Basic Thermoplasticity

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Abstract

This paper explains the basics of thermoplasticity in the context of multiplicative decomposition kinematics.

1 Introduction

The purpose of these notes is to provide some of the details of the formulation of thermoplasticity by Wright and coworkers [SW01, Wri02]. We start with the assumption that the basics of thermoelasticity and the notation used are known (the details can be found in my notes on thermoelasticity).

As the first point of departure from thermoelasticity, we consider the (now classical) multiplicative decomposition of the deformation gradient into elastic and plastic parts:

$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_p \,. \tag{1}$$

Note that even though $\boldsymbol{F}$ represents a physical gradient, neither $\boldsymbol{F}_e$ or $\boldsymbol{F}_p$ has to be a gradient of a physical quantity by itself. However, for our purposes, we require the additional restriction that

$$\det(\boldsymbol{F}_e) > 0 \qquad \text{and} \qquad \det(\boldsymbol{F}_p) > 0 \,. \tag{2}$$

This decomposition is usually interpreted to mean that there is an unstressed intermediate configuration that can be attained *instantaneously* by unloading elastically from the current configuration. Figure 1 shows a schematic of the situation.

Clearly, the intermediate configuration can be identified only modulo a rigid rotation $\boldsymbol{Q}$ since

$$\boldsymbol{F} = (\boldsymbol{F}_e \cdot \boldsymbol{Q}) \cdot (\boldsymbol{Q}^T \cdot \boldsymbol{F}_p) = (\boldsymbol{F}_e \cdot \boldsymbol{Q}_2 \cdot \boldsymbol{Q}_1) \cdot (\boldsymbol{Q}_1^T \cdot \boldsymbol{Q}_2^T \cdot \boldsymbol{F}_p) = \dots \tag{3}$$

We will not concern ourselves with such issues here. Detailed discussions can be found in many places, for example [SW01, NN04]. Instead, we will follow the lead of Wright [Wri02] and assume that the plastic component of the deformation gradient is invariant under a rigid rotation $\boldsymbol{Q}$. That is, if

$$\boldsymbol{F}^{\text{rot}} = \boldsymbol{Q} \cdot \boldsymbol{F} = \boldsymbol{Q} \cdot \boldsymbol{F}_e \cdot \boldsymbol{F}_p = \boldsymbol{F}_e^{\text{rot}} \cdot \boldsymbol{F}_p^{\text{rot}} \,. \tag{4}$$

then

$$\boldsymbol{F}_e^{\text{rot}} = \boldsymbol{Q} \cdot \boldsymbol{F}_e \qquad \text{and} \qquad \boldsymbol{F}_p^{\text{rot}} = \boldsymbol{F}_p \,. \tag{5}$$

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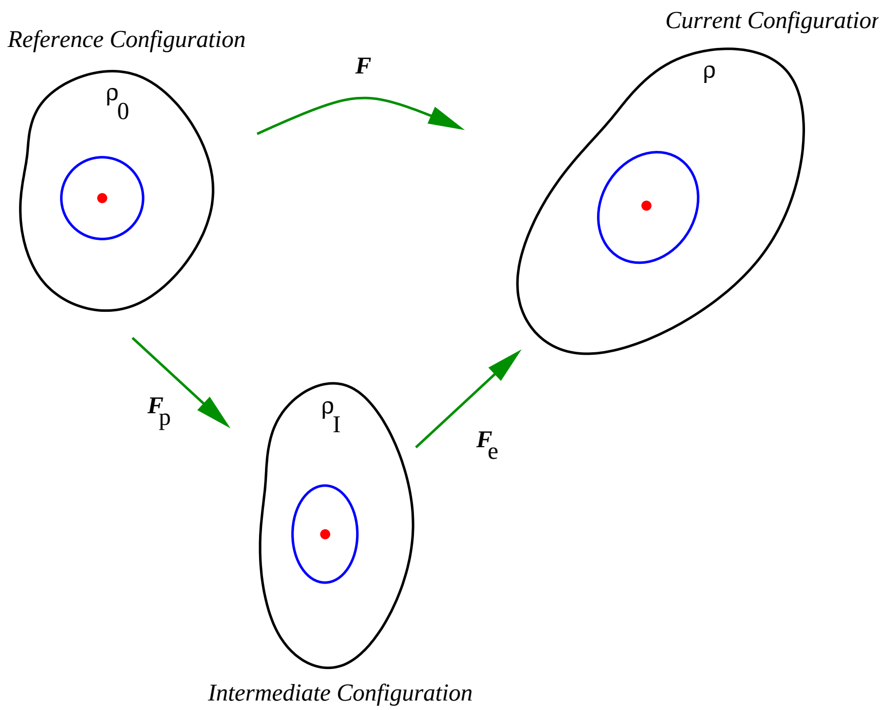

Figure 1: The initial, current, and intermediate configurations in plasticity.

2 Conservation of Mass

Experimental data strongly favors the assumption that plastic flow is volume preserving metals. However, in macroscopic scale deformations, voids and other defects in the material may cause small volume changes. To keep the analysis general we assume that plastic deformation is not necessarily volume preserving.

Thus there may be a change in volume in going from the initial to the intermediate configuration. Let the density in the initial configuration be ρ_0, that in the intermediate configuration be ρ_I, and the density in the current configuration be ρ.

Then the overall conservation of mass (from the initial to the current configuration) implies that

$$\rho \ \det(\boldsymbol{F}) = \rho_0 \ . \tag{6}$$

Plugging in the decomposition for $\boldsymbol{F}$ gives

$$\rho \ \det(\boldsymbol{F}_e \cdot \boldsymbol{F}_p) = \rho \ \det(\boldsymbol{F}_e) \ \det(\boldsymbol{F}_p) = \rho_0 \ . \tag{7}$$

If we define the density in the intermediate configuration via

$$\boxed{\rho \ \det(\boldsymbol{F}_e) =: \rho_I} \tag{8}$$

we get

$$\boxed{\rho_I \ \det(\boldsymbol{F}_p) = \rho_0 \ .} \tag{9}$$

Equation (9) represents the conservation of mass from the initial to the intermediate configuration while equation (8) represents the conservation of mass from the intermediate to the current configuration.

The quantity ρ_I is assumed to be an internal variable (or at least a function of other internal variables). In the following we often replace ρ_I with q_0, where $q_i, \ \ i = 0 \ldots n$ represents a set of internal variables.

3 Stress Tensors

Let $\boldsymbol{\sigma}$ be the Cauchy stress. Then the 2nd Piola-Kirchhoff stress is defined as

$$\boldsymbol{S} = \frac{\rho_0}{\rho}\,\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \tag{10}$$

i.e., it is the pull-back of $\boldsymbol{\sigma}$ to the initial configuration. Now, let us plug in the decomposition of $\boldsymbol{F}$ into this formula to get

$$\boldsymbol{S} = \frac{\rho_0}{\rho}\,\boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}_e^{-T} \cdot \boldsymbol{F}_p^{-T}\,. \tag{11}$$

Define the 2nd P-K stress in the intermediate configuration as the pull-back of the Cauchy stress by the elastic part of the deformation gradient, i.e.,

$$\boxed{\boldsymbol{S}_I := \frac{\rho_I}{\rho}\,\boldsymbol{F}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}_e^{-T}\,.} \tag{12}$$

Then we can write

$$\boxed{\boldsymbol{S} = \frac{\rho_0}{\rho_I}\,\boldsymbol{F}_p^{-1} \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_p^{-T}\,.} \tag{13}$$

This is equivalent to a pull-back by $\boldsymbol{F}_p$ of the 2nd P-K stress in the intermediate configuration to the initial configuration. Note that if $\rho_0 = \rho_I$, i.e., for isochoric plasticity, we can use the Kirchhoff stress

$$\boldsymbol{\tau} := \det(\boldsymbol{F})\,\boldsymbol{\sigma} = \frac{\rho_0}{\rho}\,\boldsymbol{\sigma} \tag{14}$$

instead of the Cauchy stress without any effect on our analysis.

4 Thermodynamic Potentials

Following Wright [Wri02], we assume that the Gibbs free energy is regarded as fundamental (rather than the Helmholtz free energy). Then the Gibbs free energy functional is a function of the 2nd P-K stress in the intermediate configuration ($\boldsymbol{S}_I$), the temperature (T), and a set of internal variables ($q_j \;\; j = 0 \ldots n$). That is,

$$g = g(\boldsymbol{S}_I, T, q_j) \tag{15}$$

The internal variables evolve only during the plastic part of the deformation. The intermediate configuration is considered to be the reference configuration as far as any elastic deformations are concerned.

We assume that the elastic strain ($\boldsymbol{E}_e$) and the entropy (η) are given by (see [SW01] for a detailed justification of this assumption and its connection to an instantaneous theromoelastic response)

$$\boxed{\begin{aligned} \boldsymbol{E}_e &= \rho_I\,\frac{\partial g}{\partial \boldsymbol{S}_I} \\ \eta &= \frac{\partial g}{\partial T} \end{aligned}} \tag{16}$$

We will assume that equations (16) are invertible.

The elastic strain $\boldsymbol{E}_e$ is related to the elastic part of the deformation gradient by

$$\boxed{\boldsymbol{E}_e = \frac{1}{2}\left(\boldsymbol{F}_e^T \cdot \boldsymbol{F}_e - \boldsymbol{1}\right).} \tag{17}$$

Recall that for thermoelastic processes, the Helmholtz free energy (ψ) is defined as

$$\psi = e - T\,\eta \tag{18}$$

where e is the internal energy, T is the temperature, and η is the entropy. Also, the Gibbs free energy is defined as

$$g = -e + T\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S} : \boldsymbol{E} = -\psi + \frac{1}{\rho_0}\,\boldsymbol{S} : \boldsymbol{E} \tag{19}$$

where ρ_0 is the density in the initial configuration, $\boldsymbol{S}$ is the 2nd P-K stress and $\boldsymbol{E}$ is the Green strain.

For thermoplasticity, since all elastic processes are with respect to the intermediate configuration, we may write

$$\boxed{g = -e + T\,\eta + \frac{1}{\rho_I}\,\boldsymbol{S}_I : \boldsymbol{E}_e = -\psi + \frac{1}{\rho_I}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,.} \tag{20}$$

4.1 Functional dependencies of ψ and e

Since $g = g(\boldsymbol{S}_I, T, q_j)$, we have

$$\mathrm{d}g = \frac{\partial g}{\partial \boldsymbol{S}_I} : \mathrm{d}\boldsymbol{S}_I + \frac{\partial g}{\partial T}\,\mathrm{d}T + \sum_{j=0}^{n} \frac{\partial g}{\partial q_j}\,\mathrm{d}q_j\,. \tag{21}$$

Using the relations in equation (16) we can write

$$\mathrm{d}g = \frac{1}{\rho_I}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + \eta\,\mathrm{d}T + \sum_{j=0}^{n} \frac{\partial g}{\partial q_j}\,\mathrm{d}q_j\,. \tag{22}$$

Now, from equations (20), we have

$$e = -g + T\,\eta + \frac{1}{\rho_I}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,; \quad \psi = -g + \frac{1}{\rho_I}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,. \tag{23}$$

If we set $\rho_I = q_0$ (the first internal variable in the list), we can write

$$e = -g + T\,\eta + q_0^{-1}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,; \quad \psi = -g + q_0^{-1}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,. \tag{24}$$

The differentials of equations (24) are

$$\begin{aligned} \mathrm{d}e &= -\mathrm{d}g + \eta\,\mathrm{d}T + T\,\mathrm{d}\eta - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\mathrm{d}q_0 + q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e \\ \mathrm{d}\psi &= -\mathrm{d}g - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\mathrm{d}q_0 + q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e\,. \end{aligned} \tag{25}$$

Plugging in the expression for $\mathrm{d}g$ from equation (22) gives

$$\begin{aligned} \mathrm{d}e &= -q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I - \eta\,\mathrm{d}T - \sum_{j=0}^{n} \frac{\partial g}{\partial q_j}\,\mathrm{d}q_j + \eta\,\mathrm{d}T + T\,\mathrm{d}\eta - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\mathrm{d}q_0 + q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e \\ \mathrm{d}\psi &= -q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I - \eta\,\mathrm{d}T - \sum_{j=0}^{n} \frac{\partial g}{\partial q_j}\,\mathrm{d}q_j - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\mathrm{d}q_0 + q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e \end{aligned} \tag{26}$$

or,

$$\begin{aligned} \mathrm{d}e &= -\sum_{j=0}^{n} \frac{\partial g}{\partial q_j}\,\mathrm{d}q_j + T\,\mathrm{d}\eta - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\mathrm{d}q_0 + q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e \\ \mathrm{d}\psi &= -\eta\,\mathrm{d}T - \sum_{j=0}^{n} \frac{\partial g}{\partial q_j}\,\mathrm{d}q_j - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\mathrm{d}q_0 + q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e\,. \end{aligned} \tag{27}$$

Collecting terms containing $\mathrm{d}q_0$ and rearranging, we get

$$\begin{aligned} \mathrm{d}e &= q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e + T\,\mathrm{d}\eta - \left(\frac{\partial g}{\partial q_0} + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\mathrm{d}q_0 - \sum_{j=1}^{n}\frac{\partial g}{\partial q_j}\,\mathrm{d}q_j \\ \mathrm{d}\psi &= q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e - \eta\,\mathrm{d}T - \left(\frac{\partial g}{\partial q_0} + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\mathrm{d}q_0 - \sum_{j=1}^{n}\frac{\partial g}{\partial q_j}\,\mathrm{d}q_j\,. \end{aligned} \tag{28}$$

Therefore, the differentials of the three potentials can be written as

$$\boxed{\begin{aligned} \mathrm{d}g &= q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + \eta\,\mathrm{d}T + \frac{\partial q}{\partial q_0}\,\mathrm{d}q_0 + \sum_{j=1}^{n}\frac{\partial g}{\partial q_j}\,\mathrm{d}q_j \\ \mathrm{d}e &= q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e + T\,\mathrm{d}\eta - \left(\frac{\partial g}{\partial q_0} + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\mathrm{d}q_0 - \sum_{j=1}^{n}\frac{\partial g}{\partial q_j}\,\mathrm{d}q_j \\ \mathrm{d}\psi &= q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e - \eta\,\mathrm{d}T - \left(\frac{\partial g}{\partial q_0} + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\mathrm{d}q_0 - \sum_{j=1}^{n}\frac{\partial g}{\partial q_j}\,\mathrm{d}q_j\,. \end{aligned}} \tag{29}$$

Let us define

$$Q_0 := -q_0\left(\frac{\partial g}{\partial q_0} + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right) \qquad \text{and} \qquad Q_j := -q_0\,\frac{\partial g}{\partial q_j}\ ,\ j = 1\ldots n\,. \tag{30}$$

Then we can write

$$\begin{aligned} \mathrm{d}g &= q_0^{-1}\,\boldsymbol{E}_e : \mathrm{d}\boldsymbol{S}_I + \eta\,\mathrm{d}T - \left(q_0^{-1}\,Q_0 + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\,\mathrm{d}q_0 - q_0^{-1}\,\sum_{j=1}^{n} Q_j\,\mathrm{d}q_j \\ \mathrm{d}e &= q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e + T\,\mathrm{d}\eta + q_0^{-1}\,Q_0\,\mathrm{d}q_0 + q_0^{-1}\,\sum_{j=1}^{n} Q_j\,\mathrm{d}q_j \\ \mathrm{d}\psi &= q_0^{-1}\,\boldsymbol{S}_I : \mathrm{d}\boldsymbol{E}_e - \eta\,\mathrm{d}T + q_0^{-1}\,Q_0\,\mathrm{d}q_0 + q_0^{-1}\,\sum_{j=1}^{n} Q_j\,\mathrm{d}q_j\,. \end{aligned} \tag{31}$$

The above equations suggest the following functional dependencies:

$$g = g(\boldsymbol{S}_I, T, q_0, q_j)\ ;\ \ e = e(\boldsymbol{E}_e, \eta, q_0, q_j)\ ;\ \ \psi = \psi(\boldsymbol{E}_e, T, q_0, q_j)\ ,\ \ j = 1\ldots n\,. \tag{32}$$

The partial derivatives of the potentials give us

$$\boxed{\begin{array}{llll} \dfrac{\partial g}{\partial \boldsymbol{S}_I} = q_0^{-1}\,\boldsymbol{E}_e\ ; & \dfrac{\partial g}{\partial T} = \eta\ ; & \dfrac{\partial g}{\partial q_0} = -\left(q_0^{-1}\,Q_0 + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\ ; & \dfrac{\partial g}{\partial q_j} = -q_0^{-1}\,Q_j \\ \dfrac{\partial e}{\partial \boldsymbol{E}_e} = q_0^{-1}\,\boldsymbol{S}_I\ ; & \dfrac{\partial e}{\partial \eta} = T\ ; & \dfrac{\partial e}{\partial q_0} = q_0^{-1}\,Q_0\ ; & \dfrac{\partial e}{\partial q_j} = q_0^{-1}\,Q_j \\ \dfrac{\partial \psi}{\partial \boldsymbol{E}_e} = q_0^{-1}\,\boldsymbol{S}_I\ ; & \dfrac{\partial \psi}{\partial T} = -\eta\ ; & \dfrac{\partial \psi}{\partial q_0} = q_0^{-1}\,Q_0\ ; & \dfrac{\partial \psi}{\partial q_j} = q_0^{-1}\,Q_j\,. \end{array}} \tag{33}$$

We can also find other relations between these partial derivatives. For example, equating the partial derivative of $\frac{\partial g}{\partial q_0}$ with respect to $\boldsymbol{S}_I$ with the partial derivative of $\frac{\partial g}{\partial \boldsymbol{S}_I}$ with respect to q_0, we have

$$-q_0^{-2}\,\boldsymbol{E}_e + q_0^{-1}\,\frac{\partial \boldsymbol{E}_e}{\partial q_0} = \frac{\partial^2 g}{\partial q_0 \partial \boldsymbol{S}_I} = -\left(q_0^{-1}\,\frac{\partial Q_0}{\partial \boldsymbol{S}_I} + q_0^{-2}\,\boldsymbol{E}_e + q_0^{-2}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial \boldsymbol{S}_I}\right) \tag{34}$$

or,

$$q_0^{-1}\ \frac{\partial \boldsymbol{E}_e}{\partial q_0} = -\left(q_0^{-1}\ \frac{\partial Q_0}{\partial \boldsymbol{S}_I} + q_0^{-2}\ \boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial \boldsymbol{S}_I}\right) \tag{35}$$

or,

$$\boxed{\frac{\partial \boldsymbol{E}_e}{\partial q_0} = -\frac{\partial Q_0}{\partial \boldsymbol{S}_I} - q_0^{-1}\ \boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial \boldsymbol{S}_I} = -\frac{\partial Q_0}{\partial \boldsymbol{S}_I} - \boldsymbol{S}_I : \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2}\ .} \tag{36}$$

Similarly, for the other internal variables $q_j,\ \ j = 1 \ldots n$, we have

$$q_0^{-1}\ \frac{\partial \boldsymbol{E}_e}{\partial q_j} = \frac{\partial^2 g}{\partial q_j \partial \boldsymbol{S}_I} = -q_0^{-1}\ \frac{\partial Q_j}{\partial \boldsymbol{S}_I} \tag{37}$$

or,

$$\boxed{\frac{\partial \boldsymbol{E}_e}{\partial q_j} = -\frac{\partial Q_j}{\partial \boldsymbol{S}_I}\ .} \tag{38}$$

If we consider mixed second partial derivatives of g with respect to q_0 and T, we have

$$\frac{\partial \eta}{\partial q_0} = \frac{\partial^2 g}{\partial q_0 \partial T} = -\left(q_0^{-1}\ \frac{\partial Q_0}{\partial T} + q_0^{-2}\ \boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\right) \tag{39}$$

or,

$$\boxed{q_0\ \frac{\partial \eta}{\partial q_0} = -\frac{\partial Q_0}{\partial T} - q_0^{-1}\ \boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\ .} \tag{40}$$

Similarly, derivatives with respect to the q_j s lead to

$$\frac{\partial \eta}{\partial q_j} = \frac{\partial^2 g}{\partial q_j \partial T} = -q_0^{-1}\ \frac{\partial Q_j}{\partial T} \tag{41}$$

or,

$$\boxed{q_0\ \frac{\partial \eta}{\partial q_j} = -\frac{\partial Q_j}{\partial T}\ .} \tag{42}$$

Several other relations can be derived based on mixed partial derivatives of g. For example,

$$\frac{\partial^2 g}{\partial T \partial \boldsymbol{S}_I} = \frac{\partial}{\partial T}\left(\frac{\partial g}{\partial \boldsymbol{S}_I}\right) = \frac{\partial}{\partial T}\left(q_0^{-1}\ \boldsymbol{E}_e\right) = q_0^{-1}\ \frac{\partial \boldsymbol{E}_e}{\partial T}\ . \tag{43}$$

Also,

$$\frac{\partial^2 g}{\partial T \partial \boldsymbol{S}_I} = \frac{\partial}{\partial \boldsymbol{S}_I}\left(\frac{\partial g}{\partial T}\right) = \frac{\partial \eta}{\partial \boldsymbol{S}_I}\ . \tag{44}$$

Therefore,

$$\boxed{\frac{\partial \eta}{\partial \boldsymbol{S}_I} = q_0^{-1}\ \frac{\partial \boldsymbol{E}_e}{\partial T}\ .} \tag{45}$$

We can also show that (see Appendix)

$$\boxed{\begin{aligned} \frac{\partial \eta(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} &= \frac{1}{T}\left[\frac{\partial e(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} - q_0^{-1}\ \boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\right] \\ \frac{\partial \eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} &= \frac{1}{T}\ \frac{\partial e(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} \\ \frac{\partial \eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial \boldsymbol{E}_e} &= q_0^{-1}\ \frac{\partial \boldsymbol{S}_I}{\partial T}\ . \end{aligned}} \tag{46}$$

Now if the specific heat at constant stress is defined as

$$C_p := \frac{\partial e(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} \,. \tag{47}$$

we have

$$\boxed{\frac{\partial \eta}{\partial T} = \frac{1}{T}\left[C_p - q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\right] \,.} \tag{48}$$

5 Entropy Inequality in Thermoplasticity

Recall that the entropy inequality for thermoelasticity can be written as

$$\rho\,(\dot{e} - T\,\dot{\eta}) - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} \le \frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T} \,. \tag{49}$$

Also, the internal energy for thermoplasticity can be written as (see equation (20))

$$e = -g + T\,\eta + \frac{1}{\rho_I}\,\boldsymbol{S}_I : \boldsymbol{E}_e = -g + T\,\eta + q_0^{-1}\,\boldsymbol{S}_I : \boldsymbol{E}_e \,. \tag{50}$$

Therefore,

$$\dot{e} = -\dot{g} + \dot{T}\,\eta + T\,\dot{\eta} - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\dot{q}_0 + q_0^{-1}\,\dot{\boldsymbol{S}}_I : \boldsymbol{E}_e + q_0^{-1}\,\boldsymbol{S}_I : \dot{\boldsymbol{E}}_e \,. \tag{51}$$

Now,

$$\dot{g} = \frac{\partial g}{\partial \boldsymbol{S}_I} : \dot{\boldsymbol{S}}_I + \frac{\partial g}{\partial T}\,\dot{T} + \frac{\partial g}{\partial q_0}\,\dot{q}_0 + \sum_{j=1}^{n}\frac{\partial g}{\partial q_j}\,\dot{q}_j = q_0^{-1}\,\boldsymbol{E}_e : \dot{\boldsymbol{S}}_I + \eta\,\dot{T} - q_0^{-1}\left(Q_0 + q_0^{-1}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\,\dot{q}_0 - q_0^{-1}\sum_{j=1}^{n} Q_j\,\dot{q}_j \,. \tag{52}$$

Plugging the expression for $\dot{g}$ into the expression for $\dot{e}$ gives us

$$\begin{aligned} \dot{e} = &- q_0^{-1}\,\boldsymbol{E}_e : \dot{\boldsymbol{S}}_I - \eta\,\dot{T} + q_0^{-1}\,Q_0\,\dot{q}_0 + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\dot{q}_0 + q_0^{-1}\sum_{j=1}^{n} Q_j\,\dot{q}_j \\ &+ \dot{T}\,\eta + T\,\dot{\eta} - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\,\dot{q}_0 + q_0^{-1}\,\dot{\boldsymbol{S}}_I : \boldsymbol{E}_e + q_0^{-1}\,\boldsymbol{S}_I : \dot{\boldsymbol{E}}_e \end{aligned} \tag{53}$$

or

$$\dot{e} - T\,\dot{\eta} = q_0^{-1}\,Q_0\,\dot{q}_0 + q_0^{-1}\sum_{j=1}^{n} Q_j\,\dot{q}_j + q_0^{-1}\,\boldsymbol{S}_I : \dot{\boldsymbol{E}}_e \tag{54}$$

or,

$$\dot{e} - T\,\dot{\eta} = q_0^{-1}\left(\sum_{j=0}^{n} Q_j\,\dot{q}_j + \boldsymbol{S}_I : \dot{\boldsymbol{E}}_e\right) \,. \tag{55}$$

Substituting (55) into (49) and reverting to $\rho_I \equiv q_0$ leads to

$$\frac{\rho}{\rho_I}\left(\sum_{j=0}^{n} Q_j\,\dot{q}_j + \boldsymbol{S}_I : \dot{\boldsymbol{E}}_e\right) - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} \le \frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T} \,. \tag{56}$$

Since

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \frac{1}{2}\,\boldsymbol{\sigma} : [\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T] = \boldsymbol{\sigma} : \boldsymbol{d} \tag{57}$$

we can write the entropy inequality as

$$\boxed{\boldsymbol{\sigma} : \boldsymbol{d} - \frac{\rho}{\rho_I}\,\boldsymbol{S}_I : \dot{\boldsymbol{E}}_e - \frac{\rho}{\rho_I}\sum_{j=0}^{n} Q_j\,\dot{q}_j - \frac{\mathbf{q}\cdot\boldsymbol{\nabla} T}{T} \ge 0\,.} \tag{58}$$

Note that for purely elastic deformations, we have

$$\boldsymbol{\sigma} : \boldsymbol{d} = \frac{\rho}{\rho_I}\,\boldsymbol{S}_I : \dot{\boldsymbol{E}}_e \tag{59}$$

since these terms represent the stress power. Also, if the internal variables do not evolve during such deformations, we have

$$\frac{\rho}{\rho_I}\sum_{j=0}^{n} Q_j\,\dot{q}_j = 0 \tag{60}$$

and we are left with the heat conduction inequality

$$\boxed{\frac{\mathbf{q}\cdot\boldsymbol{\nabla} T}{T} \le 0\,.} \tag{61}$$

Also, in the event that the temperature gradient vanishes, we must have

$$\boxed{\boldsymbol{\sigma} : \boldsymbol{d} - \frac{\rho}{\rho_I}\,\boldsymbol{S}_I : \dot{\boldsymbol{E}}_e - \frac{\rho}{\rho_I}\sum_{j=0}^{n} Q_j\,\dot{q}_j \ge 0\,.} \tag{62}$$

Equations (61) and (62) represent a split of the entropy inequality into purely mechanical and purely thermal parts.

5.1 Elastic-Plastic decomposition of entropy inequality

Recall that

$$\boldsymbol{d} = \frac{1}{2}\,[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T] \qquad \text{and} \qquad \boldsymbol{\nabla}\mathbf{v} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\,. \tag{63}$$

Therefore,

$$\boldsymbol{d} = \frac{1}{2}\left(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{F}}^T\right)\,. \tag{64}$$

Also,

$$\boldsymbol{F} = \boldsymbol{F}_e\cdot\boldsymbol{F}_p\,. \tag{65}$$

Hence,

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_p + \boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\,;\;\; \dot{\boldsymbol{F}}^T = \boldsymbol{F}_p^T\cdot\dot{\boldsymbol{F}}_e^T + \dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\,;\;\; \boldsymbol{F}^{-1} = \boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1}\,;\;\; \boldsymbol{F}^{-T} = \boldsymbol{F}_e^{-T}\cdot\boldsymbol{F}_p^{-T}\,. \tag{66}$$

Plugging these into the expression for $\boldsymbol{d}$, we have

$$\begin{aligned} \boldsymbol{d} &= \frac{1}{2}\left[(\dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_p + \boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p)\cdot(\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1}) + (\boldsymbol{F}_e^{-T}\cdot\boldsymbol{F}_p^{-T})\cdot(\boldsymbol{F}_p^T\cdot\dot{\boldsymbol{F}}_e^T + \dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T)\right] \\ &= \frac{1}{2}\left[\dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_e^{-1} + \boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1} + \boldsymbol{F}_e^{-T}\cdot\dot{\boldsymbol{F}}_e^T + \boldsymbol{F}_e^{-T}\cdot\boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\right] \end{aligned} \tag{67}$$

or,

$$\boxed{\boldsymbol{d} = \frac{1}{2}\left[\dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_e^{-1} + (\dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_e^{-1})^T\right] + \frac{1}{2}\left[\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1} + (\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1})^T\right]\,.} \tag{68}$$

Define

$$\boxed{\begin{aligned} \boldsymbol{d}_e &:= \frac{1}{2}\left[\dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_e^{-1} + (\dot{\boldsymbol{F}}_e\cdot\boldsymbol{F}_e^{-1})^T\right] \\ \boldsymbol{d}_p &:= \frac{1}{2}\left[\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1} + (\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1})^T\right]\ . \end{aligned}} \tag{69}$$

Then we can write

$$\boxed{\boldsymbol{d} = \boldsymbol{d}_e + \boldsymbol{d}_p\ .} \tag{70}$$

Now, form equation (12), we have

$$\boldsymbol{S}_I = \frac{\rho_I}{\rho}\,\boldsymbol{F}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}_e^{-T} \qquad \Longrightarrow \qquad \boxed{\boldsymbol{\sigma} = \frac{\rho}{\rho_I}\,\boldsymbol{F}_e\cdot\boldsymbol{S}_I\cdot\boldsymbol{F}_e^T\ .} \tag{71}$$

We can show that (see Appendix:item 5 for details)

$$\boxed{\boldsymbol{\sigma}:\boldsymbol{d} = \frac{\rho}{\rho_I}\,\boldsymbol{S}_I : \left[\frac{1}{2}\left(\boldsymbol{F}_e^T\cdot\dot{\boldsymbol{F}}_e + \dot{\boldsymbol{F}}_e^T\cdot\boldsymbol{F}_e\right) + \frac{1}{2}\left(\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1} + \boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e\right)\right]\ .} \tag{72}$$

Recall that

$$\boldsymbol{E}_e = \frac{1}{2}\left(\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e - \boldsymbol{1}\right)\ . \tag{73}$$

Hence,

$$\boxed{\boldsymbol{D}_e := \dot{\boldsymbol{E}}_e = \frac{1}{2}\left(\dot{\boldsymbol{F}}_e^T\cdot\boldsymbol{F}_e + \boldsymbol{F}_e^T\cdot\dot{\boldsymbol{F}}_e\right)\ .} \tag{74}$$

Also, define

$$\boxed{\boldsymbol{D}_p := \frac{1}{2}\left(\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1} + \boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e\right)\ .} \tag{75}$$

Then we can write equation (72) as

$$\boxed{\boldsymbol{\sigma}:\boldsymbol{d} = \frac{\rho}{\rho_I}\,\boldsymbol{S}_I:\dot{\boldsymbol{E}}_e + \frac{\rho}{\rho_I}\,\boldsymbol{S}_I:\boldsymbol{D}_p\ .} \tag{76}$$

We can also show that

$$\begin{aligned} \frac{\rho}{\rho_I}\,\boldsymbol{S}_I:\boldsymbol{D}_p &= \frac{1}{2}\,\text{tr}\left((\boldsymbol{F}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}_e^{-T})\cdot(\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1} + \boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\cdot\boldsymbol{F}_e)\right) \\ &= \frac{1}{2}\text{tr}\left(\boldsymbol{F}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\right) + \frac{1}{2}\text{tr}\left(\boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}_e^{-T}\right) \\ &= \frac{1}{2}\text{tr}\left(\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1}\cdot\boldsymbol{\sigma}\right) + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}\cdot\boldsymbol{F}_e^{-T}\cdot\boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T\right) \\ &= \frac{1}{2}\,\boldsymbol{\sigma}:(\boldsymbol{F}_e\cdot\dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\cdot\boldsymbol{F}_e^{-1} + \boldsymbol{F}_e^{-T}\cdot\boldsymbol{F}_p^{-T}\cdot\dot{\boldsymbol{F}}_p^T\cdot\boldsymbol{F}_e^T)\ . \end{aligned} \tag{77}$$

From the definition (69) we then have

$$\boxed{\frac{\rho}{\rho_I}\,\boldsymbol{S}_I:\boldsymbol{D}_p = \boldsymbol{\sigma}:\boldsymbol{d}_p\ .} \tag{78}$$

The entropy inequality in equation (62) may now be expressed as

$$\boxed{\boldsymbol{\sigma}:\boldsymbol{d}_p - \frac{\rho}{\rho_I}\sum_{j=0}^{n} Q_j\,\dot{q}_j \geq 0\ .} \tag{79}$$

where

$$\boxed{\boldsymbol{d}_p = \frac{1}{2}\left[\boldsymbol{F}_e\cdot\boldsymbol{L}_p\cdot\boldsymbol{F}_e^{-1} + (\boldsymbol{F}_e\cdot\boldsymbol{L}_p\cdot\boldsymbol{F}_e^{-1})^T\right]\ ;\ \ \boldsymbol{L}_p := \dot{\boldsymbol{F}}_p\cdot\boldsymbol{F}_p^{-1}\ .} \tag{80}$$

The quantity $\boldsymbol{d}_p$ can be interpreted as a **rate of plastic deformation**.

5.2 Is d_p really a plastic rate of deformation?

Note that

$$
\begin{aligned}
\text{tr}\,(\boldsymbol{d}_p) &= \frac{1}{2}\left[\text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{L}_p \cdot \boldsymbol{F}_e^{-1}\right) + \text{tr}\left(\boldsymbol{F}_e^{-T} \cdot \boldsymbol{L}_p^T \cdot \boldsymbol{F}_e^T\right)\right] \\
&= \frac{1}{2}\left[\text{tr}\left(\boldsymbol{F}_e^{-1} \cdot \boldsymbol{F}_e \cdot \boldsymbol{L}_p\right) + \text{tr}\left(\boldsymbol{F}_e^{T} \cdot \boldsymbol{F}_e^{-T} \cdot \boldsymbol{L}_p^T\right)\right] \\
&= \frac{1}{2}\left[\text{tr}\left(\boldsymbol{L}_p\right) + \text{tr}\left(\boldsymbol{L}_p^T\right)\right] = \text{tr}\left(\boldsymbol{L}_p\right)\ .
\end{aligned}
\tag{81}
$$

Therefore,

$$
\boxed{\text{tr}\,(\boldsymbol{d}_p) = \text{tr}\left(\dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1}\right)}
\tag{82}
$$

which is similar in form to the relation

$$
\text{tr}\,(\boldsymbol{d}) = \text{tr}\,(\boldsymbol{\nabla}\mathbf{v}) = \text{tr}\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right)\ .
\tag{83}
$$

Also, recall that the rate of change of volume is give by

$$
\dot{J} = J\ \text{tr}\,(\boldsymbol{d})\ ;\ \ J := \det(\boldsymbol{F}) = \frac{\rho_0}{\rho}\,.
\tag{84}
$$

Similarly, let us define

$$
J_p(\boldsymbol{F}_p) := \det(\boldsymbol{F}_p)\ .
\tag{85}
$$

If the tensor $\boldsymbol{F}_p$ is invertible, then the directional derivative of J_p is given by (see for instance [Gur81] p. 23)

$$
DJ_p(\boldsymbol{F}_p)[\boldsymbol{A}] = \det(\boldsymbol{F}_p)\ \text{tr}\left(\boldsymbol{A} \cdot \boldsymbol{F}_p^{-1}\right) = J_p\ \text{tr}\left(\boldsymbol{A} \cdot \boldsymbol{F}_p^{-1}\right)
\tag{86}
$$

for all tensors $\boldsymbol{A}$.

In addition, from the chain rule (see [Gur81] p.26)

$$
\frac{d}{dt} J_p(\boldsymbol{F}_p(t)) = DJ_p(\boldsymbol{F}_p)[\dot{\boldsymbol{F}}_p(t)]\ .
\tag{87}
$$

Therefore, using (82), we get

$$
\dot{J}_p = J_p\ \text{tr}\left(\dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1}\right) = J_p\ \text{tr}\,(\boldsymbol{d}_p)\ .
\tag{88}
$$

Also, from (9), we have

$$
J_p = \det(\boldsymbol{F}_p) = \frac{\rho_0}{\rho_I}\,.
\tag{89}
$$

Hence,

$$
\dot{J}_p = \frac{\rho_0}{\rho_I}\ \text{tr}\,(\boldsymbol{d}_p)\ .
\tag{90}
$$

Now,

$$
\dot{J}_p = \frac{dJ_p}{dt} = \frac{d}{dt}\left(\frac{\rho_0}{\rho_I}\right) = -\frac{\rho_0}{\rho_I^2}\,\dot{\rho}_I\ .
\tag{91}
$$

Comparing equations (90) and (91), we get

$$
\boxed{\text{tr}\,(\boldsymbol{d}_p) = -\frac{\dot{\rho}_I}{\rho_I}}
\tag{92}
$$

which has a form similar to

$$
\text{tr}\,(\boldsymbol{d}) = -\frac{\dot{\rho}}{\rho}\,.
\tag{93}
$$

These relations indicate that the quantity $\boldsymbol{d}_p$ may be considered to be a plastic rate of deformation tensor just as $\boldsymbol{d}$ is considered to be the total rate of deformation tensor.

5.3 Decomposition into volumetric and distortional components

The Cauchy stress can be decomposed into volumetric and distortional components as

$$\boxed{\boldsymbol{\sigma} = \frac{1}{3}\,\text{tr}\,(\boldsymbol{\sigma})\;\; \boldsymbol{1} + \boldsymbol{s} = -p\;\boldsymbol{1} + \boldsymbol{s}\;.} \tag{94}$$

It may not be obvious that we can do the same for the plastic rate of deformation $\boldsymbol{d}_p$. The question arises that if we decompose $\boldsymbol{d}_p$ into volumetric and deviatoric parts as

$$\boxed{\boldsymbol{d}_p = \frac{1}{3}\,\text{tr}\,(\boldsymbol{d}_p)\;\; \boldsymbol{1} + \boldsymbol{\eta}_p} \tag{95}$$

then is $\boldsymbol{\eta}_p$ truly a distortional term or does it contain a volumetric component too? Indeed, it can be shown that the deviatoric part of $\boldsymbol{d}_p$ does not contain any volumetric terms (see Appendix: item 6) .

Then we can write

$$\begin{aligned}
\boldsymbol{\sigma} : \boldsymbol{d}_p &= (-p\;\boldsymbol{1} + \boldsymbol{s}) : \left(\frac{1}{3}\,\text{tr}\,(\boldsymbol{d}_p)\;\;\boldsymbol{1} + \boldsymbol{\eta}_p\right) \\
&= -p\,\text{tr}\,(\boldsymbol{d}_p) - p\;\cancelto{0}{\text{tr}\,(\boldsymbol{\eta}_p)} + \frac{1}{3}\,\text{tr}\,(\boldsymbol{d}_p)\;\;\cancelto{0}{\text{tr}\,(\boldsymbol{s})} + \boldsymbol{s} : \boldsymbol{\eta}_p \\
&= -p\,\text{tr}\,(\boldsymbol{d}_p) + \boldsymbol{s} : \boldsymbol{\eta}_p = p\,\frac{\dot{\rho}_I}{\rho_I} + \boldsymbol{s} : \boldsymbol{\eta}_p = p\,\frac{\dot{q}_0}{q_0} + \boldsymbol{s} : \boldsymbol{\eta}_p\;.
\end{aligned} \tag{96}$$

Recall the entropy inequality in equation (79)

$$\boldsymbol{\sigma} : \boldsymbol{d}_p - \frac{\rho}{\rho_I}\sum_{j=0}^{n} Q_j\;\dot{q}_j \geq 0 \tag{97}$$

which can be written (with $\rho_I \equiv q_0$) as

$$\boldsymbol{\sigma} : \boldsymbol{d}_p - \rho\;Q_0\,\frac{\dot{q}_0}{q_0} - \frac{\rho}{q_0}\sum_{j=1}^{n} Q_j\;\dot{q}_j \geq 0 \tag{98}$$

We can now express the above equation as

$$\boldsymbol{s} : \boldsymbol{\eta}_p + (p - \rho\;Q_0)\,\frac{\dot{q}_0}{q_0} - \frac{\rho}{q_0}\sum_{j=1}^{n} Q_j\;\dot{q}_j \geq 0 \tag{99}$$

or,

$$\boxed{\boldsymbol{s} : \boldsymbol{\eta}_p + (p - \rho\;Q_0)\,\frac{\dot{\rho}_I}{\rho_I} - \frac{\rho}{\rho_I}\sum_{j=1}^{n} Q_j\;\dot{q}_j \geq 0\;.} \tag{100}$$

This is another form of **the Clausius-Duhem inequality.** From equation (100) we can see that for us to have a well posed problem it is necessary that rate equations for $\boldsymbol{\eta}_p$, $\dot{\rho}_I$, and $\dot{q}_j$ must be provided. Clearly these constitutive relations must be prescribed in such a way that the inequality in equation (100) is never violated.

5.3.1 Special cases

1. **Isochoric plastic deformation:**
   If the plastic volume change is zero, then the Clausius-Duhem inequality takes the form

$$\boldsymbol{s} : \boldsymbol{\eta}_p - \sum_{j=1}^{n} Q_j\;\dot{q}_j \geq 0\;. \tag{101}$$

2. **Internal energy and free energy do not depend in ρ_I:**
   If we have the situation

$$\rho_I = \rho_I(q_j) \tag{102}$$

   and

$$g = g(\boldsymbol{S}_I, T, q_j) \; ; \quad e = e(\boldsymbol{E}_e, \eta, q_j) \; ; \quad \psi = \psi(\boldsymbol{E}_e, T, q_j) \; , \quad j = 1 \dots n \; . \tag{103}$$

   we can show that these conditions are equivalent to assuming that $Q_0 = 0$ (see Appendix:item 7). In that case the Clausius-Duhem inequality becomes

$$\boldsymbol{s} : \boldsymbol{\eta}_p + p \, \frac{\dot{\rho}_I}{\rho_I} - \frac{\rho}{\rho_I} \sum_{j=1}^{n} Q_j \, \dot{q}_j \geq 0 \tag{104}$$

   or,

$$\boldsymbol{s} : \boldsymbol{\eta}_p + \frac{p}{\rho_I} \left(\sum_{j=1}^{n} \frac{\partial \rho_I}{\partial q_j} \, \dot{q}_j \right) - \frac{\rho}{\rho_I} \sum_{j=1}^{n} Q_j \, \dot{q}_j \geq 0 \tag{105}$$

   or,

$$\boxed{\boldsymbol{s} : \boldsymbol{\eta}_p - \frac{\rho}{\rho_I} \sum_{j=1}^{n} \left[Q_j - \frac{p}{\rho} \frac{\partial \rho_I}{\partial q_j} \right] \, \dot{q}_j \geq 0 \; .} \tag{106}$$

6 Energy Equation for Thermoplasticity

Recall that the balance of energy for thermoelastic processes is given by

$$\rho \, \dot{e} = \boldsymbol{\sigma} : \boldsymbol{\nabla} v + \rho \, s - \boldsymbol{\nabla} \cdot \mathbf{q} \; . \tag{107}$$

From equation (55) we have

$$\dot{e} = T \, \dot{\eta} + q_0^{-1} \left(\sum_{j=0}^{n} Q_j \, \dot{q}_j + \boldsymbol{S}_I : \dot{\boldsymbol{E}}_e \right) \; . \tag{108}$$

Therefore we can write the energy equation as

$$\rho \, T \, \dot{\eta} + \rho \, q_0^{-1} \left(\sum_{j=0}^{n} Q_j \, \dot{q}_j + \boldsymbol{S}_I : \dot{\boldsymbol{E}}_e \right) - \boldsymbol{\sigma} : \boldsymbol{\nabla} v - \rho \, s + \boldsymbol{\nabla} \cdot \mathbf{q} = 0 \; . \tag{109}$$

Reverting to $\rho_I \equiv q_0$, we can write

$$\rho \, T \, \dot{\eta} + \frac{\rho}{\rho_I} \left(\sum_{j=0}^{n} Q_j \, \dot{q}_j \right) + \frac{\rho}{\rho_I} \boldsymbol{S}_I : \dot{\boldsymbol{E}}_e - \boldsymbol{\sigma} : \boldsymbol{\nabla} v - \rho \, s + \boldsymbol{\nabla} \cdot \mathbf{q} = 0 \; . \tag{110}$$

From equations (76) and (78) we see that

$$\frac{\rho}{\rho_I} \boldsymbol{S}_I : \dot{\boldsymbol{E}}_e = \boldsymbol{\sigma} : \boldsymbol{d} - \boldsymbol{\sigma} : \boldsymbol{d}_p \; . \tag{111}$$

Substituting (111) into (110) gives

$$\rho \, T \, \dot{\eta} + \frac{\rho}{\rho_I} \left(\sum_{j=0}^{n} Q_j \, \dot{q}_j \right) + \boldsymbol{\sigma} : \boldsymbol{d} - \boldsymbol{\sigma} : \boldsymbol{d}_p - \boldsymbol{\sigma} : \boldsymbol{\nabla} v - \rho \, s + \boldsymbol{\nabla} \cdot \mathbf{q} = 0 \; . \tag{112}$$

Recall that

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\sigma} : \boldsymbol{d} \,. \tag{113}$$

Hence,

$$\boxed{\rho\, T\, \dot{\eta} + \frac{\rho}{\rho_I}\left(\sum_{j=0}^{n} Q_j\, \dot{q}_j\right) - \boldsymbol{\sigma} : \boldsymbol{d}_p - \rho\, s + \boldsymbol{\nabla}\cdot\mathbf{q} = 0 \,.} \tag{114}$$

Recall from equations (33) that the entropy η can be derived from a thermodynamic potential, i.e.,

$$\eta = \frac{\partial g}{\partial T} = \frac{\partial \psi}{\partial T} \tag{115}$$

where, from (32),

$$g = g(\boldsymbol{S}_I, T, q_0, q_j)\ ;\ \ \psi = \psi(\boldsymbol{E}_e, T, q_0, q_j)\ ,\ \ j = 1\ldots n\,. \tag{116}$$

There can be two forms of the energy equation based on which variables we consider to be independent.

1. If we consider the Helmholtz free energy to be fundamental, then we have

$$\eta = \eta(\boldsymbol{E}_e, T, q_0, q_j)\ ,\ \ j = 1\ldots n\,. \tag{117}$$

Therefore, using the chain rule,

$$\dot{\eta} = \frac{\partial \eta}{\partial \boldsymbol{E}_I} : \dot{\boldsymbol{E}}_e + \frac{\partial \eta}{\partial T}\,\dot{T} + \frac{\partial \eta}{\partial q_0}\,\dot{q}_0 + \sum_{j=1}^{n}\frac{\partial \eta}{\partial q_j}\,\dot{q}_j\,. \tag{118}$$

Now, from equations (33), we have

$$\frac{\partial \psi}{\partial \boldsymbol{E}_e} = q_0^{-1}\,\boldsymbol{S}_I\ ;\ \ \frac{\partial \psi}{\partial T} = -\eta\ ;\ \ \frac{\partial \psi}{\partial q_0} = q_0^{-1}\,Q_0\ ;\ \ \frac{\partial \psi}{\partial q_j} = q_0^{-1}\,Q_j\,. \tag{119}$$

Therefore,

$$\begin{aligned}\frac{\partial \eta}{\partial \boldsymbol{E}_e} &= -\frac{\partial}{\partial T}\left(\frac{\partial \psi}{\partial \boldsymbol{E}_e}\right) = -q_0^{-1}\,\frac{\partial \boldsymbol{S}_I}{\partial T}\\ \frac{\partial \eta}{\partial q_j} &= -\frac{\partial}{\partial T}\left(\frac{\partial \psi}{\partial q_j}\right) = -q_0^{-1}\,\frac{\partial Q_j}{\partial T}\,.\end{aligned} \tag{120}$$

Also, the specific heat at constant strain is defined as

$$C_v := \frac{\partial e(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} \tag{121}$$

which implies that

$$C_v = \frac{\partial}{\partial T}(\psi + T\,\eta) = \frac{\partial \psi}{\partial T} + \eta + T\,\frac{\partial \eta}{\partial T} = T\,\frac{\partial \eta}{\partial T}\,. \tag{122}$$

Hence,

$$\frac{\partial \eta}{\partial T} = \frac{C_v}{T}\,. \tag{123}$$

Plugging (120) and (123) into (118) gives

$$\dot{\eta} = -q_0^{-1}\,\frac{\partial \boldsymbol{S}_I}{\partial T} : \dot{\boldsymbol{E}}_e + \frac{C_v}{T}\,\dot{T} - q_0^{-1}\,\sum_{j=0}^{n}\frac{\partial Q_j}{\partial T}\,\dot{q}_j\,. \tag{124}$$

Substituting the expression for $\dot{\eta}$ above into equation (114) lead to

$$\rho\, C_v\, \dot{T} - \rho\, T\, q_0^{-1}\, \frac{\partial \boldsymbol{S}_I}{\partial T} : \dot{\boldsymbol{E}}_e - \rho\, T\, q_0^{-1} \sum_{j=0}^{n} \frac{\partial Q_j}{\partial T}\, \dot{q}_j + \rho\, q_0^{-1} \sum_{j=0}^{n} Q_j\, \dot{q}_j - \boldsymbol{\sigma} : \boldsymbol{d}_p - \rho\, s + \boldsymbol{\nabla} \cdot \mathbf{q} = 0 \tag{125}$$

or,

$$\rho\, C_v\, \dot{T} = \rho\, T\, q_0^{-1}\, \frac{\partial \boldsymbol{S}_I}{\partial T} : \dot{\boldsymbol{E}}_e + \rho\, q_0^{-1} \sum_{j=0}^{n} \left(T\, \frac{\partial Q_j}{\partial T} - Q_j \right)\, \dot{q}_j + \boldsymbol{\sigma} : \boldsymbol{d}_p + \rho\, s - \boldsymbol{\nabla} \cdot \mathbf{q} \tag{126}$$

Using $\rho_I \equiv q_0$ and rearranging gives us the new form of the energy equation:

$$\boxed{\rho\, C_v\, \dot{T} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s + T\, \frac{\rho}{\rho_I}\, \frac{\partial \boldsymbol{S}_I}{\partial T} : \dot{\boldsymbol{E}}_e + \frac{\rho}{\rho_I} \sum_{j=0}^{n} \left(T\, \frac{\partial Q_j}{\partial T} - Q_j \right)\, \dot{q}_j + \boldsymbol{\sigma} : \boldsymbol{d}_p\, .} \tag{127}$$

For a thermoelastic process, the last two terms above evaluate to zero and we are left with the energy equation for thermoelasticity.

If the heat flux $\mathbf{q}$ can be derived from a temperature potential T as

$$\mathbf{q} = -\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T \tag{128}$$

where $\boldsymbol{\kappa}$ is the thermal conductivity tensor, and if the coefficient of thermal stress is defined as

$$\boldsymbol{\beta}_S := \frac{\partial \boldsymbol{S}_I}{\partial T} \tag{129}$$

then we can express the energy equation as

$$\boxed{\rho\, C_v\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s + T\, \frac{\rho}{\rho_I}\, \boldsymbol{\beta}_S : \dot{\boldsymbol{E}}_e + \frac{\rho}{\rho_I} \sum_{j=0}^{n} \left(T\, \frac{\partial Q_j}{\partial T} - Q_j \right)\, \dot{q}_j + \boldsymbol{\sigma} : \boldsymbol{d}_p\, .} \tag{130}$$

The coefficient $\boldsymbol{\beta}_S$ can be difficult to determine for the general anisotropic case.

Recall from equation (96) that

$$\boldsymbol{\sigma} : \boldsymbol{d}_p = p\, \frac{\dot{q}_0}{q_0} + \boldsymbol{s} : \boldsymbol{\eta}_p = p\, \frac{\dot{\rho}_I}{\rho_I} + \boldsymbol{s} : \boldsymbol{\eta}_p \tag{131}$$

Using this relation, we can provide another form of the energy equation:

$$\boxed{\rho\, C_v\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s + T\, \frac{\rho}{\rho_I}\, \boldsymbol{\beta}_S : \dot{\boldsymbol{E}}_e + \frac{\rho}{\rho_I} \sum_{j=0}^{n} \left(T\, \frac{\partial Q_j}{\partial T} - Q_j \right)\, \dot{q}_j + p\, \frac{\dot{\rho}_I}{\rho_I} + \boldsymbol{s} : \boldsymbol{\eta}_p\, .} \tag{132}$$

2. If we consider the Gibbs free energy functional to be fundamental, then we have

$$\eta = \eta(\boldsymbol{S}_I, T, q_0, q_j)\, , \quad j = 1 \ldots n\, . \tag{133}$$

Therefore, using the chain rule,

$$\dot{\eta} = \frac{\partial \eta}{\partial \boldsymbol{S}_I} : \dot{\boldsymbol{S}}_I + \frac{\partial \eta}{\partial T}\, \dot{T} + \frac{\partial \eta}{\partial q_0}\, \dot{q}_0 + \sum_{j=1}^{n} \frac{\partial \eta}{\partial q_j}\, \dot{q}_j\, . \tag{134}$$

Now from equations (45), (40), and (42), we have

$$\begin{aligned}
\frac{\partial\eta}{\partial\boldsymbol{S}_I} &= q_0^{-1}\,\frac{\partial\boldsymbol{E}_e}{\partial T} \\
\frac{\partial\eta}{\partial q_0} &= -q_0^{-1}\,\frac{\partial Q_0}{\partial T} - q_0^{-2}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T} \\
\frac{\partial\eta}{\partial q_j} &= -q_0^{-1}\,\frac{\partial Q_j}{\partial T}\,.
\end{aligned} \tag{135}$$

Also from equation (48),

$$\frac{\partial\eta}{\partial T} = \frac{1}{T}\left[C_p - q_0^{-1}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right] \tag{136}$$

Plugging the relations in equations (135) and (48) into (134), we get

$$\dot{\eta} = q_0^{-1}\,\frac{\partial\boldsymbol{E}_e}{\partial T}:\dot{\boldsymbol{S}}_I + \frac{1}{T}\,\left(C_p - q_0^{-1}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\,\dot{T} - q_0^{-1}\left(\frac{\partial Q_0}{\partial T} + q_0^{-1}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\dot{q}_0 - q_0^{-1}\,\sum_{j=1}^{n}\frac{\partial Q_j}{\partial T}\,\dot{q}_j \tag{137}$$

or,

$$\begin{aligned}
\dot{\eta} &= \frac{1}{T}\,\left(C_p - q_0^{-1}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\,\dot{T} + q_0^{-1}\,\left(\frac{\partial\boldsymbol{E}_e}{\partial T}:\dot{\boldsymbol{S}}_I - q_0^{-1}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\,\dot{q}_0 - \sum_{j=0}^{n}\frac{\partial Q_j}{\partial T}\,\dot{q}_j\right) \\
&= \frac{1}{T}\,\left(C_p - q_0^{-1}\,\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\,\dot{T} + q_0^{-1}\,\frac{\partial\boldsymbol{E}_e}{\partial T}:\left(\dot{\boldsymbol{S}}_I - q_0^{-1}\,\boldsymbol{S}_I\,\dot{q}_0\right) - q_0^{-1}\,\sum_{j=0}^{n}\frac{\partial Q_j}{\partial T}\,\dot{q}_j\,.
\end{aligned} \tag{138}$$

Therefore (with the substitution $q_0 \equiv \rho_I$), we get

$$\rho\,T\,\dot{\eta} = \rho\,\left(C_p - \frac{1}{\rho_I}\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\,\dot{T} + T\,\frac{\rho}{\rho_I}\,\frac{\partial\boldsymbol{E}_e}{\partial T}:\left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\boldsymbol{S}_I\right) - T\,\frac{\rho}{\rho_I}\sum_{j=0}^{n}\frac{\partial Q_j}{\partial T}\,\dot{q}_j\,. \tag{139}$$

Plugging the expression in equation (139) into the energy equation (114) gives

$$\begin{aligned}
\rho\,\left(C_p - \frac{1}{\rho_I}\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\,\dot{T} + T\,\frac{\rho}{\rho_I}\,\frac{\partial\boldsymbol{E}_e}{\partial T}:\left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\boldsymbol{S}_I\right) - T\,\frac{\rho}{\rho_I}\sum_{j=0}^{n}\frac{\partial Q_j}{\partial T}\,\dot{q}_j \\
+ \frac{\rho}{\rho_I}\sum_{j=0}^{n}Q_j\,\dot{q}_j - \boldsymbol{\sigma}:\boldsymbol{d}_p - \rho\,s + \boldsymbol{\nabla}\cdot\mathbf{q} = 0
\end{aligned} \tag{140}$$

or,

$$\boxed{\begin{aligned}
\rho\,\left(C_p - \frac{1}{\rho_I}\boldsymbol{S}_I:\frac{\partial\boldsymbol{E}_e}{\partial T}\right)\,\dot{T} = -\boldsymbol{\nabla}\cdot\mathbf{q} + \rho\,s + T\,\frac{\rho}{\rho_I}\,\frac{\partial\boldsymbol{E}_e}{\partial T}:\left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\boldsymbol{S}_I\right) \\
+ \frac{\rho}{\rho_I}\sum_{j=0}^{n}\left(Q_j - T\,\frac{\partial Q_j}{\partial T}\right)\,\dot{q}_j - \boldsymbol{\sigma}:\boldsymbol{d}_p\,.
\end{aligned}} \tag{141}$$

This is a slightly more complicated version of the energy equation for thermoplasticity and the form in equation (127) is preferable since strain rates are more accessible than stress rates.

If the heat flux $\mathbf{q}$ is related to the temperature gradient by

$$\mathbf{q} = -\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}T \tag{142}$$

and if the coefficient of thermal expansion is defined as

$$\boldsymbol{\alpha}_E := \frac{\partial \boldsymbol{E}_e}{\partial T} \tag{143}$$

then the energy equation can be written as

$$\boxed{\begin{aligned}\rho\,\left(C_p - \frac{1}{\rho_I}\,\boldsymbol{S}_I : \boldsymbol{\alpha}_E\right)\,\dot{T} = -\boldsymbol{\nabla}\cdot\mathbf{q} + \rho\,s + T\,\frac{\rho}{\rho_I}\,\boldsymbol{\alpha}_E : \left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\,\boldsymbol{S}_I\right) \\ + \frac{\rho}{\rho_I}\sum_{j=0}^{n}\left(Q_j - T\,\frac{\partial Q_j}{\partial T}\right)\,\dot{q}_j - \boldsymbol{\sigma} : \boldsymbol{d}_p\,.\end{aligned}} \tag{144}$$

7 Constitutive Relations

To close the system of equations in thermoplasticity, we have to provide constitutive relations that can be used to determine the plastic part of the deformation. The usual situation is that the total deformation gradient is known and the decomposition of the deformation gradient into elastic and plastic parts have to be computed.

For elastically anisotropic materials, the orientation of the material in the intermediate configuration needs to be known before an the correct elastic stress-strain relation can be used. We will avoid the related complications for now and deal only with the elastically isotropic materials. However, it is important to know what the fuss is all about.

Consider an elastic material. The deformation gradient $\boldsymbol{F}$ can be decomposed into a pure stretch and a rotation in two ways, i.e.,

$$\boldsymbol{F}_e = \boldsymbol{R}\cdot\boldsymbol{U}_e = \boldsymbol{V}_e\cdot\boldsymbol{R} \qquad \text{where} \qquad \boldsymbol{R}\cdot\boldsymbol{R}^T = \boldsymbol{R}^T\cdot\boldsymbol{R} = \boldsymbol{1}\,. \tag{145}$$

A schematic of the polar decomposition theorem (and what the decomposition means in terms of Lagrangian triads) is shown in Figure 2. Now, suppose that we rotate the reference configuration by a rotation $\boldsymbol{Q}$ so that the deformation gradient becomes

$$\boldsymbol{F}_e^{\text{rot}} = \boldsymbol{F}_e\cdot\boldsymbol{Q}^T = \boldsymbol{R}\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q}^T = (\boldsymbol{R}\cdot\boldsymbol{Q}^T)\cdot(\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q}^T) = \boldsymbol{R}^{\text{rot}}\cdot\boldsymbol{U}_e^{\text{rot}}\,. \tag{146}$$

In this situation, the actual amount of stretch remains unchanged. However, the rotation is felt by the eigenvector triads of the stretch tensor and also be the original rotation tensor. Such a rotation of the reference configuration should not affect the Cauchy stress whether the material is isotropic or not. Let us now check how the 2nd P-K stress transforms under a rotation of the reference configuration. Recall that

$$\boldsymbol{S} = \frac{\rho_0}{\rho}\,\boldsymbol{F}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}_e^{-T}\,. \tag{147}$$

If $\boldsymbol{F}_e^{\text{rot}} = \boldsymbol{F}_e\cdot\boldsymbol{Q}^T$, then we have

$$\boldsymbol{S}^{\text{rot}} = \frac{\rho_0}{\rho}\,(\boldsymbol{F}_e\cdot\boldsymbol{Q}^T)^{-1}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{F}_e\cdot\boldsymbol{Q}^T)^{-T} = \frac{\rho_0}{\rho}\,\boldsymbol{Q}^{-T}\cdot\boldsymbol{F}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}_e^{-T}\cdot\boldsymbol{Q}^{-1} = \boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T\,. \tag{148}$$

Therefore, the 2nd P-K stress $\boldsymbol{S}$ also transforms in the same way as the elastic stretch tensor $\boldsymbol{U}_e$.

Now, suppose that the constitutive model of the material relates the 2nd P-K stress to the elastic stretch via

$$\boldsymbol{S} = \mathsf{C}(\boldsymbol{U}_e)\,. \tag{149}$$

A rotation of the initial configuration then implies that we should have

$$\boldsymbol{S}^{\text{rot}} = \mathsf{C}^{\text{rot}}(\boldsymbol{U}_e^{\text{rot}}) \tag{150}$$

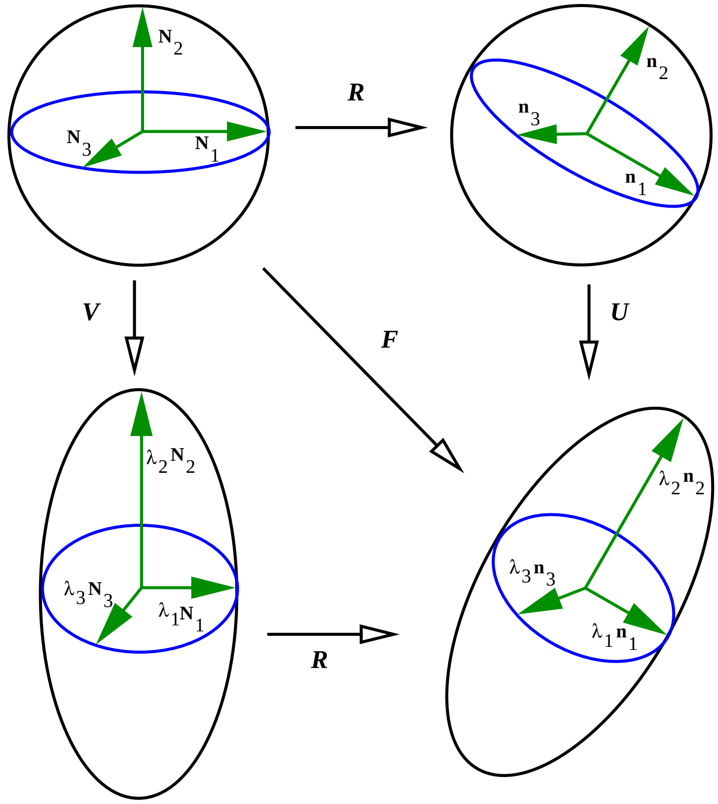

Figure 2: A schematic of the polar decomposition theorem (based on [NN04], p. 60).

or,

$$\boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T = \mathsf{C}^{\text{rot}}(\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q}) \qquad \Longrightarrow \qquad \boldsymbol{S} = \boldsymbol{Q}^T\cdot\mathsf{C}^{\text{rot}}(\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q})\cdot\boldsymbol{Q}\,. \tag{151}$$

Therefore the two constitutive models are related by

$$\mathsf{C}(\boldsymbol{U}_e) = \boldsymbol{Q}^T\cdot\mathsf{C}^{\text{rot}}(\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q})\cdot\boldsymbol{Q} \tag{152}$$

or

$$\mathsf{C}^{\text{rot}}(\boldsymbol{U}_e^{\text{rot}}) = \boldsymbol{Q}\cdot\mathsf{C}^{\text{rot}}(\boldsymbol{Q}^T\cdot\boldsymbol{U}_e^{\text{rot}}\cdot\boldsymbol{Q})\cdot\boldsymbol{Q}^T\,. \tag{153}$$

In general, $\mathsf{C}^{\text{rot}} \neq \mathsf{C}$ except for special values of the rotation $\boldsymbol{Q}$. However, if we know the constitutive relation in one configuration, we can evaluate the response in a rotated configuration by

1. Rotating the stretch back to the reference configuration.
2. Evaluating the constitutive relation in that configuration.
3. Rotating the result back to the rotated configuration.

Of course, things simplify greatly if the material is isotropic.

Similar considerations hold for elastic-plastic materials. In this case, the deformation gradient is decomposed into elastic and plastic parts with the restriction that $\det(\boldsymbol{F}_e) > 0$ and $\det(\boldsymbol{F}_p) > 0$. Hence we can use the polar decomposition theorem to write

$$\boldsymbol{F}_e = \boldsymbol{V}_e\cdot\boldsymbol{R}_e\,;\quad \boldsymbol{F}_p = \boldsymbol{R}_p\cdot\boldsymbol{U}_p\,. \tag{154}$$

Hence,

$$\boxed{\boldsymbol{F} = \boldsymbol{V}_e\cdot\boldsymbol{R}_e\cdot\boldsymbol{R}_p\cdot\boldsymbol{U}_p = \boldsymbol{V}_e\cdot\boldsymbol{Q}\cdot\boldsymbol{U}_p\,.} \tag{155}$$

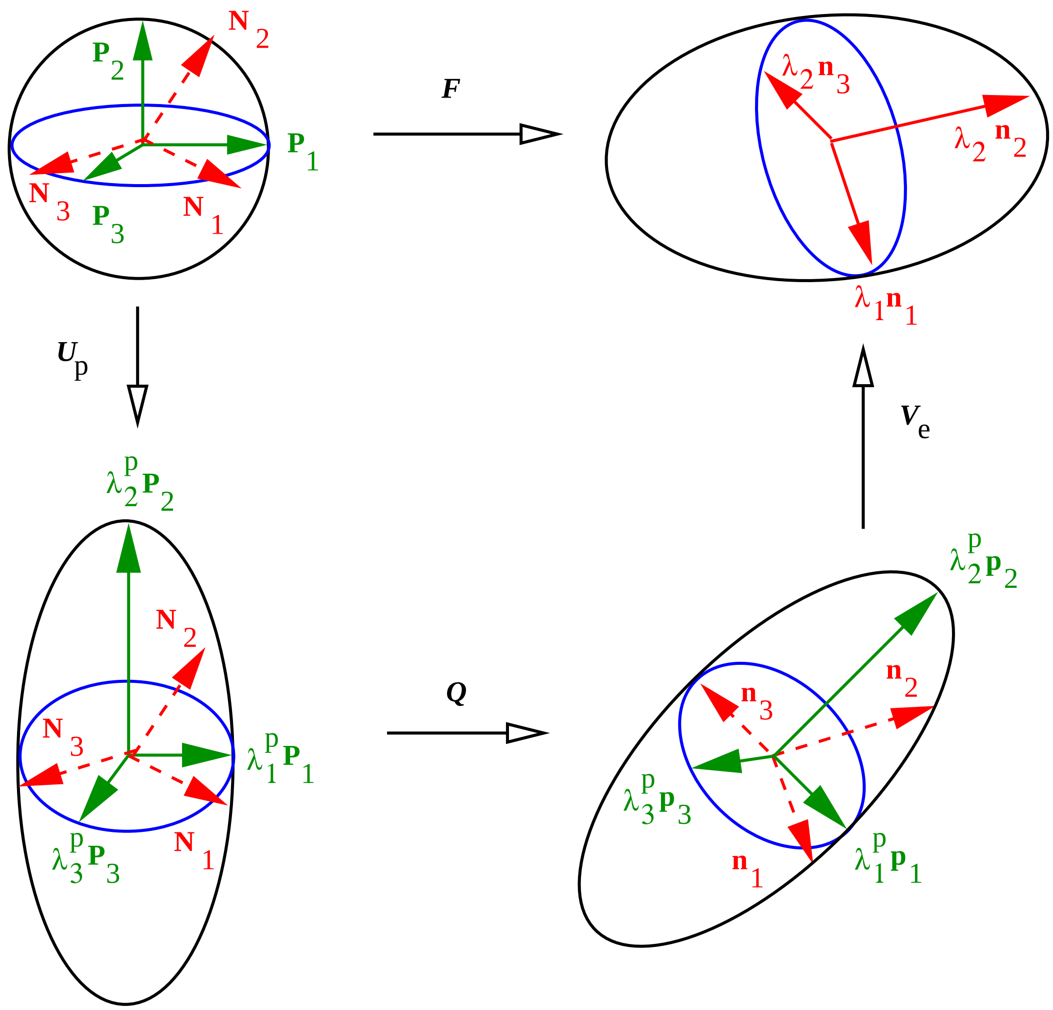

Figure 3: A decomposition of the deformation gradient into a plastic stretch $\boldsymbol{U}_p$, and rigid rotation $\boldsymbol{Q}$, and an elastic stretch $\boldsymbol{V}_e$ [NN04], p. 251).

Since the product of two orthogonal tensors is an orthogonal tensor, the tensor $\boldsymbol{Q}$ represents a rotation. Note that the rotation $\boldsymbol{Q}$ is *uniquely defined* in this decomposition. A schematic of the above decomposition is shown in Figure 3.

In the figure, $\mathbf{N}_i$ are the principal directions of $\boldsymbol{U}$ and $\mathbf{n}_i$ are the principal directions of $\boldsymbol{V}$ where $\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U} = \boldsymbol{V}\cdot\boldsymbol{R}$. The triad $\mathbf{P}_i$ represents the principal directions of the plastic stretch $\boldsymbol{U}_p$.

There are clearly two intermediate configurations in this case. The first intermediate configuration is achieved by purely plastic loading from the unloaded configuration. The second intermediate configuration is achieved by a pure rotation from the first (no elastic or plastic deformations). The final configuration is achieved by a purely elastic deformation from the second configuration. The internal variables do not evolve in this phase. For further discussions in the context of anisotropic elastic behavior see [NN04].

Also note that from (155) we have

$$\boldsymbol{V}_e^{-1}\cdot\boldsymbol{F} = \boldsymbol{Q}\cdot\boldsymbol{U}_p\,. \tag{156}$$

Since the right hand side consists of a rotation and a pure stretch, we can interpret the product to be the result of a polar decomposition of the left hand side. This implies that if we know $\boldsymbol{V}_e$ and $\boldsymbol{F}$ then we can find $\boldsymbol{U}_p$ using a polar decomposition. An additional implication is that $\det(\boldsymbol{V}_e^{-1}) = 1/\det(\boldsymbol{V}_e) > 0$, i.e., $\det(\boldsymbol{V}_e) > 0$. Another implication is that $\boldsymbol{Q}$ is fully determined if $\boldsymbol{V}_e$ and $\boldsymbol{F}$ are known.

For elastically isotropic materials, the Cauchy stress ($\boldsymbol{\sigma}$) is an isotropic function of the left stretch ($\boldsymbol{V}_e$). Then the elastic constitutive equation may be written as

$$\boldsymbol{\sigma} = \mathsf{C}(\boldsymbol{V}_e) \qquad \text{where} \qquad \mathsf{C}(\boldsymbol{Q} \cdot \boldsymbol{V}_e \cdot \boldsymbol{Q}^T) = \boldsymbol{Q} \cdot \mathsf{C}(\boldsymbol{V}_e) \cdot \boldsymbol{Q}^T \tag{157}$$

for any rigid motions $\boldsymbol{Q}$.

Also, $\boldsymbol{\sigma}$ and $\boldsymbol{V}_e$ are coaxial in the sense that their eigenvectors point in the same directions (they have the same principal directions).

7.1 Stress power in current configuration

Let us now check how the rotation $\boldsymbol{Q}$ in the decomposition (155) and its rate $\dot{\boldsymbol{Q}}$ affect the stress power in this situation. The total stress power is given by

$$\mathcal{P} = \boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\,\boldsymbol{\sigma} : [\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T] \,. \tag{158}$$

Since $\boldsymbol{\sigma}$ and $\boldsymbol{V}_e$ are coaxial, we can show that stress power can be expressed as (see Appendix, item 8)

$$\boxed{\begin{aligned} \boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\Big[&\boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) + \\ &\boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e)\Big] \,. \end{aligned}} \tag{159}$$

The first term in this expression can be interpreted as the elastic stress power while the second term is the plastic stress power. Also observe that there are no terms containing the rate of rotation $\dot{\boldsymbol{Q}}$ in the expression.

This makes life easier for us because we can now assign the entire rotation $\boldsymbol{Q}$ to the elastic part of the deformation gradient, i.e., we can write

$$\boldsymbol{F}_e = \boldsymbol{V}_e \cdot \boldsymbol{Q} \qquad \text{and} \qquad \boldsymbol{F}_p = \boldsymbol{U}_p \,. \tag{160}$$

Now, a polar decomposition of $\boldsymbol{F}_e$ also gives us

$$\boldsymbol{F}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e \,. \tag{161}$$

Hence, we can express the decomposition (155) alternatively as

$$\boxed{\boldsymbol{F} = \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \boldsymbol{U}_p} \qquad \text{and} \qquad \boldsymbol{V}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T \,. \tag{162}$$

If we know $\boldsymbol{U}_e$ and $\boldsymbol{Q}$, we can compute the Cauchy stress for isotropic elastic materials using the constitutive relation. The decomposition in (162) thus treats the plastic part of the deformation as a pure stretch and simplifies things considerably.

If we substitute $\boldsymbol{V}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T$ in equation (159) we get (see Appendix, item 9)

$$\boxed{\begin{aligned} \boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\;\Big[&\boldsymbol{\sigma} : \Big\{\boldsymbol{Q} \cdot (\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e) \cdot \boldsymbol{Q}^T\Big\} \\ &+ \boldsymbol{\sigma} : \Big\{\boldsymbol{Q} \cdot (\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e) \cdot \boldsymbol{Q}^T\Big\}\Big] . \end{aligned}} \tag{163}$$

Now recall from (69) that

$$\begin{aligned} \boldsymbol{d}_e &:= \frac{1}{2}\left[\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1} + (\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1})^T\right] \\ \boldsymbol{d}_p &:= \frac{1}{2}\left[\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} + (\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1})^T\right] \,. \end{aligned} \tag{164}$$

Substituting $\boldsymbol{F}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e$ and $\boldsymbol{F}_p = \boldsymbol{U}_p$ in these definitions can be expressed as (see Appendix, item 10)

$$\boxed{\begin{aligned} \boldsymbol{d}_e &= \frac{1}{2}\left[\boldsymbol{Q}\cdot(\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e^{-1}+\boldsymbol{U}_e^{-1}\cdot\dot{\boldsymbol{U}}_e)\cdot\boldsymbol{Q}^T\right] \\ \boldsymbol{d}_p &= \frac{1}{2}\left[\boldsymbol{Q}\cdot(\boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1}\cdot\boldsymbol{U}_e^{-1}+\boldsymbol{U}_e^{-1}\cdot\boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e)\cdot\boldsymbol{Q}^T\right]\ . \end{aligned}} \tag{165}$$

Hence we can write the stress power in equation (163) as

$$\boxed{\mathcal{P} = \boldsymbol{\sigma} : \boldsymbol{d} = \boldsymbol{\sigma} : (\boldsymbol{d}_e + \boldsymbol{d}_p)\ .} \tag{166}$$

We can show that the elastic and plastic rates of deformation transform in the same way as the Cauchy stress under rigid body motions and hence are objective rates.

7.2 Stress power in intermediate configuration

Alternatively, we can show that the stress power in the intermediate configuration can also be decomposed into a product of the tensor $\boldsymbol{S}_I$ and rates of deformation in the intermediate configuration.

We can show that the stress power can be written as (see Appendix, item 11 fro details).

$$\boxed{\mathcal{P} = J_e^{-1}\ \boldsymbol{S}_I : (\boldsymbol{D}_e + \boldsymbol{D}_p)\ .} \tag{167}$$

where

$$\boxed{\begin{aligned} \boldsymbol{D}_e &= \frac{1}{2}\ \left[\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e+\boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\right] \\ \boldsymbol{D}_p &= \frac{1}{2}\left[\boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1}+\boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2\right]\ . \end{aligned}} \tag{168}$$

If we push forward the rates of deformation $\boldsymbol{D}_e$ and $\boldsymbol{D}_p$ by $\boldsymbol{F}_e$, we get

$$\begin{aligned} \boldsymbol{F}_e^{-T}\cdot\boldsymbol{D}_e\cdot\boldsymbol{F}_e^{-1} &= \frac{1}{2}\ \left[\boldsymbol{Q}\cdot\boldsymbol{U}_e^{-1}\cdot\dot{\boldsymbol{U}}_e\cdot\boldsymbol{Q}^T+\boldsymbol{Q}\cdot\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e^{-1}\cdot\boldsymbol{Q}^T\right] = \boldsymbol{d}_e \\ \boldsymbol{F}_e^{-T}\cdot\boldsymbol{D}_p\cdot\boldsymbol{F}_e^{-1} &= \frac{1}{2}\left[\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1}\cdot\boldsymbol{U}_e^{-1}\cdot\boldsymbol{Q}^T+\boldsymbol{Q}\cdot\boldsymbol{U}_e^{-1}\cdot\boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q}^T\right] = \boldsymbol{d}_p\ . \end{aligned} \tag{169}$$

7.3 Relation between $\boldsymbol{U}_p$, $\boldsymbol{U}$, and $\boldsymbol{U}_e$

Given any deformation gradient $\boldsymbol{F}$, we can eliminate the rotation by using a deformation measure of the form

$$\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F} = \boldsymbol{U}^T\cdot\boldsymbol{U} = \boldsymbol{U}^2 \tag{170}$$

where $\boldsymbol{U}$ is a pure stretch. Substituting the decomposition

$$\boldsymbol{F} = \boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p \tag{171}$$

into (170) gives

$$\boldsymbol{U}^2 = (\boldsymbol{U}_p\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q}^T)\cdot(\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p) = \boldsymbol{U}_p\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p\ . \tag{172}$$

Hence,

$$\boldsymbol{U}_e\cdot\boldsymbol{U}^2\cdot\boldsymbol{U}_e = (\boldsymbol{U}_e\cdot\boldsymbol{U}_p\cdot\boldsymbol{U}_e)\cdot(\boldsymbol{U}_e\cdot\boldsymbol{U}_p\cdot\boldsymbol{U}_e) = (\boldsymbol{U}_e\cdot\boldsymbol{U}_p\cdot\boldsymbol{U}_e)^2\ . \tag{173}$$

This leads to

$$\boxed{\boldsymbol{U}_p = \boldsymbol{U}_e^{-1}\cdot(\boldsymbol{U}_e\cdot\boldsymbol{U}^2\cdot\boldsymbol{U}_e)^{1/2}\cdot\boldsymbol{U}_e^{-1}\ .} \tag{174}$$

7.4 Rate equation for the total stretch

Recall that

$$\boldsymbol{U}^2 = \boldsymbol{F}^T \cdot \boldsymbol{F} \; . \tag{175}$$

Differentiation with respect to time gives us

$$\frac{d}{dt}\left(\boldsymbol{U}^2\right) = \dot{\boldsymbol{F}}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}} \; . \tag{176}$$

A bit of algebra shows that (see Appendix, item 12)

$$\boxed{\frac{d}{dt}(\boldsymbol{U}^2) = \boldsymbol{U}_p \cdot (\boldsymbol{D}_e + \boldsymbol{D}_p) \cdot \boldsymbol{U}_p \; .} \tag{177}$$

If the history of $\boldsymbol{S}_I$, T, q_0, and q_j are known then we can find $\boldsymbol{D}_e$ as shown in the following section. We can then solve for $\boldsymbol{U}_e$ using the definition of $\boldsymbol{D}_e$. Since $\boldsymbol{U}_p$ can be expressed as a function of $\boldsymbol{U}$ and $\boldsymbol{U}_e$, we can use (177) to solve for $\boldsymbol{U}$. Note that this approach is different from that used in most numerical codes where the stretch $\boldsymbol{U}$ is assumed to be known and it is the stresses that are computed.

7.5 Elastic constitutive relations

To determine the elastic response of the material we need constitutive relations. In most computations, we usually know the total deformation gradient. Suppose, for the time being, that we also know $\boldsymbol{V}_e$ or $\boldsymbol{U}_e$. Then our goal is to compute the Cauchy stress $\boldsymbol{\sigma}$ (or the 2nd P-K stress $\boldsymbol{S}_I$).

The elastic part of the rate of deformation in the intermediate configuration $\boldsymbol{D}_e$ is given by (see Appendix, item 13 for proof)

$$\boxed{\boldsymbol{D}_e = \dot{\boldsymbol{E}}_e = \rho_I \, \mathsf{S} : \left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\boldsymbol{S}_I\right) + \boldsymbol{\alpha}_e \, \dot{T} - \sum_{j=0}^{n} \frac{\partial Q_j}{\partial \boldsymbol{S}_I} \, \dot{q}_j} \tag{178}$$

where the fourth-order elastic stiffness tensor and the second-order thermal expansion tensor are defined as

$$\mathsf{S} := \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} \; ; \;\; \boldsymbol{\alpha}_e := \frac{\partial \boldsymbol{E}_e}{\partial T} \; . \tag{179}$$

This is a rate equation that can be solved either for $\boldsymbol{D}_e$ or $\dot{\boldsymbol{S}}_I$.

7.6 Plastic flow rules and evolution of internal variables

We also need a relation (flow rule) to determine $\boldsymbol{D}_p$. The usual assumption is that $\boldsymbol{D}_p$ also depends on the same variables as the Gibbs function, i.e.,

$$\boldsymbol{D}_p = \widehat{\boldsymbol{D}}_p(\boldsymbol{S}_I; \;\; T, q_0, q_j) \qquad j = 1 \ldots n \; . \tag{180}$$

The evolution equations for the internal variables are also assumed to have the same dependencies, i.e.,

$$\dot{q}_i = \widehat{q}_i(\boldsymbol{S}_I; \;\; T, q_0, q_j) \qquad i = 0 \ldots n, \; j = 1 \ldots n \; . \tag{181}$$

7.6.1 Isotropic material with scalar internal variables

If the material is isotropic and all the internal variables are scalars, then the functions $\widehat{\boldsymbol{D}}_p$ and $\widehat{q}_i$ are isotropic functions of the stress $\boldsymbol{S}_I$.

Recall that

$$\boldsymbol{S}_I = J_e\, \boldsymbol{F}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}_e^{-T} \qquad \text{and} \qquad \boldsymbol{F}_e = \boldsymbol{V}_e \cdot \boldsymbol{Q}\ ;\ \ J_e = \frac{\rho_I}{\rho} = \det(\boldsymbol{F}_e) = \det(\boldsymbol{V}_e)\ . \tag{182}$$

Therefore we can write

$$\boldsymbol{S}_I = J_e\, \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q}\ . \tag{183}$$

Also, recall that

$$\boldsymbol{d}_p = \boldsymbol{F}_e^{-T} \cdot \boldsymbol{D}_p \cdot \boldsymbol{F}_e^{-1} = \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{D}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}\ . \tag{184}$$

Hence, from (180) and relations (184) and (183), we have

$$\boldsymbol{d}_p = \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \widehat{\boldsymbol{D}}_p \left(J_e\, \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q};\ \ T, q_0, q_j\right) \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}\ . \tag{185}$$

Since the function $\widehat{\boldsymbol{D}}_p$ is isotropic, we have

$$\boldsymbol{Q} \cdot \widehat{\boldsymbol{D}}_p \cdot \boldsymbol{Q}^T = \widehat{\boldsymbol{D}}_p\ . \tag{186}$$

Therefore,

$$\boldsymbol{d}_p = \boldsymbol{V}_e^{-1} \cdot \widehat{\boldsymbol{D}}_p \left(J_e\, \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q};\ \ T, q_0, q_j\right) \cdot \boldsymbol{V}_e^{-1}\ . \tag{187}$$

The isotropy of the material also implies that

$$\boldsymbol{V}_e = \widehat{\boldsymbol{V}}_e(\boldsymbol{\sigma};\ \ T, q_0, q_j)\ ;\ \ J_e = \det(\boldsymbol{V}_e) = \widehat{J}_e(\boldsymbol{\sigma};\ \ T, q_0, q_j) \tag{188}$$

where $\widehat{\boldsymbol{V}}_e$ and $\widehat{J}_e$ are isotropic functions of $\boldsymbol{\sigma}$. Hence,

$$\boxed{\boldsymbol{d}_p = \widehat{\boldsymbol{d}}_p(\boldsymbol{\sigma};\ \ T, q_0, q_j)} \tag{189}$$

where the function $\widehat{\boldsymbol{d}}_p$ is given by

$$\widehat{\boldsymbol{d}}_p(\boldsymbol{\sigma};\ \ T, q_0, q_j) = \boldsymbol{V}_e^{-1} \cdot \widehat{\boldsymbol{D}}_p \left(J_e\, \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q};\ \ T, q_0, q_j\right) \cdot \boldsymbol{V}_e^{-1} \tag{190}$$

and is an isotropic function of $\boldsymbol{\sigma}$.

Similarly, the evolution equations for the internal variables also involve isotropic functions ($\widehat{q}_i$) of $\boldsymbol{\sigma}$ and can be written as

$$\dot{q}_i = \widehat{q}_i(\boldsymbol{\sigma};\ \ T, q_0, q_j)\ . \tag{191}$$

Since these internal variables are scalars, the functions $\widehat{q}_i$ depend only on the invariants of $\boldsymbol{\sigma}$ (see Appendix, item 14 or [Gur81], p. 230 for a proof of the scalar representation theorem) which are

$$I_1(\boldsymbol{\sigma}) = \text{tr}\,(\boldsymbol{\sigma})\ ;\ \ I_2(\boldsymbol{\sigma}) = \frac{1}{2}\ \left[(\text{tr}\,(\boldsymbol{\sigma}))^2 - \text{tr}\left(\boldsymbol{\sigma}^2\right)\right]\ ;\ \ I_3(\boldsymbol{\sigma}) = \det(\boldsymbol{\sigma})\ . \tag{192}$$

The representation theorem for isotropic tensor valued functions of second order tensors implies that, since $\widehat{\boldsymbol{d}}_p$ is an isotropic function of $\boldsymbol{\sigma}$, we can write (see Appendix, item 15 or [Gur81], p. 233 for a proof)

$$\boxed{\widehat{\boldsymbol{d}}_p(\boldsymbol{\sigma};\ T, q_0, q_j) = \varphi_0(I_1, I_2, I_3;\ T, q_0, q_j)\ \boldsymbol{1} + \varphi_1(I_1, I_2, I_3;\ T, q_0, q_j)\ \boldsymbol{\sigma} + \varphi_2(I_1, I_2, I_3;\ T, q_0, q_j)\ \boldsymbol{\sigma}^2} \tag{193}$$

where $\varphi_1, \varphi_2, \varphi_3$ are scalar valued functions of the invariants of $\boldsymbol{\sigma}$, the temperature, and the internal variables.

The flow rule is commonly expressed as a function of the pressure and the invariants of the deviatoric stress.

Recall that the Cauchy stress can be decomposed as

$$\boldsymbol{\sigma} = \frac{1}{3}\,\text{tr}\,(\boldsymbol{\sigma}) + \boldsymbol{s} = -p\;\boldsymbol{1} + \boldsymbol{s}\;. \tag{194}$$

Hence,

$$\boldsymbol{\sigma}^2 = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = p^2\;\boldsymbol{1} - 2\,p\,\boldsymbol{s} + \boldsymbol{s}\cdot\boldsymbol{s}\;. \tag{195}$$

Therefore, the flow rule (193) can be written as

$$\boldsymbol{d}_p = \varphi_0\;\boldsymbol{1} + \varphi_1\,(-p\;\boldsymbol{1} + \boldsymbol{s}) + \varphi_2\,(p^2\;\boldsymbol{1} - 2\,p\,\boldsymbol{s} + \boldsymbol{s}^2)\;. \tag{196}$$

Also recall that

$$\boldsymbol{d}_p = \frac{1}{3}\,\text{tr}\,(\boldsymbol{d}_p)\;\;\boldsymbol{1} + \boldsymbol{\eta}_p\;;\;\;\text{tr}\,(\boldsymbol{d}_p) = -\frac{\dot{\rho}_I}{\rho_I} \qquad\Longrightarrow\qquad \boldsymbol{d}_p = -\frac{1}{3}\,\frac{\dot{\rho}_I}{\rho_I}\,\boldsymbol{1} + \boldsymbol{\eta}_p\;. \tag{197}$$

Combining (197) and (196) gives

$$\boldsymbol{d}_p = -\frac{\dot{\rho}_I}{3\rho_I}\,\boldsymbol{1} + \boldsymbol{\eta}_p = (\varphi_0 - p\,\varphi_1 + p^2\,\varphi_2)\;\boldsymbol{1} + (\varphi_1 - 2\,p\,\varphi_2)\,\boldsymbol{s} + \varphi_2\,\boldsymbol{s}^2 \tag{198}$$

or,

$$\boldsymbol{\eta}_p = \left(\frac{\dot{\rho}_I}{3\,\rho_I} + \varphi_0 - p\,\varphi_1 + p^2\,\varphi_2\right)\;\boldsymbol{1} + (\varphi_1 - 2\,p\,\varphi_2)\,\boldsymbol{s} + \varphi_2\,\boldsymbol{s}^2\;. \tag{199}$$

Equation (198) can then be written as

$$\boxed{\boldsymbol{d}_p = \xi_0(p, I_{\boldsymbol{\sigma}}, T, q_0, q_j)\;\boldsymbol{1} + \xi_1(p, I_{\boldsymbol{\sigma}}, T, q_0, q_j)\;\boldsymbol{s} + \xi_2(I_{\boldsymbol{\sigma}}, T, q_0, q_j)\;\boldsymbol{s}^2} \tag{200}$$

while equation (199) can be written as (recalling that $q_0 = \rho_I$)

$$\boxed{\boldsymbol{\eta}_p = \xi_0^I(p, I_{\boldsymbol{\sigma}}, T, q_0, q_j)\;\boldsymbol{1} + \xi_1(p, I_{\boldsymbol{\sigma}}, T, q_0, q_j)\;\boldsymbol{s} + \xi_2(I_{\boldsymbol{\sigma}}, T, q_0, q_j)\;\boldsymbol{s}^2\;.} \tag{201}$$

An alternative expression can be obtained by observing from equation (198) that

$$\text{tr}\,(\boldsymbol{d}_p) = -\frac{\dot{\rho}_I}{\rho_I} = 3\,(\varphi_0 - p\,\varphi_1 + p^2\,\varphi_2) + \varphi_2\;\text{tr}\left(\boldsymbol{s}^2\right) \tag{202}$$

which implies that

$$\varphi_0 - p\,\varphi_1 + p^2\,\varphi_2 = -\frac{1}{3}\,\frac{\dot{\rho}_I}{\rho_I} - \frac{1}{3}\,\varphi_2\;\text{tr}\left(\boldsymbol{s}^2\right)\;. \tag{203}$$

Now, $\dot{\rho}_I = \dot{q}_0(\boldsymbol{\sigma}; T, q_0, q_j) = \widehat{q}_0$. Hence we can write (198) as

$$\boxed{\boldsymbol{d}_p = -\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1} + (\varphi_1 - 2\,p\,\varphi_2)\,\boldsymbol{s} + \varphi_2\;\left(\boldsymbol{s}^2 - \frac{1}{3}\,\text{tr}\left(\boldsymbol{s}^2\right)\;\boldsymbol{1}\right) = -\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1} + \xi_1\;\boldsymbol{s} + \xi_2\;\left(\boldsymbol{s}^2 - \frac{1}{3}\,\text{tr}\left(\boldsymbol{s}^2\right)\;\boldsymbol{1}\right)} \tag{204}$$

or,

$$\boxed{\boldsymbol{\eta}_p = \xi_1\;\boldsymbol{s} + \xi_2\;\left(\boldsymbol{s}^2 - \frac{1}{3}\,\text{tr}\left(\boldsymbol{s}^2\right)\;\boldsymbol{1}\right)\;.} \tag{205}$$

Entropy inequality for isotropic materials with scalar internal variables. Recall from (79) that the entropy inequality can be written as

$$\boldsymbol{\sigma} : \boldsymbol{d}_p - \frac{\rho}{\rho_I} \sum_{j=0}^{n} Q_j \, \dot{q}_j \geq 0 \tag{206}$$

or alternatively from (100) as

$$\boldsymbol{s} : \boldsymbol{\eta}_p + (p - \rho \, Q_0) \, \frac{\dot{\rho}_I}{\rho_I} - \frac{\rho}{\rho_I} \sum_{j=1}^{n} Q_j \, \dot{q}_j \geq 0 \, . \tag{207}$$

If we plug the expression for $\boldsymbol{\eta}_p$ from (205) into (207) we get (using the substitutions $\dot{\rho}_I = \widehat{q}_0$, $\dot{q}_j = \widehat{q}_j$)

$$\boldsymbol{s} : \left[\xi_1 \, \boldsymbol{s} + \xi_2 \, \left(\boldsymbol{s}^2 - \frac{1}{3} \, \text{tr}\left(\boldsymbol{s}^2\right) \, \boldsymbol{1}\right)\right] + (p - \rho \, Q_0) \, \frac{\widehat{q}_0}{\rho_I} - \frac{\rho}{\rho_I} \sum_{j=1}^{n} Q_j \, \widehat{q}_j \geq 0 \tag{208}$$

or,

$$\xi_1 \, \boldsymbol{s} : \boldsymbol{s} + \xi_2 \, \left((\boldsymbol{s} \cdot \boldsymbol{s}) : \boldsymbol{s} - \frac{1}{3} \, \text{tr}\left(\boldsymbol{s}^2\right) \, \cancelto{0}{\text{tr}\,(\boldsymbol{s})}\right) + (p - \rho \, Q_0) \, \frac{\widehat{q}_0}{\rho_I} - \frac{\rho}{\rho_I} \sum_{j=1}^{n} Q_j \, \widehat{q}_j \geq 0 \tag{209}$$

or,

$$\boxed{\xi_1 \, \text{tr}\left(\boldsymbol{s}^2\right) + \xi_2 \, \text{tr}\left(\boldsymbol{s}^3\right) + (p - \rho \, Q_0) \, \frac{\widehat{q}_0}{\rho_I} - \frac{\rho}{\rho_I} \sum_{j=1}^{n} Q_j \, \widehat{q}_j \geq 0 \, .} \tag{210}$$

If the internal energy and the Gibbs free energy do not depend upon the plastic volume change we get, using (106),

$$\boxed{\xi_1 \, \text{tr}\left(\boldsymbol{s}^2\right) + \xi_2 \, \text{tr}\left(\boldsymbol{s}^3\right) - \frac{\rho}{\rho_I} \sum_{j=1}^{n} \left[Q_j - \frac{p}{\rho} \, \frac{\partial \rho_I}{\partial q_j}\right] \, \widehat{q}_j \geq 0 \, .} \tag{211}$$

Equations (210) and (211) can be used to restrict the possible forms of ξ_1 and ξ_2. If the latter, the first two terms represent the rate of deviatoric plastic work. The terms containing p represents the rate of volumetric plastic work. The terms containing Q_j represent the rate of storage of cold work.

We assume that plastic work is always positive. This constraint may be construed to imply that ξ_1 is positive and the sign of ξ_2 is always the same as that of $\text{tr}\left(\boldsymbol{s}^3\right)$ (see [Wri02], p. 47-49, p.67 for more details).

7.6.2 Isotropic material with some tensor internal variables

7.7 Plastic Yield

Recall that the 2nd P-K stress $\boldsymbol{S}_I$ refers to the intermediate configuration and is unaffected by rigid body rotations. If we define the yield function as

$$f_I(\boldsymbol{S}_I; \;\; T, q_j) \leq 0 \tag{212}$$

then the yield surface is also unaffected by rigid body rotations.

Recall that

$$\boldsymbol{S}_I = \frac{\rho_I}{\rho} \, \boldsymbol{F}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}_e^{-T} = \frac{\rho_I}{\rho} \, \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \, . \tag{213}$$

Hence the yield surface may alternatively be expressed as

$$f_I\left(\frac{\rho_I}{\rho} \, \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q}; \;\; T, q_j\right) \leq 0 \, . \tag{214}$$

7.7.1 Isotropic elastic material and scalar internal variables

If the material is isotropic and all the internal variables are scalars, then f_I is an isotropic function of $\boldsymbol{S}_I$. Also, f_I is unaffected by rigid body rotations. Hence we can write the yield function as

$$f_I\left(\frac{\rho_I}{\rho}\cdot\boldsymbol{V}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{V}_e^{-1};\;\; T,q_j\right)\le 0 \tag{215}$$

or

$$f_I\left(J_e\,\boldsymbol{V}_e^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{V}_e^{-1};\;\; T,q_j\right)\le 0\;;\;\; J_e:=\det(\boldsymbol{F}_e)=\frac{\rho_I}{\rho}\,. \tag{216}$$

Also, if the material follows an isotropic elastic constitutive relation, then the Cauchy stress ($\boldsymbol{\sigma}$) is an isotropic function of the left stretch ($\boldsymbol{V}_e$). Hence the yield function may alternatively be expressed as

$$f(\boldsymbol{\sigma};\;\; T,q_j)\le 0\,. \tag{217}$$

This function is also invariant with respect to rigid body rotations and is an isotropic function of $\boldsymbol{\sigma}$.

Recall that the plastic rate of deformation in the current configuration can be written as

$$\widehat{\boldsymbol{d}}_p(\boldsymbol{\sigma};\; T,q_0,q_j)=\varphi_0(I_1,I_2,I_3;\; T,q_0,q_j)\;\boldsymbol{1}+\varphi_1(I_1,I_2,I_3;\; T,q_0,q_j)\;\boldsymbol{\sigma}+\varphi_2(I_1,I_2,I_3;\; T,q_0,q_j)\;\boldsymbol{\sigma}^2\,. \tag{218}$$

On the yield surface, the plastic rate of deformation is zero. Hence the arbitrariness of $\boldsymbol{\sigma}$ implies that on the yield surface

$$\varphi_0=\varphi_1=\varphi_2=0\,. \tag{219}$$

Also, if we consider the alternative expression for $\boldsymbol{d}_p$ from equation (204):

$$\boldsymbol{d}_p=-\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1}+(\varphi_1-2\,p\,\varphi_2)\,\boldsymbol{s}+\varphi_2\,\left(\boldsymbol{s}^2-\frac{1}{3}\,\mathrm{tr}\left(\boldsymbol{s}^2\right)\;\boldsymbol{1}\right)=-\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1}+\xi_1\,\boldsymbol{s}+\xi_2\,\left(\boldsymbol{s}^2-\frac{1}{3}\,\mathrm{tr}\left(\boldsymbol{s}^2\right)\;\boldsymbol{1}\right) \tag{220}$$

we notice that on the yield surface we must have

$$\widehat{q}_0=\xi_1=\xi_2=0\,. \tag{221}$$

7.7.2 Isotropic elastic material and tensor internal variables

If one of the internal variables is a second order tensor, then frame indifference of the yield function is more difficult to achieve in general. More details can be found in [SW01].

7.8 Plastic flow potentials

It is common in plasticity theory for the rate of plastic deformation $\boldsymbol{d}_p$ to be derived from a flow potential ϕ such that

$$\boldsymbol{d}_p=\frac{\partial\phi}{\partial\boldsymbol{\sigma}}\,. \tag{222}$$

The principle of maximum plastic dissipation is has been used by some researchers to prove the existence of a plastic potential. However, the existence of such a potential does not have the same theoretical standing as the free energy and internal energy potentials.

7.8.1 Isotropic material with scalar internal variables

For an isotropic material with scalar internal variables, the plastic flow potential can be assumed to have the form

$$\phi \equiv \phi(p, J_2, J_3;\ \ T, q_0, q_j) \tag{223}$$

where p is the pressure and J_2, J_3 are invariants of the deviatoric stress $\boldsymbol{s}$ defined as

$$\begin{aligned} J_2 &:= -\frac{1}{2}\ \left[(\text{tr}\,(\boldsymbol{s}))^2 - \text{tr}\left(\boldsymbol{s}^2\right)\right] = \frac{1}{2}\ \text{tr}\left(\boldsymbol{s}^2\right) = \frac{1}{2}\ \boldsymbol{s} : \boldsymbol{s} \\ J_3 &:= \det(\boldsymbol{s}) = \frac{1}{3}\ \text{tr}\left(\boldsymbol{s}^3\right)\ . \end{aligned} \tag{224}$$

Then,

$$\boldsymbol{d}_p = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \frac{\partial \phi}{\partial p}\ \frac{\partial p}{\partial \boldsymbol{\sigma}} + \frac{\partial \phi}{\partial J_2}\ \frac{\partial J_2}{\partial \boldsymbol{\sigma}} + \frac{\partial \phi}{\partial J_3}\ \frac{\partial J_3}{\partial \boldsymbol{\sigma}}\ . \tag{225}$$

Now, using the expressions for the derivatives of the invariants of a second-order tensor (see [TN92], p. 26, equation 9.11 and Appendix, item 16) and noting that $\text{tr}\,(\boldsymbol{s}) = 0$, we have

$$\begin{aligned} p = -\frac{1}{3}\ \text{tr}\,(\boldsymbol{\sigma}) \qquad &\implies \qquad \frac{\partial p}{\partial \boldsymbol{\sigma}} = -\frac{1}{3}\ \boldsymbol{1} \\ J_2 = \frac{1}{2}\ \text{tr}\left(\boldsymbol{s}^2\right) \qquad &\implies \qquad \frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \boldsymbol{s} \\ J_3 = \det(\boldsymbol{s}) \qquad &\implies \qquad \frac{\partial J_3}{\partial \boldsymbol{\sigma}} = \boldsymbol{s}^2 - \frac{1}{2}\ \text{tr}\left(\boldsymbol{s}^2\right)\ \boldsymbol{1}\ . \end{aligned} \tag{226}$$

Therefore (225) can be written as

$$\boxed{\boldsymbol{d}_p = -\frac{1}{3}\ \frac{\partial \phi}{\partial p}\ \boldsymbol{1} + \frac{\partial \phi}{\partial J_2}\ \boldsymbol{s} + \frac{\partial \phi}{\partial J_3}\ \left[\boldsymbol{s}^2 - \frac{1}{2}\ \text{tr}\left(\boldsymbol{s}^2\right)\ \boldsymbol{1}\right]\ .} \tag{227}$$

Hence,

$$\text{tr}\,(\boldsymbol{d}_p) = -\frac{\partial \phi}{\partial p} \tag{228}$$

and

$$\boldsymbol{\eta}_p = \boldsymbol{d}_p - \frac{1}{3}\ \text{tr}\,(\boldsymbol{d}_p) = \frac{\partial \phi}{\partial J_2}\ \boldsymbol{s} + \frac{\partial \phi}{\partial J_3}\ \left[\boldsymbol{s}^2 - \frac{1}{2}\ \text{tr}\left(\boldsymbol{s}^2\right)\ \boldsymbol{1}\right]\ . \tag{229}$$

Clearly

$$\boldsymbol{\eta}_p = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} + \frac{1}{3}\ \frac{\partial \phi}{\partial p}\ \boldsymbol{1} \tag{230}$$

implies that the usual assumption that $\boldsymbol{\eta}_p = \partial\phi/\partial\boldsymbol{\sigma}$ only holds when $\partial\phi/\partial p = 0$.

From equation (204) we have

$$\boldsymbol{d}_p = -\frac{1}{3}\ \frac{\widehat{q}_0}{\rho_I}\ \boldsymbol{1} + \xi_1\ \boldsymbol{s} + \xi_2\ \left[\boldsymbol{s}^2 - \frac{1}{3}\ \text{tr}\left(\boldsymbol{s}^2\right)\ \boldsymbol{1}\right]\ . \tag{231}$$

Comparing expressions (225) and (231) we see that

$$\boxed{\frac{\partial \phi}{\partial p} = \frac{\widehat{q}_0}{\rho_I}\ ;\ \ \frac{\partial \phi}{\partial J_2} = \xi_i\ ;\ \ \frac{\partial \phi}{\partial J_3} = \xi_2\ .} \tag{232}$$

We can normalize equation (231) to get

$$\boldsymbol{d}_p = -\frac{1}{3}\ \frac{\widehat{q}_0}{\rho_I}\ \boldsymbol{1} + \xi_1\ (\sqrt{\boldsymbol{s} : \boldsymbol{s}})\ \left(\frac{\boldsymbol{s}}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}}\right) + \xi_2\ (\boldsymbol{s} : \boldsymbol{s})\ \left[\left(\frac{\boldsymbol{s}}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}}\right)^2 - \frac{1}{3}\ \boldsymbol{1}\right] \tag{233}$$

or,

$$\boldsymbol{d}_p = -\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1} + \xi_1\,(\sqrt{2\,J_2})\,\left(\frac{\boldsymbol{s}}{\sqrt{2\,J_2}}\right) + \xi_2\,(2\,J_2)\,\left[\left(\frac{\boldsymbol{s}}{\sqrt{2\,J_2}}\right)^2 - \frac{1}{3}\,\boldsymbol{1}\right] \tag{234}$$

or,

$$\boldsymbol{d}_p = -\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1} + \xi_1\,\sqrt{J_2}\,\frac{\boldsymbol{s}}{\sqrt{J_2}} + \xi_2\,J_2\,\left[\left(\frac{\boldsymbol{s}}{\sqrt{J_2}}\right)^2 - \frac{2}{3}\,\boldsymbol{1}\right] \tag{235}$$

Define,

$$\Gamma_1 := 2\,\xi_1\,\sqrt{J_2}\,;\;\;\Gamma_2 := 3\,\xi_2\,J_2\,. \tag{236}$$

Then

$$\boxed{\boldsymbol{d}_p = -\frac{1}{3}\,\frac{\widehat{q}_0}{\rho_I}\,\boldsymbol{1} + \frac{\Gamma_1}{2}\,\frac{\boldsymbol{s}}{\sqrt{J_2}} + \frac{\Gamma_2}{3}\,\left[\frac{\boldsymbol{s}^2}{J_2} - \frac{2}{3}\,\boldsymbol{1}\right]\,.} \tag{237}$$

Hence, the derivatives of the flow potential are also required to be related to the functions Γ_1 and Γ_2 vis

$$\frac{\partial\phi}{\partial p} = \frac{\widehat{q}_0}{\rho_I}\,;\;\;\frac{\partial\phi}{\partial J_2} = \xi_i = \frac{\Gamma_1}{2\,\sqrt{J_2}}\,;\;\;\frac{\partial\phi}{\partial J_3} = \xi_2 = \frac{\Gamma_2}{3\,J_2}\,. \tag{238}$$

8 Appendix

1. If the internal energy and the Gibbs free energy have the functional dependence

$$g = g(\boldsymbol{S}_I, T, q_0, q_j) \qquad \text{and} \qquad e = e(\boldsymbol{S}_I, T, q_0, q_j)$$

show that

$$\frac{\partial\eta}{\partial T} = \frac{1}{T}\left[\frac{\partial e}{\partial T} - q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\right]\,. \tag{239}$$

Recall that (for $j = 1 \dots n$)

$$g(\boldsymbol{S}_I, T, q_0, q_j) = -e(\boldsymbol{S}_I, T, q_0, q_j) + T\,\eta(\boldsymbol{S}_I, T, q_0, q_j) + q_0^{-1}\,\boldsymbol{S}_I : \boldsymbol{E}_e(\boldsymbol{S}_I, T, q_0, q_j)\,. \tag{240}$$

Therefore,

$$\frac{\partial g}{\partial T} = -\frac{\partial e}{\partial T} + \eta + T\,\frac{\partial\eta}{\partial T} + q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\,. \tag{241}$$

Now,

$$\frac{\partial g}{\partial T} = \eta\,. \tag{242}$$

Hence,

$$0 = -\frac{\partial e}{\partial T} + T\,\frac{\partial\eta}{\partial T} + q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T} \tag{243}$$

or,

$$\boxed{\frac{\partial\eta}{\partial T} = \frac{1}{T}\left[\frac{\partial e}{\partial T} - q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial T}\right]\,.} \tag{244}$$

2. Show that if

$$e = e(\boldsymbol{E}_e, T, q_0, q_j) \qquad \text{and} \qquad \eta = \eta(\boldsymbol{E}_e, T, q_0, q_j) \tag{245}$$

then

$$\frac{\partial\eta}{\partial T} = \frac{1}{T}\,\frac{\partial e}{\partial T}\,. \tag{246}$$

Recall that the Helmholtz free energy is given by

$$\psi(\boldsymbol{E}_e, T, q_0, q_j) = e(\boldsymbol{E}_e, T, q_0, q_j) - T\,\eta(\boldsymbol{E}_e, T, q_0, q_j)\,. \tag{247}$$

Therefore,

$$\frac{\partial\psi}{\partial T} = \frac{\partial e}{\partial T} - \eta - T\,\frac{\partial\eta}{\partial T}\,. \tag{248}$$

Now

$$\frac{\partial\psi}{\partial T} = -\eta\ . \tag{249}$$

Hence we have

$$0 = \frac{\partial e}{\partial T} - T\,\frac{\partial\eta}{\partial T} \tag{250}$$

or,

$$\boxed{\frac{\partial\eta}{\partial T} = \frac{1}{T}\,\frac{\partial e}{\partial T}\ .} \tag{251}$$

3. If the Helmholtz free energy and the entropy are given by

$$\psi = \psi(\boldsymbol{E}_e, T, q_0, q_j) \qquad \text{and} \qquad \eta = \eta(\boldsymbol{E}_e, T, q_0, q_j) \tag{252}$$

show that

$$\frac{\partial\eta}{\partial\boldsymbol{E}_e} = q_0^{-1}\,\frac{\partial\boldsymbol{S}_I}{\partial T}\ . \tag{253}$$

Recall that

$$\frac{\partial\psi}{\partial T} = -\eta\ . \tag{254}$$

Therefore,

$$\frac{\partial^2\psi}{\partial\boldsymbol{E}_e\partial T} = \frac{\partial}{\partial T}\left(\frac{\partial\psi}{\partial\boldsymbol{E}_e}\right) = -\frac{\partial\eta}{\partial\boldsymbol{E}_e}. \tag{255}$$

Now

$$\frac{\partial\psi}{\partial\boldsymbol{E}_e} = q_0^{-1}\,\boldsymbol{S}_I\ . \tag{256}$$

Hence,

$$\boxed{\frac{\partial\eta}{\partial\boldsymbol{E}_e} = -q_0^{-1}\,\frac{\partial\boldsymbol{S}_I}{\partial T}\ .} \tag{257}$$

4. For thermoplastic materials, show that the specific heats are related by the relation

$$C_p - C_v = q_0^{-1}\left(\boldsymbol{S}_I - T\,\frac{\partial\boldsymbol{S}_I}{\partial T}\right) : \frac{\partial\boldsymbol{E}_e}{\partial T}\ . \tag{258}$$

The specific heats at constant strain and constant stress are defined as

$$C_v := \frac{\partial e(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} \qquad \text{and} \qquad C_p := \frac{\partial e(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T}\ . \tag{259}$$

From the previous results in this Appendix, we have

$$\frac{\partial e(\boldsymbol{E}_e, T, q_0, q_j}{\partial T} = T\,\frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} \tag{260}$$

and

$$\frac{\partial e(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} = T\,\frac{\partial\eta(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} + q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial\boldsymbol{E}_e}{\partial T}\ . \tag{261}$$

Therefore,

$$C_p - C_v = T\,\frac{\partial\eta(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} + q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial\boldsymbol{E}_e}{\partial T} - T\,\frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T}\ . \tag{262}$$

Now, note that

$$\eta = \eta(\boldsymbol{E}_e, T, q_0, q_j) = \eta(\boldsymbol{S}_I, T, q_0, q_j)\ . \tag{263}$$

Therefore we can write

$$\frac{\partial\eta(\boldsymbol{S}_I, T, q_0, q_j)}{\partial T} = \frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial\boldsymbol{E}_e} : \frac{\partial\boldsymbol{E}_e}{\partial T} + \frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T}\ . \tag{264}$$

Hence,

$$C_p - C_v = T\,\frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial\boldsymbol{E}_e} : \frac{\partial\boldsymbol{E}_e}{\partial T} + T\,\frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} + q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial\boldsymbol{E}_e}{\partial T} - T\,\frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial T} \tag{265}$$

or,

$$C_p - C_v = \left(T\,\frac{\partial\eta(\boldsymbol{E}_e, T, q_0, q_j)}{\partial\boldsymbol{E}_e} + q_0^{-1}\,\boldsymbol{S}_I\right) : \frac{\partial\boldsymbol{E}_e}{\partial T}\ . \tag{266}$$

Since

$$\frac{\partial\eta}{\partial\boldsymbol{E}_e} = q_0^{-1}\,\frac{\partial\boldsymbol{S}_I}{\partial T}$$

we then have

$$\boxed{C_p - C_v = q_0^{-1}\, \left(-T\, \frac{\partial \boldsymbol{S}_I}{\partial T} + \boldsymbol{S}_I\right) : \frac{\partial \boldsymbol{E}_e}{\partial T}\,.} \tag{267}$$

5. Show that

$$\boldsymbol{\sigma} : \boldsymbol{d} = \frac{\rho}{\rho_I}\, \boldsymbol{S}_I : \left[\frac{1}{2}\left(\boldsymbol{F}_e^T \cdot \dot{\boldsymbol{F}}_e + \dot{\boldsymbol{F}}_e^T \cdot \boldsymbol{F}_e\right) + \frac{1}{2}\left(\boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} + \boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e\right)\right]\,. \tag{268}$$

Recall that

$$\text{tr}\left(\boldsymbol{A}^T \cdot \boldsymbol{B}\right) = \text{tr}\left(\boldsymbol{A} \cdot \boldsymbol{B}^T\right) = \text{tr}\left(\boldsymbol{B}^T \cdot \boldsymbol{A}\right) = \text{tr}\left(\boldsymbol{B} \cdot \boldsymbol{A}^T\right) = \boldsymbol{A} : \boldsymbol{B} = \boldsymbol{B} : \boldsymbol{A}\,. \tag{269}$$

Then, from equations (71) and (68) we have

$$\begin{aligned}
\boldsymbol{\sigma} : \boldsymbol{d} &= \text{tr}\left(\boldsymbol{\sigma}^T \cdot \boldsymbol{d}\right) = \text{tr}\left(\boldsymbol{\sigma} \cdot \boldsymbol{d}\right) \\
&= \frac{\rho}{2\,\rho_I}\,\text{tr}\left((\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T) \cdot (\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1} + \boldsymbol{F}_e^{-T} \cdot \dot{\boldsymbol{F}}_e^T + \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} + \boldsymbol{F}_e^{-T} \cdot \boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T)\right) \\
&= \frac{\rho}{2\,\rho_I}\,\left[\text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T \cdot \dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1}\right) + \text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \dot{\boldsymbol{F}}_e^T\right) + \text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1}\right)\right. \\
&\qquad \left. + \,\text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T)\right)\right]\,.
\end{aligned} \tag{270}$$

Since

$$\text{tr}\left(\boldsymbol{A}^T \cdot \boldsymbol{B}\right) = \text{tr}\left(\boldsymbol{B} \cdot \boldsymbol{A}^T\right) \tag{271}$$

we have

$$\begin{aligned}
\text{tr}\left([\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T] \cdot [\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1}]\right) &= \text{tr}\left([\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1}] \cdot [\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T]\right) = \text{tr}\left(\dot{\boldsymbol{F}}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T\right) \\
&= \text{tr}\left(\boldsymbol{S}_I \cdot \boldsymbol{F}_e^T \cdot \dot{\boldsymbol{F}}_e\right) = \boldsymbol{S}_I : (\boldsymbol{F}_e^T \cdot \dot{\boldsymbol{F}}_e)\,.
\end{aligned} \tag{272}$$

Also,

$$\text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \dot{\boldsymbol{F}}_e^T\right) = \text{tr}\left(\boldsymbol{S}_I \cdot \dot{\boldsymbol{F}}_e^T \cdot \boldsymbol{F}_e\right) = \boldsymbol{S}_I : (\dot{\boldsymbol{F}}_e^T \cdot \boldsymbol{F}_e)\,. \tag{273}$$

Similarly,

$$\begin{aligned}
\text{tr}\left([\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T] \cdot [\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1}]\right) &= \text{tr}\left([\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1}] \cdot [\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T]\right) \\
&= \text{tr}\left(\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T\right) \\
&= \text{tr}\left(\boldsymbol{S}_I \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1}\right) = \boldsymbol{S}_I : (\boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1})\,.
\end{aligned} \tag{274}$$

And,

$$\text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T)\right) = \text{tr}\left(\boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \boldsymbol{S}_I\right) = \boldsymbol{S}_I : (\boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e)\,. \tag{275}$$

Therefore,

$$\boxed{\boldsymbol{\sigma} : \boldsymbol{d} = \frac{\rho}{\rho_I}\, \boldsymbol{S}_I : \left[\frac{1}{2}\left(\boldsymbol{F}_e^T \cdot \dot{\boldsymbol{F}}_e + \dot{\boldsymbol{F}}_e^T \cdot \boldsymbol{F}_e\right) + \frac{1}{2}\left(\boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} + \boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e\right)\right]\,.} \tag{276}$$

6. Show that

$$\boldsymbol{\eta}_p = \frac{1}{2}\left[\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1} + (\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1})^T\right] \tag{277}$$

The volumetric/distortional split of the deformation gradient is given by

$$\boldsymbol{F} = J^{1/3}\,\widehat{\boldsymbol{F}} \qquad \text{where} \qquad J := \det(\boldsymbol{F})\,; \quad \det(\widehat{\boldsymbol{F}}) = 1 \tag{278}$$

where $\widehat{\boldsymbol{F}}$ is the distortional component of the deformation gradient. Clearly, we can perform the same decomposition for the elastic and plastic parts of the deformation gradient, i.e.,

$$\boldsymbol{F}_e = J_e^{1/3}\,\widehat{\boldsymbol{F}}_e\,; \quad \boldsymbol{F}_p = J_p^{1/3}\,\widehat{\boldsymbol{F}}_p \tag{279}$$

where

$$J_e := \det(\boldsymbol{F}_e)\,; \quad \det(\widehat{\boldsymbol{F}}_e) = 1\,; \quad J_p := \det(\boldsymbol{F}_p)\,; \quad \det(\widehat{\boldsymbol{F}}_p) = 1\,. \tag{280}$$

Since

$$J = \det(\boldsymbol{F}) = \det(\boldsymbol{F}_e \cdot \boldsymbol{F}_p) = \det(\boldsymbol{F}_e)\ \det(\boldsymbol{F}_p) = J_e\ J_p\ . \tag{281}$$

we can check that the product of the two decompositions gives us the original decomposition of $\mathbf{f}$ back, i.e.,

$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_p = (J_e^{1/3}\ \widehat{\boldsymbol{F}}_e) \cdot (J_p^{1/3}\ \widehat{\boldsymbol{F}}_p) = J^{1/3}\ \widehat{\boldsymbol{F}}_e \cdot \widehat{\boldsymbol{F}}_p = J^{1/3}\ \widehat{\boldsymbol{F}} \qquad \implies \qquad \widehat{\boldsymbol{F}} = \widehat{\boldsymbol{F}}_e \cdot \widehat{\boldsymbol{F}}_p \tag{282}$$

and

$$\det(\boldsymbol{F}) = \det(\boldsymbol{F}_e \cdot \boldsymbol{F}_p) = \det(J_e^{1/3}\ \widehat{\boldsymbol{F}}_e)\ \det(J_p^{1/3}\ \widehat{\boldsymbol{F}}_p) = J_e\ \det(\widehat{\boldsymbol{F}}_e)\ J_p\ \det(\widehat{\boldsymbol{F}}_p) = J_e\ J_p = J\ . \tag{283}$$

Recall that

$$\boldsymbol{d}_p = \frac{1}{2}\left[\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} + (\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1})^T\right]\ . \tag{284}$$

We want to express the quantities $\boldsymbol{F}_e$ and $\boldsymbol{F}_p$ in the expression in terms of their volumetric and distortional components. Let us consider the rate quantities first. We thus have

$$\boldsymbol{F}_p = J_p^{1/3}\ \widehat{\boldsymbol{F}}_p \qquad \implies \qquad \dot{\boldsymbol{F}}_p = \frac{d}{dt}(J_p^{1/3})\ \widehat{\boldsymbol{F}}_p + J_p^{1/3}\ \dot{\widehat{\boldsymbol{F}}}_p = \frac{1}{3}\ J_p^{-2/3}\ \frac{dJ_p}{dt}\ \widehat{\boldsymbol{F}}_p + J_p^{1/3}\ \dot{\widehat{\boldsymbol{F}}}_p\ . \tag{285}$$

Now, from (88) and (92), we have

$$\frac{dJ_p}{dt} = J_p\ \text{tr}\,(\boldsymbol{d}_p) = -J_p\ \frac{\dot{\rho}_I}{\rho_I}\,. \tag{286}$$

Therefore,

$$\dot{\boldsymbol{F}}_p = -\frac{1}{3}\ J_p^{1/3}\ \frac{\dot{\rho}_I}{\rho_I}\,\widehat{\boldsymbol{F}}_p + J_p^{1/3}\ \dot{\widehat{\boldsymbol{F}}}_p = J_p^{1/3}\left[-\frac{\dot{\rho}_I}{3\,\rho_I}\,\widehat{\boldsymbol{F}}_p + \dot{\widehat{\boldsymbol{F}}}_p\right]\ . \tag{287}$$

Also,

$$\boldsymbol{F}_p^{-1} = J_p^{-1/3}\ \widehat{\boldsymbol{F}}_p^{-1}\ . \tag{288}$$

Then

$$\dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} = \left[-\frac{\dot{\rho}_I}{3\,\rho_I}\,\widehat{\boldsymbol{F}}_p + \dot{\widehat{\boldsymbol{F}}}_p\right] \cdot \widehat{\boldsymbol{F}}_p^{-1} = -\frac{\dot{\rho}_I}{3\,\rho_I}\,\boldsymbol{1} + \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1}\ . \tag{289}$$

This leads to the next step

$$\begin{aligned}
\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} &= \left(J_e^{1/3}\ \widehat{\boldsymbol{F}}_e\right) \cdot \left[-\frac{\dot{\rho}_I}{3\,\rho_I}\,\boldsymbol{1} + \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1}\right] \cdot \left(J_e^{-1/3}\ \widehat{\boldsymbol{F}}_e^{-1}\right) \\
&= \widehat{\boldsymbol{F}}_e \cdot \left[-\frac{\dot{\rho}_I}{3\,\rho_I}\,\boldsymbol{1} + \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1}\right] \cdot \widehat{\boldsymbol{F}}_e^{-1} \\
&= -\frac{\dot{\rho}_I}{3\,\rho_I}\,\boldsymbol{1} + \widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1}\ .
\end{aligned} \tag{290}$$

Using equation (92) we get

$$\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} = \frac{1}{3}\ \text{tr}\,(\boldsymbol{d}_p)\ \ \boldsymbol{1} + \widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1}\ . \tag{291}$$

It follows that

$$(\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1})^T = \frac{1}{3}\ \text{tr}\,(\boldsymbol{d}_p)\ \ \boldsymbol{1} + (\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1})^T\ . \tag{292}$$

Therefore,

$$\boldsymbol{d}_p = \frac{1}{2}\left[\frac{1}{3}\ \text{tr}\,(\boldsymbol{d}_p)\ \ \boldsymbol{1} + \widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1} + \frac{1}{3}\ \text{tr}\,(\boldsymbol{d}_p)\ \ \boldsymbol{1} + (\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1})^T\right] \tag{293}$$

or,

$$\boldsymbol{d}_p = \frac{1}{3}\ \text{tr}\,(\boldsymbol{d}_p)\ \ \boldsymbol{1} + \frac{1}{2}\left[\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1} + (\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1})^T\right]\ . \tag{294}$$

Comparing the above equation with (95) we see that

$$\boxed{\boldsymbol{\eta}_p = \frac{1}{2}\left[\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1} + (\widehat{\boldsymbol{F}}_e \cdot \dot{\widehat{\boldsymbol{F}}}_p \cdot \widehat{\boldsymbol{F}}_p^{-1} \cdot \widehat{\boldsymbol{F}}_e^{-1})^T\right]} \tag{295}$$

This expression contains only distortional terms of the deformation gradients and hence must be distortional too. Therefore, the standard volumetric/deviatoric split of $\boldsymbol{d}_p$ is acceptable.

7. If we have the situation

$$\rho_I = \rho_I(q_j) \tag{296}$$

and

$$g = g(\boldsymbol{S}_I, T, q_j)\ ;\quad e = e(\boldsymbol{E}_e, \eta, q_j)\ ;\quad \psi = \psi(\boldsymbol{E}_e, T, q_j)\ ,\quad j = 1 \ldots n\ . \tag{297}$$

then we can show that $Q_0 = 0$.

To see this, let us revisit the differentials of the internal energy and free energy functions. For this situation we have

$$
\begin{aligned}
\mathsf{d}g &= \frac{\partial g}{\partial \boldsymbol{S}_I} : \mathsf{d}\boldsymbol{S}_I + \frac{\partial g}{\partial T}\,\mathsf{d}T + \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j = q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + \eta\,\mathsf{d}T + \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j \\
\mathsf{d}e &= -\mathsf{d}g + \eta\,\mathsf{d}T + T\,\mathsf{d}\eta + q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e \\
\mathsf{d}\psi &= -\mathsf{d}g + q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e\,.
\end{aligned}
\tag{298}
$$

Plugging in the expression for $\mathsf{d}g$ into those for $\mathsf{d}e$ and $\mathsf{d}\psi$ gives

$$
\begin{aligned}
\mathsf{d}e &= -q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I - \eta\,\mathsf{d}T - \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j + \eta\,\mathsf{d}T + T\,\mathsf{d}\eta + q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e \\
\mathsf{d}\psi &= -q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I - \eta\,\mathsf{d}T - \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j + q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e
\end{aligned}
\tag{299}
$$

or,

$$
\begin{aligned}
\mathsf{d}e &= -\sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j + T\,\mathsf{d}\eta + q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e \\
\mathsf{d}\psi &= -\eta\,\mathsf{d}T - \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j + q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e\,.
\end{aligned}
\tag{300}
$$

After rearranging, the differentials of the three potentials can be written as

$$
\boxed{
\begin{aligned}
\mathsf{d}g &= q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + \eta\,\mathsf{d}T + \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j \\
\mathsf{d}e &= q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e + T\,\mathsf{d}\eta - \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j \\
\mathsf{d}\psi &= q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e - \eta\,\mathsf{d}T - \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\mathsf{d}q_j\,.
\end{aligned}
}
\tag{301}
$$

For the general case discussed earlier, we had defined

$$
Q_0 := -q_0 \left(\frac{\partial g}{\partial q_0} + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right) \qquad \text{and} \qquad Q_j := -q_0\,\frac{\partial g}{\partial q_j}\ \ ,\ j = 1 \ldots n
\tag{302}
$$

so that would could write

$$
\begin{aligned}
\mathsf{d}g &= q_0^{-1}\,\boldsymbol{E}_e : \mathsf{d}\boldsymbol{S}_I + \eta\,\mathsf{d}T - \left(q_0^{-1}\,Q_0 + q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e\right)\,\mathsf{d}q_0 - q_0^{-1}\,\sum_{j=1}^{n} Q_j\,\mathsf{d}q_j \\
\mathsf{d}e &= q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e + T\,\mathsf{d}\eta + q_0^{-1}\,Q_0\,\mathsf{d}q_0 + q_0^{-1}\,\sum_{j=1}^{n} Q_j\,\mathsf{d}q_j \\
\mathsf{d}\psi &= q_0^{-1}\,\boldsymbol{S}_I : \mathsf{d}\boldsymbol{E}_e - \eta\,\mathsf{d}T + q_0^{-1}\,Q_0\,\mathsf{d}q_0 + q_0^{-1}\,\sum_{j=1}^{n} Q_j\,\mathsf{d}q_j\,.
\end{aligned}
\tag{303}
$$

Comparing (301) and (303), we clearly see that this special circumstance leads to equations that are equivalent to assuming that $Q_0 = 0$.

8. Show that the stress power can be expressed as

$$
\begin{aligned}
\boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\Big[&\boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) + \\
&\boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e)\Big]\,.
\end{aligned}
\tag{304}
$$

The stress power is given by

$$
\mathcal{P} = \boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\,\boldsymbol{\sigma} : [\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T]\,.
\tag{305}
$$

Now

$$
\boldsymbol{\nabla}\mathbf{v} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \qquad \Longrightarrow \qquad (\boldsymbol{\nabla}\mathbf{v})^T = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^T\,.
\tag{306}
$$

Using the decomposition (155), we can write

$$\boldsymbol{F}^{-1} = \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^{-1} \cdot \boldsymbol{V}_e^{-1} = \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^{T} \cdot \boldsymbol{V}_e^{-1} ~;~~ \boldsymbol{F}^{-T} = \boldsymbol{V}_e^{-T} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-T} \tag{307}$$

and

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{V}}_e \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p + \boldsymbol{V}_e \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_p + \boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p ~;~~ \dot{\boldsymbol{F}}^T = \boldsymbol{U}_p^T \cdot \boldsymbol{Q}^T \cdot \dot{\boldsymbol{V}}_e^T + \boldsymbol{U}_p^T \cdot \dot{\boldsymbol{Q}}^T \cdot \boldsymbol{V}_e^T + \dot{\boldsymbol{U}}_p^T \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^T ~. \tag{308}$$

From the symmetry of $\boldsymbol{V}_e$ and $\boldsymbol{U}_p$ we have

$$\boldsymbol{V}_e = \boldsymbol{V}_e^T \implies \boldsymbol{V}_e^{-1} = \boldsymbol{V}_e^{-T} \qquad \text{and} \qquad \boldsymbol{U}_p = \boldsymbol{U}_p^T \implies \boldsymbol{U}_p^{-1} = \boldsymbol{U}_p^{-T} ~. \tag{309}$$

Also,

$$\dot{\boldsymbol{V}}_e = \dot{\boldsymbol{V}}_e^T ~;~~ \dot{\boldsymbol{U}}_p = \dot{\boldsymbol{U}}_p^T ~; \tag{310}$$

Hence

$$\boldsymbol{F}^{-T} = \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} ~;~~ \dot{\boldsymbol{F}}^T = \boldsymbol{U}_p \cdot \boldsymbol{Q}^T \cdot \dot{\boldsymbol{V}}_e + \boldsymbol{U}_p \cdot \dot{\boldsymbol{Q}}^T \cdot \boldsymbol{V}_e + \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e ~. \tag{311}$$

The velocity gradient can then be expressed as

$$\begin{aligned} \boldsymbol{\nabla}\mathbf{v} &= (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p + \boldsymbol{V}_e \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_p + \boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p) \cdot (\boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}) \\ &= \dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} ~. \end{aligned} \tag{312}$$

Similarly, the transpose of the velocity gradient can be written as

$$\begin{aligned} (\boldsymbol{\nabla}\mathbf{v})^T &= (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1}) \cdot (\boldsymbol{U}_p \cdot \boldsymbol{Q}^T \cdot \dot{\boldsymbol{V}}_e + \boldsymbol{U}_p \cdot \dot{\boldsymbol{Q}}^T \cdot \boldsymbol{V}_e + \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e) \\ &= \boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e + \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T \cdot \boldsymbol{V}_e + \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e ~. \end{aligned} \tag{313}$$

Therefore the contraction of the velocity gradient with $\boldsymbol{\sigma}$ gives

$$\begin{aligned} \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} &= \boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1}) + \text{tr}\left(\boldsymbol{\sigma} \cdot \boldsymbol{V}_e \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}\right) + \boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}) \\ &= \boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1}) + (\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T) : (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e) + \boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}) \end{aligned} \tag{314}$$

and

$$\begin{aligned} \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v})^T &= \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) + \text{tr}\left(\boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T \cdot \boldsymbol{V}_e\right) + \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e) \\ &= \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) + (\boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T) : (\boldsymbol{V}_e \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1}) + \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e) ~. \end{aligned} \tag{315}$$

Now,

$$\boldsymbol{Q} \cdot \boldsymbol{Q}^T = \boldsymbol{1} \qquad \implies \qquad \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T = -\boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T ~. \tag{316}$$

Hence,

$$\boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v})^T = \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) - (\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T) : (\boldsymbol{V}_e \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1}) + \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e) ~. \tag{317}$$

Adding the two terms, we get

$$\begin{aligned} \boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2} \Big[& \boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) + (\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T) : (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e - \boldsymbol{V}_e \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1}) + \\ & \boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e) \Big] ~. \end{aligned} \tag{318}$$

Since $\boldsymbol{\sigma}$ and $\boldsymbol{V}_e$ are coaxial, their spectral decompositions (eigenprojections) are

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i ~\mathbf{n}_i \otimes \mathbf{n}_i ~;~~ \boldsymbol{V}_e = \sum_{j=1}^{3} \lambda_j ~\mathbf{n}_j \otimes \mathbf{n}_j \tag{319}$$

and

$$\boldsymbol{V}_e^{-1} = \sum_{k=1}^{3} \frac{1}{\lambda_k} \mathbf{n}_k \otimes \mathbf{n}_k ~. \tag{320}$$

Therefore,

$$\boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e = \left(\sum_{k=1}^{3} \frac{1}{\lambda_k} \mathbf{n}_k \otimes \mathbf{n}_k\right) \cdot \left(\sum_{i=1}^{3} \sigma_i ~\mathbf{n}_i \otimes \mathbf{n}_i\right) \cdot \left(\sum_{j=1}^{3} \lambda_j ~\mathbf{n}_j \otimes \mathbf{n}_j\right) ~; \tag{321}$$

Since

$$(\mathbf{n}_i \otimes \mathbf{n}_i) \cdot (\mathbf{n}_j \otimes \mathbf{n}_j) = \begin{cases} \boldsymbol{0} & \text{if } i \neq j \\ \mathbf{n}_i \otimes \mathbf{n}_i & \text{if } i = j \end{cases} \tag{322}$$

we have

$$\boldsymbol{V}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e = \sum_{i=1}^{3} \sigma_i ~\mathbf{n}_i \otimes \mathbf{n}_i = \boldsymbol{V}_e \cdot \boldsymbol{\sigma} \cdot \boldsymbol{V}_e^{-1} ~. \tag{323}$$

Hence the stress power takes the form

$$\boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\Big[\boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) + \boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1} + \boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e)\Big] \; . \tag{324}$$

9. Show that the stress power may also be expressed as

$$\begin{aligned}\boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\; \Big[\boldsymbol{\sigma} : \Big\{\boldsymbol{Q} \cdot (\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e) \cdot \boldsymbol{Q}^T\Big\} \\ + \boldsymbol{\sigma} : \Big\{\boldsymbol{Q} \cdot (\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e) \cdot \boldsymbol{Q}^T\Big\}\Big] \, .\end{aligned} \tag{325}$$

Since

$$\boldsymbol{V}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T \tag{326}$$

we have

$$\boldsymbol{V}_e^{-1} = \boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T \; ; \;\; \dot{\boldsymbol{V}}_e = \dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{Q}}^T \, . \tag{327}$$

Therefore,

$$\begin{aligned}\boldsymbol{\sigma} : (\dot{\boldsymbol{V}}_e \cdot \boldsymbol{V}_e^{-1}) &= \boldsymbol{\sigma} : [(\dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{Q}}^T) \cdot (\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T)] \\ &= \boldsymbol{\sigma} : [\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{Q}}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T] \\ \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \dot{\boldsymbol{V}}_e) &= \boldsymbol{\sigma} : [(\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T) \cdot (\dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{Q}}^T)] \\ &= \boldsymbol{\sigma} : [\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T] \\ \boldsymbol{\sigma} : (\boldsymbol{V}_e \cdot \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e^{-1}) &= \boldsymbol{\sigma} : [(\boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T) \cdot (\boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{Q}^T) \cdot (\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T)] \\ &= \boldsymbol{\sigma} : [\boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T] \\ \boldsymbol{\sigma} : (\boldsymbol{V}_e^{-1} \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T \cdot \boldsymbol{V}_e) &= \boldsymbol{\sigma} : [(\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T) \cdot (\boldsymbol{Q} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{Q}^T) \cdot (\boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T)] \\ &= \boldsymbol{\sigma} : [\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T] \, .\end{aligned} \tag{328}$$

Since the stress power does not contain any terms containing $\dot{\boldsymbol{Q}}$, we can ignore these terms to get

$$\begin{aligned}\boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\; \Big[\boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T) \\ + \boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T)\Big] \, .\end{aligned} \tag{329}$$

or,

$$\begin{aligned}\boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{2}\; \Big[\boldsymbol{\sigma} : \Big\{\boldsymbol{Q} \cdot (\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e) \cdot \boldsymbol{Q}^T\Big\} \\ + \boldsymbol{\sigma} : \Big\{\boldsymbol{Q} \cdot (\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e) \cdot \boldsymbol{Q}^T\Big\}\Big] \, .\end{aligned} \tag{330}$$

10. If

$$\boldsymbol{F}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e \qquad \text{and} \qquad \boldsymbol{F}_p = \boldsymbol{U}_p \tag{331}$$

show that the equations

$$\begin{aligned}\boldsymbol{d}_e &= \frac{1}{2}\left[\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1} + (\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1})^T\right] \\ \boldsymbol{d}_p &= \frac{1}{2}\left[\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1} + (\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1})^T\right] \; .\end{aligned} \tag{332}$$

can be expressed as

$$\begin{aligned}\boldsymbol{d}_e &= \frac{1}{2}\left[\boldsymbol{Q} \cdot (\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e) \cdot \boldsymbol{Q}^T\right] \\ \boldsymbol{d}_p &= \frac{1}{2}\left[\boldsymbol{Q} \cdot (\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p^T \cdot \boldsymbol{U}_e) \cdot \boldsymbol{Q}^T\right] \; .\end{aligned} \tag{333}$$

Since

$$\boldsymbol{F}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e \qquad \text{and} \qquad \boldsymbol{F}_p = \boldsymbol{U}_p \tag{334}$$

we have

$$\dot{\boldsymbol{F}}_e = \dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \; ; \;\; \boldsymbol{F}_e^{-1} = \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T \; ; \;\; \dot{\boldsymbol{F}}_p = \dot{\boldsymbol{U}}_p \; ; \;\; \boldsymbol{F}_p^{-1} = \boldsymbol{U}_p^{-1} \, . \tag{335}$$

Therefore,

$$\begin{aligned}
\boldsymbol{\sigma} : (\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1}) &= \boldsymbol{\sigma} : [(\dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e) \cdot (\boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T)] = \boldsymbol{\sigma} : (\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T) \\
\boldsymbol{\sigma} : (\dot{\boldsymbol{F}}_e \cdot \boldsymbol{F}_e^{-1})^T &= \boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^T + \boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T) \\
\boldsymbol{\sigma} : (\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1}) &= \boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T) \\
\boldsymbol{\sigma} : (\boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \cdot \boldsymbol{F}_e^{-1})^T &= \boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p^T \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T)
\end{aligned} \tag{336}$$

Since $\boldsymbol{\sigma} : \boldsymbol{d} = \boldsymbol{\sigma} : (\boldsymbol{d}_e + \boldsymbol{d}_p)$ does not have any terms containing $\dot{\boldsymbol{Q}}$, we have

$$\begin{aligned}
\boldsymbol{\sigma} : \boldsymbol{d}_e &= \frac{1}{2}\left[\boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T) + \boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T)\right] \\
\boldsymbol{\sigma} : \boldsymbol{d}_p &= \frac{1}{2}\left[\boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{Q}^T) + \boldsymbol{\sigma} : (\boldsymbol{Q} \cdot \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p^T \cdot \boldsymbol{U}_e \cdot \boldsymbol{Q}^T)\right]
\end{aligned} \tag{337}$$

Therefore,

$$\boxed{\begin{aligned}
\boldsymbol{d}_e &= \frac{1}{2}\left[\boldsymbol{Q} \cdot (\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e) \cdot \boldsymbol{Q}^T\right] \\
\boldsymbol{d}_p &= \frac{1}{2}\left[\boldsymbol{Q} \cdot (\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p^T \cdot \boldsymbol{U}_e) \cdot \boldsymbol{Q}^T\right] ~.
\end{aligned}} \tag{338}$$

11. Show that the stress power can be expressed in the intermediate configuration as

$$\mathcal{P} = J_e^{-1}\, \boldsymbol{S}_I : (\boldsymbol{D}_e + \boldsymbol{D}_p) ~. \tag{339}$$

where

$$\begin{aligned}
\boldsymbol{D}_e &= \frac{1}{2}\left[\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e + \boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_e\right] \\
\boldsymbol{D}_p &= \frac{1}{2}\left[\boldsymbol{U}_e^2 \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} + \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e^2\right] ~.
\end{aligned} \tag{340}$$

Recall that $\boldsymbol{S}_I$ and $\boldsymbol{\sigma}$ are related by

$$\boldsymbol{S}_I = J_e\, \boldsymbol{F}_e^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}_e^{-T} ~;~~ J_e := \det(\boldsymbol{F}_e) = \frac{\rho_I}{\rho} . \tag{341}$$

Therefore, the stress power can be written as

$$\begin{aligned}
\mathcal{P} = \boldsymbol{\sigma} : \boldsymbol{d} &= \left(J_e^{-1}\, \boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T\right) : (\boldsymbol{d}_e + \boldsymbol{d}_p) \\
&= J_e^{-1}\left[\text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{d}_e\right) + \text{tr}\left(\boldsymbol{F}_e \cdot \boldsymbol{S}_I \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{d}_p\right)\right] \\
&= J_e^{-1}\left[\boldsymbol{S}_I : (\boldsymbol{F}_e^T \cdot \boldsymbol{d}_e \cdot \boldsymbol{F}_e) + \boldsymbol{S}_I : (\boldsymbol{F}_e^T \cdot \boldsymbol{d}_p \cdot \boldsymbol{F}_e)\right] ~.
\end{aligned} \tag{342}$$

Since $\boldsymbol{F}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e$, we then have

$$\mathcal{P} = J_e^{-1}\, \boldsymbol{S}_I : \left(\boldsymbol{U}_e \cdot \boldsymbol{Q}^T \cdot \boldsymbol{d}_e \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_e + \boldsymbol{U}_e \cdot \boldsymbol{Q}^T \cdot \boldsymbol{d}_p \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_e\right) ~. \tag{343}$$

Using the definitions of $\boldsymbol{d}_e$ and $\boldsymbol{d}_p$ from (165) we have

$$\begin{aligned}
\boldsymbol{U}_e \cdot \boldsymbol{Q}^T \cdot \boldsymbol{d}_e \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_e &= \frac{1}{2}\left[\boldsymbol{U}_e \cdot (\dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \dot{\boldsymbol{U}}_e) \cdot \boldsymbol{U}_e\right] \\
&= \frac{1}{2}\left[\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_e + \dot{\boldsymbol{U}}_e \cdot \boldsymbol{U}_e\right] \\
\boldsymbol{U}_e \cdot \boldsymbol{Q}^T \cdot \boldsymbol{d}_p \cdot \boldsymbol{Q} \cdot \boldsymbol{U}_e &= \frac{1}{2}\left[\boldsymbol{U}_e \cdot (\boldsymbol{U}_e \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} \cdot \boldsymbol{U}_e^{-1} + \boldsymbol{U}_e^{-1} \cdot \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e) \cdot \boldsymbol{U}_e\right] \\
&= \frac{1}{2}\left[\boldsymbol{U}_e^2 \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_p^{-1} + \boldsymbol{U}_p^{-1} \cdot \dot{\boldsymbol{U}}_p \cdot \boldsymbol{U}_e^2\right] ~.
\end{aligned} \tag{344}$$

Recall from equations (74) and (75) that

$$\begin{aligned}
\boldsymbol{D}_e &= \frac{1}{2}\left(\dot{\boldsymbol{F}}_e^T \cdot \boldsymbol{F}_e + \boldsymbol{F}_e^T \cdot \dot{\boldsymbol{F}}_e\right) \\
\boldsymbol{D}_p &= \frac{1}{2}\left(\boldsymbol{F}_e^T \cdot \boldsymbol{F}_e \cdot \dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} + \boldsymbol{F}_p^{-T} \cdot \dot{\boldsymbol{F}}_p^T \cdot \boldsymbol{F}_e^T \cdot \boldsymbol{F}_e\right) ~.
\end{aligned} \tag{345}$$

Recall that $\boldsymbol{F}_e = \boldsymbol{Q} \cdot \boldsymbol{U}_e$ and $\boldsymbol{F}_p = \boldsymbol{U}_p$ which implies that

$$\dot{\boldsymbol{F}}_e = \dot{\boldsymbol{Q}} \cdot \boldsymbol{U}_e + \boldsymbol{Q} \cdot \dot{\boldsymbol{U}}_e ~;~~ \dot{\boldsymbol{F}}_e^T = \boldsymbol{U}_e \cdot \dot{\boldsymbol{Q}}^T + \dot{\boldsymbol{U}}_e \cdot \boldsymbol{Q}^T ~;~~ \boldsymbol{F}_e^T = \boldsymbol{U}_e \cdot \boldsymbol{Q}^T ~;~~ \dot{\boldsymbol{F}}_p = \dot{\boldsymbol{U}}_p ~;~~ \boldsymbol{F}_p^{-1} = \boldsymbol{U}_p^{-1} ~;~~ \boldsymbol{F}_p^{-T} = \boldsymbol{U}_p^{-1} ~. \tag{346}$$

Substituting these into the expressions for $\boldsymbol{D}_e$ and $\boldsymbol{D}_p$, we get

$$\begin{aligned}\boldsymbol{D}_e &= \frac{1}{2}\,\left[(\boldsymbol{U}_e\cdot\dot{\boldsymbol{Q}}^T + \dot{\boldsymbol{U}}_e\cdot\boldsymbol{Q}^T)\cdot(\boldsymbol{Q}\cdot\boldsymbol{U}_e) + (\boldsymbol{U}_e\cdot\boldsymbol{Q}^T)\cdot(\dot{\boldsymbol{Q}}\cdot\boldsymbol{U}_e + \boldsymbol{Q}\cdot\dot{\boldsymbol{U}}_e)\right]\\ &= \frac{1}{2}\,\left[\boldsymbol{U}_e\cdot\dot{\boldsymbol{Q}}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{U}_e + \dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\boldsymbol{Q}^T\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\right]\\ \boldsymbol{D}_p &= \frac{1}{2}\left[(\boldsymbol{U}_e\cdot\boldsymbol{Q}^T)\cdot(\boldsymbol{Q}\cdot\boldsymbol{U}_e)\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1} + \boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot(\boldsymbol{U}_e\cdot\boldsymbol{Q}^T)\cdot(\boldsymbol{Q}\cdot\boldsymbol{U}_e)\right]\\ &= \frac{1}{2}\left[\boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1} + \boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2\right]\,.\end{aligned} \tag{347}$$

Since $\dot{\boldsymbol{Q}}^T\cdot\boldsymbol{Q} = -\boldsymbol{Q}^T\cdot\dot{\boldsymbol{Q}}$, we get

$$\boxed{\begin{aligned}\boldsymbol{D}_e &= \frac{1}{2}\,\left[\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\right]\\ \boldsymbol{D}_p &= \frac{1}{2}\left[\boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1} + \boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2\right]\,.\end{aligned}} \tag{348}$$

Comparing (348) with (344) and (343), we see that

$$\boxed{\mathcal{P} = J_e^{-1}\,\boldsymbol{S}_I : (\boldsymbol{D}_e + \boldsymbol{D}_p)\,.} \tag{349}$$

12. If $\boldsymbol{U}^2 = \boldsymbol{F}^T\cdot\boldsymbol{F}$, show that

$$\frac{d}{dt}(\boldsymbol{U}^2) = \boldsymbol{U}_p\cdot(\boldsymbol{D}_p + \boldsymbol{D}_e)\cdot\boldsymbol{U}_p\,. \tag{350}$$

Differentiation with respect to time gives us

$$\frac{d}{dt}\left(\boldsymbol{U}^2\right) = \dot{\boldsymbol{F}}^T\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\dot{\boldsymbol{F}}\,. \tag{351}$$

Recall that

$$\boldsymbol{F} = \boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p \quad\implies\quad \boldsymbol{F}^T = \boldsymbol{U}_p^T\cdot\boldsymbol{U}_e^T\cdot\boldsymbol{Q}^T\,. \tag{352}$$

Hence,

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{Q}}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p + \boldsymbol{Q}\cdot\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_p + \boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_p \quad\text{and}\quad \dot{\boldsymbol{F}}^T = \dot{\boldsymbol{U}}_p^T\cdot\boldsymbol{U}_e^T\cdot\boldsymbol{Q}^T + \boldsymbol{U}_p^T\cdot\dot{\boldsymbol{U}}_e^T\cdot\boldsymbol{Q}^T + \boldsymbol{U}_p^T\cdot\boldsymbol{U}_e^T\cdot\dot{\boldsymbol{Q}}^T\,. \tag{353}$$

Taking products gives us

$$\begin{aligned}\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F} &= \left(\dot{\boldsymbol{U}}_p^T\cdot\boldsymbol{U}_e^T\cdot\boldsymbol{Q}^T + \boldsymbol{U}_p^T\cdot\dot{\boldsymbol{U}}_e^T\cdot\boldsymbol{Q}^T + \boldsymbol{U}_p^T\cdot\boldsymbol{U}_e^T\cdot\dot{\boldsymbol{Q}}^T\right)\cdot(\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p)\\ &= \dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\boldsymbol{U}_e\cdot\dot{\boldsymbol{Q}}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p\,.\\ \boldsymbol{F}^T\cdot\dot{\boldsymbol{F}} &= \left(\boldsymbol{U}_p^T\cdot\boldsymbol{U}_e^T\cdot\boldsymbol{Q}^T\right)\cdot\left(\dot{\boldsymbol{Q}}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p + \boldsymbol{Q}\cdot\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_p + \boldsymbol{Q}\cdot\boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_p\right)\\ &= \boldsymbol{U}_p\cdot\boldsymbol{U}_e\cdot\boldsymbol{Q}^T\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{U}_e\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p\,.\end{aligned} \tag{354}$$

Adding the two terms, we get

$$\begin{aligned}\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\dot{\boldsymbol{F}} = \dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p + \boldsymbol{U}_p\cdot\left(\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\right)\cdot\boldsymbol{U}_p\\ + \boldsymbol{U}_p\cdot\left(\boldsymbol{U}_e\cdot\dot{\boldsymbol{Q}}^T\cdot\boldsymbol{Q}\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\boldsymbol{Q}^T\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{U}_e\right)\cdot\boldsymbol{U}_p\,.\end{aligned} \tag{355}$$

Since $\boldsymbol{Q}^T\cdot\boldsymbol{Q} = \mathit{1}$ we have $\boldsymbol{Q}^T\cdot\dot{\boldsymbol{Q}} = -\dot{\boldsymbol{Q}}^T\cdot\boldsymbol{Q}$. Hence,

$$\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\dot{\boldsymbol{F}} = \dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p + \boldsymbol{U}_p\cdot\left(\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\right)\cdot\boldsymbol{U}_p \tag{356}$$

or,

$$\frac{d}{dt}(\boldsymbol{U}^2) = \boldsymbol{U}_p\cdot(\boldsymbol{U}_p^{-1}\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_e^2 + \boldsymbol{U}_e^2\cdot\dot{\boldsymbol{U}}_p\cdot\boldsymbol{U}_p^{-1})\cdot\boldsymbol{U}_p + \boldsymbol{U}_p\cdot\left(\dot{\boldsymbol{U}}_e\cdot\boldsymbol{U}_e + \boldsymbol{U}_e\cdot\dot{\boldsymbol{U}}_e\right)\cdot\boldsymbol{U}_p\,. \tag{357}$$

Comparing with equation (168) we see that

$$\boxed{\frac{d}{dt}(\boldsymbol{U}^2) = \boldsymbol{U}_p\cdot(\boldsymbol{D}_p + \boldsymbol{D}_e)\cdot\boldsymbol{U}_p\,.} \tag{358}$$

13. Show that the strain rate in the intermediate configuration can be expressed as

$$\boldsymbol{D}_e = \dot{\boldsymbol{E}}_e = \rho_I \, \mathsf{S} : \left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\,\boldsymbol{S}_I\right) + \boldsymbol{\alpha}_e\,\dot{T} - \sum_{j=0}^{n} \frac{\partial Q_j}{\partial \boldsymbol{S}_I}\,\dot{q}_j \tag{359}$$

where

$$\mathsf{S} := \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} \; ; \;\; \boldsymbol{\alpha}_e := \frac{\partial \boldsymbol{E}_e}{\partial T} \; . \tag{360}$$

Now, from (74) we have $\boldsymbol{D}_e = \dot{\boldsymbol{E}}_e$ and from (33) we have

$$\boldsymbol{E}_e = \rho_I \, \frac{\partial g}{\partial \boldsymbol{S}_I} \; . \tag{361}$$

Hence $\boldsymbol{D}_e$ is completely determined by the Gibbs free energy functional g. Also

$$\dot{\boldsymbol{E}}_e = \dot{\rho}_I \, \frac{\partial g}{\partial \boldsymbol{S}_I} + \rho_I \, \frac{\partial^2 g}{\partial t \partial \boldsymbol{S}_I} = \frac{\dot{\rho}_I}{\rho_I}\,\boldsymbol{E}_e + \rho_I \, \frac{\partial}{\partial \boldsymbol{S}_I}\left(\frac{\partial g}{\partial t}\right) \; . \tag{362}$$

Since the Gibbs free energy functional is given by $g = g(\boldsymbol{S}_I, T, q_0, q_j)$ we have

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial \boldsymbol{S}_I} : \dot{\boldsymbol{S}}_I + \frac{\partial g}{\partial T}\,\dot{T} + \frac{\partial g}{\partial q_0}\,\dot{q}_0 + \sum_{j=1}^{n} \frac{\partial g}{\partial q_j}\,\dot{q}_j \; . \tag{363}$$

From equations (33) we have

$$\frac{\partial g}{\partial T} = \eta \; ; \;\; \frac{\partial g}{\partial q_0} = -q_0^{-1}\,Q_0 - q_0^{-2}\,\boldsymbol{S}_I : \boldsymbol{E}_e \; ; \;\; \frac{\partial g}{\partial q_j} = -q_0^{-1}\,Q_j \; . \tag{364}$$

Hence,

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial \boldsymbol{S}_I} : \dot{\boldsymbol{S}}_I + \eta\,\dot{T} - q_0^{-1}\,(\,Q_0 + q_0^{-1}\,\boldsymbol{S}_I : \boldsymbol{E}_e)\,\dot{q}_0 - \sum_{j=1}^{n} q_0^{-1}\,Q_j\,\dot{q}_j \; . \tag{365}$$

Taking the derivative with respect to $\boldsymbol{S}_I$ gives us

$$\frac{\partial}{\partial \boldsymbol{S}_I}\left(\frac{\partial g}{\partial t}\right) = \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} : \dot{\boldsymbol{S}}_I + \frac{\partial \eta}{\partial \boldsymbol{S}_I}\,\dot{T} - q_0^{-1}\,\left(\frac{\partial Q_0}{\partial \boldsymbol{S}_I} + q_0^{-1}\,\boldsymbol{E}_e + q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial \boldsymbol{E}_e}{\partial \boldsymbol{S}_I}\right)\,\dot{q}_0 - \sum_{j=1}^{n} q_0^{-1}\,\frac{\partial Q_j}{\partial \boldsymbol{S}_I}\,\dot{q}_j \; . \tag{366}$$

Now,

$$\boldsymbol{E}_e = \rho_I \, \frac{\partial g}{\partial \boldsymbol{S}_I} \qquad \implies \qquad \frac{\partial \boldsymbol{E}_e}{\partial \boldsymbol{S}_I} = q_0 \, \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} \tag{367}$$

and from equation (45) we have

$$\frac{\partial \eta}{\partial \boldsymbol{S}_I} = q_0^{-1}\,\frac{\partial \boldsymbol{E}_e}{\partial T} \; . \tag{368}$$

Hence we can write

$$\frac{\partial}{\partial \boldsymbol{S}_I}\left(\frac{\partial g}{\partial t}\right) = \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} : \dot{\boldsymbol{S}}_I + q_0^{-1}\,\frac{\partial \boldsymbol{E}_e}{\partial T}\,\dot{T} - q_0^{-2}\,\boldsymbol{E}_e\,\dot{q}_0 - q_0^{-1}\,\boldsymbol{S}_I : \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2}\,\dot{q}_0 - \sum_{j=0}^{n} q_0^{-1}\,\frac{\partial Q_j}{\partial \boldsymbol{S}_I}\,\dot{q}_j \; . \tag{369}$$

Recall that

$$\dot{\boldsymbol{E}}_e = q_0^{-1}\,\boldsymbol{E}_e\,\dot{q}_0 + q_0\,\frac{\partial}{\partial \boldsymbol{S}_I}\left(\frac{\partial g}{\partial t}\right) \; . \tag{370}$$

Plugging the expression from (369) into (370), we get

$$\dot{\boldsymbol{E}}_e = q_0\,\frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} : \dot{\boldsymbol{S}}_I + \frac{\partial \boldsymbol{E}_e}{\partial T}\,\dot{T} - \boldsymbol{S}_I : \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2}\,\dot{q}_0 - \sum_{j=0}^{n} \frac{\partial Q_j}{\partial \boldsymbol{S}_I}\,\dot{q}_j \; . \tag{371}$$

Reverting back to $\rho_I = q_0$ and collecting terms gives

$$\boldsymbol{D}_e = \dot{\boldsymbol{E}}_e = \rho_I\,\frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} : \left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\,\boldsymbol{S}_I\right) + \frac{\partial \boldsymbol{E}_e}{\partial T}\,\dot{T} - \sum_{j=0}^{n} \frac{\partial Q_j}{\partial \boldsymbol{S}_I}\,\dot{q}_j \tag{372}$$

If we define the fourth-order elastic stiffness tensor and the second-order thermal expansion tensor as

$$\mathsf{S} := \frac{\partial^2 g}{\partial \boldsymbol{S}_I^2} \; ; \;\; \boldsymbol{\alpha}_e := \frac{\partial \boldsymbol{E}_e}{\partial T} \tag{373}$$

we then have

$$\boxed{\boldsymbol{D}_e = \dot{\boldsymbol{E}}_e = \rho_I \, \mathsf{S} : \left(\dot{\boldsymbol{S}}_I - \frac{\dot{\rho}_I}{\rho_I}\,\boldsymbol{S}_I\right) + \boldsymbol{\alpha}_e\,\dot{T} - \sum_{j=0}^{n} \frac{\partial Q_j}{\partial \boldsymbol{S}_I}\,\dot{q}_j \; .} \tag{374}$$

14. Let $\boldsymbol{S}$ be a second-order tensor and let $\widehat{q}(\boldsymbol{S})$ be a scalar valued function of $\boldsymbol{S}$. Show that $\widehat{q}$ is an isotropic function of $\boldsymbol{S}$ if and only if there exists a function $\widetilde{q}(I_1, I_2, I_3)$ such that

$$\widehat{q}(\boldsymbol{S}) = \widetilde{q}(I_1, I_2, I_3) \tag{375}$$

where I_1, I_2, I_3 are the invariants of $\boldsymbol{S}$ defined as

$$I_1(\boldsymbol{S}) = \text{tr}\,(\boldsymbol{S}) \;\; ; \;\; I_2(\boldsymbol{S}) = \frac{1}{2}\,\left[(\text{tr}\,(\boldsymbol{S}))^2 - \text{tr}\,\left(\boldsymbol{S}^2\right)\right] \;\; ; \;\; I_3(\boldsymbol{S}) = \det(\boldsymbol{S})\,. \tag{376}$$

The scalar function $\widehat{q}$ is invariant under rigid body rotations $\boldsymbol{R}$ if

$$\widehat{q}(\boldsymbol{S}) = \widehat{q}(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T)\,. \tag{377}$$

An isotropic function is a function that is invariant under rigid body rotations. We will first show that the three invariants of $\boldsymbol{S}$ are isotropic functions.

Thus,

$$\begin{aligned} I_1(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T) &= \text{tr}\left(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T\right) = \text{tr}\left(\boldsymbol{S}\cdot\boldsymbol{R}^T\cdot\boldsymbol{R}\right) = \text{tr}\,(\boldsymbol{S}) = I_1(\boldsymbol{S}) \\ I_2(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T) &= \frac{1}{2}\,\left[(\text{tr}\left(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T\right))^2 - \text{tr}\left((\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T)^2\right)\right] = \frac{1}{2}\,\left[(\text{tr}\,(\boldsymbol{S}))^2 - \text{tr}\left(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T\right)\right] \\ &= \frac{1}{2}\,\left[(\text{tr}\,(\boldsymbol{S}))^2 - \text{tr}\left(\boldsymbol{S}^2\right)\right] = I_2(\boldsymbol{S}) \\ I_3(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T) &= \det(\boldsymbol{R}\cdot\boldsymbol{S}\cdot\boldsymbol{R}^T) = \det(\boldsymbol{R})\,\det(\boldsymbol{S})\,\det(\boldsymbol{R}^T) = \det(\boldsymbol{S}) = I_3(\boldsymbol{S})\,. \end{aligned} \tag{378}$$

Hence I_1, I_2, I_3 are isotropic functions of $\boldsymbol{S}$. Therefore, any function $\widetilde{q}(I_1, I_2, I_3)$ is also an isotropic function of $\boldsymbol{S}$.

Conversely, we can establish the requirement in equation (375) by assuming that $\widehat{q}$ is isotropic and then showing that $\widehat{q}(\boldsymbol{S}) = \widehat{q}(\boldsymbol{B})$ whenever the invariants of $\boldsymbol{S}$ and $\boldsymbol{B}$ are identical and vice versa.

If the tensor $\boldsymbol{S}$ is symmetric then it is completely characterized by its invariants via the spectral decomposition theorem. Let us therefore assume that $\boldsymbol{S}$ and $\boldsymbol{B}$ are symmetric. Then we can write

$$\boldsymbol{S} = \sum_{i=1}^{3}\lambda_i\,\mathbf{s}_i\otimes\mathbf{s}_i\,; \;\; \boldsymbol{B} = \sum_{i=1}^{3}\lambda_i\,\mathbf{b}_i\otimes\mathbf{b}_i\,. \tag{379}$$

Also let $\mathbf{s}_i = \boldsymbol{Q}\cdot\mathbf{b}_i$ where $\boldsymbol{Q}$ is a rotation. Then

$$\boldsymbol{Q}\cdot(\mathbf{b}_i\otimes\mathbf{b}_i)\cdot\boldsymbol{Q}^T = Q_{mj}\,b_j\,b_k\,Q_{lk} = (\boldsymbol{Q}\cdot\mathbf{b}_i)\otimes(\boldsymbol{Q}\cdot\mathbf{b}_i) = \mathbf{s}_i\otimes\mathbf{s}_i\,. \tag{380}$$

Therefore

$$\boldsymbol{Q}\cdot\boldsymbol{B}\cdot\boldsymbol{Q}^T = \sum_{i=1}^{3}\lambda_i\,\boldsymbol{Q}\cdot(\mathbf{b}_i\otimes\mathbf{b}_i)\cdot\boldsymbol{Q}^T = \sum_{i=1}^{3}\lambda_i\,\mathbf{s}_i\otimes\mathbf{s}_i) = \boldsymbol{S}\,. \tag{381}$$

But $\widehat{q}$ is isotropic, i.e. $\widehat{q}(\boldsymbol{Q}\cdot\boldsymbol{B}\cdot\boldsymbol{Q}^T) = \widehat{q}(\boldsymbol{B})$. Therefore $\widehat{q}(\boldsymbol{S}) = \widehat{q}(\boldsymbol{B})$ and hence the representation (375) holds.

15. Suppose that $\boldsymbol{\sigma}$ is a symmetric second-order tensor and let $\widehat{\boldsymbol{d}}(\boldsymbol{\sigma})$ be a symmetric second-order tensor valued function of $\boldsymbol{\sigma}$. Show that the function $\widehat{\boldsymbol{d}}$ is isotropic if an only if there exist scalar functions $\varphi_0, \varphi_1, \varphi_2$ of the invariants $I_{\boldsymbol{\sigma}} = (I_1, I_2, I_3)$ of $\boldsymbol{\sigma}$ such that

$$\widehat{\boldsymbol{d}}(\boldsymbol{\sigma}) = \varphi_0(I_1, I_2, I_3)\;\boldsymbol{1} + \varphi_1(I_1, I_2, I_3)\;\boldsymbol{\sigma} + \varphi_2(I_1, I_2, I_3)\;\boldsymbol{\sigma}^2\,. \tag{382}$$

Recall that a tensor valued function of a second order tensor is isotropic if

$$\boldsymbol{R}\cdot\widehat{\boldsymbol{d}}(\boldsymbol{\sigma})\cdot\boldsymbol{R}^T = \widehat{\boldsymbol{d}}(\boldsymbol{R}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{R}^T) \tag{383}$$

for all rigid body rotations $\boldsymbol{R}$.

Let us assume that the representation (382) holds. Let $\boldsymbol{B} = \boldsymbol{R}\cdot\boldsymbol{\sigma}\boldsymbol{R}^T$ where $\boldsymbol{R}$ is a rotation tensor. Then

$$\begin{aligned} \boldsymbol{R}\cdot\widehat{\boldsymbol{d}}(\boldsymbol{\sigma})\cdot\boldsymbol{R}^T &= \varphi_0(I_{\boldsymbol{\sigma}})\;\boldsymbol{R}\cdot\boldsymbol{R}^T + \varphi_1(I_{\boldsymbol{\sigma}})\;\boldsymbol{R}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{R}^T + \varphi_2(I_{\boldsymbol{\sigma}})\;\boldsymbol{R}\cdot\boldsymbol{\sigma}^2\cdot\boldsymbol{R}^T \\ &= \varphi_0(I_{\boldsymbol{\sigma}})\;\boldsymbol{1} + \varphi_1(I_{\boldsymbol{\sigma}})\;\boldsymbol{R}\cdot\boldsymbol{R}^T\cdot\boldsymbol{B}\cdot\boldsymbol{R}\cdot\boldsymbol{R}^T + \varphi_2(I_{\boldsymbol{\sigma}})\;\boldsymbol{R}\cdot\boldsymbol{R}^T\cdot\boldsymbol{B}\cdot\boldsymbol{R}\cdot\boldsymbol{R}^T\cdot\boldsymbol{B}\cdot\boldsymbol{R}\cdot\boldsymbol{R}^T \\ &= \varphi_0(I_{\boldsymbol{\sigma}})\;\boldsymbol{1} + \varphi_1(I_{\boldsymbol{\sigma}})\;\boldsymbol{B} + \varphi_2(I_{\boldsymbol{\sigma}})\;\boldsymbol{B}^2\,. \end{aligned} \tag{384}$$

Since the invariants of $\boldsymbol{\sigma}$ are isotropic functions, we have

$$I_{\boldsymbol{\sigma}} = I_{\boldsymbol{B}}\,. \tag{385}$$

Therefore,

$$\boldsymbol{R}\cdot\widehat{\boldsymbol{d}}(\boldsymbol{\sigma})\cdot\boldsymbol{R}^T = \varphi_0(I_{\boldsymbol{B}})\;\boldsymbol{1} + \varphi_1(I_{\boldsymbol{B}})\;\boldsymbol{B} + \varphi_2(I_{\boldsymbol{B}})\;\boldsymbol{B}^2 = \widehat{\boldsymbol{d}}(\boldsymbol{B}) = \widehat{\boldsymbol{d}}(\boldsymbol{R}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{R}^T)\,. \tag{386}$$

This implies that the function $\widehat{\boldsymbol{d}}(\boldsymbol{\sigma})$ is isotropic.

We prove the converse by using the spectral representation of $\boldsymbol{\sigma}$. Special treatments are needed for the cases where eigenvalues are repeated. Let us examine the case where the eigenvalues are distinct (for the other cases consult [Gur81], p. 234).

Let the spectral representation of $\boldsymbol{\sigma}$ be

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \, \mathbf{n}_i \otimes \mathbf{n}_i \; . \tag{387}$$

Then, assuming that the function $\widehat{\boldsymbol{d}}$ is isotropic, we have

$$\widehat{\boldsymbol{d}}(\boldsymbol{\sigma}) = \sum_{i=1}^{3} d_i \, \mathbf{n}_i \otimes \mathbf{n}_i \; . \tag{388}$$

Since the eigenvalues are distinct, we have

$$\text{span}(\boldsymbol{1}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^2) = \text{span}(\mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2, \mathbf{n}_3 \otimes \mathbf{n}_3) \; . \tag{389}$$

Hence we have the alternative representation of $\widehat{\boldsymbol{d}}(\boldsymbol{\sigma})$ as a sum of the three linearly independent bases, i.e.,

$$\widehat{\boldsymbol{d}}(\boldsymbol{\sigma}) = \alpha_0(\boldsymbol{\sigma}) \; \boldsymbol{1} + \alpha_1(\boldsymbol{\sigma}) \; \boldsymbol{\sigma} + \alpha_2(\boldsymbol{\sigma}) \; \boldsymbol{\sigma}^2 \tag{390}$$

where $\alpha_1, \alpha_2, \alpha_3$ are scalar valued functions of $\boldsymbol{\sigma}$ and hence of the invariants of $\boldsymbol{\sigma}$.

16. Let $\boldsymbol{A}$ be a second order tensor and let I_1, I_2, I_3 be its principal invariants. Show that

$$\frac{\partial I_1}{\partial \boldsymbol{A}} = \boldsymbol{1} \; ; \;\; \frac{\partial I_2}{\partial \boldsymbol{A}} = I_1 \; \boldsymbol{1} - \boldsymbol{A}^T \; ; \;\; \frac{\partial I_3}{\partial \boldsymbol{A}} = (\boldsymbol{A}^2 - I_1 \; \boldsymbol{A} + I_2 \; \boldsymbol{1})^T \; . \tag{391}$$

The derivative of a scalar valued function $\phi(\boldsymbol{A})$ of a second order tensor $\boldsymbol{A}$ can be defined via the directional derivative using

$$\frac{\partial \phi}{\partial \boldsymbol{A}} : \boldsymbol{B} = \left. \frac{d}{ds} \phi(\boldsymbol{A} + s \; \boldsymbol{B}) \right|_{s=0} \tag{392}$$

where $\boldsymbol{B}$ is an arbitrary second order tensor.

The invariant I_3 is given by

$$I_3(\boldsymbol{A}) = \det(\boldsymbol{A}) \; . \tag{393}$$

Therefore, from the definition of the derivative,

$$\begin{aligned} \frac{\partial I_3}{\partial \boldsymbol{A}} : \boldsymbol{B} &= \left. \frac{d}{ds} \det(\boldsymbol{A} + s \; \boldsymbol{B}) \right|_{s=0} \\ &= \left. \frac{d}{ds} \det \left[s \; \boldsymbol{A} \left(\frac{1}{s} \; \boldsymbol{1} + \boldsymbol{A}^{-1} \cdot \boldsymbol{B} \right) \right] \right|_{s=0} \\ &= \left. \frac{d}{ds} \left[s^3 \; \det(\boldsymbol{A}) \; \det \left(\frac{1}{s} \; \boldsymbol{1} + \boldsymbol{A}^{-1} \cdot \boldsymbol{B} \right) \right] \right|_{s=0} \; . \end{aligned} \tag{394}$$

Recall that we can expand the determinant of a tensor in the form of a characteristic equation in terms of the invariants I_1, I_2, I_3 using

$$\det(\lambda \; \boldsymbol{1} + \boldsymbol{A}) = \lambda^3 + I_1(\boldsymbol{A}) \; \lambda^2 + I_2(\boldsymbol{A}) \; \lambda + I_3(\boldsymbol{A}) \; . \tag{395}$$

Using this expansion we can write

$$\begin{aligned} \frac{\partial I_3}{\partial \boldsymbol{A}} : \boldsymbol{B} &= \left. \frac{d}{ds} \left[s^3 \; \det(\boldsymbol{A}) \; \left(\frac{1}{s^3} + I_1(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; \frac{1}{s^2} + I_2(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; \frac{1}{s} + I_3(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \right) \right] \right|_{s=0} \\ &= \det(\boldsymbol{A}) \; \frac{d}{ds} \left. \left[1 + I_1(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; s + I_2(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; s^2 + I_3(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; s^3 \right] \right|_{s=0} \\ &= \det(\boldsymbol{A}) \; \left. \left[I_1(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) + 2 \; I_2(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; s + 3 \; I_3(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; s^2 \right] \right|_{s=0} \\ &= \det(\boldsymbol{A}) \; I_1(\boldsymbol{A}^{-1} \cdot \boldsymbol{B}) \; . \end{aligned} \tag{396}$$

Recall that the invariant I_1 is given by

$$I_1(\boldsymbol{A}) = \text{tr}\,(\boldsymbol{A}) \; . \tag{397}$$

Hence,

$$\frac{\partial I_3}{\partial \boldsymbol{A}} : \boldsymbol{B} = \det(\boldsymbol{A}) \; \text{tr} \left(\boldsymbol{A}^{-1} \cdot \boldsymbol{B} \right) = \det(\boldsymbol{A}) \; [\boldsymbol{A}^{-1}]^T : \boldsymbol{B} \; . \tag{398}$$

Invoking the arbitrariness of $\boldsymbol{B}$ we then have

$$\boxed{\frac{\partial I_3}{\partial \boldsymbol{A}} = \det(\boldsymbol{A})\,[\boldsymbol{A}^{-1}]^T\,.} \tag{399}$$

In an orthonormal basis the components of $\boldsymbol{A}$ can be written as a matrix A. In that case, the right hand side corresponds the cofactors of the matrix.

For the derivatives of the other two invariants, let us go back to the characteristic equation

$$\det(\lambda\,\boldsymbol{1} + \boldsymbol{A}) = \lambda^3 + I_1(\boldsymbol{A})\,\lambda^2 + I_2(\boldsymbol{A})\,\lambda + I_3(\boldsymbol{A})\,. \tag{400}$$

Using the same approach as before, we can show that

$$\frac{\partial}{\partial \boldsymbol{A}}\det(\lambda\,\boldsymbol{1} + \boldsymbol{A}) = \det(\lambda\,\boldsymbol{1} + \boldsymbol{A})\,[(\lambda\,\boldsymbol{1} + \boldsymbol{A})^{-1}]^T\,. \tag{401}$$

Now the left hand side can be expanded as

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{A}}\det(\lambda\,\boldsymbol{1} + \boldsymbol{A}) &= \frac{\partial}{\partial \boldsymbol{A}}\left[\lambda^3 + I_1(\boldsymbol{A})\,\lambda^2 + I_2(\boldsymbol{A})\,\lambda + I_3(\boldsymbol{A})\right]\\ &= \frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^2 + \frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda + \frac{\partial I_3}{\partial \boldsymbol{A}}\,.\end{aligned} \tag{402}$$

Hence

$$\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^2 + \frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda + \frac{\partial I_3}{\partial \boldsymbol{A}} = \det(\lambda\,\boldsymbol{1} + \boldsymbol{A})\,[(\lambda\,\boldsymbol{1} + \boldsymbol{A})^{-1}]^T \tag{403}$$

or,

$$(\lambda\,\boldsymbol{1} + \boldsymbol{A})^T\cdot\left[\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^2 + \frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda + \frac{\partial I_3}{\partial \boldsymbol{A}}\right] = \det(\lambda\,\boldsymbol{1} + \boldsymbol{A})\,\boldsymbol{1}\,. \tag{404}$$

Expanding the right hand side and separating terms on the left hand side gives

$$(\lambda\,\boldsymbol{1} + \boldsymbol{A}^T)\cdot\left[\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^2 + \frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda + \frac{\partial I_3}{\partial \boldsymbol{A}}\right] = \left[\lambda^3 + I_1\,\lambda^2 + I_2\,\lambda + I_3\right]\boldsymbol{1} \tag{405}$$

or,

$$\left[\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^3 + \frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda^2 + \frac{\partial I_3}{\partial \boldsymbol{A}}\,\lambda\right]\boldsymbol{1} + +\boldsymbol{A}^T\cdot\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^2 + \boldsymbol{A}^T\cdot\frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda + \boldsymbol{A}^T\cdot\frac{\partial I_3}{\partial \boldsymbol{A}} = \left[\lambda^3 + I_1\,\lambda^2 + I_2\,\lambda + I_3\right]\boldsymbol{1}\,. \tag{406}$$

If we define $I_0 := 1$ and $I_4 := 0$, we can write the above as

$$\left[\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^3 + \frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda^2 + \frac{\partial I_3}{\partial \boldsymbol{A}}\,\lambda + \frac{\partial I_4}{\partial \boldsymbol{A}}\right]\boldsymbol{1} + \boldsymbol{A}^T\cdot\frac{\partial I_0}{\partial \boldsymbol{A}}\,\lambda^3 + \boldsymbol{A}^T\cdot\frac{\partial I_1}{\partial \boldsymbol{A}}\,\lambda^2 + \boldsymbol{A}^T\cdot\frac{\partial I_2}{\partial \boldsymbol{A}}\,\lambda + \boldsymbol{A}^T\cdot\frac{\partial I_3}{\partial \boldsymbol{A}} = \left[I_0\,\lambda^3 + I_1\,\lambda^2 + I_2\,\lambda + I_3\right]\boldsymbol{1}\,. \tag{407}$$

Collecting terms containing various powers of λ, we get

$$\begin{aligned}\lambda^3\left(I_0\,\boldsymbol{1} - \frac{\partial I_1}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_0}{\partial \boldsymbol{A}}\right) + \lambda^2\left(I_1\,\boldsymbol{1} - \frac{\partial I_2}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_1}{\partial \boldsymbol{A}}\right) +\\ \lambda\left(I_2\,\boldsymbol{1} - \frac{\partial I_3}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_2}{\partial \boldsymbol{A}}\right) + \left(I_3\,\boldsymbol{1} - \frac{\partial I_4}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_3}{\partial \boldsymbol{A}}\right) = 0\,.\end{aligned} \tag{408}$$

Then, invoking the arbitrariness of λ, we have

$$\begin{aligned}I_0\,\boldsymbol{1} - \frac{\partial I_1}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_0}{\partial \boldsymbol{A}} &= 0\\ I_1\,\boldsymbol{1} - \frac{\partial I_2}{\partial \boldsymbol{A}}\,\boldsymbol{1} - I_2\,\boldsymbol{1} - \frac{\partial I_3}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_2}{\partial \boldsymbol{A}} &= 0\\ I_3\,\boldsymbol{1} - \frac{\partial I_4}{\partial \boldsymbol{A}}\,\boldsymbol{1} - \boldsymbol{A}^T\cdot\frac{\partial I_3}{\partial \boldsymbol{A}} &= 0\,.\end{aligned} \tag{409}$$

This implies that

$$\boxed{\frac{\partial I_1}{\partial \boldsymbol{A}} = \boldsymbol{1}\,;\ \ \frac{\partial I_2}{\partial \boldsymbol{A}} = I_1\,\boldsymbol{1} - \boldsymbol{A}^T\,;\ \ \frac{\partial I_3}{\partial \boldsymbol{A}} = I_2\,\boldsymbol{1} - \boldsymbol{A}^T\,(I_1\,\boldsymbol{1} - \boldsymbol{A}^T) = (\boldsymbol{A}^2 - I_1\,\boldsymbol{A} + I_2\,\boldsymbol{1})^T\,.} \tag{410}$$

Other interesting relations that can be inferred based on the above are

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\left[\boldsymbol{A}^2 - I_1\,\boldsymbol{A} + I_2\,\boldsymbol{1}\right] \tag{411}$$

and

$$\frac{\partial I_3}{\partial \boldsymbol{A}} = I_3\,[\boldsymbol{A}^T]^{-1}\,. \tag{412}$$

Recall that

$$p = -\frac{1}{3}\,\text{tr}\,(\boldsymbol{\sigma}) = -\frac{1}{3}\,I_1(\boldsymbol{\sigma})\,. \tag{413}$$

Therefore,

$$\frac{\partial p}{\partial \boldsymbol{\sigma}} = -\frac{1}{3}\,\frac{\partial I_1(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = -\frac{1}{3}\,\boldsymbol{\mathit{1}}\,. \tag{414}$$

Also recall that

$$J_2 = \frac{1}{2}\,\text{tr}\left(\boldsymbol{s}^2\right) = -\frac{1}{2}\,\left[\{\text{tr}\,(\boldsymbol{s})\}^2 - \text{tr}\left(\boldsymbol{s}^2\right)\right] = -I_2(\boldsymbol{s})\,. \tag{415}$$

Hence,

$$\frac{\partial J_2}{\partial \boldsymbol{s}} = -\frac{\partial I_2(\boldsymbol{s})}{\partial \boldsymbol{s}} = -I_1(\boldsymbol{s})\,\boldsymbol{\mathit{1}} + \boldsymbol{s}^T = -\cancelto{0}{\text{tr}\,(\boldsymbol{s})}\,\boldsymbol{\mathit{1}} + \boldsymbol{s} = \boldsymbol{s}\,. \tag{416}$$

Finally,

$$J_3 = \det(\boldsymbol{s}) = I_3(\boldsymbol{s})\,. \tag{417}$$

Therefore,

$$\frac{\partial J_3}{\partial \boldsymbol{s}} = \frac{\partial I_3(\boldsymbol{s})}{\partial \boldsymbol{s}} = [\boldsymbol{s}^2 - I_1(\boldsymbol{s})\,\boldsymbol{s} + I_2(\boldsymbol{s})\,\boldsymbol{\mathit{1}}]^T = \boldsymbol{s}^2 - \frac{1}{2}\,\text{tr}\left(\boldsymbol{s}^2\right)\,\boldsymbol{\mathit{1}}\,. \tag{418}$$

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