}_B \delta_{ab} u^b{}_{|c} \\ &= F_0{}^c{}_A F_0{}^b{}_B u_{b|c} + F_0{}^a{}_A F_0{}^c{}_B u_{a|c} \\ &= 2F_0{}^a{}_A F_0{}^b{}_B \epsilon_{ab} , \end{aligned} \tag{3.62}$$

where $\epsilon_{ab} = \frac{1}{2}(u_{a|b} + u_{b|a})$ is the linearized strain. At $\epsilon = 0$, we consider a natural embedding such that $F_0{}^a{}_A = \delta^a{}_A$ $(J_0 = 1)$. It follows that

$$\frac{d}{dt}\left(\frac{dC_{\epsilon AB}}{d\epsilon}\Big|_{\epsilon=0}\right) = 2\delta^a{}_A \delta^b{}_B \dot{\epsilon}_{ab} , \tag{3.63}$$

and the Piola transform gives us

$$\left(k_0 \delta^{AB} \frac{\partial(\delta T)}{\partial X^B}\right)_{|A} = \left(k_0 \delta^{ab} \frac{\partial(\delta T)}{\partial x^b}\right)_{|a} . \tag{3.64}$$

Furthermore, since $S^{AB} F^a{}_A F^b{}_B = J\sigma^{ab}$, at $\epsilon = 0$ we have $S_0{}^{AB} \delta^a{}_A \delta^b{}_B = \sigma_0{}^{ab}$. Therefore, the linearized heat equation (3.59) is simplified to read

$$\left(k_0 \delta^{ab} \frac{\partial(\delta T)}{\partial x^b}\right)_{|a} = \rho_0 c_{E0} \frac{\partial(\delta T)}{\partial t} - T_0 \frac{\partial \sigma_0{}^{ab}}{\partial T} \dot{\epsilon}_{ab} . \tag{3.65}$$

Recalling that the linearized temperature change is δT , we recover the classical linear coupled heat equation for a constant thermal conduction coefficient k_0 [Boley and Weiner, 1960; Dillon Jr, 1962; Hetnarski and Eslami, 2008]

$$k_0 \Delta T = \rho_0 c_{E0} \dot{T} - T_0 \frac{\partial \sigma^{ab}}{\partial T} \dot{\epsilon}_{ab} , \tag{3.66}$$

where Δ denotes the Laplacian operator in spatial Cartesian coordinates, i.e., $\Delta T = \delta^{ab} \frac{\partial^2 T}{\partial x^a \partial x^b}$.

4 Examples

As applications of the geometric theory, we study in the following the nonlinear thermoelastic problem in the case of an infinitely long solid circular cylinder and a spherical ball. We formulate the governing equations and analytically solve for the thermal stress field for an arbitrary incompressible isotropic hyperelastic solid with a radially-symmetric temperature distribution. Then, we restrict the problem to the thermally isotropic and homogeneous solids following the thermoelastic constitutive model for rubber-like materials described in Appendix A to numerically solve for the static and time-dependent temperature and the thermal stress fields induced by a thermal inclusion. We will also compare the nonlinear solutions with their corresponding linear elasticity solutions.

4.1 An infinitely long circular cylindrical bar made of an incompressible isotropic solid

In this section we consider the static problem (i.e., zero acceleration) in the absence of body forces for an infinitely long solid circular cylinder of radius R_o made of an incompressible isotropic solid under uniform normal traction on its boundary. Let (R, Θ, Z) be the cylindrical coordinate system for which $R \geq 0$, $0 \leq \Theta \leq 2\pi$, and $Z \in \mathbb{R}$. In cylindrical coordinates, the material metric for the configuration with the stress-free temperature field $T_0 = T_0(R)$ reads

$$\boldsymbol{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.1}$$

We assume a radially-symmetric temperature field $T = T(R,t)$ and let $\alpha_R = \alpha_R(R,T)$ be the radial thermal expansion coefficient and $\alpha_\Theta = \alpha_\Theta(R,T)$ be the circumferential thermal expansion coefficient. As the cylinder is infinite in the Z direction any dilatation in the Z direction leaves the cylinder unchanged. We can therefore assume a zero thermal expansion coefficient in the Z direction, i.e., $\alpha_Z = 0$. The temperature-dependent material metric for the cylinder, as introduced in Section 2.1, reads[9]

$$\boldsymbol{G} = \begin{pmatrix} e^{2\omega_R} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.2}$$

where for $K \in \{R, \Theta\}$, $\omega_K(R, T(R,t)) = \displaystyle\int_{T_0}^{T(R,t)} \alpha_K(R,\tau)d\tau$. The Christoffel symbols for $\boldsymbol{G}$ are given as

$$\begin{aligned} \boldsymbol{\Gamma}^R = \left[\Gamma^R{}_{AB}\right] &= \begin{pmatrix} \frac{\partial \omega_R}{\partial R}\big|_t & 0 & 0 \\ 0 & -e^{2(\omega_\Theta-\omega_R)}\left(R + R^2\, \frac{\partial \omega_\Theta}{\partial R}\big|_t\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \boldsymbol{\Gamma}^\Theta = \left[\Gamma^\Theta{}_{AB}\right] = \begin{pmatrix} 0 & \frac{1}{R} + \frac{\partial \omega_\Theta}{\partial R}\big|_t & 0 \\ \frac{1}{R} + \frac{\partial \omega_\Theta}{\partial R}\big|_t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}&, \quad \boldsymbol{\Gamma}^Z = \left[\Gamma^Z{}_{AB}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.3}$$

where $\frac{\partial \omega_K}{\partial R}\big|_t = \frac{\partial \omega_K}{\partial R}\big|_{t,T} + \alpha_K \frac{\partial T}{\partial R}$. We endow the ambient space with the following flat metric in cylindrical coordinates (r, θ, z).

$$\boldsymbol{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

The Christoffel symbols for $\boldsymbol{g}$ read

$$\boldsymbol{\gamma}^r = \left[\gamma^r{}_{ab}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^\theta = \left[\gamma^\theta{}_{ab}\right] = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^z = \left[\gamma^z{}_{ab}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

Thermal stresses. In the following we solve for the stress field for an arbitrary radial temperature field. In order to calculate the thermal stresses, we embed the material manifold into the ambient space and look for solutions of the form $(r, \theta, z) = (r(R,t), \Theta, Z)$. The deformation gradient reads

$$\boldsymbol{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.6}$$

For an incompressible solid, we have

$$J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}}\det \boldsymbol{F} = \frac{r\frac{\partial r}{\partial R}}{Re^{\omega_R+\omega_\Theta}} = 1, \tag{4.7}$$

and hence, assuming $r(0,t) = 0$ (to eliminate rigid translations), we find

$$r(R,t) = \left(\int_0^R 2\xi e^{\omega_R(\xi,T(\xi,t))+\omega_\Theta(\xi,T(\xi,t))}d\xi\right)^{\frac{1}{2}}. \tag{4.8}$$

The right Cauchy-Green deformation tensor reads

$$\boldsymbol{C} = \begin{pmatrix} \frac{R^2}{r^2}e^{2\omega_\Theta} & 0 & 0 \\ 0 & \frac{r^2}{R^2}e^{-2\omega_\Theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

[9]Note that because $\frac{\partial \omega_Z(R,T)}{\partial T} = \alpha_Z = 0$ and $\omega_Z(R,T_0) = 0$, we have $\omega_Z(R,T) = 0$.

For an incompressible isotropic solid, the free energy density per unit undeformed volume $\psi = \rho\Psi$ is expressed as a function of $\mathrm{I} = \mathrm{tr}\boldsymbol{C}$ and $\mathrm{II} = \det \boldsymbol{C} \mathrm{tr}\boldsymbol{C}^{-1} = \frac{1}{2}(\mathrm{tr}(\boldsymbol{C}^2) - \mathrm{tr}(\boldsymbol{C})^2)$, i.e., $\psi = \psi(R, T, \mathrm{I}, \mathrm{II}, J = 1)$. Therefore, we can write [Doyle and Ericksen, 1956]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B \left[(\psi_{\mathrm{I}} + \mathrm{I}\psi_{\mathrm{II}})\, G^{AB} - \psi_{\mathrm{II}} C^{AB} \right] - p g^{ab} \,, \tag{4.10}$$

where $\psi_{\mathrm{I}} = \frac{\partial \psi}{\partial \mathrm{I}}$, $\psi_{\mathrm{II}} = \frac{\partial \psi}{\partial \mathrm{II}}$, and $p = p(R, t)$ is the pressure field due to the incompressibility constraint. Thus, the non-zero components of the Cauchy stress tensor are given by

$$\sigma^{rr} = 2\frac{R^2 e^{2\omega_\Theta}}{r^2} \left[\psi_{\mathrm{I}} + \left(1 + \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) \psi_{\mathrm{II}} \right] - p \,, \tag{4.11a}$$

$$\sigma^{\theta\theta} = \frac{1}{r^2} \left\{ 2\frac{r^2 e^{-2\omega_\Theta}}{R^2} \left[\psi_{\mathrm{I}} + \left(1 + \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) \psi_{\mathrm{II}} \right] - p \right\} \,, \tag{4.11b}$$

$$\sigma^{zz} = 2 \left[\psi_{\mathrm{I}} + \left(\frac{R^2 e^{2\omega_\Theta}}{r^2} + \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) \psi_{\mathrm{II}} \right] - p \,. \tag{4.11c}$$

The only non-trivial equilibrium equation is $\sigma^{ra}{}_{|a} = 0$, which is simplified to read

$$\frac{r e^{-\omega_R - \omega_\Theta}}{R} \sigma^{rr}{}_{,R} + \frac{1}{r}\sigma^{rr} - r\sigma^{\theta\theta} = 0 \,. \tag{4.12}$$

This yields

$$\frac{\partial p}{\partial R} = \frac{\partial}{\partial R} \left\{ 2\frac{R^2 e^{2\omega_\Theta}}{r^2} \left[\psi_{\mathrm{I}} + \left(1 + \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) \psi_{\mathrm{II}} \right] \right\} - 2\frac{e^{\omega_R}}{r} \frac{R e^{\omega_\Theta}}{r} \left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \,. \tag{4.13}$$

Assuming that the boundary of the cylinder is under uniform normal traction, i.e., $\sigma^{rr}(R_o, T(R_o)) = -\sigma_o$, the pressure field at the boundary is

$$p(R_o) = \left\{ 2\frac{R^2 e^{2\omega_\Theta}}{r^2} \left[\psi_{\mathrm{I}} + \left(1 + \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) \psi_{\mathrm{II}} \right] \right\} \Big|_{R=R_o} + \sigma_o \,, \tag{4.14}$$

and it follows that

$$p(R,t) = 2\frac{R^2 e^{2\omega_\Theta}}{r^2} \left[\psi_{\mathrm{I}} + \left(1 + \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) \psi_{\mathrm{II}} \right] + \int_R^{R_o} 2\frac{\xi}{r^2} e^{\omega_R + \omega_\Theta} \left(\frac{r^2 e^{-2\omega_\Theta}}{\xi^2} - \frac{\xi^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}})\, d\xi + \sigma_o \,. \tag{4.15}$$

Finally, given a radially-symmetric temperature field $T = T(R, t)$, the thermal stress field is given by the following non-zero components of the Cauchy stress tensor

$$\sigma^{rr} = -\int_R^{R_o} 2\frac{\xi}{r^2} e^{\omega_R + \omega_\Theta} \left(\frac{r^2 e^{-2\omega_\Theta}}{\xi^2} - \frac{\xi^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}})\, d\xi - \sigma_o \,, \tag{4.16a}$$

$$\sigma^{\theta\theta} = \frac{1}{r^2}\sigma^{rr} + \frac{2}{r^2} \left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \,, \tag{4.16b}$$

$$\sigma^{zz} = \sigma^{rr} + 2 \left(1 - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) \left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) . \tag{4.16c}$$

Alternatively, the non-zero components of the second Piola-Kirchhoff stress tensor read

$$S^{RR} = -\frac{r^2 e^{-2(\omega_\Theta + \omega_R)}}{R^2} \int_R^{R_o} 2\frac{\xi}{r^2} e^{\omega_R + \omega_\Theta} \left(\frac{r^2 e^{-2\omega_\Theta}}{\xi^2} - \frac{\xi^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}})\, d\xi \,, \tag{4.17a}$$

$$S^{\Theta\Theta} = \frac{R^2 e^{2(\omega_\Theta + \omega_R)}}{r^4} S^{RR} + \frac{2}{r^2} \left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \,, \tag{4.17b}$$

$$S^{ZZ} = \frac{R^2 e^{2(\omega_\Theta + \omega_R)}}{r^2} S^{RR} + 2 \left(1 - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) \left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) . \tag{4.17c}$$

Heat equation. We assume that there is no external heat supply, i.e., $R = 0$, and that the heat conduction in the material is isotropic, i.e., $\boldsymbol{K} = k\boldsymbol{G}^{-1}$, where $k = k(R, T(R,t))$ is a scalar-valued function. Therefore, using (3.19), the coupled heat equation (3.49) reads

$$\left[\frac{\partial^2 T}{\partial R^2} + \left(\frac{1}{R} + \frac{1}{k}\left.\frac{\partial k}{\partial R}\right|_t\right)\frac{\partial T}{\partial R} + \left.\frac{\partial\left(\omega_\Theta - \omega_R\right)}{\partial R}\right|_t \frac{\partial T}{\partial R}\right] k e^{-2\omega_R} = \rho_o c_E \dot{T} e^{-\omega_\Theta - \omega_R} - \frac{1}{2} T \frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,, \qquad (4.18)$$

where $\rho_o(X) = \rho(X, \boldsymbol{G}_o)$ is the mass density in the stress-free configuration with uniform temperature T_0. If we further assume that the material is thermally isotropic (i.e., $\omega_R = \omega_\Theta = \omega(R, T(R,t))$), then (4.18) reduces to

$$\frac{k}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) + \left.\frac{\partial k}{\partial R}\right|_t \frac{\partial T}{\partial R} = \rho_o c_E \dot{T} - \frac{e^{2\omega}}{2} T \frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,. \qquad (4.19)$$

Before solving the problem, we find the stress-free temperature fields and obtain expressions for ω_R and ω_Θ to explicitly specify the material metric $\boldsymbol{G}$ (cf. (4.2)). A temperature field $T_0(R)$ is stress-free if and only if the Riemann curvature tensor of the cylinder is identically zero at $T = T_0$. The non-trivially non-zero components of curvature tensor for the cylinder with the metric $\boldsymbol{G}$ are

$$\mathcal{R}^R{}_{\Theta\Theta R} = -\mathcal{R}^R{}_{\Theta R\Theta} = R\left[2\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t - \left.\frac{\partial\omega_R}{\partial R}\right|_t + R\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t\left(\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t - \left.\frac{\partial\omega_R}{\partial R}\right|_t\right) + R\left.\frac{\partial^2\omega_\Theta}{\partial R^2}\right|_t\right] e^{2(\omega_\Theta - \omega_R)}\,, \qquad (4.20a)$$

$$\mathcal{R}^\Theta{}_{RR\Theta} = -\mathcal{R}^\Theta{}_{R\Theta R} = \frac{1}{R}\left[2\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t - \left.\frac{\partial\omega_R}{\partial R}\right|_t + R\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t\left(\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t - \left.\frac{\partial\omega_R}{\partial R}\right|_t\right) + R\left.\frac{\partial^2\omega_\Theta}{\partial R^2}\right|_t\right]. \qquad (4.20b)$$

Therefore, $T_0(R)$ is stress free if and only if

$$\left.\left[2\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t - \left.\frac{\partial\omega_R}{\partial R}\right|_t + R\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t\left(\left.\frac{\partial\omega_\Theta}{\partial R}\right|_t - \left.\frac{\partial\omega_R}{\partial R}\right|_t\right) + R\left.\frac{\partial^2\omega_\Theta}{\partial R^2}\right|_t\right]\right|_{(R,T_0)} = 0\,. \qquad (4.21)$$

If we assume that the material is thermally homogeneous, i.e., for $K \in \{R, \Theta\}$, we have $\alpha_K = \alpha_K(T)$ independent of position, then $\left.\frac{\partial\omega_K}{\partial R}\right|_t = \alpha_K\frac{\partial T}{\partial R}$. Therefore, (4.21) becomes

$$(2\alpha_\Theta - \alpha_R)\frac{dT_0}{dR} + R\alpha_\Theta\left(\alpha_\Theta - \alpha_R\right)\left(\frac{dT_0}{dR}\right)^2 + R\alpha_\Theta\frac{d^2T_0}{dR^2} = 0\,. \qquad (4.22)$$

We observe that a uniform temperature field is stress-free in the general anisotropic case. In the particular case of a thermally homogeneous and isotropic material with a constant linear expansion coefficient α, we find two possible stress-free temperature fields

$$T_0(R) = T_0\,, \qquad (4.23a)$$

$$T_0(R) = a\ln R + b\,, \qquad (4.23b)$$

for some constants a and b. We assume in the remainder of this section that the material is thermally homogeneous and that the metric $\boldsymbol{G}_0$ corresponds to the uniform stress-free temperature field of the cylinder $T_0(R) = T_o$ (i.e., $\boldsymbol{G}|_{T=T_o} = \boldsymbol{G}_0$), where T_o is the temperature of the outside medium surrounding the cylinder. For $K \in \{R, \Theta\}$ we write following (2.20)

$$\omega_K(R, T(R,t)) = \int_{T_o}^{T(R,t)} \alpha_K(\tau)d\tau\,, \quad K \in \{R, \Theta\}\,. \qquad (4.24)$$

Example 4.1. In this example we solve for the stress field induced by a thermal inclusion in a homogeneous, isotropic, thermally homogeneous and anisotropic solid cylinder. We then assume that the material is thermally isotropic to compare with the linear elasticity solution. We consider a thermal inclusion of radius $R_i < R_o$:

$$T(R) = \begin{cases} T_i & R \le R_i\,, \\ T_o & R > R_i\,. \end{cases} \qquad (4.25)$$

Thus, for $K \in \{R, \Theta\}$ we have

$$\omega_K(R) = \begin{cases} \displaystyle\int_{T_o}^{T_i} \alpha_K(\tau) d\tau & R \leq R_i \,, \\ 0 & R > R_i \,. \end{cases} \tag{4.26}$$

We substitute (4.26) into (4.8) to find

$$r(R) = \begin{cases} R e^{\frac{1}{2}(\omega_R(R_i)+\omega_\Theta(R_i))} & R \leq R_i \,, \\ \left[R_i^2 \left(e^{(\omega_R(R_i)+\omega_\Theta(R_i))} - 1 \right) + R^2 \right]^{\frac{1}{2}} & R \geq R_i \,. \end{cases} \tag{4.27}$$

Following (4.16), the physical components of the Cauchy stress tensor read

$$\hat{\sigma}^{rr} = -\int_R^{R_o} 2 \frac{\xi}{r^2} e^{(\omega_R+\omega_\Theta)} \left(\frac{r^2 e^{-2\omega_\Theta}}{\xi^2} - \frac{\xi^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \, d\xi - \sigma_o \,, \tag{4.28a}$$

$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{rr} + 2 \left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \,, \tag{4.28b}$$

$$\hat{\sigma}^{zz} = \hat{\sigma}^{rr} + 2 \left(1 - \frac{R^2 e^{2\omega_\Theta}}{r^2} \right) \left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) . \tag{4.28c}$$

For $R \leq R_i$, following (4.9), we have

$$\boldsymbol{C} = \begin{pmatrix} e^{\omega_\Theta(R_i)-\omega_R(R_i)} & 0 & 0 \\ 0 & e^{\omega_R(R_i)-\omega_\Theta(R_i)} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.29}$$

Therefore, I and II are constant inside the inclusion. As the material is homogeneous and isotropic (i.e., $\psi = \psi(T, \mathrm{I}, \mathrm{II})$), the terms ψ_{I} and ψ_{II} are constant inside the inclusion and it follows that for $R \leq R_i$, we have

$$\hat{\sigma}^{rr}(R) = 2 \left[e^{(\omega_R(R_i)-\omega_\Theta(R_i))} - e^{-(\omega_R(R_i)-\omega_\Theta(R_i))} \right] (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \ln\left(\frac{R}{R_i} \right) + \hat{\sigma}_c \,, \tag{4.30a}$$

$$\hat{\sigma}^{\theta\theta}(R) = 2 \left[e^{(\omega_R(R_i)-\omega_\Theta(R_i))} - e^{-(\omega_R(R_i)-\omega_\Theta(R_i))} \right] (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \left[\ln\left(\frac{R}{R_i} \right) + 1 \right] + \hat{\sigma}_c \,, \tag{4.30b}$$

$$\begin{aligned} \hat{\sigma}^{zz}(R) = {} & 2 \left[e^{(\omega_R(R_i)-\omega_\Theta(R_i))} - e^{-(\omega_R(R_i)-\omega_\Theta(R_i))} \right] (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \ln\left(\frac{R}{R_i} \right) \\ & + \left(e^{-\omega_\Theta(R_i)} - e^{-\omega_R(R_i)} \right) \left(\psi_{\mathrm{I}} e^{\omega_\Theta(R_i)} + \psi_{\mathrm{II}} e^{\omega_R(R_i)} \right) + \hat{\sigma}_c \,, \end{aligned} \tag{4.30c}$$

where $\hat{\sigma}_c$ is a constant given by

$$\hat{\sigma}_c = -2 \int_{R_i}^{R_o} \left(\frac{1}{\xi} - \frac{\xi^3}{r^4(\xi)} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \, d\xi - \sigma_o \,. \tag{4.31}$$

On the other hand, for $R > R_i$, we have

$$\hat{\sigma}^{rr}(R) = -2 \int_R^{R_o} \left(\frac{1}{\xi} - \frac{\xi^3}{r^4(\xi)} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \, d\xi - \sigma_o \,, \tag{4.32a}$$

$$\hat{\sigma}^{\theta\theta}(R) = \hat{\sigma}^{rr} + 2 \left(\frac{r^2(R)}{R^2} - \frac{R^2}{r^2(R)} \right) (\psi_{\mathrm{I}} + \psi_{\mathrm{II}}) \,, \tag{4.32b}$$

$$\hat{\sigma}^{zz}(R) = \hat{\sigma}^{rr} + 2 \left(1 - \frac{R^2}{r^2(R)} \right) \left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} \frac{r^2(R)}{R^2} \right) . \tag{4.32c}$$

Remark 4.1. Note that in the absence of body forces, for a homogeneous, isotropic, thermally homogeneous and isotropic infinite solid cylinder with a uniform normal traction on its boundary and with a radially-symmetric thermal inclusion, the thermal stress inside the inclusion is uniform and hydrostatic. However, if the material is thermally anisotropic (i.e., $\omega_R \neq \omega_\Theta$) the thermal stress field has a logarithmic singularity on the axis of the cylinder. Similar results have been observed for distributed point defects in [Yavari and Goriely, 2012b] and for distributed eigenstrains in [Yavari and Goriely, 2013, 2015].

Comparison with the linear solution. Next, we compare the static thermal stress field with the classical linear elasticity solution. We consider a homogeneous, isotropic, thermally homogeneous and isotropic and traction-free infinite solid circular cylinder. Note that the infinite cylinder corresponds to the case of a disk in plane strain. In classical linearized elasticity, the solution in plane strain for a solid disk with a radial temperature field and constant μ_o and α_o reads [Boley and Weiner, 1960; Hetnarski and Eslami, 2008][10]

$$\hat{\sigma}^{rr} = 2\alpha_o \frac{\mu_o}{1-\nu}\left[\frac{1}{R_o^2}\int_0^{R_o} T(\xi)\xi d\xi - \frac{1}{R^2}\int_0^{R} T(\xi)\xi d\xi\right], \tag{4.33a}$$

$$\hat{\sigma}^{\theta\theta} = 2\alpha_o \frac{\mu_o}{1-\nu}\left[\frac{1}{R_o^2}\int_0^{R_o} T(\xi)\xi d\xi + \frac{1}{R^2}\int_0^{R} T(\xi)\xi d\xi - T\right], \tag{4.33b}$$

$$\hat{\sigma}^{zz} = \nu\left(\hat{\sigma}^{rr} + \hat{\sigma}^{\theta\theta}\right). \tag{4.33c}$$

Incompressible linearized elasticity corresponds to $\nu = 0.5$, and considering the thermal inclusion (4.25), we find

$$\begin{aligned}
&R \leq R_i : \ \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = -2\mu_o\alpha_o\Delta_i T\left(1 - \frac{R_i^2}{R_o^2}\right), \\
&R > R_i : \ \begin{cases} \hat{\sigma}^{rr} = -2\mu_o\alpha_o\Delta_i T\left(\dfrac{R_i^2}{R^2} - \dfrac{R_i^2}{R_o^2}\right), \\ \hat{\sigma}^{\theta\theta} = 2\mu_o\alpha_o\Delta_i T\left(\dfrac{R_i^2}{R^2} + \dfrac{R_i^2}{R_o^2}\right), \\ \hat{\sigma}^{zz} = 2\mu_o\alpha_o\Delta_i T\dfrac{R_i^2}{R_o^2}, \end{cases}
\end{aligned} \tag{4.34}$$

where $\Delta_i T = T_i - T_o$.

In order to compare the nonlinear solution with the linear one, we consider the thermoelastic model presented in Appendix A and enforce the incompressibility condition. Hence, following (A.11), we find for the thermal inclusion (4.25)

$$\omega(R) = \begin{cases} \dfrac{1}{2}\ln\left[\dfrac{(1+2\alpha_o T_o)\Delta_i T + T_o}{\Delta_i T + T_o}\right] & R \leq R_i, \\ 0 & R > R_i, \end{cases} \tag{4.35}$$

and from (4.28) and (A.7) the thermal stress field reads

$$\begin{aligned}
&R \leq R_i : \ \left\{ \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = -\mu_o\int_{R_i}^{R_o}\left(\frac{1}{\xi} - \frac{\xi^3}{r^4(\xi)}\right)d\xi, \right. \\
&R > R_i : \ \begin{cases} \hat{\sigma}^{rr} = -\mu_o\displaystyle\int_{R}^{R_o}\left(\frac{1}{\xi} - \frac{\xi^3}{r^4(\xi)}\right)d\xi, \\ \hat{\sigma}^{\theta\theta} = \mu_o\left(\dfrac{r^2(R)}{R^2} - \dfrac{R^2}{r^2(R)}\right) + \hat{\sigma}^{rr}, \\ \hat{\sigma}^{zz} = \mu_o\left(1 - \dfrac{R^2}{r^2(R)}\right) + \hat{\sigma}^{rr}, \end{cases}
\end{aligned} \tag{4.36}$$

where

$$r(R) = \begin{cases} \left[\dfrac{(1+2\alpha_o T_o)\Delta_i T + T_o}{\Delta_i T + T_o}\right]^{\frac{1}{2}} R & R \leq R_i, \\ \left[2\alpha_o T_o\dfrac{\Delta_i T}{\Delta_i T + T_o}R_i^2 + R^2\right]^{\frac{1}{2}} & R \geq R_i. \end{cases} \tag{4.37}$$

[10] Eq. (4.33) is obtained by replacing E by $\frac{E}{1-\nu^2} = \frac{2\mu_o}{1-\nu}$ in equation (9.10.5) in [Boley and Weiner, 1960] to recover the plane strain solution. Note that α_o remains unchanged because we have an infinite cylinder where the thermal longitudinal stretch does not affect the solution (c.f. discussion in the beginning of this section regarding $\alpha_z = 0$).

Therefore

$$
R \leq R_i \ : \ \begin{cases} \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = \mu_o \ln\left(\dfrac{R_i}{R_o}\right) + \dfrac{\mu_o}{2} \ln\left[\dfrac{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + 1}{\frac{(1+2\alpha_o T_o)\Delta_i T + T_o}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2}\right] \\ \qquad\qquad + \dfrac{\mu_o}{2} \dfrac{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2}{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + 1} - \dfrac{2\alpha_o T_o \Delta_i T}{(1+2\alpha_o T_o)\Delta_i T + T_o} \,, \end{cases}
$$

$$
R > R_i \ : \ \begin{cases} \hat{\sigma}^{rr} = \mu_o \ln\left(\dfrac{R}{R_o}\right) + \dfrac{\mu_o}{2} \ln\left[\dfrac{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + 1}{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + \left(\frac{R}{R_o}\right)^2}\right] \\ \qquad + \dfrac{\mu_o}{2} \dfrac{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2}{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + 1} - \dfrac{\mu_o}{2} \dfrac{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2}{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + \left(\frac{R}{R_o}\right)^2} \,, \\ \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{rr} + \mu_o \dfrac{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + \left(\frac{R}{R_o}\right)^2}{\left(\frac{R}{R_o}\right)^2} - \mu_o \dfrac{\left(\frac{R}{R_o}\right)^2}{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + \left(\frac{R}{R_o}\right)^2} \,, \\ \hat{\sigma}^{zz} = \hat{\sigma}^{rr} + \mu_o - \mu_o \dfrac{\left(\frac{R}{R_o}\right)^2}{2\alpha_o T_o \frac{\Delta_i T}{\Delta_i T + T_o}\left(\frac{R_i}{R_o}\right)^2 + \left(\frac{R}{R_o}\right)^2} \,. \end{cases} \tag{4.38}
$$

For small $\Delta_i T$, we have the following asymptotic expansions:

$$
\begin{aligned} & R \leq R_i : \\ & \begin{cases} \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = -2\mu_o \left(1 - \dfrac{R_i^2}{R_o^2}\right) \alpha_o \Delta_i T + \mu_o \left[\dfrac{2}{\alpha_o T_o}\left(1 - \dfrac{R_i^2}{R_o^2}\right) + 3\left(1 - \dfrac{R_i^4}{R_o^4}\right)\right] (\alpha_o \Delta_i T)^2 \\ \qquad\qquad + o\left((\Delta_i T)^3\right) \,, \end{cases} \\ & R > R_i : \\ & \begin{cases} \hat{\sigma}^{rr} = -2\mu_o \left(\dfrac{R_i^2}{R^2} - \dfrac{R_i^2}{R_o^2}\right) \alpha_o \Delta_i T + \mu_o \left[\dfrac{2}{\alpha_o T_o}\left(\dfrac{R_i^2}{R^2} - \dfrac{R_i^2}{R_o^2}\right) + 3\left(\dfrac{R_i^4}{R^4} - \dfrac{R_i^4}{R_o^4}\right)\right] (\alpha_o \Delta_i T)^2 \\ \qquad + o\left((\Delta_i T)^3\right) \,, \\ \hat{\sigma}^{\theta\theta} = 2\mu_o \left(\dfrac{R_i^2}{R^2} + \dfrac{R_i^2}{R_o^2}\right) \alpha_o \Delta_i T - \mu_o \left[\dfrac{2}{\alpha_o T_o}\left(\dfrac{R_i^2}{R^2} + \dfrac{R_i^2}{R_o^2}\right) + \dfrac{R_i^4}{R^4} + 3\dfrac{R_i^4}{R_o^4}\right] (\alpha_o \Delta_i T)^2 + o\left((\Delta_i T)^3\right) \,, \\ \hat{\sigma}^{zz} = 2\mu_o \dfrac{R_i^2}{R_o^2} \alpha_o \Delta_i T - \mu_o \left(\dfrac{2}{\alpha_o T_o}\dfrac{R_i^2}{R_o^2} + 3\dfrac{R_i^4}{R_o^4} + \dfrac{R_i^4}{R^4}\right) (\alpha_o \Delta_i T)^2 + o\left((\Delta_i T)^3\right) \,. \end{cases} \end{aligned} \tag{4.39}
$$

We have therefore recovered, up to the first order in $\Delta_i T$, the classical linear elasticity solution.

We consider the case of rubber-like solids for which we typically have $\alpha_o = 6 \times 10^{-4}\,\mathrm{K}^{-1}$ at 300°K, i.e., $\alpha_o T_o = 0.18$. In Figures 1 and 2, we plot the static thermal stresses for different values of the initial relative temperature difference $\delta_T = \frac{\Delta_i T}{T_o}$ in the thermal inclusion (4.25). We see, in Figure 1, that the two solutions for the fields σ^{rr} and $\sigma^{\theta\theta}$ are very close for small values of δ_T (i.e., in the range of validity of linearized elasticity). However, for larger values of δ_T, even though linearized elasticity captures the overall stress behavior, it fails by overestimating the stresses σ^{rr} and $\sigma^{\theta\theta}$ (the relative difference of stress reaches 38% inside the inclusion for $\delta_T = 30\%$). In Figure 2, we show the longitudinal stress field σ^{zz} and note that, outside the inclusion, it is two orders of magnitude smaller than the uniform hydrostatic stress inside the inclusion. We also note that, outside the inclusion, the nonlinear solution predicts a non-constant σ^{zz} stress field unlike the constant stress predicted by the linear solution.

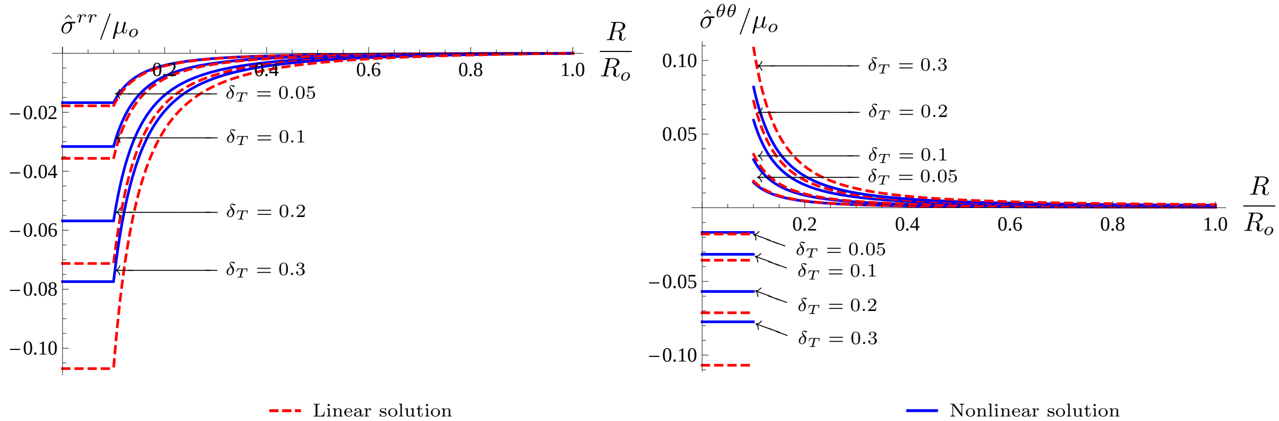

Figure 1: *Nonlinear and linear solutions for* σ^{rr} *and* $\sigma^{\theta\theta}$ *for* $\frac{R_i}{R_o} = 0.1$, $\alpha_o T_o = 0.18$ *and different values of* $\delta_T = \frac{\Delta_i T}{T_o}$.

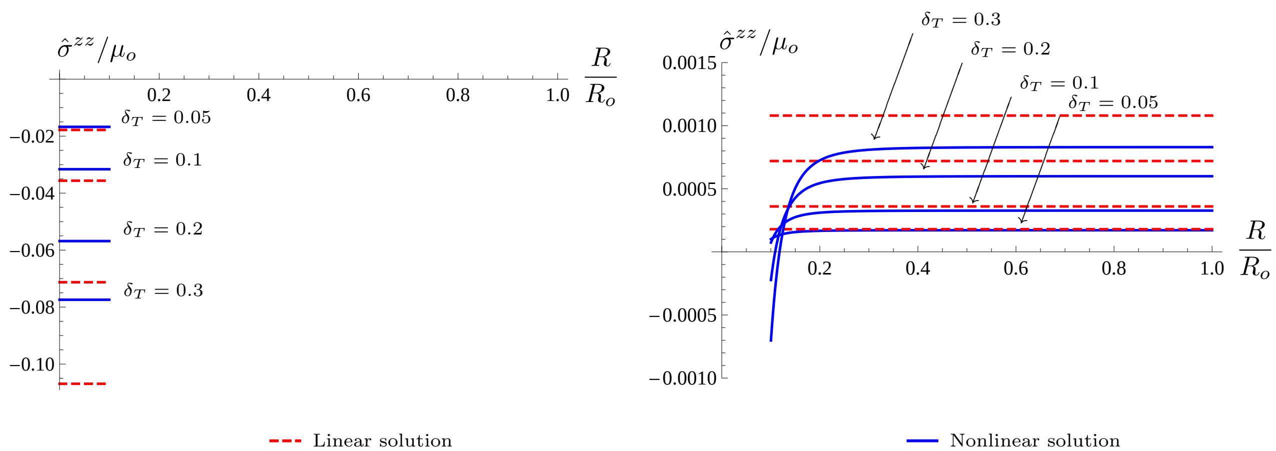

Figure 2: *Nonlinear and linear solutions for* σ^{zz} *for* $\frac{R_i}{R_o} = 0.1$, $\alpha_o T_o = 0.18$ *and different values of* $\delta_T = \frac{\Delta_i T}{T_o}$ *(left: stresses inside the thermal inclusion; right: stresses outside the thermal inclusion).*

Example 4.2. In this example we numerically solve for the evolution of temperature and thermal stress fields for a homogeneous, isotropic, thermally homogeneous and isotropic solid cylinder for which we assume the thermoelastic model described in Appendix A. Following (A.11), we find

$$\omega(R, T(R,t)) = \frac{1}{2} \ln \left[1 + 2\alpha_0 T_0 \left(1 - \frac{T_0}{T(R,t)}\right)\right] . \tag{4.40}$$

Thus

$$r(R,t) = \left\{ \int_0^R 2\xi \left[1 + 2\alpha_0 T_0 \left(1 - \frac{T_0}{T(\xi,t)}\right)\right] d\xi \right\}^{\frac{1}{2}} = \left[\left(1 + 2\alpha_0 T_0\right) R^2 - 4\alpha_0 T_0 \int_0^R \frac{T_0 \xi}{T(\xi,t)} d\xi \right]^{\frac{1}{2}} . \tag{4.41}$$

Following (4.16) and the free energy density (A.7), the physical components of the Cauchy stress field are given by

$$\hat{\sigma}^{rr} = -\mu_o \int_R^{R_o} \left[\frac{1}{\xi} - \frac{\left(1+2\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right)^2 \xi^3}{r^4}\right] \frac{T}{T_o} d\xi\,, \tag{4.42a}$$

$$\hat{\sigma}^{\theta\theta} = \mu_o \left[\frac{r^2}{\left(1+2\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right) R^2} - \frac{\left(1+2\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right) R^2}{r^2}\right] \frac{T}{T_o} + \hat{\sigma}^{rr}\,, \tag{4.42b}$$

$$\hat{\sigma}^{zz} = \mu_o \left[1 - \frac{\left(1+2\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right) R^2}{r^2}\right] \frac{T}{T_o} + \hat{\sigma}^{rr}\,. \tag{4.42c}$$

Now, let us find the time-dependent temperature field in order to evaluate thermal stresses by solving the coupled heat equation (4.19). We assume that the heat conduction coefficient depends only on temperature and consider the following empirical model for elastomer vulcanizates (cf. [Sircar and Wells, 1982]), suggesting that the heat conduction coefficient decreases with temperature

$$k(T(R,t)) = k_o\left[1 - s(T(R,t) - T_o)\right], \tag{4.43}$$

where $k_o = k(T_o)$ and s is a softening parameter. Therefore, (4.19) reads

$$k_o\left[1 - s(T-T_o)\right] \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) - k_o s \left(\frac{\partial T}{\partial R}\right)^2 = \rho_o c_E \dot{T} - \frac{e^{2\omega}}{2} T \frac{\partial S^{AB}}{\partial T} \dot{C}_{AB}\,. \tag{4.44}$$

Following (4.17), the non-zero components of the second Piola-Kirchhoff stress tensor read

$$S^{RR} = -\mu_o \frac{r^2 e^{-4\omega}}{R^2} \int_R^{R_o} \frac{T(\xi,t)}{T_o}\left(\frac{1}{\xi} - \frac{\xi^3 e^{4\omega(\xi,T(\xi,t))}}{r^4(\xi,t)}\right) d\xi\,, \tag{4.45a}$$

$$S^{\Theta\Theta} = \frac{R^2 e^{4\omega}}{r^6} S^{RR} + \mu_o \frac{T}{T_o}\left(\frac{e^{-2\omega}}{R^2} - \frac{R^2 e^{2\omega}}{r^4}\right), \tag{4.45b}$$

$$S^{ZZ} = \frac{R^2 e^{4\omega}}{r^2} S^{RR} + \mu_o \frac{T}{T_o}\left(1 - \frac{R^2 e^{2\omega}}{r^2}\right). \tag{4.45c}$$

Hence[11]

$$\frac{\partial S^{RR}}{\partial T} = 0\,, \tag{4.46a}$$

$$\frac{\partial S^{\Theta\Theta}}{\partial T} = \frac{\mu_o}{T_o}\left(\frac{e^{-2\omega}}{R^2} - \frac{R^2 e^{2\omega}}{r^4}\right), \tag{4.46b}$$

$$\frac{\partial S^{ZZ}}{\partial T} = \frac{\mu_o}{T_o}\left(1 - \frac{R^2 e^{2\omega}}{r^2}\right). \tag{4.46c}$$

The non-vanishing components of $\dot{\boldsymbol{C}}$ are

$$\dot{C}_{RR} = \frac{\partial}{\partial t}\left(\frac{Re^{2\omega}}{r}\right)^2 = 2\left(\frac{Re^{2\omega}}{r}\right)^2\left(2\alpha\dot{T} - \frac{1}{r}\frac{\partial r}{\partial t}\right), \quad \dot{C}_{\Theta\Theta} = \frac{\partial r^2}{\partial t} = 2r\frac{\partial r}{\partial t}\,. \tag{4.47}$$

Also, from (4.41), we find

$$\frac{\partial r}{\partial t} = \frac{2\alpha_o}{r}\int_0^R \xi\left(\frac{T_o}{T(\xi,t)}\right)^2 \dot{T}(\xi,t) d\xi\,. \tag{4.48}$$

Thus, the coupled heat equation (4.19) reads

$$\begin{aligned} &\left[1 - s(T-T_o)\right]\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) - s\left(\frac{\partial T}{\partial R}\right)^2 = \frac{\rho_o c_E}{k_o}\dot{T} \\ &-2\frac{\mu_o}{k_o}\alpha_o T_o T\left[\frac{1}{R^2} - \frac{\left(1+2\alpha_0 T_0\left(1-T_0/T\right)\right)^2 R^2}{\left(\left(1+2\alpha_0 T_0\right) R^2 - 4\alpha_0 T_0 \int_0^R \xi T_0/T d\xi\right)^2}\right]\int_0^R \frac{\xi\dot{T}}{T^2} d\xi\,. \end{aligned} \tag{4.49}$$

[11] Recall that $\frac{\partial}{\partial T} = \frac{\partial}{\partial T}|_{\boldsymbol{C},\boldsymbol{G}}$ is a partial derivative with respect to T with $\boldsymbol{C}$ and $\boldsymbol{G}$ fixed (cf. Remark 3.3).

On the boundary of the cylinder, we consider a convection boundary condition, i.e.

$$k_o \left[1 - s(T(R_o,t) - T_o)\right] \frac{\partial T}{\partial R}\bigg|_{(R_o,t)} = h_o[T_o - T(R_o,t)]\,, \tag{4.50}$$

where h_o is the surface heat transfer coefficient at the boundary of the cylinder. We assume that h_o is constant and introduce the parameter $\gamma = h_o R_o / k_o$. As an initial temperature field, we consider a thermal inclusion of radius R_i, i.e.

$$T_{\text{init}}(R) = \begin{cases} T_i & R \leq R_i\,, \\ T_o & R > R_i\,. \end{cases} \tag{4.51}$$

In the scope of the classical theory of linearized elasticity, the thermal stresses are given by [Boley and Weiner, 1960]:

$$\hat{\sigma}^{rr} = 4\alpha_o\mu_o \left[\frac{1}{R_o^2}\int_0^{R_o} T(\xi,t)\xi d\xi - \frac{1}{R^2}\int_0^{R} T(\xi,t)\xi d\xi\right], \tag{4.52a}$$

$$\hat{\sigma}^{\theta\theta} = 4\alpha_o\mu_o \left[\frac{1}{R_o^2}\int_0^{R_o} T(\xi,t)\xi d\xi + \frac{1}{R^2}\int_0^{R} T(\xi,t)\xi d\xi - T\right], \tag{4.52b}$$

$$\hat{\sigma}^{zz} = \frac{1}{2}\left(\hat{\sigma}^{rr} + \hat{\sigma}^{\theta\theta}\right). \tag{4.52c}$$

In the classical linearized elasticity literature, the coupling term is neglected and the linearized heat equation problem for the cylinder reads

$$\begin{cases} \dfrac{1}{R}\dfrac{\partial}{\partial R}\left(R\dfrac{\partial T}{\partial R}\right) = \dfrac{\rho_o c_E}{k_o}\dot{T}\,, \\ T(R,0) = T_{\text{init}}(R)\,, \\ \dfrac{\partial T}{\partial R}\bigg|_{(R_o,t)} = \dfrac{\gamma}{R_o}[T_o - T(R_o,t)]\,. \end{cases} \tag{4.53}$$

The solution to (4.53) can be found analytically by using the Dini series expansion (see [Carslaw and Jaeger, 1986; Watson, 1944] for a detailed derivation.)

$$T(R,t) = 2\Delta_i T \sum_{n=1}^{\infty} \frac{\zeta_n J_0\left(\zeta_n \frac{R}{R_o}\right) J_1\left(\zeta_n \frac{R_i}{R_o}\right)}{(\gamma^2 + \zeta_n^2) J_0^2(\zeta_n)} \frac{R_i}{R_o} e^{-\frac{\zeta_n^2 t}{\tau}} + T_o\,, \tag{4.54}$$

where ζ_n are the positive solutions of $\zeta J_1(\zeta) = \gamma J_0(\zeta)$ and J_k is the Bessel function of the first kind of order $k = 1, 2$.

We consider a rubber cylinder of radius $R_o = 15\,\text{cm}$ for which the surface heat transfer coefficient for the rubber-air convection is $h_o = 10\,\text{W/m}^2\text{.K}$. We let $T_o = 300°\text{K}$ and $\delta_T = \frac{\Delta_i T}{T_o} = 30\%$ and assume the following typical values of rubber-like materials: $\rho_o = 10^3\,\text{kg/m}^3$, $c_E = 1800\,\text{J/kg.m}$, $k_o = 0.15\,\text{W/m.K}$, $s = 0{,}004\,\text{K}^{-1}$, $\alpha_o = 6\times 10^{-4}\,\text{K}^{-1}$, and $\mu_o = 0.54\,\text{GPa}$. We numerically solve the initial/boundary value problem (4.49), (4.50), (4.51) for the temperature field $T(R,t)$ and show its evolution in Figure 3 by plotting $\frac{T(R,t)-T_o}{T_o}$ at different values of t/τ, where τ is a characteristic time defined as $\tau = \rho_o c_E R_o^2 / k_o$. In Figures 4-6, we show the evolution of nonlinear thermal stresses (4.42). For comparison purposes, we also show the evolution of the linearized solution (4.52) and (4.54) in Figures 3-6. We observe that the initial irregularities in the initial temperature and thermal stress fields are smoothed out, and at large times, both the temperature difference $T - T_o$ and thermal stress fields tend to zero. The nonlinear and linearized solutions for the temperature field (Figure 3) show a similar trend. However, we observe a significant difference for thermal stress fields between the linear (4.52) and nonlinear (4.42) solutions (the maximum relative difference is 38% in the core of the inclusion for $\hat{\sigma}^{rr}$, $\hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{zz}$, see Figures 3-6). We also observe that in the nonlinear solution the maximum thermal stress does not necessarily correspond to $t = 0$, i.e. we observe the maximum stress at a later time $t > 0$.

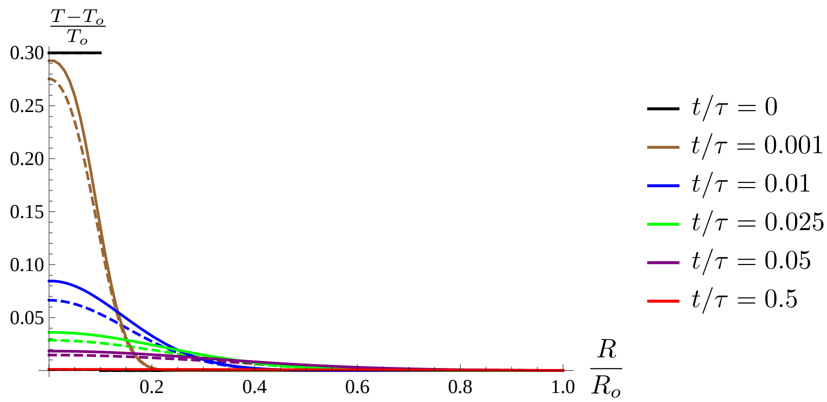

Figure 3: *Temperature field in the cylinder for* $\frac{R_i}{R_o} = 0.1$, $\gamma = 10$, $\alpha_o T_o = 0.18$, $\mu_o/\rho_o c_E T_o = 0.001$ *and* $\delta_T = 30\%$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

Remark 4.2. We observe numerically that the coupling term in the nonlinear heat equation (4.49) is negligeable. In fact, if we neglect the coupling term in (4.49), the resulting solution is only affected in the order of 10^{-6}. We further note that if we neglect the coupling term and assume a constant heat conduction coefficient, the nonlinear and linear uncoupled heat equations would be identical in the case of the infinite cylinder geometry (we will see later in this section that this is not the case in the solid sphere geometry, cf. Remark 4.4). Note however that the stress formulas are not affected by this simplification and that we would still observe a significant difference in the thermal stresses between the linear and nonlinear solutions.

4.2 A spherical ball made of an incompressible isotropic solid

In this section we consider an incompressible isotropic solid sphere of radius R_o under uniform normal traction on its boundary and ignore body forces. In spherical coordinates (R, Θ, Φ), for which $R \geq 0$, $0 \leq \Theta \leq \pi$, and $0 \leq \Phi < 2\pi$, the material metric for the configuration with the stress-free temperature field $T_0 = T_0(R)$, reads

$$\boldsymbol{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2 \Theta \end{pmatrix} . \tag{4.55}$$

We assume a radially-symmetric temperature field $T = T(R,t)$ in the ball and let $\alpha_R = \alpha_R(R,T)$ be the radial thermal expansion coefficient and $\alpha_\Theta = \alpha_\Theta(R,T)$ be the circumferential thermal expansion coefficient of the ball. The temperature-dependent material metric of the ball, as introduced in Section 2.1, reads

$$\boldsymbol{G} = \begin{pmatrix} e^{2\omega_R} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta} & 0 \\ 0 & 0 & R^2 e^{2\omega_\Theta} \sin^2 \Theta \end{pmatrix} , \tag{4.56}$$

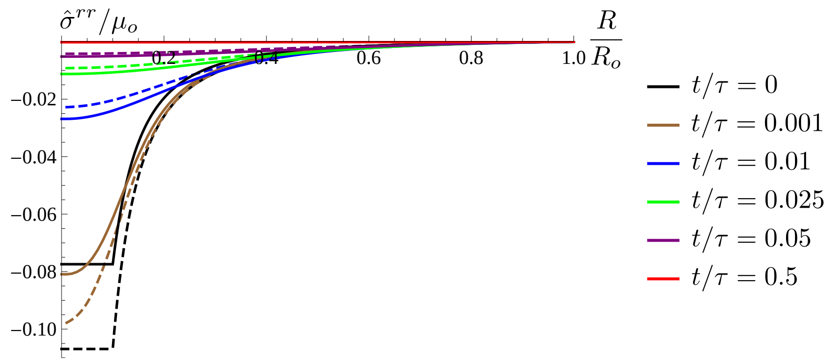

Figure 4: *Stress field* σ^{rr} *for* $\frac{R_i}{R_o} = 0.1$, $\gamma = 10$, $\alpha_o T_o = 0.18$ *and* $\mu_o/\rho_o c_E T_o = 0.001$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

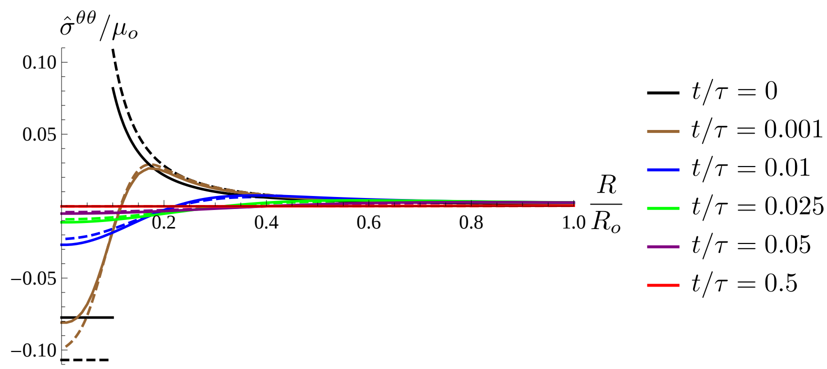

Figure 5: *Stress field* $\sigma^{\theta\theta}$ *for* $\frac{R_i}{R_o} = 0.1$, $\gamma = 10$, $\alpha_o T_o = 0.18$ *and* $\mu_o/\rho_o c_E T_o = 0.001$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

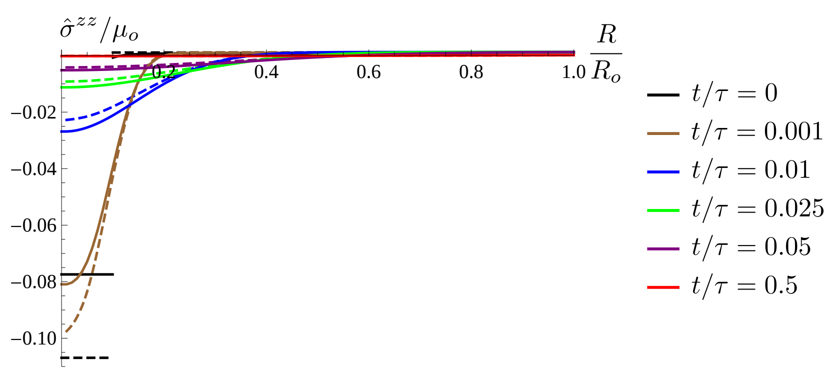

Figure 6: *Stress field* σ^{zz} *for* $\frac{R_i}{R_o} = 0.1$, $\gamma = 10$, $\alpha_o T_o = 0.18$ *and* $\mu_o/\rho_o c_E T_o = 0.001$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

where for $K \in \{R, \Theta\}$, $\omega_K(R, T(R,t)) = \displaystyle\int_{T_0}^{T(R,t)} \alpha_K(R,\tau) d\tau$. The Christoffel symbol matrices of $\boldsymbol{G}$ read

$$
\boldsymbol{\Gamma}^R = \left[\Gamma^R{}_{AB}\right] = \begin{pmatrix} \frac{\partial \omega_R}{\partial R}\big|_t & 0 & 0 \\ 0 & -e^{2(\omega_\Theta - \omega_R)}(R + R^2 \ \frac{\partial \omega_\Theta}{\partial R}\big|_t) & 0 \\ 0 & 0 & -e^{2(\omega_\Theta-\omega_R)}\left(R + R^2 \ \frac{\partial \omega_\Theta}{\partial R}\big|_t\right)\sin^2\Theta \end{pmatrix} ,
$$

$$
\boldsymbol{\Gamma}^\Theta = \left[\Gamma^\Theta{}_{AB}\right] = \begin{pmatrix} 0 & \frac{1}{R} + \frac{\partial \omega_\Theta}{\partial R}\big|_t & 0 \\ \frac{1}{R} + \frac{\partial \omega_\Theta}{\partial R}\big|_t & 0 & 0 \\ 0 & 0 & -\sin\Theta\cos\Theta \end{pmatrix} , \tag{4.57}
$$

$$
\boldsymbol{\Gamma}^\Phi = \left[\Gamma^\Phi{}_{AB}\right] = \begin{pmatrix} 0 & 0 & \frac{1}{R} + \frac{\partial \omega_\Theta}{\partial R}\big|_t \\ 0 & 0 & 1/\tan\Theta \\ \frac{1}{R} + \frac{\partial \omega_\Theta}{\partial R}\big|_t & 1/\tan\Theta & 0 \end{pmatrix} ,
$$

where $\frac{\partial \omega_K}{\partial R}\big|_t = \frac{\partial \omega_K}{\partial R}\big|_{t,T} + \alpha_K \frac{\partial T}{\partial R}$. We equip the ambient space with the following flat metric in the spherical coordinates (r, θ, ϕ) .

$$
\boldsymbol{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} . \tag{4.58}
$$

The Christoffel symbol matrices for $\boldsymbol{g}$ read

$$
\begin{gathered}
\boldsymbol{\gamma}^r = \left[\gamma^r{}_{ab}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{pmatrix} , \ \boldsymbol{\gamma}^\theta = \left[\gamma^\theta{}_{ab}\right] = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} , \\
\boldsymbol{\gamma}^\phi = \left[\gamma^\phi{}_{ab}\right] = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & 1/\tan\Theta \\ \frac{1}{r} & 1/\tan\Theta & 0 \end{pmatrix} .
\end{gathered} \tag{4.59}
$$

Thermal stresses. We next solve for the thermal stress field when the solid sphere is made of an arbitrary incompressible isotropic solid and the temperature field is radially symmetric. In order to calculate the thermal stresses, we embed the material manifold into the ambient space and look for solutions of the form $(r, \theta, \phi) = (r(R,t), \Theta, \Phi)$. The deformation gradient reads

$$
\boldsymbol{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.60}
$$

For an incompressible solid, we have

$$
J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F} = \frac{r^2}{R^2 e^{\omega_R + 2\omega_\Theta}} r' = 1 , \tag{4.61}
$$

and hence, assuming $r(0,T) = 0$ (to eliminate rigid translations), we find

$$
r(R,t) = \left(\int_0^R 3\xi^2 e^{\omega_R(\xi, T(\xi,t)) + 2\omega_\Theta(\xi, T(\xi,t))} d\xi \right)^{\frac{1}{3}} . \tag{4.62}
$$

The right Cauchy-Green deformation tensor reads

$$
\boldsymbol{C} = \begin{pmatrix} \frac{R^4}{r^4} e^{4\omega_\Theta} & 0 & 0 \\ 0 & \frac{r^2}{R^2} e^{-2\omega_\Theta} & 0 \\ 0 & 0 & \frac{r^2}{R^2} e^{-2\omega_\Theta} \end{pmatrix} . \tag{4.63}
$$

Following (4.10), the non-zero components of the Cauchy stress tensor are given by

$$\sigma^{rr} = 2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) - p\,, \tag{4.64a}$$

$$\sigma^{\theta\theta} = \frac{1}{r^2}\left\{2\frac{r^2 e^{-2\omega_\Theta}}{R^2}\left[\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} + \frac{R^4 e^{4\omega_\Theta}}{r^4}\right)\right] - p\right\}\,, \tag{4.64b}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\theta}\sigma^{\theta\theta}\,, \tag{4.64c}$$

where $p = p(R,t)$ is the pressure field due to the incompressibility constraint. The only non-trivial equilibrium equation is $\sigma^{ra}{}_{|a} = 0$, which is simplified to read

$$\frac{r^2 e^{-\omega_R - 2\omega_\Theta}}{R^2}\sigma^{rr}{}_{,R} + \frac{2}{r}\sigma^{rr} - 2r\sigma^{\theta\theta} = 0\,. \tag{4.65}$$

This yields

$$\frac{\partial p}{\partial R} = \frac{\partial}{\partial R}\left[2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)\right] - 4\frac{e^{\omega_R}}{r}\left(1 - \frac{R^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)\,. \tag{4.66}$$

Assuming that the boundary of the solid sphere is under uniform normal traction, i.e., $\sigma^{rr}(R_o, T(R_o)) = -\sigma_o$, the pressure field at the boundary is

$$p(R_o) = \left[2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)\right]\Bigg|_{R=R_o} + \sigma_o\,, \tag{4.67}$$

and it follows that

$$\begin{aligned} p(R,t) &= 2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) \\ &\quad + \int_R^{R_o} 4\frac{e^{\omega_R}}{r}\left(1 - \frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{\xi^2}\right) d\xi + \sigma_o\,. \end{aligned} \tag{4.68}$$

Finally, given a radially-symmetric temperature field $T = T(R,t)$, the thermal stress field is given in terms of the non-zero components of the Cauchy stress tensor as

$$\sigma^{rr} = -4\int_R^{R_o}\frac{e^{\omega_R}}{r}\left(1 - \frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{\xi^2}\right) d\xi - \sigma_o\,, \tag{4.69a}$$

$$\sigma^{\theta\theta} = 2\frac{1}{r^2}\left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} - \frac{R^4 e^{4\omega_\Theta}}{r^4}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) + \frac{1}{r^2}\sigma^{rr}\,, \tag{4.69b}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\theta}\sigma^{\theta\theta}\,. \tag{4.69c}$$

Alternatively, the non-zero components of the second Piola-Kirchhoff stress tensor read

$$S^{RR} = -4\frac{r^4 e^{-4\omega_\Theta - 2\omega_R}}{R^4}\int_R^{R_o}\frac{e^{\omega_R}}{r}\left(1 - \frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{\xi^2}\right) d\xi\,, \tag{4.70a}$$

$$S^{\Theta\Theta} = \frac{2}{r^2}\left(\frac{r^2}{R^2 e^{6\omega_\Theta}} - \frac{R^4 e^{4\omega_\Theta}}{r^4}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) + \frac{R^4 e^{4\omega_\Theta + 2\omega_R}}{r^6}S^{RR}\,, \tag{4.70b}$$

$$S^{\Phi\Phi} = \frac{1}{\sin^2\Theta}S^{\Theta\Theta}\,. \tag{4.70c}$$

Heat equation. We assume that there is no external heat supply, i.e., $R = 0$, and that the heat conduction in the material is isotropic, i.e., $\boldsymbol{K} = k\boldsymbol{G}^{-1}$, where $k = k(R, T(R,t))$ is a scalar valued function. Therefore, recalling (3.19), the coupled heat equation (3.49) is rewritten as

$$\left[\frac{\partial^2 T}{\partial R^2} + \left(\frac{2}{R} + \frac{1}{k}\left.\frac{\partial k}{\partial R}\right|_t\right)\frac{\partial T}{\partial R} + \left.\frac{\partial\left(2\omega_\Theta - \omega_R\right)}{\partial R}\right|_t\frac{\partial T}{\partial R}\right] k e^{-2\omega_R} = \rho_o c_E e^{-\omega_R - 2\omega_\Theta}\dot{T} - \frac{1}{2}T\frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,, \tag{4.71}$$

where $\rho_o(X)=\rho(X,\boldsymbol{G}_o)$ is the mass density in the stress-free configuration with uniform temperature T_0. If we further assume that the material is thermally isotropic (i.e., $\omega_R=\omega_\Theta=\omega(R,T(R,t))$), (4.71) reduces to

$$\left[\frac{k}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\left(\frac{\partial k}{\partial R}\bigg|_t+k\,\frac{\partial \omega}{\partial R}\bigg|_t\right)\frac{\partial T}{\partial R}\right]e^{\omega}=\rho_o c_E\dot{T}-\frac{e^{3\omega}}{2}T\frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,. \tag{4.72}$$

In order to solve for the temperature and stress fields, one needs to find ω_R and ω_Θ to explicitly specify the material metric (cf. (4.56)). We first look for the stress-free temperature fields for the ball. A temperature field $T_0(R)$ is stress-free if and only if the curvature tensor of the ball with the metric $\boldsymbol{G}$ is identically zero at $T=T_0$. The non-trivially non-zero components of the curvature tensor for the ball (cf. (4.56)) are

$$\mathcal{R}^R{}_{\Theta\Theta R}=-\mathcal{R}^R{}_{\Theta R\Theta}=R\left[2\,\frac{\partial\omega_\Theta}{\partial R}\bigg|_t-\frac{\partial\omega_R}{\partial R}\bigg|_t+R\,\frac{\partial\omega_\Theta}{\partial R}\bigg|_t\left(\frac{\partial\omega_\Theta}{\partial R}\bigg|_t-\frac{\partial\omega_R}{\partial R}\bigg|_t\right)+R\,\frac{\partial^2\omega_\Theta}{\partial R^2}\bigg|_t\right]e^{2(\omega_\Theta-\omega_R)}\,, \tag{4.73a}$$

$$\mathcal{R}^R{}_{\Phi\Phi R}=-\mathcal{R}^R{}_{\Phi R\Phi}=R\left[2\,\frac{\partial\omega_\Theta}{\partial R}\bigg|_t-\frac{\partial\omega_R}{\partial R}\bigg|_t+R\,\frac{\partial\omega_\Theta}{\partial R}\bigg|_t\left(\frac{\partial\omega_\Theta}{\partial R}\bigg|_t-\frac{\partial\omega_R}{\partial R}\bigg|_t\right)+R\,\frac{\partial^2\omega_\Theta}{\partial R^2}\bigg|_t\right]e^{2(\omega_\Theta-\omega_R)}\sin^2\Theta\,, \tag{4.73b}$$

$$\mathcal{R}^\Theta{}_{RR\Theta}=-\mathcal{R}^\Theta{}_{R\Theta R}=\left(\frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R}\bigg|_t\right)\left(\frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R}\bigg|_t-\frac{\partial\omega_R}{\partial R}\bigg|_t\right)+\frac{\partial^2\omega_\Theta}{\partial R^2}\bigg|_t-\frac{1}{R^2}\,, \tag{4.73c}$$

$$\mathcal{R}^\Theta{}_{\Phi\Phi\Theta}=-\mathcal{R}^\Theta{}_{\Phi\Theta\Phi}=\left[1-e^{2(\omega_\Theta-\omega_R)}\left(1+R\,\frac{\partial\omega_\Theta}{\partial R}\bigg|_t\right)^2\right]\sin^2\Theta\,, \tag{4.73d}$$

$$\mathcal{R}^\Phi{}_{RR\Phi}=-\mathcal{R}^\Phi{}_{R\Phi R}=\left(\frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R}\bigg|_t\right)\left(\frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R}\bigg|_t-\frac{\partial\omega_R}{\partial R}\bigg|_t\right)+\frac{\partial^2\omega_\Theta}{\partial R^2}\bigg|_t-\frac{1}{R^2}\,, \tag{4.73e}$$

$$\mathcal{R}^\Phi{}_{\Theta\Theta\Phi}=-\mathcal{R}^\Phi{}_{\Theta\Phi\Theta}=1-e^{2(\omega_\Theta-\omega_R)}\left(1+R\,\frac{\partial\omega_\Theta}{\partial R}\bigg|_t\right)^2\,. \tag{4.73f}$$

The temperature field $T_0(R)$ is stress-free if and only if all the components of the curvature tensor are identically zero. In particular, if the material is thermally homogeneous, i.e., for $K\in\{R,\Theta\}$, we have $\alpha_K=\alpha_K(T)$, then $\frac{\partial\omega_K}{\partial R}\big|_t=\alpha_K\frac{\partial T}{\partial R}$ and it follows that a uniform temperature field is stress-free in any arbitrary thermally homogeneous anisotropic body. If the material is homogeneous and thermally isotropic, i.e., $\omega_R=\omega_\Theta=\omega(T)$, then $T_0(R)$ is stress free if and only if

$$\frac{dT_0}{dR}\left(2+\alpha R\frac{dT_0}{dR}\right)\bigg|_{(R,T_0)}=0\,. \tag{4.74}$$

Equivalently

$$\frac{dT_0}{dR}\bigg|_{(R,T_0)}=0\quad\text{or}\quad\frac{dT_0}{dR}\bigg|_{(R,T_0)}=-\frac{2}{\alpha R}\,. \tag{4.75}$$

In the particular case of a thermally homogeneous and isotropic material with a constant linear expansion coefficient α, we find two possible stress-free temperature fields

$$T_0(R)=T_0(R_o)\,, \tag{4.76a}$$

$$T_0(R)=-\frac{2}{\alpha}\ln R+b\,, \tag{4.76b}$$

for some contant b. We assume in the remainder of this section that the material is thermally homogeneous and that the metric $\boldsymbol{G}_0$ corresponds to the uniform stress-free temperature field of the ball $T_0(R)=T_o$ (i.e., $\boldsymbol{G}|_{T=T_o}=\boldsymbol{G}_0$), where T_o is the temperature of the outside medium surrounding the ball. For $K\in\{R,\Theta\}$, we have

$$\omega_K(R,T(R,t))=\int_{T_o}^{T(R,t)}\alpha_K(T)dT\,. \tag{4.77}$$

Example 4.3. In this example we solve for the stress field induced by a static thermal inclusion in a homogeneous, isotropic, thermally homogeneous and anisotropic ball. Next, we assume a thermally isotropic material to compare with the linearized elasticity solution. We consider a thermal inclusion of radius $R_i < R_o$:

$$T(R) = \begin{cases} T_i & R \leq R_i\,, \\ T_o & R \geq R_i\,. \end{cases} \tag{4.78}$$

Thus, for $K \in \{R, \Theta\}$, we have

$$\omega_K(R) = \begin{cases} \displaystyle\int_{T_o}^{T_i} \alpha_K(T)dT & R \leq R_i\,, \\ 0 & R > R_i\,. \end{cases} \tag{4.79}$$

We substitute (4.79) into (4.62) to find

$$r(R) = \begin{cases} Re^{\frac{1}{3}(\omega_R(R_i)+2\omega_\Theta(R_i))} & R \leq R_i\,, \\ \left[R_i^3\left(e^{(\omega_R(R_i)+2\omega_\Theta(R_i))}-1\right)+R^3\right]^{\frac{1}{3}} & R > R_i\,. \end{cases} \tag{4.80}$$

Following (4.69), the physical components of the thermal stress field read

$$\hat{\sigma}^{rr} = -\int_R^{R_o} 4\frac{e^{\omega_R}}{r}\left(1-\frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2e^{-2\omega_\Theta}}{\xi^2}\right)d\xi - \sigma_o\,, \tag{4.81a}$$

$$\hat{\sigma}^{\theta\theta} = 2\left(\frac{r^2e^{-2\omega_\Theta}}{R^2}-\frac{R^4e^{4\omega_\Theta}}{r^4}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2e^{-2\omega_\Theta}}{R^2}\right)+\hat{\sigma}^{rr}\,, \tag{4.81b}$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}\,. \tag{4.81c}$$

For $R \leq R_i$, we have

$$\boldsymbol{C} = \begin{pmatrix} e^{-\frac{4}{3}(\omega_R(R_i)-\omega_\Theta(R_i))} & 0 & 0 \\ 0 & e^{\frac{2}{3}(\omega_R(R_i)-\omega_\Theta(R_i))} & 0 \\ 0 & 0 & e^{\frac{2}{3}(\omega_R(R_i)-\omega_\Theta(R_i))} \end{pmatrix}. \tag{4.82}$$

Therefore, I and II are both constant inside the inclusion, and because the material is homogeneous and isotropic, $\psi = \psi(T_i, \mathrm{I}, \mathrm{II})$. Thus, the terms ψ_{I} and ψ_{II} are both constant and it follows that for $R \leq R_i$, we have

$$\hat{\sigma}^{rr} = 4e^{\frac{2}{3}(\omega_R(R_i)+2\omega_\Theta(R_i))}\left(e^{-2\omega_\Theta(R_i)}-e^{-2\omega_R(R_i)}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}e^{\frac{2}{3}(\omega_R(R_i)-\omega_\Theta(R_i))}\right)\ln\left(\frac{R}{R_i}\right)+\hat{\sigma}_c\,, \tag{4.83a}$$

$$\hat{\sigma}^{\theta\theta} = 2e^{\frac{2}{3}(\omega_R(R_i)+2\omega_\Theta(R_i))}\left(e^{-2\omega_\Theta(R_i)}-e^{-2\omega_R(R_i)}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}e^{\frac{2}{3}(\omega_R(R_i)-\omega_\Theta(R_i))}\right)\left[2\ln\left(\frac{R}{R_i}\right)+1\right]+\hat{\sigma}_c\,, \tag{4.83b}$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}\,, \tag{4.83c}$$

where $\hat{\sigma}_c$ is a constant given by

$$\hat{\sigma}_c = -4\int_{R_i}^{R_o} \frac{1}{r(\xi)}\left(1-\frac{\xi^6}{r^6(\xi)}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2(\xi)}{\xi^2}\right)d\xi - \sigma_o\,. \tag{4.84}$$

On the other hand, for $R > R_i$, we have

$$\hat{\sigma}^{rr} = -\int_R^{R_o} \frac{4}{r(\xi)}\left(1-\frac{\xi^6}{r^6(\xi)}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2(\xi)}{\xi^2}\right)d\xi - \sigma_o\,, \tag{4.85a}$$

$$\hat{\sigma}^{\theta\theta} = 2\left(\frac{r^2(R)}{R^2}-\frac{R^4}{r^4(R)}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2(R)}{R^2}\right)+\hat{\sigma}^{rr}\,, \tag{4.85b}$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}\,. \tag{4.85c}$$

Remark 4.3. Note that in the absence of body forces, for a homogeneous, isotropic, thermally homogeneous and isotropic solid sphere with a uniform normal traction on its boundary and with a radially-symmetric thermal inclusion, the stress inside the inclusion is uniform and hydrostatic. However, if the material is thermally anisotropic (i.e., $\omega_R \neq \omega_\Theta$), the stress field has a logarithmic singularity at the center of the ball.[12]

Comparison with the linear case. Next, we compare the thermal stress field with the classical linear elasticity solution. We consider a homogeneous, isotropic, thermally homogeneous and isotropic and traction-free solid sphere. The classical linear elasticity solution of the sphere problem for constant μ_o and α_o reads [Boley and Weiner, 1960; Hetnarski and Eslami, 2008]

$$\hat{\sigma}^{rr} = 4\mu_o\alpha_o \frac{3\lambda+2\mu_o}{\lambda+2\mu_o}\left[\frac{1}{R_o^3}\int_0^{R_o} T(\xi)\xi^2 d\xi - \frac{1}{R^3}\int_0^{R} T(\xi)\xi^2 d\xi\right], \tag{4.86a}$$

$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\mu_o\alpha_o \frac{3\lambda+2\mu_o}{\lambda+2\mu_o}\left[\frac{2}{R_o^3}\int_0^{R_o} T(\xi)\xi^2 d\xi + \frac{1}{R^3}\int_0^{R} T(\xi)\xi^2 d\xi - T\right]. \tag{4.86b}$$

Incompressible linearized elasticity corresponds to $\nu = 0.5$, i.e., $\lambda \to \infty$. Thus, in the case of the thermal inclusion (4.78), we find

$$\begin{aligned} R \le R_i :&\quad \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = -4\mu_o\left(1-\frac{R_i^3}{R_o^3}\right)\alpha_o\Delta_i T\,, \\ R > R_i :&\quad \begin{cases} \hat{\sigma}^{rr} = -4\mu_o\left(\dfrac{R_i^3}{R^3}-\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_i T\,, \\ \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\mu_o\left(\dfrac{R_i^3}{R^3}+2\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_i T\,, \end{cases} \end{aligned} \tag{4.87}$$

where $\Delta_i T = T_i - T_o$.

In order to compare the nonlinear solution with the linearized one, we consider the thermoelastic model presented in Appendix A and enforce the incompressibility condition. Hence, following (A.11), we find for the thermal inclusion (4.78)

$$\omega(R) = \begin{cases} \dfrac{1}{3}\ln\left[\dfrac{(1+3\alpha_o T_o)\Delta_i T + T_o}{\Delta_i T + T_o}\right] & R \le R_i\,, \\ 0 & R > R_i\,, \end{cases} \tag{4.88}$$

and from (4.81) and (A.7) the thermal stress field reads

$$\begin{aligned} R \le R_i :&\quad \left\{\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = -2\mu_o\int_{R_i}^{R_o}\frac{1}{r(\xi)}\left(1-\frac{\xi^6}{r^6(\xi)}\right)d\xi\,,\right. \\ R > R_i :&\quad \begin{cases} \hat{\sigma}^{rr} = -2\mu_o\displaystyle\int_{R}^{R_o}\frac{1}{r(\xi)}\left(1-\frac{\xi^6}{r^6(\xi)}\right)d\xi\,, \\ \hat{\sigma}^{\theta\theta} = \mu_o\left(\dfrac{r^2(R)}{R^2}-\dfrac{R^4}{r^4(R)}\right)+\hat{\sigma}^{rr}\,, \\ \hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}\,, \end{cases} \end{aligned} \tag{4.89}$$

where

$$r(R) = \begin{cases} \left[\dfrac{(1+3\alpha_o T_o)\Delta_i T + T_o}{\Delta_i T + T_o}\right]^{\frac{1}{3}} R & R \le R_i\,, \\ \left[3\alpha_o T_o\dfrac{\Delta_i T}{\Delta_i T + T_o}R_i^3 + R^3\right]^{\frac{1}{3}} & R \ge R_i\,. \end{cases} \tag{4.90}$$

[12]Similar results were obtained for distributed point defects in [Yavari and Goriely, 2012b] and for distributed eigenstrains in [Yavari and Goriely, 2013, 2015].

For small $\Delta_i T$, we have the following asymptotic expansions:

$$R \le R_i: \quad \begin{cases} \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = -4\mu_o \left(1 - \frac{R_i^3}{R_o^3}\right) \alpha_o \Delta_i T + \left(1 - \frac{R_i^3}{R_o^3}\right) \left[4 + 11 \left(1 + \frac{R_i^3}{R_o^3}\right) \alpha_o T_o\right] \frac{\alpha_o}{T_o} (\Delta_i T)^2 \\ \qquad + o\left((\Delta_i T)^3\right), \end{cases}$$

$$R > R_i: \quad \begin{cases} \hat{\sigma}^{rr} = -4\mu_o \left(\frac{R_i^3}{R^3} - \frac{R_i^3}{R_o^3}\right) \alpha_o \Delta_i T - \mu_o \left(\frac{R_i^3}{R^3} - \frac{R_i^3}{R_o^3}\right) \left[4 + 11 \left(\frac{R_i^3}{R^3} + \frac{R_i^3}{R_o^3} + 1\right) \alpha_o T_o\right] \frac{\alpha_o}{T_o} (\Delta_i T)^2 \\ \quad + o\left((\Delta_i T)^3\right), \\ \hat{\sigma}^{\theta\theta} = 2\mu_o \left(\frac{R_i^3}{R^3} + 2\frac{R_i^3}{R_o^3}\right) \alpha_o \Delta_i T - \mu_o \left[2\left(\frac{R_i^3}{R^3} + \frac{R_i^3}{R_o^3}\right) + \left(4\frac{R_i^6}{R^6} + 11\frac{R_i^6}{R_o^6}\right) \alpha_o T_o\right] \frac{\alpha_o}{T_o} (\Delta_i T)^2 \\ \quad + o\left((\Delta_i T)^3\right), \\ \hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}. \end{cases} \tag{4.91}$$

We have thus recovered, up to the first order in $\Delta_i T$, the classical linearized elasticity solution.

We consider the case of rubber-like solids for which we typically have $\alpha_o = 6 \times 10^{-4}\,\mathrm{K}^{-1}$ at $300°\mathrm{K}$, i.e., $\alpha_o T_o = 0.18$. In Figure 7, we plot the static thermal stresses for different values of the initial relative temperature difference $\delta_T = \frac{\Delta_i T}{T_o}$ in the thermal inclusion (4.78). The two solutions for the stress field are very close for small values of δ_T (i.e., in the range of validity of linearized elasticity). For larger values of δ_T, even though linearized elasticity captures the overall behavior of $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$, it fails by overestimating their values (the relative difference of stress reaches 45% inside the inclusion for $\delta_T = 30\%$).

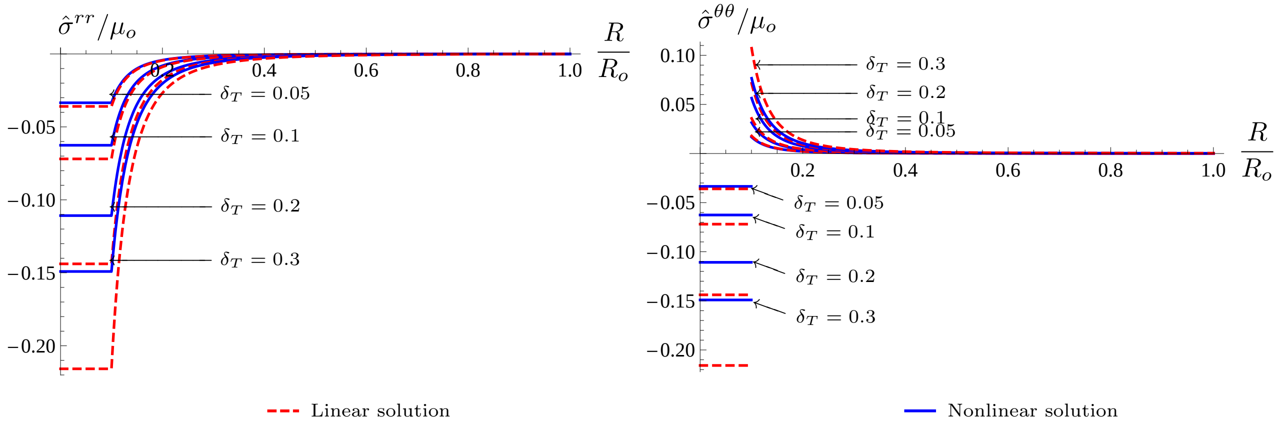

Figure 7: *Nonlinear and linear solutions for* σ^{rr} *and* $\sigma^{\theta\theta}$ *stress fields for* $\frac{R_i}{R_o} = 0.1$, $\alpha_o T_o = 0.18$ *and different values of* $\delta_T = \frac{\Delta_i T}{T_o}$.

Example 4.4. In this example we numerically solve for the evolution of temperature and thermal stress fields for a homogeneous, isotropic, thermally homogeneous and isotropic ball for which we assume the thermoelastic model described in Appendix A. Following (A.11), we find

$$\omega(R, T(R,t)) = \frac{1}{3} \ln \left[1 + 3\alpha_0 T_0 \left(1 - \frac{T_0}{T(R,t)}\right)\right]. \tag{4.92}$$

Thus

$$r(R,t) = \left\{\int_0^R 3\xi^2 \left[1 + 3\alpha_0 T_0 \left(1 - \frac{T_0}{T(\xi,t)}\right)\right] d\xi\right\}^{\frac{1}{3}} = \left[(1 + 3\alpha_0 T_0) R^3 - 9\alpha_0 T_0 \int_0^R \frac{T_0 \xi^2}{T(\xi,t)} d\xi\right]^{\frac{1}{3}}. \tag{4.93}$$

Given the free energy density (A.7), it follows from (4.69) that the physical components of the Cauchy stress field are

$$\hat{\sigma}^{rr} = -2\mu_o \int_R^{R_o} \frac{\left(1+3\alpha_0 T_0 \left(1-\frac{T_0}{T}\right)\right)^{1/3}}{r} \left(1 - \frac{\xi^6 \left[1+3\alpha_0 T_0 \left(1-\frac{T_0}{T}\right)\right]^2}{r^6}\right) \frac{T}{T_o} d\xi - \sigma_o \,, \tag{4.94a}$$

$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{rr} + \mu_o \left[\frac{r^2\left(1+3\alpha_0 T_0 \left(1-\frac{T_0}{T}\right)\right)^{-2/3}}{R^2} - \frac{R^4\left[1+3\alpha_0 T_0 \left(1-\frac{T_0}{T}\right)\right]^{4/3}}{r^4}\right] \frac{T}{T_o} \,, \tag{4.94b}$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta} \,. \tag{4.94c}$$

Now, let us find the time-dependent temperature field in order to evaluate the thermal stress field by solving the coupled heat equation (4.72). We assume the model (4.43) for heat conduction and hence the heat equation (4.72) reads

$$\left\{[1-s(T-T_o)]\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right) + \alpha\left(\frac{\partial T}{\partial R}\right)^2\right] - s\left(\frac{\partial T}{\partial R}\right)^2\right\} e^{\omega} = \frac{\rho_o c_E}{k_o}\dot{T} - \frac{e^{3\omega}}{2k_o} T \frac{\partial S^{AB}}{\partial T}\dot{C}_{AB} \,. \tag{4.95}$$

Following (4.70), the non-zero components of the second Piola-Kirchhoff tensor are

$$S^{RR} = -2\mu_o \frac{r^4 e^{-6\omega}}{R^4} \int_R^{R_o} \frac{T}{T_o} \frac{e^{\omega(\xi,T(\xi,t))}}{r(\xi,t)} \left(1 - \frac{\xi^6 e^{6\omega(\xi,T(\xi,t))}}{r^6(\xi,t)}\right) d\xi \,, \tag{4.96a}$$

$$S^{\Theta\Theta} = \frac{R^4 e^{6\omega}}{r^6} S^{RR} + \mu_o \frac{T}{T_o}\frac{1}{r^2}\left(\frac{r^2}{R^2 e^{2\omega}} - \frac{R^4 e^{4\omega}}{r^4}\right) \,, \tag{4.96b}$$

$$S^{\Phi\Phi} = \frac{1}{\sin^2\Theta} S^{\Theta\Theta} \,. \tag{4.96c}$$

Hence[13]

$$\frac{\partial S^{RR}}{\partial T} = 0 \,, \tag{4.97a}$$

$$\frac{\partial S^{\Theta\Theta}}{\partial T} = \frac{\mu_o}{T_o}\left(\frac{1}{R^2 e^{2\omega}} - \frac{R^4 e^{4\omega}}{r^6}\right) \,, \tag{4.97b}$$

$$\frac{\partial S^{\Phi\Phi}}{\partial T} = \frac{1}{\sin^2\Theta}\frac{\partial S^{\Theta\Theta}}{\partial T} \,. \tag{4.97c}$$

The non-vanishing components of $\dot{\boldsymbol{C}}$ are

$$\dot{C}_{RR} = \frac{\partial}{\partial t}\left(\frac{R^4 e^{6\omega}}{r^4}\right) = 2\frac{R^4 e^{6\omega}}{r^4}\left(3\alpha\dot{T} - \frac{2}{r}\frac{\partial r}{\partial t}\right) \,, \tag{4.98a}$$

$$\dot{C}_{\Theta\Theta} = \frac{\partial r^2}{\partial t} = 2r\frac{\partial r}{\partial t} \,, \tag{4.98b}$$

$$\dot{C}_{\Phi\Phi} = \dot{C}_{\Theta\Theta}\sin^2\Theta \,, \tag{4.98c}$$

and following (4.93), we find

$$\frac{\partial r}{\partial t} = \frac{3\alpha_o}{r^2}\int_0^R \xi^2 \left(\frac{T_o}{T(\xi,t)}\right)^2 \dot{T}(\xi,t) d\xi \,. \tag{4.99}$$

[13] Recall that $\frac{\partial}{\partial T} = \frac{\partial}{\partial T}|_{\boldsymbol{C},\boldsymbol{G}}$ is a partial derivative with respect to T with $\boldsymbol{C}$ and $\boldsymbol{G}$ fixed (cf. Remark 3.3).

Thus, the coupled heat equation (4.72) reads

$$
\begin{aligned}
&\left\{[1-s(T-T_o)]\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\alpha\left(\frac{\partial T}{\partial R}\right)^2\right]-s\left(\frac{\partial T}{\partial R}\right)^2\right\}\sqrt[3]{1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)} \\
&\quad=\frac{\rho_o c_E}{k_o}\dot{T}-6\frac{\mu_o}{k_o}\alpha_o T_o T\left[\frac{1}{R^2}-\frac{\left(1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right)^2 R^4}{\left(\left(1+3\alpha_0 T_0\right)R^3-9\alpha_0 T_0\int_0^R \xi^2\frac{T_0}{T}d\xi\right)^2}\right] \\
&\quad\quad\times\left[\frac{1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)}{\left(1+3\alpha_0 T_0\right)R^3-9\alpha_0 T_0\int_0^R \xi^2\frac{T_0}{T}d\xi}\right]^{1/3}\int_0^R\frac{\xi^2\dot{T}}{T^2}d\xi\,.
\end{aligned}
\tag{4.100}
$$

On the boundary of the ball, we consider a convection boundary condition, i.e.

$$
k_o\left[1-s(T(R_o,t)-T_o)\right]\left.\frac{\partial T}{\partial R}\right|_{(R_o,t)}=h_o[T_o-T(R_o,t)]\,,\tag{4.101}
$$

where h_o is the surface heat transfer coefficient at the boundary of the ball. We assume that h_o is constant and introduce the parameter $\gamma=h_oR_o/k_o$. As an initial temperature field, we consider a thermal inclusion of radius R_i, i.e.

$$
T_{\text{init}}(R)=\begin{cases}T_i & R\leq R_i\,,\\ T_o & R>R_i\,.\end{cases}\tag{4.102}
$$

In the scope of the classical theory of linearized elasticity, the thermal stresses are given by [Boley and Weiner, 1960]:

$$
\hat{\sigma}^{rr}=12\mu_o\alpha_o\left[\frac{1}{R_o^3}\int_0^{R_o}T(\xi,t)\xi^2d\xi-\frac{1}{R^3}\int_0^R T(\xi,t)\xi^2d\xi\right]\,,\tag{4.103a}
$$

$$
\hat{\sigma}^{\theta\theta}=\hat{\sigma}^{\phi\phi}=6\mu_o\alpha_o\left[\frac{2}{R_o^3}\int_0^{R_o}T(\xi,t)\xi^2d\xi+\frac{1}{R^3}\int_0^R T(\xi,t)\xi^2d\xi-T(R,t)\right]\,.\tag{4.103b}
$$

In the classical linearized elasticity literature, the coupling term is neglected and the linearized heat equation problem for the sphere reads

$$
\begin{cases}
\dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2\dfrac{\partial T}{\partial R}\right)=\dfrac{\rho_o c_E}{k_o}\dot{T}\,,\\
T(R,0)=T_{\text{init}}(R)\,,\\
\left.\dfrac{\partial T}{\partial R}\right|_{(R_o,t)}=\dfrac{\gamma}{R_o}[T_o-T(R_o,t)]\,.
\end{cases}\tag{4.104}
$$

The solution to (4.104) can be found analytically using a Fourier series expansion (see [Carslaw and Jaeger, 1986] for a detailed derivation.)

$$
T(R,t)=2\Delta_i T\sum_{n=1}^{\infty}\frac{\zeta_n^2+(\gamma-1)^2}{\zeta_n^2+\gamma(\gamma-1)}\sin\left(\zeta_n\frac{R}{R_o}\right)\left[\frac{1}{\zeta_n^2}\frac{R_o}{R}\sin\left(\zeta_n\frac{R_i}{R_o}\right)-\frac{1}{\zeta_n}\frac{R_i}{R}\cos\left(\zeta_n\frac{R_i}{R_o}\right)\right]e^{-\frac{\zeta_n^2 t}{\tau}}+T_o\,,\tag{4.105}
$$

where ζ_n are the positive solutions of $\zeta\cot\zeta=1-\gamma$.

We consider a rubber sphere of radius $R_o=15\,\text{cm}$ for which the surface heat transfer coefficient for the rubber-air convection is $h_o=10\,\text{W/m}^2.\text{K}$. We let $T_o=300°\text{K}$ and $\delta_T=\frac{\Delta_i T}{T_o}=30\%$ and assume the following typical values for rubber-like materials: $\rho_o=10^3\,\text{kg/m}^3$, $c_E=1800\,\text{J/kg.m}$, $k_o=0.15\,\text{W/m.K}$, $s=0,004\,\text{K}^{-1}$, $\alpha_o=6\times10^{-4}\,\text{K}^{-1}$, and $\mu_o=0.54\,\text{GPa}$. We numerically solve the initial/boundary value problem (4.100), (4.101), (4.102) for the temperature field $T(R,t)$ and show its evolution in Figure 8 by plotting $\frac{T(R,t)-T_o}{T_o}$ at different values of t/τ, where τ is a characteristic time defined as $\tau=\rho_o c_E R_o^2/k_o$. In Figures 9-10, we show the nonlinear thermal stresses (4.94). For comparison purposes, we also show the linearized solution (4.103) and

(4.105) in Figures 8-10. We observe that the initial irregularities in the initial temperature and thermal stress fields are smoothed out; at large times both the temperature difference $T - T_o$ and thermal stress fields tend to zero. The nonlinear and linear solutions for the temperature field (Figure 8) show a similar trend but we observe a significant difference for thermal stress fields between the linear (4.103) and nonlinear (4.94) solutions (the maximum relative difference is of 38% inside the inclusion for $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$, see Figures 8-10). We also observe that in the nonlinear solution the maximum thermal stress does not necessarily correspond to $t = 0$, i.e. maximum stress occurs at a later time $t > 0$.

Remark 4.4. We observe that the coupling term in the nonlinear heat equation (4.100) is negligeable. In fact, if we neglect the coupling term in (4.100), the resulting solution is only affected in the order of 10^{-6} . However, unlike the cylinder case (cf. Remark 4.2), even when k is assumed to be constant, the uncoupled nonlinear heat equation does not reduce to the classical heat equation.

5 Concluding Remarks

In this paper we presented a geometric theory of nonlinear thermoelasticity to study the coupling of nonlinear elasticity with heat conduction. We assumed that the material metric explicitly depends on temperature and the thermal expansion properties of the body. In this geometric framework the body is always stress free in the material manifold and there is no need, in the scope of this theory, to introduce a multiplicative decomposition of the deformation gradient and the so-called intermediate configuration. We obtained the temperature-dependent governing equations of motion and presented a modified energy balance equation and Clausius-Duhem inequality to include the rate of change of the evolving material metric. We derived the governing equation of the evolution of temperature in the form of a generalized coupled heat equation. We showed that by linearization of the geometric theory, we recover the classical coupled heat equation from linearized elasticity. Finally, in order to illustrate the capability of our method, we considered the cases of an infinite circular cylinder and a spherical ball made of an isotropic, homogeneous, and thermally homogeneous hyperelastic material (a constitutive model is presented in Appendix A). We showed that for a thermal inclusion, if the material thermal expansion properties are anisotropic, the stress field inside the inclusion develops a logarithmic singularity but if the material thermal expansion properties are isotropic, the stress field inside the inclusion is uniform and hydrostatic. Assuming further that the material is thermally isotropic and initially contains a thermal inclusion, we numerically solved for the evolution of temperature and thermal stress fields for a rubber-like material and demonstrated the significant differences between the results of the nonlinear theory and those of the classical linearized thermoelasticity.

Acknowledgments

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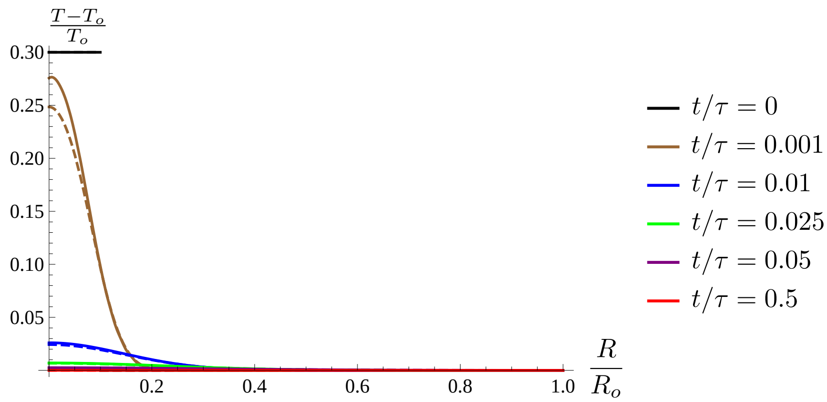

Figure 8: *Temperature field in the sphere for* $\frac{R_i}{R_o} = 0.1$ *,* $\gamma = 1$ *,* $\alpha_o T_o = 0.18$ *,* $\mu_o/\rho_o c_E T_o = 0.001$ *and* $\delta_T = 30\%$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

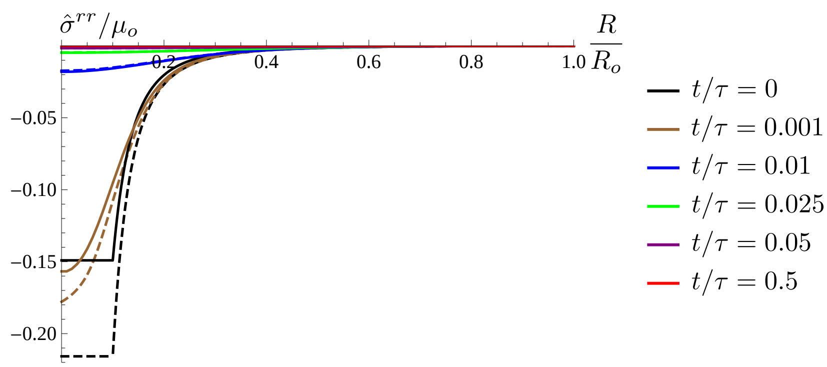

Figure 9: *Radial stress field for* $\frac{R_i}{R_o} = 0.1$ *,* $\gamma = 1$ *,* $\alpha_o T_o = 0.18$ *and* $\mu_o/\rho_o c_E T_o = 0.001$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

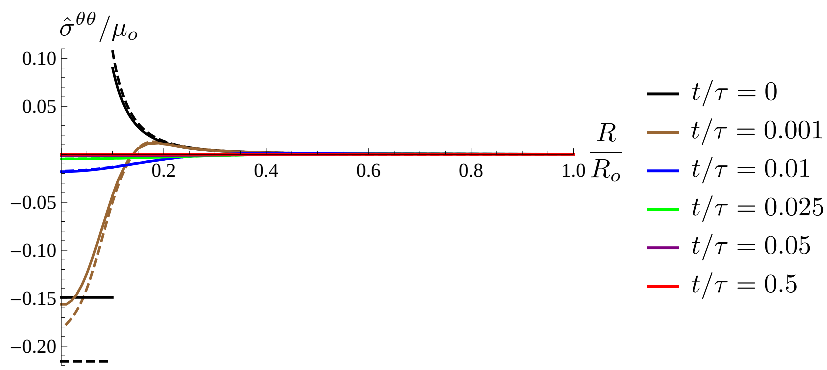

Figure 10: *Circumferential stress field for* $\frac{R_i}{R_o} = 0.1$ *,* $\gamma = 1$ *,* $\alpha_o T_o = 0.18$ *and* $\mu_o/\rho_o c_E T_o = 0.001$ *at different* $\frac{t}{\tau}$ *(Solid lines: nonlinear solution; dashed lines: linear solution).*

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Appendices

A A nonlinear thermoelastic constitutive model

In this appendix we present, in the context of the proposed geometric theory with a temperature-dependent material metric, a thermoelastic model for rubber-like materials following the models proposed by Chadwick [1974], Ogden [1972a,b, 1992], and Holzapfel and Simo [1996].

For a hyperelastic solid, the free energy provides the material constitutive information given by the independent variables $(X,T,\boldsymbol{C},\boldsymbol{G})$. The specific free energy is[14] $\psi = E - TN$ (E is the internal energy density and N is the entropy density) and the (hyperelastic) constitutive model reads $\psi = \psi(X,T,\boldsymbol{C},\boldsymbol{G})$. The specific heat capacity at constant strain c_E is defined as (cf. (3.47))

$$c_E = -T\frac{\partial^2(\psi/\rho)}{\partial T^2} = T\frac{\partial(N/\rho)}{\partial T} = \frac{\partial(E/\rho)}{\partial T}\,. \tag{A.1}$$

If we assume that c_E depends only on temperature, then we can write[15]

$$E(X,T,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) - E(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) = \rho(X,\boldsymbol{G}(X,T))\int_{T_0}^{T} c_E(\tau)d\tau\,, \tag{A.2}$$

and

$$N(X,T,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) - N(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) = \rho(X,\boldsymbol{G}(X,T))\int_{T_0}^{T} c_E(\tau)\frac{d\tau}{\tau}\,. \tag{A.3}$$

We can therefore write

$$\begin{aligned}\psi(X,T,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) = \frac{T}{T_0}\psi(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) - \left(\frac{T}{T_0}-1\right)E(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T))\\ - \rho(X,\boldsymbol{G}(X,T))\int_{T_0}^{T} c_E(\tau)\frac{T-\tau}{\tau}d\tau\,.\end{aligned} \tag{A.4}$$

In the case of an isotropic solid, the specific free energy ψ depends on $\mathrm{I} = \mathrm{tr}(\boldsymbol{C})$, $\mathrm{II} = \det(\boldsymbol{C})\,\mathrm{tr}(\boldsymbol{C}^{-1}) = \frac{1}{2}(\mathrm{tr}(\boldsymbol{C}^2) - \mathrm{tr}(\boldsymbol{C})^2)$, and $J = \sqrt{\det \boldsymbol{C}}$, which are the principal invariants of $\boldsymbol{C}$. Furthermore, experiments suggest that for rubber-like materials, the internal energy density depends only on the volumetric part of deformation [Treloar, 2005], i.e, we can write $E = E(T,J)$. Note that this confirms the assumption made on $c_E = c_E(T)$. Following the works cited above, if we denote by κ_0, μ_0 and β_0, the bulk modulus, the shear modulus, and the volumetric coefficient of thermal expansion at T_0, respectively, we consider the following constitutive model for a homogeneous isotropic rubber-like material

$$\begin{aligned}\psi(T_0,\tilde{\mathrm{I}}(X,T),J(X,T)) &= \frac{\mu_0}{2}(\tilde{\mathrm{I}}-3) + \frac{\kappa_0}{2}(J-1)^2\,,\\ E(T_0,J) &= \kappa_0\beta_0 T_0(J-1),\end{aligned} \tag{A.5}$$

where $\tilde{\mathrm{I}} = J^{-2/3}\mathrm{I}$. It follows that [Holzapfel and Simo, 1996]

$$\psi(T,\tilde{\mathrm{I}},J) = \frac{\mu_0}{2}\frac{T}{T_0}(\tilde{\mathrm{I}}-3) + \frac{\kappa_0}{2}\frac{T}{T_0}(J-1)^2 - \kappa_0\beta_0(J-1)\,(T-T_0) - \rho\int_{T_0}^{T} c_E(\tau)\frac{T-\tau}{\tau}d\tau\,. \tag{A.6}$$

[14] Note that $\psi = \rho\Psi$.

[15] Note that the material mass density ρ depends only implicitly on temperature via the material metric (cf. (3.19)), i.e., $\frac{\partial\rho}{\partial T} = 0$ but $\frac{d\rho}{dT} = -\beta\rho$.

In the incompressible case, we have the constraint $J-1=0$ associated with the pressure field p as the Lagrange multiplier

$$\psi(T,\mathrm{I},J)=\frac{\mu_o}{2}\frac{T}{T_o}(\mathrm{I}-3)-\rho\int_{T_o}^{T}c_E(\tau)\frac{T-\tau}{\tau}d\tau-p(J-1)\,. \tag{A.7}$$

One may now ask if it is possible to find a relation between the function $\boldsymbol{\omega}(T)$ appearing in the material metric and the free energy (A.6). The answer is affirmative. Let us consider a homogeneous body modeled by the free energy density (A.6) and assume that it is stress free at the uniform temperature T_0. Now let us assume that the temperature of the body is changed to another uniform temperature T. The body undergos a purely volumetric deformation and remains stress free. Note that the mean Cauchy stress $\sigma=\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$ is given by

$$\sigma=\frac{\partial\psi}{\partial J}=\kappa_0\frac{T}{T_0}\,(J-1)-\kappa_0\beta_0(T-T_0)=0. \tag{A.8}$$

Therefore

$$J=1+\beta_0T_0\left(1-\frac{T_0}{T}\right). \tag{A.9}$$

On the other hand, note that

$$J=\frac{\rho(\boldsymbol{G}_0)}{\rho(\boldsymbol{G})}=e^{\mathrm{tr}(\boldsymbol{\omega}(T))}\,. \tag{A.10}$$

It follows from (A.9) and (A.10) that

$$\mathrm{tr}(\boldsymbol{\omega}(T))=\ln\left[1+\beta_0T_0\left(1-\frac{T_0}{T}\right)\right], \tag{A.11}$$

and hence[16]

$$\beta(T)=\frac{\beta_0\frac{T_0^2}{T^2}}{1+\beta_0T_0\left(1-\frac{T_0}{T}\right)}\,. \tag{A.12}$$

B Lagrangian field theory of nonlinear thermoelasticity

In this appendix we derive the governing equations of motion using a modified Hamilton's least action principle for non-conservative systems in the form of Lagrange-d'Alembert's principle as nonlinear thermoelasticity is, in general, dissipative. We define the Lagrangian to be a map $L:T\mathcal{C}\times\mathbb{R}\to\mathbb{R}$ such that for a motion φ_t of $\mathcal{B}$

$$L(\varphi_t,\dot{\varphi}_t,T)=\int_{\mathcal{B}}\mathcal{L}(X,T,\varphi_t(X),\dot{\varphi}_t(X),\boldsymbol{C}^\flat(X,t),\boldsymbol{G}(X,T))dV(X,\boldsymbol{G})\,, \tag{B.1}$$

where we assume the Lagrangian density $\mathcal{L}=\mathcal{L}(X,T,\varphi,\dot{\varphi},\boldsymbol{C}^\flat,\boldsymbol{G})$ is given by

$$\mathcal{L}=\frac{1}{2}\rho g_{ab}\dot{\varphi}^a\dot{\varphi}^b-\rho\Psi(X,T,\varphi,\boldsymbol{C}^\flat,\boldsymbol{G})\,. \tag{B.2}$$

The action functional is defined as a map $S:\mathcal{C}\times\mathbb{R}\to\mathbb{R}$ such that for a motion φ of $\mathcal{B}$

$$S(\varphi,T)=\int_{t_1}^{t_2}L(\varphi_t,\dot{\varphi}_t,T)dt\,. \tag{B.3}$$

In order to take variations, we let φ_ϵ be a 1-parameter family of motions such that[17]

$$\begin{aligned}\varphi_{0,t}&=\varphi_t\,,\\ \varphi_{\epsilon,t}|_{\partial\mathcal{B}}&=\varphi_t|_{\partial\mathcal{B}}\,,\\ \varphi_{\epsilon,t_1}=\varphi_{t_1}\,,&\quad\varphi_{\epsilon,t_2}=\varphi_{t_2}\,.\end{aligned} \tag{B.4}$$

[16]Note that we recover the result derived by Ogden [1992], Eq. (102-c).
[17]For fixed t and ϵ, we let $\varphi_{\epsilon,t}(X):=\varphi_\epsilon(X,t)$.

For fixed X and t, we consider the curve $\varphi_{t,X} : \epsilon \to \varphi_{t,X}(\epsilon) := \varphi_{\epsilon,t}(X)$, and define the variation of motion as the spatial vector given by

$$\delta\varphi(X,t) = d_\epsilon \varphi_{t,X}\left[\partial_\epsilon\right]\Big|_{\epsilon=0} . \tag{B.5}$$

Let $T_\epsilon(X,t)$ be a 1-parameter family of temperature fields such that[18]

$$\begin{aligned} T_{0,t} &= T_t \,, \\ T_{\epsilon,t}|_{\partial\mathcal{B}} &= T_t|_{\partial\mathcal{B}} \,, \\ T_{\epsilon,t_1} = T_{t_1} \,, &\quad T_{\epsilon,t_2} = T_{t_2} \,. \end{aligned} \tag{B.6}$$

We define the variation of temperature as the scalar field

$$\delta T = \frac{dT_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} . \tag{B.7}$$

We write the variation of S as a total derivative along the curve $\varphi_{t,X}$ evaluated at $\epsilon = 0$:

$$\delta S(\varphi, T) = \frac{d}{d\epsilon} S(\varphi_\epsilon, T_\epsilon)\bigg|_{\epsilon=0} . \tag{B.8}$$

For a conservative system, Hamilton's least action principle states that the physical motion φ and temperature evolution of $\mathcal{B}$ between t_1 and t_2 is the critical point for the action functional, i.e., the variation of S at (φ, T) vanishes

$$\delta S(\varphi, T) = 0 \,. \tag{B.9}$$

However, nonlinear thermoelasticity is, in general, dissipative. We assume the existence of a Rayleigh dissipation potential $\mathcal{R} = \mathcal{R}(\varphi, \dot{\varphi}, T, \dot{T})$ such that

$$\boldsymbol{F} = -\frac{\partial \mathcal{R}}{\partial \dot{\varphi}} \quad \text{and} \quad F_T = -\frac{\partial \mathcal{R}}{\partial \dot{T}} \,, \tag{B.10}$$

where $\boldsymbol{F}$ represent dissipation by damping and F_T represents thermal dissipation. The Lagrange-d'Alembert's principle (see [Marsden and Ratiu, 1999; Yavari, 2010]) states in the case of nonlinear thermoelasticity that

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{B}} \mathcal{L}(X, T, \varphi, \dot{\varphi}, \boldsymbol{C}^\flat, \boldsymbol{G}) dV dt + \int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\boldsymbol{F}.\delta\varphi + F_T \delta T\right) dV dt = 0 \,. \tag{B.11}$$

It follows by Lagrange-d'Alembert's principle that

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} . \delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{C}^\flat} : \delta \boldsymbol{C}^\flat + \frac{1}{\sqrt{\det \boldsymbol{G}}} \frac{\partial \sqrt{\det \boldsymbol{G}} \mathcal{L}}{\partial \boldsymbol{G}} : \delta \boldsymbol{G} + \frac{\partial \mathcal{L}}{\partial T} \delta T \right) \sqrt{\det \boldsymbol{G}} dV_0 dt \\ = \int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\frac{\partial \mathcal{R}}{\partial \dot{\varphi}} . \delta\varphi + \frac{\partial \mathcal{R}}{\partial \dot{T}} \delta T \right) dV dt \,. \end{aligned} \tag{B.12}$$

For different values of ϵ, the velocity vector field $\dot{\varphi}_\epsilon$ lies in different tangent spaces $T_{\varphi_\epsilon(X,t)}\mathcal{S}$. Therefore, the variation of the velocity is given by its covariant derivative along the curve $\varphi_{t,X}$ in $\mathcal{S}$ evaluated at $\epsilon = 0$[19]

$$\delta\dot{\varphi} = \bar{\nabla}_{\delta\varphi} \dot{\varphi} = \frac{D\dot{\varphi}_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} = \frac{D\delta\varphi}{dt} \,. \tag{B.13}$$

Unlike the velocity vector field, the material tensor fields $\boldsymbol{G}$ and $\boldsymbol{C}^\flat$ lie in the same space when ϵ is varied. Therefore, their variations are given by the total derivative with respect to ϵ evaluated at $\epsilon = 0$

$$\delta \boldsymbol{G} = \frac{d\boldsymbol{G}_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d\boldsymbol{G}}{dT} \delta T \,, \tag{B.14}$$

[18] For fixed t and ϵ, we let $T_{\epsilon,t}(X) := T_\epsilon(X,t)$.

[19] Note that we use the symmetry lemma, See [do Carmo, 1992; Nishikawa, 2002].

$$\delta \boldsymbol{C}^\flat = \frac{d\boldsymbol{C}^\flat_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\varphi^*_\epsilon \boldsymbol{g}_\epsilon\right)\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\varphi^* \varphi_* \varphi^*_\epsilon \boldsymbol{g}_\epsilon\right)\Big|_{\epsilon=0} = \varphi^* \boldsymbol{L}_{\delta\varphi} \boldsymbol{g} \,. \tag{B.15}$$

In components, (B.15) reads

$$\delta C_{AB} = F^a{}_A g_{ac} \delta\varphi^c{}_{|B} + F^b{}_B g_{bc} \delta\varphi^c{}_{|A} \,. \tag{B.16}$$

Let us first consider the variation of motion only, and from (B.12), it follows by Stokes' theorem and arbitrariness of $\delta\varphi$ that

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} - \frac{1}{\sqrt{\det \boldsymbol{G}}} \frac{D}{dt}\left(\sqrt{\det \boldsymbol{G}}\, \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a}\right) - 2\left(\frac{\partial \mathcal{L}}{\partial C_{AB}} F^b{}_B g_{ab}\right)_{|A} = \frac{\partial \mathcal{R}}{\partial \dot{\varphi}}. \tag{B.17}$$

For the Lagrangian density (B.1), we have[20]

$$\frac{1}{\sqrt{\det \boldsymbol{G}}} \frac{D}{dt}\left(\sqrt{\det \boldsymbol{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right)_a = \rho g_{ab} A^b \,. \tag{B.18}$$

Note that from (3.40) we can write

$$\boldsymbol{S} = -2 \frac{\partial \mathcal{L}}{\partial \boldsymbol{C}^\flat} = 2\rho \frac{\partial \Psi}{\partial \boldsymbol{C}^\flat} \,, \tag{B.19}$$

Therefore, (B.17) yields the local form of the balance of linear momentum

$$\rho \boldsymbol{B} + \mathrm{Div} \boldsymbol{P} = \rho \boldsymbol{A} \,, \tag{B.20}$$

where $\boldsymbol{B} = -\frac{\partial \Psi}{\partial \varphi^a} - \frac{1}{\rho}\frac{\partial \mathcal{R}}{\partial \dot{\varphi}}$ is the total body force per unit undeformed mass, and $\boldsymbol{P}(X,t)$ the first Piola-Kirchhoff stress tensor $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$. Note, however, that we obtain the local form of the balance of angular momentum as a consequence of (B.19) and the symmetry of the right Cauchy-Green deformation tensor

$$\boldsymbol{P}\boldsymbol{F}^\mathsf{T} = \boldsymbol{F}\boldsymbol{P}^\mathsf{T} \,. \tag{B.21}$$

Next, we consider the variation of temperature only, and from (B.12), it follows by arbitrariness of δT that

$$\frac{\partial \mathcal{L}}{\partial T} + \frac{1}{\sqrt{\det \boldsymbol{G}}} \frac{\partial \sqrt{\det \boldsymbol{G}}\mathcal{L}}{\partial \boldsymbol{G}} : \frac{d\boldsymbol{G}}{dT} = \frac{\partial \mathcal{R}}{\partial \dot{T}} \,. \tag{B.22}$$

Therefore, (B.22) is simplified to read[21]

$$-\rho \frac{\partial \Psi}{\partial T} - \rho \frac{\partial \Psi}{\partial \boldsymbol{G}} : \frac{d\boldsymbol{G}}{dT} = \frac{\partial \mathcal{R}}{\partial \dot{T}} \,. \tag{B.23}$$

By integration of (B.23) with respect to $\dot{T}$, we find that

$$\mathcal{R}(\varphi, \dot{\varphi}, T, \dot{T}) = -\rho \frac{\partial \Psi}{\partial T} \dot{T} - \rho \frac{\partial \Psi}{\partial \boldsymbol{G}} : \dot{\boldsymbol{G}} + g(\varphi, \dot{\varphi}) \,, \tag{B.24}$$

where $g = g(\varphi, \dot{\varphi})$ is an arbitrary function. Therefore

$$F_T = \rho \frac{\partial \Psi}{\partial T} + \rho \frac{\partial \Psi}{\partial \boldsymbol{G}} : \frac{d\boldsymbol{G}}{dT}, \quad \boldsymbol{F} = -\frac{\partial g}{\partial \dot{\varphi}}. \tag{B.25}$$

In the case of thermoelasticity we can further assume that dissipation is only due to temperature and take the Rayleigh potential as a function of $\boldsymbol{G}$ and $\dot{\boldsymbol{G}}$ only, i.e., $g(\varphi, \dot{\varphi}) = 0$, and hence

$$\mathcal{R}(T, \dot{T}) = -\rho \frac{\partial \Psi}{\partial T} \dot{T} - \rho \frac{\partial \Psi}{\partial \boldsymbol{G}} : \dot{\boldsymbol{G}}. \tag{B.26}$$

[20] Recall that mass conservation (3.18) can be written as $\frac{d\rho}{dT} + \frac{1}{2}\rho \,\mathrm{tr}\left(\frac{\partial \boldsymbol{G}}{\partial T}\right) = 0$.

[21] Recall (3.18) as in footnote 20.