Basic Thermoelasticity

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Abstract

This paper explains the basics of nonlinear thermoelasticity. The governing balance equations are derived and several useful identities are explained.

1 Introduction

The purpose of these notes is to explain the basis of nonlinear thermoelasticity in modern notation.

2 Governing Equations

The equations that govern the motion of a thermoelastic solid include the balance laws for mass, momentum, and energy. Kinematic equations and constitutive relations are needed to complete the system of equations. Physical restrictions on the form of the constitutive relations are imposed by an entropy inequality that expresses the second law of thermodynamics in mathematical form.

The balance laws express the idea that the rate of change of a quantity (mass, momentum, energy) in a volume must arise from three causes:

1. the physical quantity itself flows through the surface that bounds the volume,
2. there is a source of the physical quantity on the surface of the volume, or/and,
3. there is a source of the physical quantity inside the volume.

Let Ω be the body (an open subset of Euclidean space) and let $\partial\Omega$ be its surface (the boundary of Ω).

Let the motion of material points in the body be described by the map

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) \tag{1}$$

where $\mathbf{X}$ is the position of a point in the initial configuration and $\mathbf{x}$ is the location of the same point in the deformed configuration. The deformation gradient ($\boldsymbol{F}$) is given by

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \boldsymbol{\nabla}_0\, \mathbf{x}\,. \tag{2}$$

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2.1 Balance Laws

Let $f(\mathbf{x},t)$ be a physical quantity that is flowing through the body. Let $g(\mathbf{x},t)$ be sources on the surface of the body and let $h(\mathbf{x},t)$ be sources inside the body. Let $\mathbf{n}(\mathbf{x},t)$ be the outward unit normal to the surface $\partial\Omega$. Let $\mathbf{v}(\mathbf{x},t)$ be the velocity of the physical particles that carry the physical quantity that is flowing. Also, let the speed at which the bounding surface $\partial\Omega$ is moving be u_n (in the direction $\mathbf{n}$).

Then, balance laws can be expressed in the general form ([1])

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\ \text{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\ \text{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\ \text{dA} + \int_\Omega h(\mathbf{x},t)\ \text{dV}\ . \tag{3}$$

Note that the functions $f(\mathbf{x},t)$, $g(\mathbf{x},t)$, and $h(\mathbf{x},t)$ can be scalar valued, vector valued, or tensor valued - depending on the physical quantity that the balance equation deals with.

It can be shown that the balance laws of mass, momentum, and energy can be written as (see Appendix):

$$\boxed{\begin{aligned} \dot\rho + \rho\ \boldsymbol{\nabla}\cdot\mathbf{v} &= 0 && \text{Balance of Mass}\\ \rho\ \dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\ \mathbf{b} &= 0 && \text{Balance of Linear Momentum}\\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}^T && \text{Balance of Angular Momentum}\\ \rho\ \dot e - \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\ s &= 0 && \text{Balance of Energy.}\end{aligned}} \tag{4}$$

In the above equations $\rho(\mathbf{x},t)$ is the mass density (current), $\dot\rho$ is the material time derivative of ρ, $\mathbf{v}(\mathbf{x},t)$ is the particle velocity, $\dot{\mathbf{v}}$ is the material time derivative of $\mathbf{v}$, $\boldsymbol{\sigma}(\mathbf{x},t)$ is the Cauchy stress tensor, $\mathbf{b}(\mathbf{x},t)$ is the body force density, $e(\mathbf{x},t)$ is the internal energy per unit mass, $\dot e$ is the material time derivative of e, $\mathbf{q}(\mathbf{x},t)$ is the heat flux vector, and $s(\mathbf{x},t)$ is an energy source per unit mass.

With respect to the reference configuration, the balance laws can be written as

$$\boxed{\begin{aligned} \rho\ \det(\boldsymbol{F}) - \rho_0 &= 0 && \text{Balance of Mass}\\ \rho_0\ \ddot{\mathbf{x}} - \boldsymbol{\nabla}_0\cdot\boldsymbol{P}^T - \rho_0\ \mathbf{b} &= 0 && \text{Balance of Linear Momentum}\\ \boldsymbol{F}\cdot\boldsymbol{P} &= \boldsymbol{P}^T\cdot\boldsymbol{F}^T && \text{Balance of Angular Momentum}\\ \rho_0\ \dot e - \boldsymbol{P}^T : \dot{\boldsymbol{F}} + \boldsymbol{\nabla}_0\cdot\mathbf{q} - \rho_0\ s &= 0 && \text{Balance of Energy.}\end{aligned}} \tag{5}$$

In the above, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, and ρ_0 is the mass density in the reference configuration. The first Piola-Kirchhoff stress tensor is related to the Cauchy stress tensor by

$$\boldsymbol{P} = \det(\boldsymbol{F})\ \boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\ . \tag{6}$$

We can alternatively define the nominal stress tensor $\boldsymbol{N}$ which is the transpose of the first Piola-Kirchhoff stress tensor such that

$$\boldsymbol{N} := \boldsymbol{P}^T = \det(\boldsymbol{F})\ (\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma})^T = \det(\boldsymbol{F})\ \boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\ . \tag{7}$$

Then the balance laws become

$$\boxed{\begin{aligned} \rho\ \det(\boldsymbol{F}) - \rho_0 &= 0 && \text{Balance of Mass}\\ \rho_0\ \ddot{\mathbf{x}} - \boldsymbol{\nabla}_0\cdot\boldsymbol{N} - \rho_0\ \mathbf{b} &= 0 && \text{Balance of Linear Momentum}\\ \boldsymbol{F}\cdot\boldsymbol{N}^T &= \boldsymbol{N}\cdot\boldsymbol{F}^T && \text{Balance of Angular Momentum}\\ \rho_0\ \dot e - \boldsymbol{N} : \dot{\boldsymbol{F}} + \boldsymbol{\nabla}_0\cdot\mathbf{q} - \rho_0\ s &= 0 && \text{Balance of Energy.}\end{aligned}} \tag{8}$$

The gradient and divergence operators are defined such that

$$\boldsymbol{\nabla}\mathbf{v} = \sum_{i,j=1}^{3}\frac{\partial v_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j = v_{i,j}\mathbf{e}_i\otimes\mathbf{e}_j\ ;\ \ \boldsymbol{\nabla}\cdot\mathbf{v} = \sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i} = v_{i,i}\ ;\ \ \boldsymbol{\nabla}\cdot\boldsymbol{S} = \sum_{i,j=1}^{3}\frac{\partial S_{ij}}{\partial x_j}\ \mathbf{e}_i = \sigma_{ij,j}\ \mathbf{e}_i\ . \tag{9}$$

where $\mathbf{v}$ is a vector field, BS is a second-order tensor field, and $\mathbf{e}_i$ are the components of an orthonormal basis in the current configuration. Also,

$$\boldsymbol{\nabla}_0\,\mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial X_j}\mathbf{E}_i\otimes\mathbf{E}_j = v_{i,j}\mathbf{E}_i\otimes\mathbf{E}_j\;;\;\; \boldsymbol{\nabla}_0\cdot\mathbf{v} = \sum_{i=1}^{3}\frac{\partial v_i}{\partial X_i} = v_{i,i}\;;\;\; \boldsymbol{\nabla}_0\cdot\boldsymbol{S} = \sum_{i,j=1}^{3}\frac{\partial S_{ij}}{\partial X_j}\,\mathbf{E}_i = S_{ij,j}\,\mathbf{E}_i \quad (10)$$

where $\mathbf{v}$ is a vector field, BS is a second-order tensor field, and $\mathbf{E}_i$ are the components of an orthonormal basis in the reference configuration.

The contraction operation is defined as

$$\boldsymbol{A}:\boldsymbol{B} = \sum_{i,j=1}^{3} A_{ij}\,B_{ij} = A_{ij}\,B_{ij}\,. \quad (11)$$

2.2 The Clausius-Duhem Inequality

The Clausius-Duhem inequality can be used to express the second law of thermodynamics for elastic-plastic materials. This inequality is a statement concerning the irreversibility of natural processes, especially when energy dissipation is involved.

Just like in the balance laws in the previous section, we assume that there is a flux of a quantity, a source of the quantity, and an internal density of the quantity per unit mass. The quantity of interest in this case is the entropy. Thus, we assume that there is an entropy flux, an entropy source, and an internal entropy density per unit mass (η) in the region of interest.

Let Ω be such a region and let $\partial\Omega$ be its boundary. Then the second law of thermodynamics states that the rate of increase of η in this region is greater than or equal to the sum of that supplied to Ω (as a flux or from internal sources) and the change of the internal entropy density due to material flowing in and out of the region.

Let $\partial\Omega$ move with a velocity u_n and let particles inside Ω have velocities $\mathbf{v}$. Let $\mathbf{n}$ be the unit outward normal to the surface $\partial\Omega$. Let ρ be the density of matter in the region, $\bar{q}$ be the entropy flux at the surface, and r be the entropy source per unit mass. Then (see [1, 2]), the entropy inequality may be written as

$$\frac{d}{dt}\left(\int_\Omega \rho\,\eta\,\mathrm{dV}\right) \geq \int_{\partial\Omega}\rho\,\eta\,(u_n - \mathbf{v}\cdot\mathbf{n})\,\mathrm{dA} + \int_{\partial\Omega}\bar{q}\,\mathrm{dA} + \int_\Omega \rho\,r\,\mathrm{dV}\,. \quad (12)$$

The scalar entropy flux can be related to the vector flux at the surface by the relation $\bar{q} = -\boldsymbol{\psi}(\mathbf{x})\cdot\mathbf{n}$. Under the assumption of incrementally isothermal conditions (see [3] for a detailed discussion of the assumptions involved), we have

$$\boldsymbol{\psi}(\mathbf{x}) = \frac{\mathbf{q}(\mathbf{x})}{T}\,;\;\; r = \frac{s}{T}$$

where $\mathbf{q}$ is the heat flux vector, s is a energy source per unit mass, and T is the absolute temperature of a material point at $\mathbf{x}$ at time t.

We then have the Clausius-Duhem inequality in integral form:

$$\boxed{\frac{d}{dt}\left(\int_\Omega \rho\,\eta\,\mathrm{dV}\right) \geq \int_{\partial\Omega}\rho\,\eta\,(u_n - \mathbf{v}\cdot\mathbf{n})\,\mathrm{dA} - \int_{\partial\Omega}\frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{dA} + \int_\Omega \frac{\rho\,s}{T}\,\mathrm{dV}\,.} \quad (13)$$

We can show that (see Appendix) the entropy inequality may be written in differential form as

$$\boxed{\rho\,\dot{\eta} \geq -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\,s}{T}\,.} \quad (14)$$

In terms of the Cauchy stress and the internal energy, the Clausius-Duhem inequality may be written as (see Appendix)

$$\boxed{\rho\,(\dot{e} - T\,\dot{\eta}) - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \leq -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.} \quad (15)$$

2.3 Constitutive Relations

A set of constitutive equations is required to close to system of balance laws. For large deformation plasticity, we have to define appropriate kinematic quantities and stress measures so that constitutive relations between them may have a physical meaning.

Let the fundamental kinematic quantity be the deformation gradient ($\boldsymbol{F}$) which is given by

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \boldsymbol{\nabla}_0 \, \mathbf{x} \; ; \quad \det \boldsymbol{F} > 0 \; .$$

A thermoelastic material is one in which the internal energy (e) is a function only of $\boldsymbol{F}$ and the specific entropy (η), that is

$$e = \bar{e}(\boldsymbol{F}, \eta) \; .$$

For a thermoelastic material, we can show that the entropy inequality can be written as (see Appendix)

$$\boxed{\rho \left(\frac{\partial \bar{e}}{\partial \eta} - T\right) \dot{\eta} + \left(\rho \frac{\partial \bar{e}}{\partial \boldsymbol{F}} - \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\right) : \dot{\boldsymbol{F}} + \frac{\mathbf{q} \cdot \boldsymbol{\nabla} T}{T} \le 0 \; .} \tag{16}$$

At this stage, we make the following constitutive assumptions:

1. Like the internal energy, we assume that $\boldsymbol{\sigma}$ and T are also functions only of $\boldsymbol{F}$ and η, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{F}, \eta) \; ; \; \; T = T(\boldsymbol{F}, \eta) \; .$$

2. The heat flux $\mathbf{q}$ satisfies the thermal conductivity inequality and if $\mathbf{q}$ is independent of $\dot{\eta}$ and $\dot{\boldsymbol{F}}$, we have

$$\mathbf{q} \cdot \boldsymbol{\nabla} T \le 0 \qquad \Longrightarrow \qquad -(\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) \cdot \boldsymbol{\nabla} T \le 0 \qquad \Longrightarrow \qquad \boldsymbol{\kappa} \ge \boldsymbol{0}$$

   i.e., the thermal conductivity $\boldsymbol{\kappa}$ is positive semidefinite.

Therefore, the entropy inequality may be written as

$$\rho \left(\frac{\partial \bar{e}}{\partial \eta} - T\right) \dot{\eta} + \left(\rho \frac{\partial \bar{e}}{\partial \boldsymbol{F}} - \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\right) : \dot{\boldsymbol{F}} \le 0 \; .$$

Since $\dot{\eta}$ and $\dot{\boldsymbol{F}}$ are arbitrary, the entropy inequality will be satisfied if and only if

$$\frac{\partial \bar{e}}{\partial \eta} - T = 0 \implies T = \frac{\partial \bar{e}}{\partial \eta} \qquad \text{and} \qquad \rho \frac{\partial \bar{e}}{\partial \boldsymbol{F}} - \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \boldsymbol{0} \implies \boldsymbol{\sigma} = \rho \frac{\partial \bar{e}}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^T \; .$$

Therefore,

$$\boxed{T = \frac{\partial \bar{e}}{\partial \eta}} \qquad \text{and} \qquad \boxed{\boldsymbol{\sigma} = \rho \frac{\partial \bar{e}}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^T \; .} \tag{17}$$

Given the above relations, the energy equation may expressed in terms of the specific entropy as (see Appendix)

$$\boxed{\rho \, T \, \dot{\eta} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \rho \, s \; .} \tag{18}$$

- **Effect of a rigid body rotation of the internal energy:**
  If a thermoelastic body is subjected to a rigid body rotation $\boldsymbol{Q}$, then its internal energy should not change. After a rotation, the new deformation gradient ($\hat{\boldsymbol{F}}$) is given by

$$\hat{\boldsymbol{F}} = \boldsymbol{Q} \cdot \boldsymbol{F} \; .$$

  Since the internal energy does not change, we must have

$$e = \bar{e}(\hat{\boldsymbol{F}}, \eta) = \bar{e}(\boldsymbol{F}, \eta) \; .$$

  Now, from the polar decomposition theorem, $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$ where $\boldsymbol{R}$ is the orthogonal rotation tensor (i.e., $\boldsymbol{R} \cdot \boldsymbol{R}^T = \boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{1}$) and $\boldsymbol{U}$ is the symmetric right stretch tensor. Therefore,

$$\bar{e}(\boldsymbol{Q} \cdot \boldsymbol{R} \cdot \boldsymbol{U}, \eta) = \bar{e}(\boldsymbol{F}, \eta) \; .$$

  Now, we can choose any rotation $\boldsymbol{Q}$. In particular, if we choose $\boldsymbol{Q} = \boldsymbol{R}^T$, we have

$$\bar{e}(\boldsymbol{R}^T \cdot \boldsymbol{R} \cdot \boldsymbol{U}, \eta) = \bar{e}(\boldsymbol{1} \cdot \boldsymbol{U}, \eta) = \tilde{e}(\boldsymbol{U}, \eta) \; .$$

  Therefore,

$$\bar{e}(\boldsymbol{U}, \eta) = \bar{e}(\boldsymbol{F}, \eta) \; .$$

  This means that *the internal energy depends only on the stretch* $\boldsymbol{U}$ *and not on the orientation of the body.*

2.3.1 Other strain and stress measures

The internal energy depends on $\boldsymbol{F}$ only through the stretch $\boldsymbol{U}$. A strain measure that reflects this fact and also vanishes in the reference configuration is the Green strain

$$\boxed{\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{1}) = \frac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{1})\,.} \tag{19}$$

Recall that the Cauchy stress is given by

$$\boldsymbol{\sigma} = \rho\,\frac{\partial \bar{e}}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^T\,.$$

We can show that the Cauchy stress can be expressed in terms of the Green strain as (see Appendix)

$$\boxed{\boldsymbol{\sigma} = \rho\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T\,.} \tag{20}$$

Recall that the nominal stress tensor is defined as

$$\boldsymbol{N} = \det \boldsymbol{F}\,(\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T})\,.$$

From the conservation of mass, we have $\rho_0 = \rho\ \det \boldsymbol{F}$. Hence,

$$\boxed{\boldsymbol{N} = \frac{\rho_0}{\rho}\,\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\,.} \tag{21}$$

The nominal stress is unsymmetric. We can define a symmetric stress measure with respect to the reference configuration call the second Piola-Kirchhoff stress tensor ($\boldsymbol{S}$):

$$\boxed{\boldsymbol{S} := \boldsymbol{F}^{-1} \cdot \boldsymbol{N} = \boldsymbol{P} \cdot \boldsymbol{F}^{-T} = \frac{\rho_0}{\rho}\,\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\,.} \tag{22}$$

In terms of the derivatives of the internal energy, we have

$$\boldsymbol{S} = \frac{\rho_0}{\rho}\,\boldsymbol{F}^{-1} \cdot \left(\rho\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T\right) \cdot \boldsymbol{F}^{-T} = \rho_0\,\frac{\partial \bar{e}}{\partial \boldsymbol{E}}$$

and

$$\boldsymbol{N} = \frac{\rho_0}{\rho}\,\left(\rho\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T\right) \cdot \boldsymbol{F}^{-T} = \rho_0\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}}\,.$$

That is,

$$\boxed{\boldsymbol{S} = \rho_0\,\frac{\partial \bar{e}}{\partial \boldsymbol{E}}} \quad \text{and} \quad \boxed{\boldsymbol{N} = \rho_0\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}}}\,. \tag{23}$$

2.3.2 Stress Power

The stress power per unit volume is given by $\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v}$. In terms of the stress measures in the reference configuration, we have

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \left(\rho\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T\right) : (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})\,.$$

Using the identity $\boldsymbol{A} : (\boldsymbol{B} \cdot \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C}^T) : \boldsymbol{B}$, we have

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \left[\left(\rho\,\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T\right) \cdot \boldsymbol{F}^{-T}\right] : \dot{\boldsymbol{F}} = \rho\,\left(\boldsymbol{F} \cdot \frac{\partial \bar{e}}{\partial \boldsymbol{E}}\right) : \dot{\boldsymbol{F}} = \frac{\rho}{\rho_0}\,\boldsymbol{N} : \dot{\boldsymbol{F}}\,.$$

We can alternatively express the stress power in terms of $\boldsymbol{S}$ and $\dot{\boldsymbol{E}}$. Taking the material time derivative of $\boldsymbol{E}$ we have

$$\dot{\boldsymbol{E}} = \frac{1}{2}(\dot{\boldsymbol{F}^T} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}) .$$

Therefore,

$$\boldsymbol{S} : \dot{\boldsymbol{E}} = \frac{1}{2}[\boldsymbol{S} : (\dot{\boldsymbol{F}^T} \cdot \boldsymbol{F}) + \boldsymbol{S} : (\boldsymbol{F}^T \cdot \dot{\boldsymbol{F}})] .$$

Using the identities $\boldsymbol{A} : (\boldsymbol{B} \cdot \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C}^T) : \boldsymbol{B} = (\boldsymbol{B}^T \cdot \boldsymbol{A}) : \boldsymbol{C}$ and $\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{A}^T : \boldsymbol{B}^T$ and using the symmetry of $\boldsymbol{S}$, we have

$$\boldsymbol{S} : \dot{\boldsymbol{E}} = \frac{1}{2}[(\boldsymbol{S} \cdot \boldsymbol{F}^T) : \dot{\boldsymbol{F}}^T + (\boldsymbol{F} \cdot \boldsymbol{S}) : \dot{\boldsymbol{F}}] = \frac{1}{2}[(\boldsymbol{F} \cdot \boldsymbol{S}^T) : \dot{\boldsymbol{F}} + (\boldsymbol{F} \cdot \boldsymbol{S}) : \dot{\boldsymbol{F}}] = (\boldsymbol{F} \cdot \boldsymbol{S}) : \dot{\boldsymbol{F}} .$$

Now, $\boldsymbol{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{N}$. Therefore, $\boldsymbol{S} : \dot{\boldsymbol{E}} = \boldsymbol{N} : \dot{\boldsymbol{F}}$. Hence, the stress power can be expressed as

$$\boxed{\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \boldsymbol{N} : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{E}} .} \tag{24}$$

If we split the velocity gradient into symmetric and skew parts using

$$\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w}$$

where $\boldsymbol{d}$ is the rate of deformation tensor and $\boldsymbol{w}$ is the spin tensor, we have

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\sigma} : \boldsymbol{d} + \boldsymbol{\sigma} : \boldsymbol{w} = \text{tr}\left(\boldsymbol{\sigma}^T \cdot \boldsymbol{d}\right) + \text{tr}\left(\boldsymbol{\sigma}^T \cdot \boldsymbol{w}\right) = \text{tr}\left(\boldsymbol{\sigma} \cdot \boldsymbol{d}\right) + \text{tr}\left(\boldsymbol{\sigma} \cdot \boldsymbol{w}\right) .$$

Since $\boldsymbol{\sigma}$ is symmetric and $\boldsymbol{w}$ is skew, we have $\text{tr}\,(\boldsymbol{\sigma} \cdot \boldsymbol{w}) = 0$. Therefore, $\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \text{tr}\,(\boldsymbol{\sigma} \cdot \boldsymbol{d})$. Hence, we may also express the stress power as

$$\boxed{\text{tr}\left(\boldsymbol{\sigma} \cdot \boldsymbol{d}\right) = \text{tr}\left(\boldsymbol{N}^T \cdot \dot{\boldsymbol{F}}\right) = \text{tr}\left(\boldsymbol{S} \cdot \dot{\boldsymbol{E}}\right) .} \tag{25}$$

2.3.3 Helmholtz and Gibbs free energy

Recall that

$$\boldsymbol{S} = \rho_0 \, \frac{\partial \bar{e}}{\partial \boldsymbol{E}} .$$

Therefore,

$$\frac{\partial \bar{e}}{\partial \boldsymbol{E}} = \frac{1}{\rho_0} \boldsymbol{S} .$$

Also recall that

$$\frac{\partial \bar{e}}{\partial \eta} = T .$$

Now, the internal energy $e = \bar{e}(\boldsymbol{E}, \eta)$ is a function only of the Green strain and the specific entropy. Let us assume, that the above relations can be uniquely inverted locally at a material point so that we have

$$\boldsymbol{E} = \tilde{\boldsymbol{E}}(\boldsymbol{S}, T) \qquad \text{and} \qquad \eta = \tilde{\eta}(\boldsymbol{S}, T) .$$

Then the specific internal energy, the specific entropy, and the stress can also be expressed as functions of $\boldsymbol{S}$ and T, or $\boldsymbol{E}$ and T, i.e.,

$$e = \bar{e}(\boldsymbol{E}, \eta) = \tilde{e}(\boldsymbol{S}, T) = \hat{e}(\boldsymbol{E}, T) \; ; \qquad \eta = \tilde{\eta}(\boldsymbol{S}, T) = \hat{\eta}(\boldsymbol{E}, T) \; ; \qquad \text{and} \qquad \boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E}, T)$$

We can show that (see Appendix)

$$\frac{d}{dt}(e - T\,\eta) = -\dot{T}\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S} : \dot{\boldsymbol{E}} \qquad \text{or} \qquad \frac{d\psi}{dt} = -\dot{T}\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S} : \dot{\boldsymbol{E}} . \tag{26}$$

and

$$\frac{d}{dt}(e - T\,\eta - \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E}) = -\dot{T}\,\eta - \frac{1}{\rho_0}\,\dot{\boldsymbol{S}}:\boldsymbol{E} \qquad \text{or} \qquad \frac{dg}{dt} = \dot{T}\,\eta + \frac{1}{\rho_0}\,\dot{\boldsymbol{S}}:\boldsymbol{E}\,. \tag{27}$$

We define the **Helmholtz free energy** as

$$\boxed{\psi = \hat{\psi}(\boldsymbol{E}, T) := e - T\,\eta\,.} \tag{28}$$

We define the **Gibbs free energy** as

$$\boxed{g = \tilde{g}(\boldsymbol{S}, T) := -e + T\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E}\,.} \tag{29}$$

The functions $\hat{\psi}(\boldsymbol{E}, T)$ and $\tilde{g}(\boldsymbol{S}, T)$ are unique. Using these definitions it can be showed that (see Appendix)

$$\frac{\partial\hat{\psi}}{\partial\boldsymbol{E}} = \frac{1}{\rho_0}\,\hat{\boldsymbol{S}}(\boldsymbol{E}, T)\,;\;\; \frac{\partial\hat{\psi}}{\partial T} = -\hat{\eta}(\boldsymbol{E}, T)\,;\;\; \frac{\partial\tilde{g}}{\partial\boldsymbol{S}} = \frac{1}{\rho_0}\,\tilde{\boldsymbol{E}}(\boldsymbol{S}, T)\,;\;\; \frac{\partial\tilde{g}}{\partial T} = \tilde{\eta}(\boldsymbol{S}, T) \tag{30}$$

and

$$\frac{\partial\hat{\boldsymbol{S}}}{\partial T} = -\rho_0\,\frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} \qquad \text{and} \qquad \frac{\partial\tilde{\boldsymbol{E}}}{\partial T} = \rho_0\,\frac{\partial\tilde{\eta}}{\partial\boldsymbol{S}}\,. \tag{31}$$

2.3.4 Specific Heats

The **specific heat at constant strain** (or constant volume) is defined as

$$\boxed{C_v := \frac{\partial\hat{e}(\boldsymbol{E}, T)}{\partial T}\,.} \tag{32}$$

The **specific heat at constant stress** (or constant pressure) is defined as

$$\boxed{C_p := \frac{\partial\tilde{e}(\boldsymbol{S}, T)}{\partial T}\,.} \tag{33}$$

We can show that (see Appendix)

$$C_v = T\,\frac{\partial\hat{\eta}}{\partial T} = -T\,\frac{\partial^2\hat{\psi}}{\partial T^2} \tag{34}$$

and

$$C_p = T\,\frac{\partial\tilde{\eta}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S}:\frac{\partial\tilde{\boldsymbol{E}}}{\partial T} = T\,\frac{\partial^2\tilde{g}}{\partial T^2} + \boldsymbol{S}:\frac{\partial^2\tilde{g}}{\partial\boldsymbol{S}\partial T}\,. \tag{35}$$

Also the equation for the balance of energy can be expressed in terms of the specific heats as (see Appendix)

$$\boxed{\begin{aligned} \rho\,C_v\,\dot{T} &= \boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}T) + \rho\,s + \frac{\rho}{\rho_0}\,T\,\boldsymbol{\beta}_S:\dot{\boldsymbol{E}} \\ \rho\,\left(C_p - \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{\alpha}_E\right)\,\dot{T} &= \boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}T) + \rho\,s - \frac{\rho}{\rho_0}\,T\,\boldsymbol{\alpha}_E:\dot{\boldsymbol{S}} \end{aligned}} \tag{36}$$

where

$$\boldsymbol{\beta}_S := \frac{\partial\hat{\boldsymbol{S}}}{\partial T} \qquad \text{and} \qquad \boldsymbol{\alpha}_E := \frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,. \tag{37}$$

The quantity $\boldsymbol{\beta}_S$ is called the **coefficient of thermal stress** and the quantity $\boldsymbol{\alpha}_E$ is called the **coefficient of thermal expansion**.

The difference between C_p and C_v can be expressed as

$$C_p - C_v = \frac{1}{\rho_0}\left(\boldsymbol{S} - T\,\frac{\partial\hat{\boldsymbol{S}}}{\partial T}\right) : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,. \tag{38}$$

However, it is more common to express the above relation in terms of the elastic modulus tensor as (see Appendix for proof)

$$\boxed{C_p - C_v = \frac{1}{\rho_0}\,\boldsymbol{S} : \boldsymbol{\alpha}_E + \frac{T}{\rho_0}\,\boldsymbol{\alpha}_E : \mathsf{C} : \boldsymbol{\alpha}_E} \tag{39}$$

where the **fourth-order tensor of elastic moduli** is defined as

$$\mathsf{C} := \frac{\partial\hat{\boldsymbol{S}}}{\partial\tilde{\boldsymbol{E}}} = \rho_0\,\frac{\partial^2\hat{\psi}}{\partial\tilde{\boldsymbol{E}}\partial\tilde{\boldsymbol{E}}}\,. \tag{40}$$

For isotropic materials with a constant coefficient of thermal expansion that follow the St. Venant-Kirchhoff material model, we can show that (see Appendix)

$$C_p - C_v = \frac{1}{\rho_0}\left[\alpha\,\text{tr}\,(\boldsymbol{S}) + 9\,\alpha^2\,K\,T\right]\,.$$

3 Appendix

1. The integral

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\,\mathsf{dx}$$

is a function of the parameter t. Show that the derivative of F is given by

$$\frac{dF}{dt} = \frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathsf{dx}\right) = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\mathsf{dx} + f[b(t),t]\,\frac{\partial b(t)}{\partial t} - f[a(t),t]\,\frac{\partial a(t)}{\partial t}\,.$$

This relation is also known as the **Leibniz rule**.

The following proof is taken from [4].

We have,

$$\frac{dF}{dt} = \lim_{\Delta t\to 0}\frac{F(t+\Delta t) - F(t)}{\Delta t}\,.$$

Now,

$$\begin{aligned}
\frac{F(t+\Delta t) - F(t)}{\Delta t} &= \frac{1}{\Delta t}\left[\int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x,t+\Delta t)\,\text{dx} - \int_{a(t)}^{b(t)} f(x,t)\,\text{dx}\right] \\
&\equiv \frac{1}{\Delta t}\left[\int_{a+\Delta a}^{b+\Delta b} f(x,t+\Delta t)\,\text{dx} - \int_a^b f(x,t)\,\text{dx}\right] \\
&= \frac{1}{\Delta t}\left[-\int_a^{a+\Delta a} f(x,t+\Delta t)\,\text{dx} + \int_a^{b+\Delta b} f(x,t+\Delta t)\,\text{dx} - \int_a^b f(x,t)\,\text{dx}\right] \\
&= \frac{1}{\Delta t}\left[-\int_a^{a+\Delta a} f(x,t+\Delta t)\,\text{dx} + \int_a^b f(x,t+\Delta t)\,\text{dx} + \int_b^{b+\Delta b} f(x,t+\Delta t)\,\text{dx} - \int_a^b f(x,t)\,\text{dx}\right] \\
&= \int_a^b \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}\,\text{dx} + \frac{1}{\Delta t}\int_b^{b+\Delta b} f(x,t+\Delta t)\,\text{dx} - \frac{1}{\Delta t}\int_a^{a+\Delta a} f(x,t+\Delta t)\,\text{dx}\,.
\end{aligned}$$

Since $f(x,t)$ is essentially constant over the infinitesimal intervals $a < x < a + \Delta a$ and $b < x < b + \Delta b$, we may write

$$\frac{F(t+\Delta t) - F(t)}{\Delta t} \approx \int_a^b \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}\,\text{dx} + f(b,t+\Delta t)\,\frac{\Delta b}{\Delta t} - f(a,t+\Delta t)\,\frac{\Delta a}{\Delta t}\,.$$

Taking the limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t\to 0}\left[\frac{F(t+\Delta t) - F(t)}{\Delta t}\right] = \lim_{\Delta t\to 0}\left[\int_a^b \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}\,\text{dx}\right] + \lim_{\Delta t\to 0}\left[f(b,t+\Delta t)\,\frac{\Delta b}{\Delta t}\right] - \lim_{\Delta t\to 0}\left[f(a,t+\Delta t)\,\frac{\Delta a}{\Delta t}\right]$$

or,

$$\boxed{\frac{dF(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\text{dx} + f[b(t),t]\,\frac{\partial b(t)}{\partial t} - f[a(t),t]\,\frac{\partial a(t)}{\partial t}\,.} \qquad \square$$

2. Let $\Omega(t)$ be a region in Euclidean space with boundary $\partial\Omega(t)$. Let $\mathbf{x}(t)$ be the positions of points in the region and let $\mathbf{v}(\mathbf{x}, t)$ be the velocity field in the region. Let $\mathbf{n}(\mathbf{x}, t)$ be the outward unit normal to the boundary. Let $\mathbf{f}(\mathbf{x}, t)$ be a vector field in the region (it may also be a scalar field). Show that

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f} \text{ dV}\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t} \text{ dV} + \int_{\partial\Omega(t)} (\mathbf{v} \cdot \mathbf{n})\mathbf{f} \text{ dA} .$$

This relation is also known as the **Reynold's Transport Theorem** and is a generalization of the Leibniz rule.

This proof is taken from [5] (also see [6]).

Let Ω_0 be reference configuration of the region $\Omega(t)$. Let the motion and the deformation gradient be given by

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t) ; \qquad \boldsymbol{F}(\mathbf{X}, t) = \boldsymbol{\nabla}_0 \, \boldsymbol{\varphi} .$$

Let $J(\mathbf{X}, t) = \det[\boldsymbol{F}(\mathbf{x}, t)]$. Then, integrals in the current and the reference configurations are related by

$$\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV} = \int_{\Omega_0} \mathbf{f}[\boldsymbol{\varphi}(\mathbf{X}, t), t] \; J(\mathbf{X}, t) \text{ dV} = \int_{\Omega_0} \hat{\mathbf{f}}(\mathbf{X}, t) \; J(\mathbf{X}, t) \text{ dV} .$$

The time derivative of an integral over a volume is defined as

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV}\right) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{\Omega(t+\Delta t)} \mathbf{f}(\mathbf{x}, t + \Delta t) \text{ dV} - \int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV}\right) .$$

Converting into integrals over the reference configuration, we get

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV}\right) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{\Omega_0} \hat{\mathbf{f}}(\mathbf{X}, t + \Delta t) \; J(\mathbf{X}, t + \Delta t) \text{ dV} - \int_{\Omega_0} \hat{\mathbf{f}}(\mathbf{X}, t) \; J(\mathbf{X}, t) \text{ dV}\right) .$$

Since Ω_0 is independent of time, we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV}\right) &= \int_{\Omega_0} \left[\lim_{\Delta t \to 0} \frac{\hat{\mathbf{f}}(\mathbf{X}, t + \Delta t) \; J(\mathbf{X}, t + \Delta t) - \hat{\mathbf{f}}(\mathbf{X}, t) \; J(\mathbf{X}, t)}{\Delta t}\right] \text{ dV} \\
&= \int_{\Omega_0} \frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X}, t) \; J(\mathbf{X}, t)] \text{ dV} \\
&= \int_{\Omega_0} \left(\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X}, t)] \; J(\mathbf{X}, t) + \hat{\mathbf{f}}(\mathbf{X}, t) \; \frac{\partial}{\partial t}[J(\mathbf{X}, t)]\right) \text{ dV}
\end{aligned}$$

Now, the time derivative of $\det \boldsymbol{F}$ is given by (see [6], p. 77)

$$\frac{\partial J(\mathbf{X}, t)}{\partial t} = \frac{\partial}{\partial t}(\det \boldsymbol{F}) = (\det \boldsymbol{F})(\boldsymbol{\nabla} \cdot \mathbf{v}) = J(\mathbf{X}, t) \; \boldsymbol{\nabla} \cdot \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) = J(\mathbf{X}, t) \; \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}, t) .$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV}\right) &= \int_{\Omega_0} \left(\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X}, t)] \; J(\mathbf{X}, t) + \hat{\mathbf{f}}(\mathbf{X}, t) \; J(\mathbf{X}, t) \; \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}, t)\right) \text{ dV} \\
&= \int_{\Omega_0} \left(\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X}, t)] + \hat{\mathbf{f}}(\mathbf{X}, t) \; \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}, t)\right) \; J(\mathbf{X}, t) \text{ dV} \\
&= \int_{\Omega(t)} \left(\dot{\mathbf{f}}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) \; \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}, t)\right) \text{ dV}
\end{aligned}$$

where $\dot{\mathbf{f}}$ is the material time derivative of $\mathbf{f}$. Now, the material derivative is given by

$$\dot{\mathbf{f}}(\mathbf{x}, t) = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t} + [\boldsymbol{\nabla} \mathbf{f}(\mathbf{x}, t)] \cdot \mathbf{v}(\mathbf{x}, t) .$$

Therefore,

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t) \text{ dV}\right) = \int_{\Omega(t)} \left(\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t} + [\boldsymbol{\nabla} \mathbf{f}(\mathbf{x}, t)] \cdot \mathbf{v}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) \; \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}, t)\right) \text{ dV}$$

or,

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f} \text{ dV}\right) = \int_{\Omega(t)} \left(\frac{\partial \mathbf{f}}{\partial t} + \boldsymbol{\nabla} \mathbf{f} \cdot \mathbf{v} + \mathbf{f} \; \boldsymbol{\nabla} \cdot \mathbf{v}\right) \text{ dV} .$$

Using the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{v} \otimes \mathbf{w}) = \mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{w}) + \boldsymbol{\nabla} \mathbf{v} \cdot \mathbf{w}$$

we then have

$$\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}\ \mathrm{dV}\right) = \int_{\Omega(t)} \left(\frac{\partial \mathbf{f}}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{f}\otimes\mathbf{v})\right)\ \mathrm{dV}\ .$$

Using the divergence theorem and the identity $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{n} = (\mathbf{b}\cdot\mathbf{n})\mathbf{a}$ we have

$$\boxed{\frac{d}{dt}\left(\int_{\Omega(t)} \mathbf{f}\ \mathrm{dV}\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\ \mathrm{dV} + \int_{\partial\Omega(t)} (\mathbf{f}\otimes\mathbf{v})\cdot\mathbf{n}\ \mathrm{dV} = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\ \mathrm{dV} + \int_{\partial\Omega(t)} (\mathbf{v}\cdot\mathbf{n})\mathbf{f}\ \mathrm{dV}\ .}$$

3. Show that the balance of mass can be expressed as:

$$\dot{\rho} + \rho\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

where $\rho(\mathbf{x},t)$ is the current mass density, $\dot{\rho}$ is the material time derivative of ρ, and $\mathbf{v}(\mathbf{x},t)$ is the velocity of physical particles in the body Ω bounded by the surface $\partial\Omega$.

Recall that the general equation for the balance of a physical quantity $f(\mathbf{x},t)$ is given by

$$\frac{d}{dt}\left[\int_{\Omega} f(\mathbf{x},t)\ \mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\ \mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\ \mathrm{dA} + \int_{\Omega} h(\mathbf{x},t)\ \mathrm{dV}\ .$$

To derive the equation for the balance of mass, we assume that the physical quantity of interest is the mass density $\rho(\mathbf{x},t)$. Since mass is neither created or destroyed, the surface and interior sources are zero, i.e., $g(\mathbf{x},t) = h(\mathbf{x},t) = 0$. Therefore, we have

$$\frac{d}{dt}\left[\int_{\Omega} \rho(\mathbf{x},t)\ \mathrm{dV}\right] = \int_{\partial\Omega} \rho(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\ \mathrm{dA}\ .$$

Let us assume that the volume Ω is a control volume (i.e., it does not change with time). Then the surface $\partial\Omega$ has a zero velocity ($u_n = 0$) and we get

$$\int_{\Omega} \frac{\partial \rho}{\partial t}\ \mathrm{dV} = -\int_{\partial\Omega} \rho\ (\mathbf{v}\cdot\mathbf{n})\ \mathrm{dA}\ .$$

Using the divergence theorem

$$\int_{\Omega} \boldsymbol{\nabla}\cdot\mathbf{v}\ \mathrm{dV} = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\ \mathrm{dA}$$

we get

$$\int_{\Omega} \frac{\partial \rho}{\partial t}\ \mathrm{dV} = -\int_{\Omega} \boldsymbol{\nabla}\cdot(\rho\ \mathbf{v})\ \mathrm{dV}.$$

or,

$$\int_{\Omega} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\ \mathbf{v})\right]\ \mathrm{dV} = 0\ .$$

Since Ω is arbitrary, we must have

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\ \mathbf{v}) = 0\ .$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\varphi\ \mathbf{v}) = \varphi\ \boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\varphi\cdot\mathbf{v}$$

we have

$$\frac{\partial \rho}{\partial t} + \rho\ \boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\rho\cdot\mathbf{v} = 0\ .$$

Now, the material time derivative of ρ is defined as

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}\ .$$

Therefore,

$$\boxed{\dot{\rho} + \rho\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0\ .}$$

4. Show that the balance of linear momentum can be expressed as:

$$\rho\ \dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\ \mathbf{b} = 0$$

where $\rho(\mathbf{x},t)$ is the mass density, $\mathbf{v}(\mathbf{x},t)$ is the velocity, $\boldsymbol{\sigma}(\mathbf{x},t)$ is the Cauchy stress, and $\rho\ \mathbf{b}$ is the body force density.

Recall the general equation for the balance of a physical quantity

$$\frac{d}{dt}\left[\int_{\Omega} f(\mathbf{x},t)\ \mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\ \mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\ \mathrm{dA} + \int_{\Omega} h(\mathbf{x},t)\ \mathrm{dV}\ .$$

In this case the physical quantity of interest is the momentum density, i.e., $f(\mathbf{x},t) = \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)$. The source of momentum flux at the surface is the surface traction, i.e., $g(\mathbf{x},t) = \mathbf{t}$. The source of momentum inside the body is the body force, i.e., $h(\mathbf{x},t) = \rho(\mathbf{x},t)\,\mathbf{b}(\mathbf{x},t)$. Therefore, we have

$$\frac{d}{dt}\left[\int_\Omega \rho\,\mathbf{v}\,\text{dV}\right] = \int_{\partial\Omega} \rho\,\mathbf{v}[u_n - \mathbf{v}\cdot\mathbf{n}]\,\text{dA} + \int_{\partial\Omega} \mathbf{t}\,\text{dA} + \int_\Omega \rho\,\mathbf{b}\,\text{dV}\,.$$

The surface tractions are related to the Cauchy stress by

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}\,.$$

Therefore,

$$\frac{d}{dt}\left[\int_\Omega \rho\,\mathbf{v}\,\text{dV}\right] = \int_{\partial\Omega} \rho\,\mathbf{v}[u_n - \mathbf{v}\cdot\mathbf{n}]\,\text{dA} + \int_{\partial\Omega} \boldsymbol{\sigma}\cdot\mathbf{n}\,\text{dA} + \int_\Omega \rho\,\mathbf{b}\,\text{dV}\,.$$

Let us assume that Ω is an arbitrary fixed control volume. Then,

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\text{dV} = -\int_{\partial\Omega} \rho\,\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,\text{dA} + \int_{\partial\Omega} \boldsymbol{\sigma}\cdot\mathbf{n}\,\text{dA} + \int_\Omega \rho\,\mathbf{b}\,\text{dV}\,.$$

Now, from the definition of the tensor product we have (for all vectors $\mathbf{a}$)

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\,\mathbf{u}\,.$$

Therefore,

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\text{dV} = -\int_{\partial\Omega} \rho\,(\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{n}\,\text{dA} + \int_{\partial\Omega} \boldsymbol{\sigma}\cdot\mathbf{n}\,\text{dA} + \int_\Omega \rho\,\mathbf{b}\,\text{dV}\,.$$

Using the divergence theorem

$$\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{v}\,\text{dV} = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,\text{dA}$$

we have

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\text{dV} = -\int_\Omega \boldsymbol{\nabla}\cdot[\rho\,(\mathbf{v}\otimes\mathbf{v})]\,\text{dV} + \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\,\text{dV} + \int_\Omega \rho\,\mathbf{b}\,\text{dV}$$

or,

$$\int_\Omega \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \boldsymbol{\nabla}\cdot[(\rho\,\mathbf{v})\otimes\mathbf{v}] - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right]\,\text{dV} = 0\,.$$

Since Ω is arbitrary, we have

$$\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \boldsymbol{\nabla}\cdot[(\rho\,\mathbf{v})\otimes\mathbf{v}] - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{v}$$

we get

$$\frac{\partial\rho}{\partial t}\,\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + (\boldsymbol{\nabla}\cdot\mathbf{v})(\rho\mathbf{v}) + \boldsymbol{\nabla}(\rho\,\mathbf{v})\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

or,

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\rho\,\mathbf{v})\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

Using the identity

$$\boldsymbol{\nabla}(\varphi\,\mathbf{v}) = \varphi\,\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes(\boldsymbol{\nabla}\varphi)$$

we get

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + [\rho\,\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes(\boldsymbol{\nabla}\rho)]\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

From the definition

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\,\mathbf{u}$$

we have

$$[\mathbf{v}\otimes(\boldsymbol{\nabla}\rho)]\cdot\mathbf{v} = [\mathbf{v}\cdot(\boldsymbol{\nabla}\rho)]\,\mathbf{v}\,.$$

Hence,

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} + [\mathbf{v}\cdot(\boldsymbol{\nabla}\rho)]\,\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0$$

or,

$$\left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

The material time derivative of ρ is defined as

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}\,.$$

Therefore,

$$[\dot{\rho} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}]\,\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,.$$

From the balance of mass, we have

$$\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} = 0\,.$$

Therefore,

$$\rho\, \frac{\partial \mathbf{v}}{\partial t} + \rho\, \boldsymbol{\nabla}\mathbf{v} \cdot \mathbf{v} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b} = 0\,.$$

The material time derivative of $\mathbf{v}$ is defined as

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}\mathbf{v} \cdot \mathbf{v}\,.$$

Hence,

$$\boxed{\rho\, \dot{\mathbf{v}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b} = 0\,.}$$

5. Show that the balance of angular momentum can be expressed as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

We assume that there are no surface couples on $\partial\Omega$ or body couples in Ω. Recall the general balance equation

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\ \mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\ \mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\ \mathrm{dA} + \int_\Omega h(\mathbf{x},t)\ \mathrm{dV}\,.$$

In this case, the physical quantity to be conserved the angular momentum density, i.e., $f = \mathbf{x} \times (\rho\, \mathbf{v})$. The angular momentum source at the surface is then $g = \mathbf{x} \times \mathbf{t}$ and the angular momentum source inside the body is $h = \mathbf{x} \times (\rho\, \mathbf{b})$. The angular momentum and moments are calculated with respect to a fixed origin. Hence we have

$$\frac{d}{dt}\left[\int_\Omega \mathbf{x} \times (\rho\, \mathbf{v})\ \mathrm{dV}\right] = \int_{\partial\Omega} [\mathbf{x} \times (\rho\, \mathbf{v})][u_n - \mathbf{v}\cdot\mathbf{n}]\ \mathrm{dA} + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}\ \mathrm{dA} + \int_\Omega \mathbf{x} \times (\rho\, \mathbf{b})\ \mathrm{dV}\,.$$

Assuming that Ω is a control volume, we have

$$\int_\Omega \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\, \mathbf{v})\right]\ \mathrm{dV} = -\int_{\partial\Omega} [\mathbf{x} \times (\rho\, \mathbf{v})][\mathbf{v}\cdot\mathbf{n}]\ \mathrm{dA} + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}\ \mathrm{dA} + \int_\Omega \mathbf{x} \times (\rho\, \mathbf{b})\ \mathrm{dV}\,.$$

Using the definition of a tensor product we can write

$$[\mathbf{x} \times (\rho\, \mathbf{v})][\mathbf{v}\cdot\mathbf{n}] = [[\mathbf{x} \times (\rho\, \mathbf{v})] \otimes \mathbf{v}] \cdot \mathbf{n}\,.$$

Also, $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$. Therefore we have

$$\int_\Omega \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\, \mathbf{v})\right]\ \mathrm{dV} = -\int_{\partial\Omega} [[\mathbf{x} \times (\rho\, \mathbf{v})] \otimes \mathbf{v}] \cdot \mathbf{n}\ \mathrm{dA} + \int_{\partial\Omega} \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\ \mathrm{dA} + \int_\Omega \mathbf{x} \times (\rho\, \mathbf{b})\ \mathrm{dV}\,.$$

Using the divergence theorem, we get

$$\int_\Omega \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\, \mathbf{v})\right]\ \mathrm{dV} = -\int_\Omega \boldsymbol{\nabla} \cdot [[\mathbf{x} \times (\rho\, \mathbf{v})] \otimes \mathbf{v}]\ \mathrm{dV} + \int_{\partial\Omega} \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\ \mathrm{dA} + \int_\Omega \mathbf{x} \times (\rho\, \mathbf{b})\ \mathrm{dV}\,.$$

To convert the surface integral in the above equation into a volume integral, it is convenient to use index notation. Thus,

$$\left[\int_{\partial\Omega} \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\ \mathrm{dA}\right]_i = \int_{\partial\Omega} e_{ijk}\ x_j\ \sigma_{kl}\ n_l\ \mathrm{dA} = \int_{\partial\Omega} A_{il}\ n_l\ \mathrm{dA} = \int_{\partial\Omega} \boldsymbol{A}\cdot\mathbf{n}\,\mathrm{dA}$$

where $[\;]_i$ represents the i-th component of the vector. Using the divergence theorem

$$\int_{\partial\Omega} \boldsymbol{A}\cdot\mathbf{n}\ \mathrm{dA} = \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{A}\ \mathrm{dV} = \int_\Omega \frac{\partial A_{il}}{\partial x_l}\ \mathrm{dV} = \int_\Omega \frac{\partial}{\partial x_l}(e_{ijk}\ x_j\ \sigma_{kl})\ \mathrm{dV}\,.$$

Differentiating,

$$\int_{\partial\Omega} \boldsymbol{A}\cdot\mathbf{n}\ \mathrm{dA} = \int_\Omega \left[e_{ijk}\ \delta_{jl}\ \sigma_{kl} + e_{ijk}\ x_j\ \frac{\partial \sigma_{kl}}{\partial x_l}\right]\ \mathrm{dV} = \int_\Omega \left[e_{ijk}\ \sigma_{kj} + e_{ijk}\ x_j\ \frac{\partial \sigma_{kl}}{\partial x_l}\right]\ \mathrm{dV} = \int_\Omega \left[e_{ijk}\ \sigma_{kj} + e_{ijk}\ x_j\ [\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}]_l\right]\ \mathrm{dV}\,.$$

Expressed in direct tensor notation,

$$\int_{\partial\Omega} \boldsymbol{A}\cdot\mathbf{n}\ \mathrm{dA} = \int_\Omega \left[[\mathcal{E} : \boldsymbol{\sigma}^T]_i + [\mathbf{x} \times (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})]_i\right]\ \mathrm{dV}$$

where $\mathcal{E}$ is the third-order permutation tensor. Therefore,

$$\left[\int_{\partial\Omega} \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\ \mathrm{dA}\right]_i == \int_\Omega \left[[\mathcal{E} : \boldsymbol{\sigma}^T]_i + [\mathbf{x} \times (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})]_i\right]\ \mathrm{dV}$$

or,

$$\int_{\partial\Omega} \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})~\text{dA} == \int_{\Omega} \left[\mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right]~\text{dV}~.$$

The balance of angular momentum can then be written as

$$\int_{\Omega} \mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]~\text{dV} = -\int_{\Omega} \boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}]~\text{dV} + \int_{\Omega}\left[\mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right]~\text{dV} + \int_{\Omega}\mathbf{x}\times(\rho\,\mathbf{b})~\text{dV}~.$$

Since Ω is an arbitrary volume, we have

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right] = -\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] + \mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}) + \mathbf{x}\times(\rho\,\mathbf{b})$$

or,

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] + \mathcal{E}:\boldsymbol{\sigma}^T~.$$

Using the identity,

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{v}$$

we get

$$\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] = (\boldsymbol{\nabla}\cdot\mathbf{v})[\mathbf{x}\times(\rho\,\mathbf{v})] + (\boldsymbol{\nabla}[\mathbf{x}\times(\rho\,\mathbf{v})])\cdot\mathbf{v}~.$$

The second term on the right can be further simplified using index notation as follows.

$$\begin{aligned}
[(\boldsymbol{\nabla}[\mathbf{x}\times(\rho\,\mathbf{v})])\cdot\mathbf{v}]_i = [(\boldsymbol{\nabla}[\rho\,(\mathbf{x}\times\mathbf{v})])\cdot\mathbf{v}]_i &= \frac{\partial}{\partial x_l}(\rho\,e_{ijk}\,x_j\,v_k)\,v_l \\
&= e_{ijk}\left[\frac{\partial\rho}{\partial x_l}\,x_j\,v_k\,v_l + \rho\,\frac{\partial x_j}{\partial x_l}\,v_k\,v_l + \rho\,x_j\,\frac{\partial v_k}{\partial x_l}\,v_l\right] \\
&= (e_{ijk}\,x_j\,v_k)\,\left(\frac{\partial\rho}{\partial x_l}\,v_l\right) + \rho\,(e_{ijk}\,\delta_{jl}\,v_k\,v_l) + e_{ijk}\,x_j\,\left(\rho\,\frac{\partial v_k}{\partial x_l}\,v_l\right) \\
&= [(\mathbf{x}\times\mathbf{v})(\boldsymbol{\nabla}\rho\cdot\mathbf{v}) + \rho\,\mathbf{v}\times\mathbf{v} + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]_i \\
&= [(\mathbf{x}\times\mathbf{v})(\boldsymbol{\nabla}\rho\cdot\mathbf{v}) + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]_i~.
\end{aligned}$$

Therefore we can write

$$\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] = (\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times~\mathbf{v}) + (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})~.$$

The balance of angular momentum then takes the form

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times~\mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T$$

or,

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times~\mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T$$

or,

$$\mathbf{x}\times\left[\rho\frac{\partial\mathbf{v}}{\partial t} + \frac{\partial\rho}{\partial t}\,\mathbf{v} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times~\mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T$$

The material time derivative of $\mathbf{v}$ is defined as

$$\dot{\mathbf{v}} = \frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}~.$$

Therefore,

$$\mathbf{x}\times[\rho\,\dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}] = -\mathbf{x}\times\frac{\partial\rho}{\partial t}\,\mathbf{v} + -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times~\mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T~.$$

Also, from the conservation of linear momentum

$$\rho\,\dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0~.$$

Hence,

$$\begin{aligned}
0 &= \mathbf{x}\times\frac{\partial\rho}{\partial t}\,\mathbf{v} + (\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times~\mathbf{v}) + (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - \mathcal{E}:\boldsymbol{\sigma}^T \\
&= \left(\frac{\partial\rho}{\partial t} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}\right)(\mathbf{x}\times\mathbf{v}) - \mathcal{E}:\boldsymbol{\sigma}^T~.
\end{aligned}$$

The material time derivative of ρ is defined as

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}~.$$

Hence,

$$(\dot{\rho} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - \mathcal{E}:\boldsymbol{\sigma}^T = 0~.$$

From the balance of mass

$$\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \, .$$

Therefore,

$$\mathcal{E} : \boldsymbol{\sigma}^T = 0 \, .$$

In index notation,

$$e_{ijk}\, \sigma_{kj} = 0 \, .$$

Expanding out, we get

$$\sigma_{12} - \sigma_{21} = 0 \, ; \;\; \sigma_{23} - \sigma_{32} = 0 \, ; \;\; \sigma_{31} - \sigma_{13} = 0 \, .$$

Hence,

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T}$$

6. Show that the balance of energy can be expressed as:

$$\rho\, \dot{e} - \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) + \boldsymbol{\nabla} \cdot \mathbf{q} - \rho\, s = 0$$

where $\rho(\mathbf{x}, t)$ is the mass density, $e(\mathbf{x}, t)$ is the internal energy per unit mass, $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress, $\mathbf{v}(\mathbf{x}, t)$ is the particle velocity, $\mathbf{q}$ is the heat flux vector, and s is the rate at which energy is generated by sources inside the volume (per unit mass).

Recall the general balance equation

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x}, t) \text{ dV}\right] = \int_{\partial\Omega} f(\mathbf{x}, t)[u_n(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t)] \text{ dA} + \int_{\partial\Omega} g(\mathbf{x}, t) \text{ dA} + \int_\Omega h(\mathbf{x}, t) \text{ dV} \, .$$

In this case, the physical quantity to be conserved the total energy density which is the sum of the internal energy density and the kinetic energy density, i.e., $f = \rho\, e + 1/2\, \rho\, |\mathbf{v} \cdot \mathbf{v}|$. The energy source at the surface is a sum of the rate of work done by the applied tractions and the rate of heat leaving the volume (per unit area), i.e, $g = \mathbf{v} \cdot \mathbf{t} - \mathbf{q} \cdot \mathbf{n}$ where $\mathbf{n}$ is the outward unit normal to the surface. The energy source inside the body is the sum of the rate of work done by the body forces and the rate of energy generated by internal sources, i.e., $h = \mathbf{v} \cdot (\rho\mathbf{b}) + \rho\, s$.

Hence we have

$$\frac{d}{dt}\left[\int_\Omega \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \text{ dV}\right] = \int_{\partial\Omega} \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) (u_n - \mathbf{v} \cdot \mathbf{n}) \text{ dA} + \int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{t} - \mathbf{q} \cdot \mathbf{n}) \text{ dA} + \int_\Omega \rho\, (\mathbf{v} \cdot \mathbf{b} + s) \text{ dV} \, .$$

Let Ω be a control volume that does not change with time. Then we get

$$\int_\Omega \frac{\partial}{\partial t}\left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\right] \text{ dV} = -\int_{\partial\Omega} \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) (\mathbf{v} \cdot \mathbf{n}) \text{ dA} + \int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{t} - \mathbf{q} \cdot \mathbf{n}) \text{ dA} + \int_\Omega \rho\, (\mathbf{v} \cdot \mathbf{b} + s) \text{ dV} \, .$$

Using the relation $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$, the identity $\mathbf{v} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) = (\boldsymbol{\sigma}^T \cdot \mathbf{v}) \cdot \mathbf{n}$, and invoking the symmetry of the stress tensor, we get

$$\int_\Omega \frac{\partial}{\partial t}\left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\right] \text{ dV} = -\int_{\partial\Omega} \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) (\mathbf{v} \cdot \mathbf{n}) \text{ dA} + \int_{\partial\Omega} (\boldsymbol{\sigma} \cdot \mathbf{v} - \mathbf{q}) \cdot \mathbf{n} \text{ dA} + \int_\Omega \rho\, (\mathbf{v} \cdot \mathbf{b} + s) \text{ dV} \, .$$

We now apply the divergence theorem to the surface integrals to get

$$\int_\Omega \frac{\partial}{\partial t}\left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\right] \text{ dV} = -\int_\Omega \boldsymbol{\nabla} \cdot \left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \mathbf{v}\right] \text{ dA} + \int_\Omega \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) \text{ dA} - \int_\Omega \boldsymbol{\nabla} \cdot \mathbf{q} \text{ dA} + \int_\Omega \rho\, (\mathbf{v} \cdot \mathbf{b} + s) \text{ dV} \, .$$

Since Ω is arbitrary, we have

$$\frac{\partial}{\partial t}\left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\right] = -\boldsymbol{\nabla} \cdot \left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \mathbf{v}\right] + \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) - \boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, (\mathbf{v} \cdot \mathbf{b} + s) \, .$$

Expanding out the left hand side, we have

$$\begin{aligned} \frac{\partial}{\partial t}\left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\right] &= \frac{\partial \rho}{\partial t}\left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) + \rho\, \left(\frac{\partial e}{\partial t} + \frac{1}{2}\, \frac{\partial}{\partial t}(\mathbf{v} \cdot \mathbf{v})\right) \\ &= \frac{\partial \rho}{\partial t}\left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) + \rho\, \frac{\partial e}{\partial t} + \rho\, \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} \, . \end{aligned}$$

For the first term on the right hand side, we use the identity $\boldsymbol{\nabla} \cdot (\varphi\, \mathbf{v}) = \varphi\, \boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla}\varphi \cdot \mathbf{v}$ to get

$$\begin{aligned} \boldsymbol{\nabla} \cdot \left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \mathbf{v}\right] &= \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\, \boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla}\left[\rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\right] \cdot \mathbf{v} \\ &= \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\, \boldsymbol{\nabla} \cdot \mathbf{v} + \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \boldsymbol{\nabla}\rho \cdot \mathbf{v} + \rho\, \boldsymbol{\nabla}\left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \cdot \mathbf{v} \\ &= \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\, \boldsymbol{\nabla} \cdot \mathbf{v} + \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \boldsymbol{\nabla}\rho \cdot \mathbf{v} + \rho\, \boldsymbol{\nabla}e \cdot \mathbf{v} + \frac{1}{2}\, \rho\, \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{v}) \cdot \mathbf{v} \\ &= \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\, \boldsymbol{\nabla} \cdot \mathbf{v} + \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \boldsymbol{\nabla}\rho \cdot \mathbf{v} + \rho\, \boldsymbol{\nabla}e \cdot \mathbf{v} + \rho\, (\boldsymbol{\nabla}\mathbf{v}^T \cdot \mathbf{v}) \cdot \mathbf{v} \\ &= \rho\, \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right)\, \boldsymbol{\nabla} \cdot \mathbf{v} + \left(e + \frac{1}{2}\, \mathbf{v} \cdot \mathbf{v}\right) \boldsymbol{\nabla}\rho \cdot \mathbf{v} + \rho\, \boldsymbol{\nabla}e \cdot \mathbf{v} + \rho\, (\boldsymbol{\nabla}\mathbf{v} \cdot \mathbf{v}) \cdot \mathbf{v} \, . \end{aligned}$$

For the second term on the right we use the identity $\boldsymbol{\nabla} \cdot (\boldsymbol{S}^T \cdot \mathbf{v}) = \boldsymbol{S} : \boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla} \cdot \boldsymbol{S}) \cdot \mathbf{v}$ and the symmetry of the Cauchy stress tensor to get

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) = \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{v} \,.$$

After collecting terms and rearranging, we get

$$\left(\frac{\partial \rho}{\partial t} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla}\rho \cdot \mathbf{v}\right)\left(e + \frac{1}{2}\,\mathbf{v} \cdot \mathbf{v}\right) + \left(\rho\, \frac{\partial \mathbf{v}}{\partial t} + \rho\, \boldsymbol{\nabla}\mathbf{v} \cdot \mathbf{v} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b}\right) \cdot \mathbf{v} + \rho\, \left(\frac{\partial e}{\partial t} + \boldsymbol{\nabla} e \cdot \mathbf{v}\right) + -\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla} \cdot \mathbf{q} - \rho\, s = 0 \,.$$

Applying the balance of mass to the first term and the balance of linear momentum to the second term, and using the material time derivative of the internal energy

$$\dot{e} = \frac{\partial e}{\partial t} + \boldsymbol{\nabla} e \cdot \mathbf{v}$$

we get the final form of the balance of energy:

$$\boxed{\rho\, \dot{e} - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla} \cdot \mathbf{q} - \rho\, s = 0 \,.}$$

7. Show that the Clausius-Duhem inequality in integral form:

$$\frac{d}{dt}\left(\int_{\Omega} \rho\, \eta\, \mathsf{dV}\right) \geq \int_{\partial\Omega} \rho\, \eta\, (u_n - \mathbf{v} \cdot \mathbf{n})\, \mathsf{dA} - \int_{\partial\Omega} \frac{\mathbf{q} \cdot \mathbf{n}}{T}\, \mathsf{dA} + \int_{\Omega} \frac{\rho\, s}{T}\, \mathsf{dV} \,.$$

can be written in differential form as

$$\rho\, \dot{\eta} \geq -\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T} \,.$$

Assume that Ω is an arbitrary fixed control volume. Then $u_n = 0$ and the derivative can be take inside the integral to give

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho\, \eta)\, \mathrm{dV} \geq -\int_{\partial\Omega} \rho\, \eta\, (\mathbf{v} \cdot \mathbf{n})\, \mathrm{dA} - \int_{\partial\Omega} \frac{\mathbf{q} \cdot \mathbf{n}}{T}\, \mathrm{dA} + \int_{\Omega} \frac{\rho\, s}{T}\, \mathrm{dV} \,.$$

Using the divergence theorem, we get

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho\, \eta)\, \mathrm{dV} \geq -\int_{\Omega} \boldsymbol{\nabla} \cdot (\rho\, \eta\, \mathbf{v})\, \mathrm{dV} - \int_{\Omega} \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right)\, \mathrm{dV} + \int_{\Omega} \frac{\rho\, s}{T}\, \mathrm{dV} \,.$$

Since Ω is arbitrary, we must have

$$\frac{\partial}{\partial t}(\rho\, \eta) \geq -\boldsymbol{\nabla} \cdot (\rho\, \eta\, \mathbf{v}) - \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T} \,.$$

Expanding out

$$\frac{\partial \rho}{\partial t}\, \eta + \rho\, \frac{\partial \eta}{\partial t} \geq -\boldsymbol{\nabla}(\rho_\eta) \cdot \mathbf{v} - \rho\, \eta\, (\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T}$$

or,

$$\frac{\partial \rho}{\partial t}\, \eta + \rho\, \frac{\partial \eta}{\partial t} \geq -\eta\, \boldsymbol{\nabla}\rho \cdot \mathbf{v} - \rho\, \boldsymbol{\nabla}\eta \cdot \mathbf{v} - \rho\, \eta\, (\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T}$$

or,

$$\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\rho \cdot \mathbf{v} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v}\right)\, \eta + \rho\, \left(\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}\eta \cdot \mathbf{v}\right) \geq -\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T} \,.$$

Now, the material time derivatives of ρ and η are given by

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\rho \cdot \mathbf{v} \,; \quad \dot{\eta} = \frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}\eta \cdot \mathbf{v} \,.$$

Therefore,

$$(\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v})\, \eta + \rho\, \dot{\eta} \geq -\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T} \,.$$

From the conservation of mass $\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} = 0$. Hence,

$$\boxed{\rho\, \dot{\eta} \geq -\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\, s}{T} \,.}$$

8. Show that the Clausius-Duhem inequality

$$\rho\,\dot{\eta} \ge -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\,s}{T}$$

can be expressed in terms of the internal energy as

$$\rho\,(\dot{e} - T\,\dot{\eta}) - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.$$

Using the identity $\boldsymbol{\nabla}\cdot(\varphi\,\mathbf{v}) = \varphi\,\boldsymbol{\nabla}\cdot\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\varphi$ in the Clausius-Duhem inequality, we get

$$\rho\,\dot{\eta} \ge -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\,s}{T} \qquad \text{or} \qquad \rho\,\dot{\eta} \ge -\frac{1}{T}\,\boldsymbol{\nabla}\cdot\mathbf{q} - \mathbf{q}\cdot\boldsymbol{\nabla}\left(\frac{1}{T}\right) + \frac{\rho\,s}{T}\,.$$

Now, using index notation with respect to a Cartesian basis $\mathbf{e}_j$,

$$\boldsymbol{\nabla}\left(\frac{1}{T}\right) = \frac{\partial}{\partial x_j}\left(T^{-1}\right)\,\mathbf{e}_j = -\left(T^{-2}\right)\,\frac{\partial T}{\partial x_j}\,\mathbf{e}_j = -\frac{1}{T^2}\,\boldsymbol{\nabla}T\,.$$

Hence,

$$\rho\,\dot{\eta} \ge -\frac{1}{T}\,\boldsymbol{\nabla}\cdot\mathbf{q} + \frac{1}{T^2}\,\mathbf{q}\cdot\boldsymbol{\nabla}T + \frac{\rho\,s}{T} \qquad \text{or} \qquad \rho\,\dot{\eta} \ge -\frac{1}{T}(\boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s) + \frac{1}{T^2}\,\mathbf{q}\cdot\boldsymbol{\nabla}T\,.$$

Recall the balance of energy

$$\rho\,\dot{e} - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0 \qquad \Longrightarrow \qquad \rho\,\dot{e} - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} = -(\boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s)\,.$$

Therefore,

$$\rho\,\dot{\eta} \ge \frac{1}{T}(\rho\,\dot{e} - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}) + \frac{1}{T^2}\,\mathbf{q}\cdot\boldsymbol{\nabla}T \qquad \Longrightarrow \qquad \rho\,\dot{\eta}\,T \ge \rho\,\dot{e} - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} + \frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.$$

Rearranging,

$$\boxed{\rho\,(\dot{e} - T\,\dot{\eta}) - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.}$$

9. For thermoelastic materials, the internal energy is a function only of the deformation gradient and the temperature, i.e., $e = e(\boldsymbol{F}, T)$. Show that, for thermoelastic materials, the Clausius-Duhem inequality

$$\rho\,(\dot{e} - T\,\dot{\eta}) - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}$$

can be expressed as

$$\rho\,\left(\frac{\partial e}{\partial\eta} - T\right)\,\dot{\eta} + \left(\rho\,\frac{\partial e}{\partial\boldsymbol{F}} - \boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\right):\dot{\boldsymbol{F}} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.$$

Since $e = e(\boldsymbol{F}, T)$, we have

$$\dot{e} = \frac{\partial e}{\partial\boldsymbol{F}}:\dot{\boldsymbol{F}} + \frac{\partial e}{\partial\eta}\,\dot{\eta}\,.$$

Therefore,

$$\rho\,\left(\frac{\partial e}{\partial\boldsymbol{F}}:\dot{\boldsymbol{F}} + \frac{\partial e}{\partial\eta}\,\dot{\eta} - T\,\dot{\eta}\right) - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T} \qquad \text{or} \qquad \rho\,\left(\frac{\partial e}{\partial\eta} - T\right)\,\dot{\eta} + \rho\,\frac{\partial e}{\partial\boldsymbol{F}}:\dot{\boldsymbol{F}} - \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.$$

Now, $\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{l} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}$. Therefore, using the identity $\boldsymbol{A}:(\boldsymbol{B}\cdot\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C}^T):\boldsymbol{B}$, we have

$$\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\sigma}:(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}) = (\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}):\dot{\boldsymbol{F}}\,.$$

Hence,

$$\rho\left(\frac{\partial e}{\partial\eta} - T\right)\,\dot{\eta} + \rho\,\frac{\partial e}{\partial\boldsymbol{F}}:\dot{\boldsymbol{F}} - (\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}):\dot{\boldsymbol{F}} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}$$

or,

$$\boxed{\rho\,\left(\frac{\partial e}{\partial\eta} - T\right)\,\dot{\eta} + \left(\rho\,\frac{\partial e}{\partial\boldsymbol{F}} - \boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\right):\dot{\boldsymbol{F}} \le -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}\,.}$$

10. Show that, for thermoelastic materials, the balance of energy

$$\rho\,\dot{e} - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0\,.$$

can be expressed as

$$\rho\,T\,\dot{\eta} = -\boldsymbol{\nabla}\cdot\mathbf{q} + \rho\,s\,.$$

Since $e = e(\boldsymbol{F}, T)$, we have

$$\dot{e} = \frac{\partial e}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial e}{\partial \eta}\,\dot{\eta}\,.$$

Plug into energy equation to get

$$\rho\,\frac{\partial e}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \rho\,\frac{\partial e}{\partial \eta}\,\dot{\eta} - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0\,.$$

Recall,

$$\frac{\partial e}{\partial \eta} = T \qquad \text{and} \qquad \rho\,\frac{\partial e}{\partial \boldsymbol{F}} = \boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\,.$$

Hence,

$$(\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}) : \dot{\boldsymbol{F}} + \rho\,T\,\dot{\eta} - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0\,.$$

Now, $\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{l} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}$. Therefore, using the identity $\boldsymbol{A} : (\boldsymbol{B}\cdot\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C}^T) : \boldsymbol{B}$, we have

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\sigma} : (\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}) = (\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}) : \dot{\boldsymbol{F}}\,.$$

Plugging into the energy equation, we have

$$\boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \rho\,T\,\dot{\eta} - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\,s = 0$$

or,

$$\boxed{\rho\,T\,\dot{\eta} = -\boldsymbol{\nabla}\cdot\mathbf{q} + \rho\,s\,.}$$

11. Show that, for thermoelastic materials, the Cauchy stress can be expressed in terms of the Green strain as

$$\boldsymbol{\sigma} = \rho\,\boldsymbol{F}\cdot\frac{\partial e}{\partial \boldsymbol{E}}\cdot\boldsymbol{F}^T\,.$$

Recall that the Cauchy stress is given by

$$\boldsymbol{\sigma} = \rho\,\frac{\partial e}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^T \qquad \Longrightarrow \qquad \sigma_{ij} = \rho\,\frac{\partial e}{\partial F_{ik}}F^T_{kj} = \rho\,\frac{\partial e}{\partial F_{ik}}F_{jk}\,.$$

The Green strain $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{F}) = \boldsymbol{E}(\boldsymbol{U})$ and $e = e(\boldsymbol{F}, \eta) = e(\boldsymbol{U}, \eta)$. Hence, using the chain rule,

$$\frac{\partial e}{\partial \boldsymbol{F}} = \frac{\partial e}{\partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{F}} \qquad \Longrightarrow \qquad \frac{\partial e}{\partial F_{ik}} = \frac{\partial e}{\partial E_{lm}}\,\frac{\partial E_{lm}}{\partial F_{ik}}\,.$$

Now,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\cdot\boldsymbol{F} - \boldsymbol{1}) \qquad \Longrightarrow \qquad E_{lm} = \frac{1}{2}(F^T_{lp}\,F_{pm} - \delta_{lm}) = \frac{1}{2}(F_{pl}\,F_{pm} - \delta_{lm})\,.$$

Taking the derivative with respect to $\boldsymbol{F}$, we get

$$\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{F}} = \frac{1}{2}\left(\frac{\partial \boldsymbol{F}^T}{\partial \boldsymbol{F}}\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}}\right) \qquad \Longrightarrow \qquad \frac{\partial E_{lm}}{\partial F_{ik}} = \frac{1}{2}\left(\frac{\partial F_{pl}}{\partial F_{ik}}\,F_{pm} + F_{pl}\,\frac{\partial F_{pm}}{\partial F_{ik}}\right)\,.$$

Therefore,

$$\boldsymbol{\sigma} = \frac{1}{2}\,\rho\,\left[\frac{\partial e}{\partial \boldsymbol{E}} : \left(\frac{\partial \boldsymbol{F}^T}{\partial \boldsymbol{F}}\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}}\right)\right]\cdot\boldsymbol{F}^T \qquad \Longrightarrow \qquad \sigma_{ij} = \frac{1}{2}\,\rho\,\left[\frac{\partial e}{\partial E_{lm}}\left(\frac{\partial F_{pl}}{\partial F_{ik}}\,F_{pm} + F_{pl}\,\frac{\partial F_{pm}}{\partial F_{ik}}\right)\right]\,F_{jk}\,.$$

Recall,

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} \equiv \frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\,\delta_{jl} \qquad \text{and} \qquad \frac{\partial \boldsymbol{A}^T}{\partial \boldsymbol{A}} \equiv \frac{\partial A_{ji}}{\partial A_{kl}} = \delta_{jk}\,\delta_{il}\,.$$

Therefore,

$$\sigma_{ij} = \frac{1}{2}\,\rho\,\left[\frac{\partial e}{\partial E_{lm}}\left(\delta_{pi}\,\delta_{lk}\,F_{pm} + F_{pl}\,\delta_{pi}\,\delta_{mk}\right)\right]\,F_{jk} = \frac{1}{2}\,\rho\,\left[\frac{\partial e}{\partial E_{lm}}\left(\delta_{lk}\,F_{im} + F_{il}\,\delta_{mk}\right)\right]\,F_{jk}$$

or,

$$\sigma_{ij} = \frac{1}{2}\,\rho\,\left[\frac{\partial e}{\partial E_{km}}\,F_{im} + \frac{\partial e}{\partial E_{lk}}\,F_{il}\right]\,F_{jk} \qquad \Longrightarrow \qquad \boldsymbol{\sigma} = \frac{1}{2}\,\rho\,\left[\boldsymbol{F}\cdot\left(\frac{\partial e}{\partial \boldsymbol{E}}\right)^T + \boldsymbol{F}\cdot\frac{\partial e}{\partial \boldsymbol{E}}\right]\cdot\boldsymbol{F}^T$$

or,

$$\boldsymbol{\sigma} = \frac{1}{2}\,\rho\,\boldsymbol{F}\cdot\left[\left(\frac{\partial e}{\partial \boldsymbol{E}}\right)^T + \frac{\partial e}{\partial \boldsymbol{E}}\right]\cdot\boldsymbol{F}^T\,.$$

From the symmetry of the Cauchy stress, we have

$$\boldsymbol{\sigma} = (\boldsymbol{F}\cdot\boldsymbol{A})\cdot\boldsymbol{F}^T \quad \text{and} \quad \boldsymbol{\sigma}^T = \boldsymbol{F}\cdot(\boldsymbol{F}\cdot\boldsymbol{A})^T = \boldsymbol{F}\cdot\boldsymbol{A}^T\cdot\boldsymbol{F}^T \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \implies \boldsymbol{A} = \boldsymbol{A}^T\,.$$

Therefore,

$$\frac{\partial e}{\partial \boldsymbol{E}} = \left(\frac{\partial e}{\partial \boldsymbol{E}}\right)^T$$

and we get

$$\boxed{\boldsymbol{\sigma} = \rho\,\boldsymbol{F}\cdot\frac{\partial e}{\partial \boldsymbol{E}}\cdot\boldsymbol{F}^T\,.}$$

12. For thermoelastic materials, the specific internal energy is given by

$$e = \bar{e}(\boldsymbol{E},\eta)$$

where $\boldsymbol{E}$ is the Green strain and η is the specific entropy. Show that

$$\frac{d}{dt}(e - T\,\eta) = -\dot{T}\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S}:\dot{\boldsymbol{E}} \quad \text{and} \quad \frac{d}{dt}(e - T\,\eta - \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E}) = -\dot{T}\,\eta - \frac{1}{\rho_0}\,\dot{\boldsymbol{S}}:\boldsymbol{E}$$

where ρ_0 is the initial density, T is the absolute temperature, $\boldsymbol{S}$ is the 2nd Piola-Kirchhoff stress, and a dot over a quantity indicates the material time derivative.

Taking the material time derivative of the specific internal energy, we get

$$\dot{e} = \frac{\partial \bar{e}}{\partial \boldsymbol{E}}:\dot{\boldsymbol{E}} + \frac{\partial \bar{e}}{\partial \eta}\,\dot{\eta}\,.$$

Now, for thermoelastic materials,

$$T = \frac{\partial \bar{e}}{\partial \eta} \quad \text{and} \quad \boldsymbol{S} = \rho_0\,\frac{\partial \bar{e}}{\partial \boldsymbol{E}}\,.$$

Therefore,

$$\dot{e} = \frac{1}{\rho_0}\,\boldsymbol{S}:\dot{\boldsymbol{E}} + T\,\dot{\eta}\,. \quad \implies \quad \dot{e} - T\,\dot{\eta} = \frac{1}{\rho_0}\,\boldsymbol{S}:\dot{\boldsymbol{E}}\,.$$

Now,

$$\frac{d}{dt}(T\,\eta) = \dot{T}\,\eta + T\,\dot{\eta}\,.$$

Therefore,

$$\dot{e} - \frac{d}{dt}(T\,\eta) + \dot{T}\,\eta = \frac{1}{\rho_0}\,\boldsymbol{S}:\dot{\boldsymbol{E}} \quad \implies \quad \boxed{\frac{d}{dt}(e - T\,\eta) = -\dot{T}\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S}:\dot{\boldsymbol{E}}\,.}$$

Also,

$$\frac{d}{dt}\left(\frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E}\right) = \frac{1}{\rho_0}\,\boldsymbol{S}:\dot{\boldsymbol{E}} + \frac{1}{\rho_0}\,\dot{\boldsymbol{S}}:\boldsymbol{E}\,.$$

Hence,

$$\dot{e} - \frac{d}{dt}(T\,\eta) + \dot{T}\,\eta = \frac{d}{dt}\left(\frac{1}{\rho_0}\boldsymbol{S}:\boldsymbol{E}\right) - \frac{1}{\rho_0}\,\dot{\boldsymbol{S}}:\boldsymbol{E} \quad \implies \quad \boxed{\frac{d}{dt}\left(e - T\,\eta - \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E}\right) = -\dot{T}\,\eta - \frac{1}{\rho_0}\,\dot{\boldsymbol{S}}:\boldsymbol{E}\,.}$$

13. For thermoelastic materials, show that the following relations hold:

$$\frac{\partial \psi}{\partial \boldsymbol{E}} = \frac{1}{\rho_0}\,\hat{\boldsymbol{S}}(\boldsymbol{E},T)\,;\;\; \frac{\partial \psi}{\partial T} = -\hat{\eta}(\boldsymbol{E},T)\,;\;\; \frac{\partial g}{\partial \boldsymbol{S}} = \frac{1}{\rho_0}\,\tilde{\boldsymbol{E}}(\boldsymbol{S},T)\,;\;\; \frac{\partial g}{\partial T} = \tilde{\eta}(\boldsymbol{S},T)$$

where $\psi(\boldsymbol{E},T)$ is the Helmholtz free energy and $g(\boldsymbol{S},T)$ is the Gibbs free energy.

Also show that

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial T} = -\rho_0\,\frac{\partial \hat{\eta}}{\partial \boldsymbol{E}} \quad \text{and} \quad \frac{\partial \tilde{\boldsymbol{E}}}{\partial T} = \rho_0\,\frac{\partial \tilde{\eta}}{\partial \boldsymbol{S}}\,.$$

Recall that

$$\psi(\boldsymbol{E},T) = e - T\,\eta = \bar{e}(\boldsymbol{E},\eta) - T\,\eta\,.$$

and

$$g(\boldsymbol{S},T) = -e + T\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E}\,.$$

(Note that we can choose any functional dependence that we like, because the quantities e, η, $\boldsymbol{E}$ are the actual quantities and not any particular functional relations).

The derivatives are

$$\frac{\partial\psi}{\partial\boldsymbol{E}} = \frac{\partial\bar{e}}{\partial\boldsymbol{E}} = \frac{1}{\rho_0}\,\boldsymbol{S}\,; \qquad \frac{\partial\psi}{\partial T} = -\eta\,.$$

and

$$\frac{\partial g}{\partial\boldsymbol{S}} = \frac{1}{\rho_0}\,\frac{\partial\boldsymbol{S}}{\partial\boldsymbol{S}}:\boldsymbol{E} = \frac{1}{\rho_0}\,\boldsymbol{E}\,; \qquad \frac{\partial g}{\partial T} = \eta\,.$$

Hence,

$$\boxed{\frac{\partial\psi}{\partial\boldsymbol{E}} = \frac{1}{\rho_0}\,\hat{\boldsymbol{S}}(\boldsymbol{E},T)\,;\;\; \frac{\partial\psi}{\partial T} = -\hat{\eta}(\boldsymbol{E},T)\,;\;\; \frac{\partial g}{\partial\boldsymbol{S}} = \frac{1}{\rho_0}\,\tilde{\boldsymbol{E}}(\boldsymbol{S},T)\,;\;\; \frac{\partial g}{\partial T} = \tilde{\eta}(\boldsymbol{S},T)}$$

From the above, we have

$$\frac{\partial^2\psi}{\partial T\partial\boldsymbol{E}} = \frac{\partial^2\psi}{\partial\boldsymbol{E}\partial T} \qquad \Longrightarrow \qquad -\frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} = \frac{1}{\rho_0}\frac{\partial\hat{\boldsymbol{S}}}{\partial T}\,.$$

and

$$\frac{\partial^2 g}{\partial T\partial\boldsymbol{S}} = \frac{\partial^2 g}{\partial\boldsymbol{S}\partial T} \qquad \Longrightarrow \qquad \frac{\partial\tilde{\eta}}{\partial\boldsymbol{S}} = \frac{1}{\rho_0}\frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,.$$

Hence,

$$\boxed{\frac{\partial\hat{\boldsymbol{S}}}{\partial T} = -\rho_0\,\frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} \qquad \text{and} \qquad \frac{\partial\tilde{\boldsymbol{E}}}{\partial T} = \rho_0\,\frac{\partial\tilde{\eta}}{\partial\boldsymbol{S}}\,.}$$

14. For thermoelastic materials, show that the following relations hold:

$$\frac{\partial\hat{e}(\boldsymbol{E},T)}{\partial T} = T\,\frac{\partial\hat{\eta}}{\partial T} = -T\,\frac{\partial^2\hat{\psi}}{\partial T^2}$$

and

$$\frac{\partial\tilde{e}(\boldsymbol{S},T)}{\partial T} = T\,\frac{\partial\tilde{\eta}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S}:\frac{\partial\tilde{\boldsymbol{E}}}{\partial T} = T\,\frac{\partial^2\tilde{g}}{\partial T^2} + \boldsymbol{S}:\frac{\partial^2\tilde{g}}{\partial\boldsymbol{S}\partial T}\,.$$

Recall,

$$\hat{\psi}(\boldsymbol{E},T) = \psi = e - T\,\eta = \hat{e}(\boldsymbol{E},T) - T\,\hat{\eta}(\boldsymbol{E},T)$$

and

$$\tilde{g}(\boldsymbol{S},T) = g = -e + T\,\eta + \frac{1}{\rho_0}\,\boldsymbol{S}:\boldsymbol{E} = -\tilde{e}(\boldsymbol{S},T) + T\,\tilde{\eta}(\boldsymbol{S},T) + \frac{1}{\rho_0}\,\boldsymbol{S}:\tilde{\boldsymbol{E}}(\boldsymbol{S},T)\,.$$

Therefore,

$$\frac{\partial\hat{e}(\boldsymbol{E},T)}{\partial T} = \frac{\partial\hat{\psi}}{\partial T} + \hat{\eta}(\boldsymbol{E},T) + T\,\frac{\partial\hat{\eta}}{\partial T}$$

and

$$\frac{\partial\tilde{e}(\boldsymbol{S},T)}{\partial T} = -\frac{\partial\tilde{g}}{\partial T} + \tilde{\eta}(\boldsymbol{S},T) + T\,\frac{\partial\tilde{\eta}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S}:\frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,.$$

Also, recall that

$$\hat{\eta}(\boldsymbol{E},T) = -\frac{\partial\hat{\psi}}{\partial T} \qquad \Longrightarrow \qquad \frac{\partial\hat{\eta}}{\partial T} = -\frac{\partial^2\hat{\psi}}{\partial T^2}\,,$$

$$\tilde{\eta}(\boldsymbol{S},T) = \frac{\partial\tilde{g}}{\partial T} \qquad \Longrightarrow \qquad \frac{\partial\tilde{\eta}}{\partial T} = \frac{\partial^2\tilde{g}}{\partial T^2}\,,$$

and

$$\tilde{\boldsymbol{E}}(\boldsymbol{S},T) = \rho_0\,\frac{\partial\tilde{g}}{\partial\boldsymbol{S}} \qquad \Longrightarrow \qquad \frac{\partial\tilde{\boldsymbol{E}}}{\partial T} = \rho_0\,\frac{\partial^2\tilde{g}}{\partial\boldsymbol{S}\partial T}\,.$$

Hence,

$$\boxed{\frac{\partial\hat{e}(\boldsymbol{E},T)}{\partial T} = T\,\frac{\partial\hat{\eta}}{\partial T} = -T\,\frac{\partial^2\hat{\psi}}{\partial T^2}}$$

and

$$\boxed{\frac{\partial\tilde{e}(\boldsymbol{S},T)}{\partial T} = T\,\frac{\partial\tilde{\eta}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S}:\frac{\partial\tilde{\boldsymbol{E}}}{\partial T} = T\,\frac{\partial^2\tilde{g}}{\partial T^2} + \boldsymbol{S}:\frac{\partial^2\tilde{g}}{\partial\boldsymbol{S}\partial T}\,.}$$

15. For thermoelastic materials, show that the balance of energy equation

$$\rho\, T\, \dot{\eta} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s$$

can be expressed as either

$$\rho\, C_v\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s + \frac{\rho}{\rho_0}\, T\, \frac{\partial \hat{\boldsymbol{S}}}{\partial T} : \dot{\boldsymbol{E}}$$

or

$$\rho\, \left(C_p - \frac{1}{\rho_0}\, \boldsymbol{S} : \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\right)\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s - \frac{\rho}{\rho_0}\, T\, \frac{\partial \tilde{\boldsymbol{E}}}{\partial T} : \dot{\boldsymbol{S}}$$

where

$$C_v = \frac{\partial \hat{e}(\boldsymbol{E}, T)}{\partial T} \qquad \text{and} \qquad C_p = \frac{\partial \tilde{e}(\boldsymbol{S}, T)}{\partial T}\ .$$

If the independent variables are $\boldsymbol{E}$ and T, then

$$\eta = \hat{\eta}(\boldsymbol{E}, T) \qquad \implies \qquad \dot{\eta} = \frac{\partial \hat{\eta}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \hat{\eta}}{\partial T}\, \dot{T}\ .$$

On the other hand, if we consider $\boldsymbol{S}$ and T to be the independent variables

$$\eta = \tilde{\eta}(\boldsymbol{S}, T) \qquad \implies \qquad \dot{\eta} = \frac{\partial \tilde{\eta}}{\partial \boldsymbol{S}} : \dot{\boldsymbol{S}} + \frac{\partial \tilde{\eta}}{\partial T}\, \dot{T}\ .$$

Since

$$\frac{\partial \hat{\eta}}{\partial \boldsymbol{E}} = -\frac{1}{\rho_0}\, \frac{\partial \hat{\boldsymbol{S}}}{\partial T}\ ; \quad \frac{\partial \hat{\eta}}{\partial T} = \frac{C_v}{T}\ ; \quad \frac{\partial \tilde{\eta}}{\partial \boldsymbol{S}} = \frac{1}{\rho_0}\, \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\ ; \ \text{and} \ \frac{\partial \tilde{\eta}}{\partial T} = \frac{1}{T}\left(C_p - \frac{1}{\rho_0} \boldsymbol{S} : \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\right)$$

we have, either

$$\dot{\eta} = -\frac{1}{\rho_0}\, \frac{\partial \hat{\boldsymbol{S}}}{\partial T} : \dot{\boldsymbol{E}} + \frac{C_v}{T}\, \dot{T}$$

or

$$\dot{\eta} = \frac{1}{\rho_0}\, \frac{\partial \tilde{\boldsymbol{E}}}{\partial T} : \dot{\boldsymbol{S}} + \frac{1}{T}\left(C_p - \frac{1}{\rho_0} \boldsymbol{S} : \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\right)\, \dot{T}\ .$$

The equation for balance of energy in terms of the specific entropy is

$$\rho\, T\, \dot{\eta} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s\ .$$

Using the two forms of $\dot{\eta}$, we get two forms of the energy equation:

$$-\frac{\rho}{\rho_0}\, T\, \frac{\partial \hat{\boldsymbol{S}}}{\partial T} : \dot{\boldsymbol{E}} + \rho\, C_v\, \dot{T} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s$$

and

$$\frac{\rho}{\rho_0}\, T\, \frac{\partial \tilde{\boldsymbol{E}}}{\partial T} : \dot{\boldsymbol{S}} + \rho\, C_p\, \dot{T} - \frac{\rho}{\rho_0}\, \boldsymbol{S} : \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\, \dot{T} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, s\ .$$

From Fourier's law of heat conduction

$$\mathbf{q} = -\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T\ .$$

Therefore,

$$-\frac{\rho}{\rho_0}\, T\, \frac{\partial \hat{\boldsymbol{S}}}{\partial T} : \dot{\boldsymbol{E}} + \rho\, C_v\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s$$

and

$$\frac{\rho}{\rho_0}\, T\, \frac{\partial \tilde{\boldsymbol{E}}}{\partial T} : \dot{\boldsymbol{S}} + \rho\, C_p\, \dot{T} - \frac{\rho}{\rho_0}\, \boldsymbol{S} : \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s\ .$$

Rearranging,

$$\boxed{\rho\, C_v\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s + \frac{\rho}{\rho_0}\, T\, \frac{\partial \hat{\boldsymbol{S}}}{\partial T} : \dot{\boldsymbol{E}}}$$

or,

$$\boxed{\rho\, \left(C_p - \frac{1}{\rho_0}\, \boldsymbol{S} : \frac{\partial \tilde{\boldsymbol{E}}}{\partial T}\right)\, \dot{T} = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} T) + \rho\, s - \frac{\rho}{\rho_0}\, T\, \frac{\partial \tilde{\boldsymbol{E}}}{\partial T} : \dot{\boldsymbol{S}}\ .}$$

16. For thermoelastic materials, show that the specific heats are related by the relation

$$C_p - C_v = \frac{1}{\rho_0}\left(\boldsymbol{S} - T\,\frac{\partial\hat{\boldsymbol{S}}}{\partial T}\right) : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,.$$

Recall that

$$C_v := \frac{\partial\hat{e}(\boldsymbol{E},T)}{\partial T} = T\,\frac{\partial\hat{\eta}}{\partial T}$$

and

$$C_p := \frac{\partial\tilde{e}(\boldsymbol{S},T)}{\partial T} = T\,\frac{\partial\tilde{\eta}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S} : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,.$$

Therefore,

$$C_p - C_v = T\,\frac{\partial\tilde{\eta}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S} : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T} - T\,\frac{\partial\hat{\eta}}{\partial T}\,.$$

Also recall that

$$\eta = \hat{\eta}(\boldsymbol{E},T) = \tilde{\eta}(\boldsymbol{S},T)\,.$$

Therefore, keeping $\boldsymbol{S}$ constant while differentiating, we have

$$\frac{\partial\tilde{\eta}}{\partial T} = \frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} : \frac{\partial\boldsymbol{E}}{\partial T} + \frac{\partial\hat{\eta}}{\partial T}\,.$$

Noting that $\boldsymbol{E} = \tilde{\boldsymbol{E}}(\boldsymbol{S},T)$, and plugging back into the equation for the difference between the two specific heats, we have

$$C_p - C_v = T\,\frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T} + \frac{1}{\rho_0}\,\boldsymbol{S} : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,.$$

Recalling that

$$\frac{\partial\hat{\eta}}{\partial\boldsymbol{E}} = -\frac{1}{\rho_0}\,\frac{\partial\hat{\boldsymbol{S}}}{\partial T}$$

we get

$$\boxed{C_p - C_v = \frac{1}{\rho_0}\left(\boldsymbol{S} - T\,\frac{\partial\hat{\boldsymbol{S}}}{\partial T}\right) : \frac{\partial\tilde{\boldsymbol{E}}}{\partial T}\,.}$$

17. For thermoelastic materials, show that the specific heats can also be related by the equations

$$C_p - C_v = \frac{1}{\rho_0}\,\boldsymbol{S} : \frac{\partial\boldsymbol{E}}{\partial T} + \frac{\partial\boldsymbol{E}}{\partial T} : \left(\frac{\partial^2\psi}{\partial\boldsymbol{E}\partial\boldsymbol{E}} : \frac{\partial\boldsymbol{E}}{\partial T}\right) = \frac{1}{\rho_0}\,\boldsymbol{S} : \frac{\partial\boldsymbol{E}}{\partial T} + \frac{T}{\rho_0}\,\frac{\partial\boldsymbol{E}}{\partial T} : \left(\frac{\partial\boldsymbol{S}}{\partial\boldsymbol{E}} : \frac{\partial\boldsymbol{E}}{\partial T}\right)\,.$$

Recall that

$$\boldsymbol{S} = \rho_0\,\frac{\partial\psi}{\partial\boldsymbol{E}} = \rho_0\,\boldsymbol{f}(\boldsymbol{E}(\boldsymbol{S},T),T)\,.$$

Recall the chain rule which states that if

$$g(u,t) = f(x(u,t), y(u,t))$$

then, if we keep u fixed, the partial derivative of g with respect to t is given by

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}\,.$$

In our case,

$$u = \boldsymbol{S},\;\; t = T,\;\; g(\boldsymbol{S},T) = \boldsymbol{S},\;\; x(\boldsymbol{S},T) = \boldsymbol{E}(\boldsymbol{S},T),\;\; y(\boldsymbol{S},T) = T,\;\; \text{and}\;\; f = \rho_0\,\boldsymbol{f}.$$

Hence, we have

$$\boldsymbol{S} = g(\boldsymbol{S},T) = f(\boldsymbol{E}(\boldsymbol{S},T),T) = \rho_0\,\boldsymbol{f}(\boldsymbol{E}(\boldsymbol{S},T),T)\,.$$

Taking the derivative with respect to T keeping $\boldsymbol{S}$ constant, we have

$$\frac{\partial g}{\partial T} = \cancelto{0}{\frac{\partial\boldsymbol{S}}{\partial T}} = \rho_0\,\left[\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{E}} : \frac{\partial\boldsymbol{E}}{\partial T} + \frac{\partial\boldsymbol{f}}{\partial T}\,\cancelto{1}{\frac{\partial T}{\partial T}}\right]$$

or,

$$\boldsymbol{0} = \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{E}} : \frac{\partial\boldsymbol{E}}{\partial T} + \frac{\partial\boldsymbol{f}}{\partial T}\,.$$

Now,

$$\boldsymbol{f} = \frac{\partial \psi}{\partial \boldsymbol{E}} \qquad \Longrightarrow \qquad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{E}} = \frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} \quad \text{and} \quad \frac{\partial \boldsymbol{f}}{\partial T} = \frac{\partial^2 \psi}{\partial T \partial \boldsymbol{E}} \, .$$

Therefore,

$$\boldsymbol{0} = \frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} + \frac{\partial^2 \psi}{\partial T \partial \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}} \left(\frac{\partial \psi}{\partial \boldsymbol{E}} \right) : \frac{\partial \boldsymbol{E}}{\partial T} + \frac{\partial}{\partial T} \left(\frac{\partial \psi}{\partial \boldsymbol{E}} \right) \, .$$

Again recall that,

$$\frac{\partial \psi}{\partial \boldsymbol{E}} = \frac{1}{\rho_0} \boldsymbol{S} \, .$$

Plugging into the above, we get

$$\boldsymbol{0} = \frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} + \frac{1}{\rho_0} \frac{\partial \boldsymbol{S}}{\partial T} = \frac{1}{\rho_0} \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} + \frac{1}{\rho_0} \frac{\partial \boldsymbol{S}}{\partial T} \, .$$

Therefore, we get the following relation for $\partial \boldsymbol{S}/\partial T$:

$$\frac{\partial \boldsymbol{S}}{\partial T} = -\rho_0 \, \frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} = -\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \, .$$

Recall that

$$C_p - C_v = \frac{1}{\rho_0} \left(\boldsymbol{S} - T \, \frac{\partial \boldsymbol{S}}{\partial T} \right) : \frac{\partial \boldsymbol{E}}{\partial T} \, .$$

Plugging in the expressions for $\partial \boldsymbol{S}/\partial T$ we get:

$$C_p - C_v = \frac{1}{\rho_0} \left(\boldsymbol{S} + T \, \rho_0 \, \frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \right) : \frac{\partial \boldsymbol{E}}{\partial T} = \frac{1}{\rho_0} \left(\boldsymbol{S} + T \, \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \right) : \frac{\partial \boldsymbol{E}}{\partial T} \, .$$

Therefore,

$$C_p - C_v = \frac{1}{\rho_0} \boldsymbol{S} : \frac{\partial \boldsymbol{E}}{\partial T} + T \, \left(\frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \right) : \frac{\partial \boldsymbol{E}}{\partial T} = \frac{1}{\rho_0} \boldsymbol{S} : \frac{\partial \boldsymbol{E}}{\partial T} + \frac{T}{\rho_0} \, \left(\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \right) : \frac{\partial \boldsymbol{E}}{\partial T} \, .$$

Using the identity $(\mathbf{A} : \boldsymbol{B}) : \boldsymbol{C} = \boldsymbol{C} : (\mathbf{A} : \boldsymbol{B})$, we have

$$\boxed{C_p - C_v = \frac{1}{\rho_0} \boldsymbol{S} : \frac{\partial \boldsymbol{E}}{\partial T} + T \, \frac{\partial \boldsymbol{E}}{\partial T} : \left(\frac{\partial^2 \psi}{\partial \boldsymbol{E} \partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \right) = \frac{1}{\rho_0} \boldsymbol{S} : \frac{\partial \boldsymbol{E}}{\partial T} + \frac{T}{\rho_0} \, \frac{\partial \boldsymbol{E}}{\partial T} : \left(\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} : \frac{\partial \boldsymbol{E}}{\partial T} \right) \, .}$$

18. Consider an isotropic thermoelastic material that has a constant coefficient of thermal expansion and which follows the St-Venant Kirchhoff model, i.e,

$$\boldsymbol{\alpha}_E = \alpha \, \boldsymbol{1} \qquad \text{and} \qquad \mathbf{C} = \lambda \, \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \, \mathbf{I}$$

where α is the coefficient of thermal expansion and $3 \, \lambda = 3 \, K - 2 \, \mu$ where K, μ are the bulk and shear moduli, respectively.

Show that the specific heats related by the equation

$$C_p - C_v = \frac{1}{\rho_0} \left[\alpha \, \text{tr} \, (\boldsymbol{S}) + 9 \, \alpha^2 \, K \, T \right] \, .$$

Recall that,

$$C_p - C_v = \frac{1}{\rho_0} \boldsymbol{S} : \boldsymbol{\alpha}_E + \frac{T}{\rho_0} \boldsymbol{\alpha}_E : \mathbf{C} : \boldsymbol{\alpha}_E \, .$$

Plugging the expressions of $\boldsymbol{\alpha}_E$ and $\mathbf{C}$ into the above equation, we have

$$\begin{aligned} C_p - C_v &= \frac{1}{\rho_0} \boldsymbol{S} : (\alpha \, \boldsymbol{1}) + \frac{T}{\rho_0} (\alpha \, \boldsymbol{1}) : (\lambda \, \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \, \mathbf{I}) : (\alpha \, \boldsymbol{1}) \\ &= \frac{\alpha}{\rho_0} \, \text{tr} \, (\boldsymbol{S}) + \frac{\alpha^2 \, T}{\rho_0} \, \boldsymbol{1} : (\lambda \, \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \, \mathbf{I}) : \boldsymbol{1} \\ &= \frac{\alpha}{\rho_0} \, \text{tr} \, (\boldsymbol{S}) + \frac{\alpha^2 \, T}{\rho_0} \, \boldsymbol{1} : (\lambda \, \text{tr} \, (\boldsymbol{1}) \, \, \boldsymbol{1} + 2\mu \, \boldsymbol{1}) \\ &= \frac{\alpha}{\rho_0} \, \text{tr} \, (\boldsymbol{S}) + \frac{\alpha^2 \, T}{\rho_0} \, (3 \, \lambda \, \text{tr} \, (\boldsymbol{1}) + 2\mu \, \text{tr} \, (\boldsymbol{1})) \\ &= \frac{\alpha}{\rho_0} \, \text{tr} \, (\boldsymbol{S}) + \frac{3 \, \alpha^2 \, T}{\rho_0} \, (3 \, \lambda + 2\mu) \\ &= \frac{\alpha \, \text{tr} \, (\boldsymbol{S})}{\rho_0} + \frac{9 \, \alpha^2 \, K \, T}{\rho_0} \, . \end{aligned}$$

Therefore,

$$\boxed{C_p - C_v = \frac{1}{\rho_0} \left[\alpha \, \text{tr} \, (\boldsymbol{S}) + 9 \, \alpha^2 \, K \, T \right] \, .}$$

References

[1] T. W. Wright. *The Physics and Mathematics of Adiabatic Shear Bands*. Cambridge University Press, Cambridge, UK, 2002.

[2] R. C. Batra. *Elements of Continuum Mechanics*. AIAA, Reston, VA., 2006.

[3] G. A. Maugin. *The Thermomechanics of Nonlinear Irreversible Behaviors: An Introduction*. World Scientific, Singapore, 1999.

[4] M. D. Greenberg. *Foundations of Applied Mathematics*. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1978.

[5] T. Belytschko, W. K. Liu, and B. Moran. *Nonlinear Finite Elements for Continua and Structures*. John Wiley and Sons, Ltd., New York, 2000.

[6] M. E. Gurtin. *An Introduction to Continuum Mechanics*. Academic Press, New York, 1981.