T-STRESS OF A BI-MATERIAL STRIP UNDER GENERALIZED EDGE LOADS

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Abstract. An expression for the T-stress in a bi-material strip with a semi-finite interfacial crack is derived for general edge loads using a conservation integral method. The expression is explicit except for two non-dimensional constants, which are determined from a finite element analysis and which are tabulated as a function of the Dundurs' parameters and the thickness ratio of the strip.

Keywords: T-stress, bi-material, conservation integral, interface crack.

1. Introduction. The T-stress is the first non-singular term in William's eigen-function expansion of a crack tip stress field. It is an important parameter that together with the stress intensity factor characterizes the crack field. Its value, for instance, affects both shape and size of the plastic zone at the crack tip. The T-stress also governs the configurational stability of a growing crack (Cotterell and Rice, 1980). One of the most promising methods for evaluating the T-stress makes use of a conservation integral first introduced by Chen and Shield (1977). Beom and Earmme (1993) and Cho et al. (1994) obtained an explicit expression for the T-stress for a semi-infinite interfacial crack in an infinite isotropic bimaterial solid using this approach. Sladek and Sladek (1997) used the conservation integral method combined with a boundary element analysis to compute the T-stress of an interfacial crack in a finite body. Moon and Earmme (1998) obtained expressions for the T-stress under in-plane and anti-plane loading conditions. Kim et al. (2001) calculated the T-stress for anisotropic bimaterials using complex potentials.

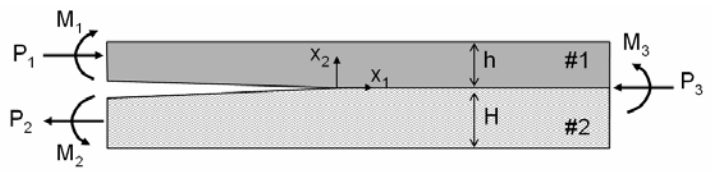

Figure 1. A bimaterial strip specimen under generalized edge loads.

The bi-material strip specimen (shown in Fig. 1) has been used extensively in fracture mechanics tests such as the four point bending test and the double cantilever beam (DCB) test. The stress intensity factor for the bi-material strip geometry is readily available (Suo and Hutchinson, 1990), but evaluating its T-stress is not trivial and requires careful numerical analysis. In this study, an expression for the T-stress of the bi-material strip specimen is obtained for general edge loads using the conservation integral method. The thicknesses of the two layers composing the strip are taken to be finite, while the interfacial crack between them is semi-infinite. The expression is explicit except for two non-dimensional constants that are determined from a finite element analysis. It is assumed throughout this paper that the materials of the specimen are linearly elastic and isotropic.

2. J-based mutual integral method for calculation of the T-stress. Consider an arbitrary path Γ around the tip of a crack aligned with the x_1-axis. The J-integral is then defined as (Rice, 1968)

$$J = \int_\Gamma \left(w n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds\,. \qquad (1)$$

Here w is the strain energy density, n_i the unit normal vector component, t_i the traction vector component, and u_i the displacement vector component. A repeated index implies summation over this index running from one to three. The strain energy density for a linearly elastic material is given by

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\,, \qquad (2)$$

where σ_{ij} and ε_{ij} are the stress and strain components, respectively. The J-integral is path-independent and is a generalization of the energy release rate concept. Now consider two independent equilibrium stress states in a cracked body, identified by (A) and (B), respectively. The J-based mutual integral proposed by Chen and Shield (1977) is then defined by

$$\begin{aligned} I_\Gamma^{(A,B)} &= J_\Gamma^{(A+B)} - J_\Gamma^{(A)} - J_\Gamma^{(B)} \\ &= \int_\Gamma \left(2w^{(A,B)} n_1 - t_i^{(A)} \frac{\partial u_i^{(B)}}{\partial x_1} - t_i^{(B)} \frac{\partial u_i^{(A)}}{\partial x_1} \right) ds, \end{aligned} \qquad (3)$$

where $2w^{(A,B)} = \sigma_{ij}^{(A)}\varepsilon_{ij}^{(B)} = \sigma_{ij}^{(B)}\varepsilon_{ij}^{(A)}$. Note that the J-based mutual integral has the same path-independence as the J-integral.

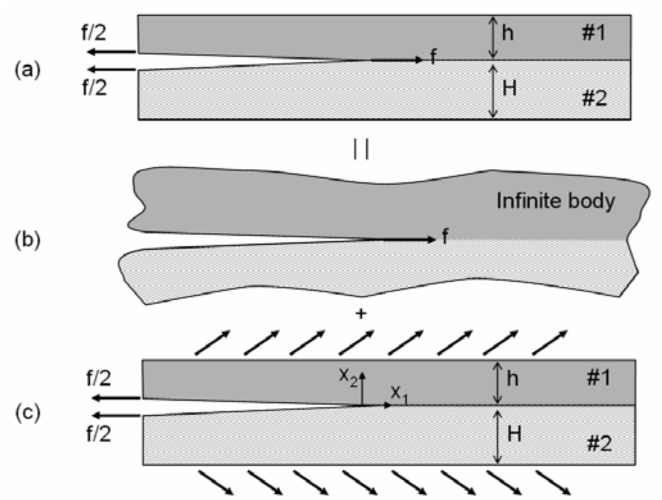

Figure 2. Analysis of the auxiliary field for T-stress evaluation; (a) auxiliary field for a bimaterial strip specimen, (b) point force at a semi-infinite crack tip in an infinite bimaterial body, (c) a bimaterial strip specimen under prescribed traction and point forces.

Consider as state (A) the problem to be analyzed (Fig. 1) and as state (B) the auxiliary stress state that arises when a point force f is applied at the crack tip (Fig. 2(a)). The asymptotic stress field at the tip of an interfacial crack (Fig. 1) can be expressed as

$$\left(\sigma_{ij}\right)_m = \frac{\mathrm{Re}(Kr^{i\varepsilon})}{\sqrt{2\pi r}}\sigma_{ij}^{I}\left(\theta,\varepsilon\right) + \frac{\mathrm{Im}(Kr^{i\varepsilon})}{\sqrt{2\pi r}}\sigma_{ij}^{II}\left(\theta,\varepsilon\right) + T_m\delta_{1i}\delta_{1j}\,. \tag{4}$$

Here, the subscript m (=1,2) identifies the upper and lower strips, respectively, $K(=K_1 + iK_2)$ is the complex stress intensity factor, and T_m is the T-stress. The angular functions $\sigma_{ij}^{I}\left(\theta,\varepsilon\right)$ and $\sigma_{ij}^{II}\left(\theta,\varepsilon\right)$ can be found in (Sladek and Sladek, 1997) and δ_{ij} is the Kronecker delta. Dundurs' parameters for a bimaterial structure with prescribed traction are defined by

$$\alpha = \frac{\mu_1(\kappa_2+1)-\mu_2(\kappa_1+1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)}\,,\ \beta = \frac{\mu_1(\kappa_2-1)-\mu_2(\kappa_1-1)}{\mu_1(\kappa_2+1)+\mu_2(\kappa_1+1)}\,, \tag{5}$$

where μ is the shear modulus, $\kappa = 3-4\nu$ for plane strain and $\kappa = \left(3-\nu\right)/\left(1+\nu\right)$ for plane stress; ν is Poisson's ratio. The oscillatory index, ε, is given by

$$\varepsilon = \frac{1}{2\pi}\ln\left(\frac{1-\beta}{1+\beta}\right). \tag{6}$$

The asymptotic stress field for the auxiliary problem shown in Fig. 2(a) is given by (Moon and Earmme, 1998)

$$\left(\sigma_{ij}\right)_m = -\frac{(1+(-1)^m\alpha)f}{\pi r}\sigma_{ij}^f(\theta) + \frac{\mathrm{Re}(K_f r^{i\varepsilon})}{\sqrt{2\pi r}}\sigma_{ij}^I(\theta,\varepsilon)$$
$$+\frac{\mathrm{Im}(K_f r^{i\varepsilon})}{\sqrt{2\pi r}}\sigma_{ij}^{II}(\theta,\varepsilon) + T_{fm}\delta_{1i}\delta_{1j}\,. \tag{7}$$

Here $\sigma_{11}^f = \cos^3\theta$, $\sigma_{12}^f = \sigma_{21}^f = \cos^2\theta\sin\theta$, and $\sigma_{22}^f = \cos\theta\sin^2\theta$. Using Eqs. (4) and (7), the J-based mutual integral for a path Γ_δ enclosing the crack tip with a vanishingly small radius can be written as (Moon and Earmme, 1998)

$$I_{\Gamma_\delta}^{(A,B)} = \frac{c_1}{8}\left\{T_1 f + \frac{1-\beta^2}{1-\alpha}\left(K\bar{K}_f + K_f\bar{K}\right)\right\}$$
$$= \frac{c_2}{8}\left\{T_2 f + \frac{1-\beta^2}{1+\alpha}\left(K\bar{K}_f + K_f\bar{K}\right)\right\}, \tag{8}$$

where $c = (1+\kappa)/\mu$. Note that the J-based mutual integral does not depend on the T-stresses T_{f1} and T_{f2} of the auxiliary field. When K_f is equal to zero – say, for a semi-infinite crack in an infinite body – the J-based mutual integral is linearly proportional to the T-stress of interest. More generally, Eq. (8) allows evaluation of the T-stress if K and K_f are known. Since the J-based mutual integral is path-independent, its value can also be determined for an integration path Γ along the surface of the specimen. Such a path allows expression of the integral in terms of the edge loads as

$$I_\Gamma^{(A,B)} = \left\{\frac{c_1(-Ph+6M)+c_2(4hP\eta+6M\eta^2+3hP\eta^2)}{16h^2} - \frac{c_2}{8}\left(\frac{P_3}{hA_0} + \frac{M_3}{h^3 I_0}(h/\eta-\delta)\right)\right\}f\,. \tag{9}$$

In this equation, $\eta = h/H$, $P = P_1 - C_1P_3 - C_2M_3/h$, and $M = M_1 - C_3M_3$. The constants C_1, C_2, C_3, A_0, I_0, and δ are defined in (Suo and Hutchinson, 1990). Note that there are a total of six forces and moments in Fig. 1, but that the mutual integral $I_\Gamma^{(A,B)}$ can be expressed in terms of just four loading parameters because of the force and moment equilibria. The complex stress intensity factor for a strip specimen is expressed by (Suo and Hutchinson, 1990)

$$K = \left(\frac{P}{\sqrt{Ah}} - ie^{i\gamma}\frac{M}{\sqrt{Ih^3}}\right)\frac{e^{i(\omega-\varepsilon\ln h)}}{\sqrt{2}}\sqrt{\frac{1-\alpha}{1-\beta^2}}\,, \tag{10}$$

where A, I, γ, and ω can be found in (Suo and Hutchinson, 1990). This expression contains only two loading parameters as discussed in (Suo and Hutchinson, 1990). From dimensional considerations, the complex stress intensity factor of the auxiliary field can be written as

$$K_f = \frac{\xi f}{\sqrt{h}}\sqrt{\frac{1-\alpha}{1-\beta^2}}e^{i(\omega_f - \varepsilon \ln h)}, \tag{11}$$

where ξ and ω_f are real non-dimensional constants that depend on the Dundurs' parameters and on the thickness ratio η. By combining Eqns. (8)-(11), the T-stress can be obtained explicitly as a function of the two constants ξ and ω_f. After some algebra, one finds

$$T_1 = \frac{c_2(4hP\eta + 6M\eta^2 + 3hP\eta^2) + c_1(6M - Ph)}{2h^2 c_1} - \frac{c_2}{c_1}\left(\frac{P_3}{hA_0} + \frac{M_3}{h^3 I_0}(h/\eta - \delta)\right)$$
$$- \xi\sqrt{2}\left(\frac{P\cos(\omega - \omega_f)}{h\sqrt{A}} + \frac{M\sin(\gamma + \omega - \omega_f)}{h^2\sqrt{I}}\right), \tag{12}$$

The T-stress in the lower strip, T_2 is easily obtained from $T_2 = T_1 c_1 / c_2$. Note that the expression for the T-stress contains four loading parameters, P_3, M_3, P and M, unlike the stress intensity factor, which depends on two parameters, P and M, only.

3. T-stress for a homogeneous strip specimen with equal layer thicknesses. The homogeneous strip specimen with equal layer thicknesses is widely utilized in fracture toughness testing. The T-stress expression for this type of specimen is a special case of Eq. (12) and is practically useful. By taking $c_2 = c_1$ and $\eta = 1$, the T-stress is obtained as

$$T_1 = T_2 = \frac{(3 - 2\sqrt{3}\xi)(Ph + 2M)}{h^2} - \frac{P_3}{2h}, \tag{13}$$

Note that $\omega_f = 0$, $\cos(\omega) = \sqrt{3/7}$, and $\sin(\gamma + \omega) = 1$ because of the symmetry of the strip specimen in this case. It is noted that this expression for T-stress of a homogeneous strip contains three loading parameters. The real constant ξ can be found in Table 1 (the calculation details are described in the next section), and is independent of elastic modulus and strip thickness. For example, the T-stress for a DCB test specimen under applied moment M_o is given by $T = (6 - 4\sqrt{3}\xi)M_0 h^{-2}$, and that for a 4-point bending specimen is $T = (3 - 2\sqrt{3}\xi)M_0 h^{-2}$.

Table 1. Two non-dimensional constants ξ and ω_f

α	β	η=1		η=0.5		η=0.1	
		ξ	ω_f	ξ	ω_f	ξ	ω_f
−0.80	−0.40	0.5974	−0.9565	0.6075	−0.9570	0.6189	−0.9837
	−0.30	0.5988	−1.0477	0.6078	−1.0444	0.6189	−1.0691
	−0.20	0.6000	−1.1345	0.6085	−1.1280	0.6190	−1.1509
	−0.10	0.6012	−1.2187	0.6090	−1.2096	0.6193	−1.2313
	0.00	0.5928	−1.3109	0.6099	−1.2903	0.6394	−1.2918
−0.60	−0.40	0.4764	−0.9040	0.5008	−0.9302	0.5264	−1.0011
	−0.30	0.4800	−1.0087	0.5018	−1.0191	0.5265	−1.0856
	−0.20	0.4822	−1.0989	0.5029	−1.1041	0.5268	−1.1667
	−0.10	0.4843	−1.1859	0.5040	−1.1867	0.5273	−1.2464
	0.00	0.4760	−1.2762	0.5054	−1.2682	0.5475	−1.3019
−0.40	−0.30	0.3508	−0.9112	0.3872	−0.9657	0.4289	−1.1091
	−0.20	0.3535	−1.0055	0.3886	−1.0526	0.4294	−1.1895
	−0.10	0.3562	−1.0958	0.3901	−1.1369	0.4300	−1.2679
	0.00	0.3459	−1.1598	0.3920	−1.2197	0.4500	−1.3159
	0.10	0.3520	−1.2839	0.3939	−1.3029	0.4513	−1.3947
−0.20	−0.30	0.2171	−0.6379	0.2638	−0.8338	0.3243	−1.1463
	−0.20	0.2197	−0.7316	0.2653	−0.9254	0.3251	−1.2257
	−0.10	0.2223	−0.8305	0.2670	−1.0140	0.3261	−1.3033
	0.00	0.2250	−0.9262	0.2689	−1.1005	0.3277	−1.3799
	0.10	0.2280	−1.0201	0.2712	−1.1869	0.3293	−1.4582
0.00	−0.20	0.1453	0.2293	0.1444	−0.4800	0.2121	−1.2968
	−0.10	0.1445	0.1138	0.1450	−0.5825	0.2134	−1.3732
	0.00	0.1442	0.0000	0.1461	−0.6832	0.2152	−1.4486
	0.10	0.1446	−0.1138	0.1476	−0.7836	0.2176	−1.5237
	0.20	0.1454	−0.2298	0.1496	−0.8847	0.2203	−1.6011
0.20	−0.20	0.3013	1.1016	0.1466	0.8276	0.0889	−1.5555
	−0.10	0.2794	1.0198	0.1429	0.7390	0.0905	−1.6262
	0.00	0.2756	0.9258	0.1394	0.6482	0.0926	−1.6948
	0.10	0.2722	0.8305	0.1363	0.5533	0.0952	−1.7599
	0.20	0.2689	0.7317	0.1332	0.4514	0.0981	−1.8300
0.40	−0.10	0.5524	1.2713	0.3360	1.2731	0.0749	2.3909
	0.00	0.5483	1.1841	0.3316	1.1988	0.0730	2.3399
	0.10	0.5441	1.0957	0.3271	1.1227	0.0707	2.2932
	0.20	0.5399	1.0055	0.3224	1.0431	0.0676	2.2510
	0.30	0.5358	0.9111	0.3175	0.9588	0.0641	2.2161
0.60	−0.10	0.9952	1.3465	0.6532	1.4011	0.2604	1.9943
	0.00	0.9729	1.2710	0.6486	1.3303	0.2577	1.9291
	0.10	0.9684	1.1860	0.6438	1.2583	0.2545	1.8638
	0.20	0.9643	1.0991	0.6389	1.1843	0.2506	1.7964
	0.30	0.9601	1.0089	0.6336	1.1067	0.2462	1.7279
0.80	0.00	1.8075	1.3019	1.2580	1.3579	0.5717	1.7837
	0.10	1.8036	1.2187	1.2538	1.2901	0.5684	1.7222
	0.20	1.8004	1.1345	1.2495	1.2210	0.5647	1.6592
	0.30	1.7960	1.0478	1.2441	1.1492	0.5596	1.5934
	0.40	1.7927	0.9565	1.2389	1.0733	0.5544	1.5244

4. Analysis of auxiliary field. In order to determine the non-dimensional constants, ξ and ω_f, the elastic solution to the auxiliary problem shown in Fig. 2(a) is needed. The solution could be obtained directly using finite element analysis, but one has to be careful when calculating the stress intensity factor because of the combined $1/r$ and $1/\sqrt{r}$ stress singularities. Alternatively, the solution for this boundary value problem can be obtained as the superposition of the solutions for the problems shown in Fig. 2(b) and Fig. 2(c). In Fig. 2(c), the tractions on the top and bottom surfaces are prescribed such that the tractions on the top and bottom surfaces of the strip specimen in Fig. 2(a) are zero; i.e., they are equal and opposite to the tractions on $x_2 = h$ and $x_2 = -H$ in Fig. 2(b). The elastic solution for Fig. 2(b) is well known and can be expressed as (Moon and Earmme, 1998)

$$\left(\sigma_{ij}\right)_m = -\frac{(1-(-1)^m \alpha) f}{\pi r} \sigma_{ij}^f(\theta), \tag{14}$$

where σ_{ij}^f has been defined before. The solution for the problem shown in Fig. 2(c) is not trivial and can be obtained using the finite element method or through the numerical solution of an integral equation (Suo and Hutchinson, 1990). In this study, the solution was obtained using the finite element code ABAQUS. The calculations were performed using second-order continuum elements with reduced integration and a mesh with at least 60000 elements depending on the thickness ratio of the strip. The complex stress intensity factor was calculated using the ABAQUS contour integral option. The complex stress intensity factor for the problem shown in Fig. 2(a) can then be found through superposition. In fact, it is equal to the stress intensity factor for Fig. 2(c), because the stress intensity factor for Fig 2(b) is equal to zero. Finally, the two non-dimensional constants, ξ and ω_f are determined by comparing the value of the stress intensity factor obtained from the finite element analysis with Eq. (11). They are tabulated in Table 1 as a function of the Dundurs' parameters and the thickness ratio of the strip.

5. Summary and conclusion.

Bi-material strip specimens are widely utilized in interfacial fracture toughness measurement and sub-critical fracture tests. The complex stress intensity factor for a bi-material strip can be found in Suo and Hutchinson (1990), but the T-stress is not readily available. Since the T-stress is an important crack-tip parameter that affects the plastic zone at the crack tip as well as crack stability, it is necessary to evaluate the T-stress for this specimen geometry. In this study, the T-stress of a bi-material strip under general edge loads is explicitly obtained as a function of two real, non-dimensional constants. The two non-dimensional constants are calculated using a finite element analysis, and are tabulated as a function of the Dundurs' parameters and the thickness ratio of the strip.

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