# PLATE BENDING ELEMENTS

# Behavior of Plates

- The behavior of plates is similar to that of beams. They both carry transverse loads by bending action.
  - Plates carry transverse loads by bending and shear just like beams, but they have some peculiarities

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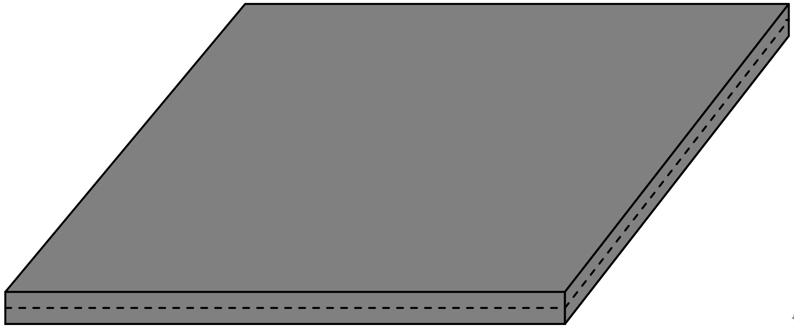
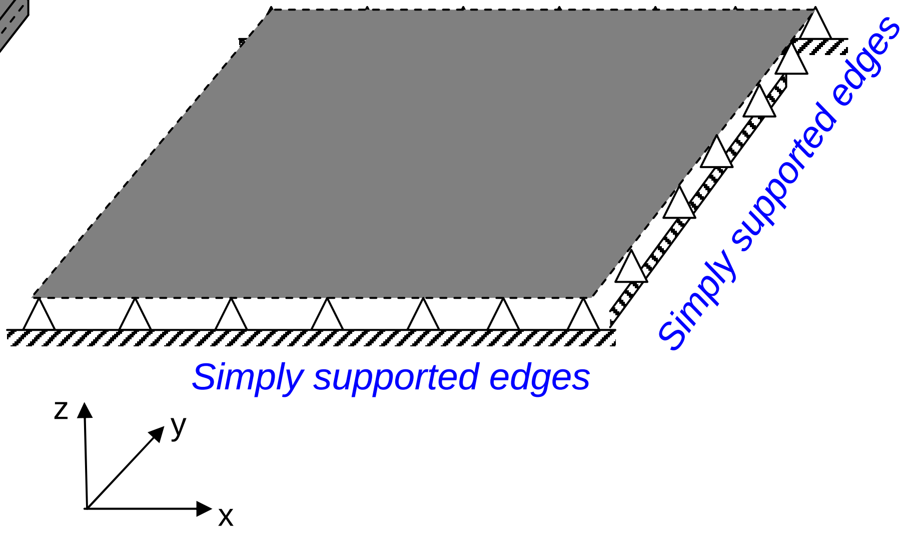
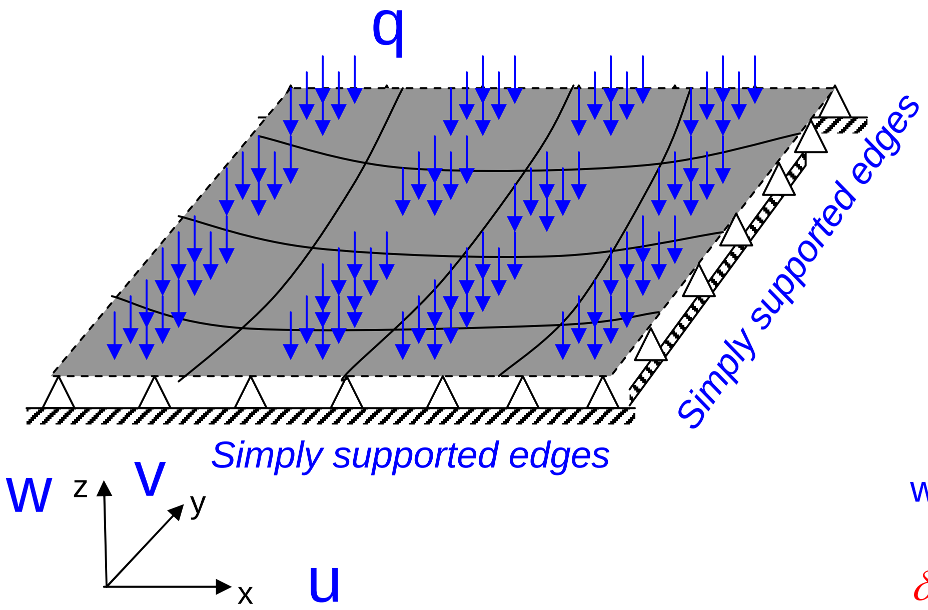
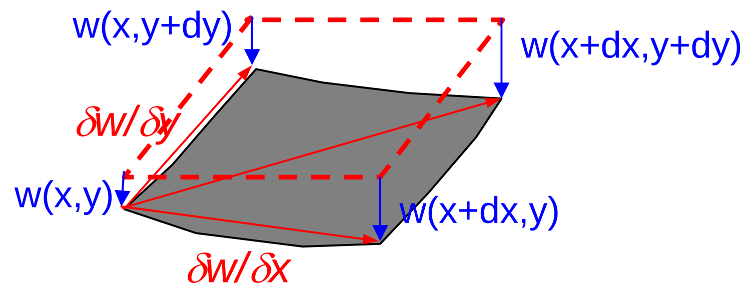




# Behavior of Plates



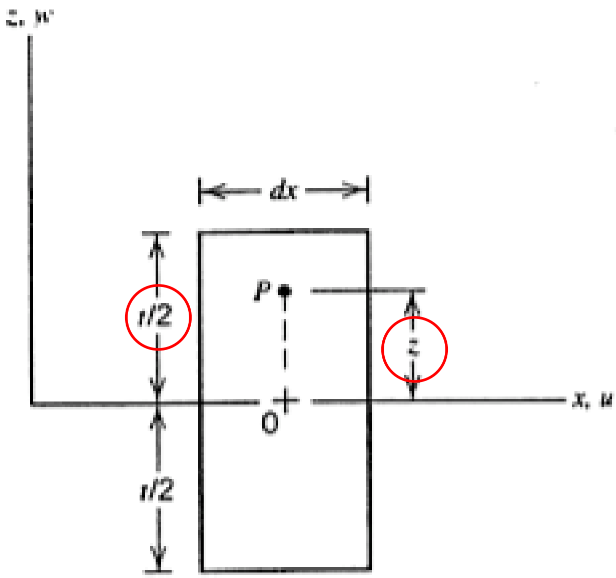

- Plates undergo bending which can be represented by the deflection (*w*) of the middle plane of the plate

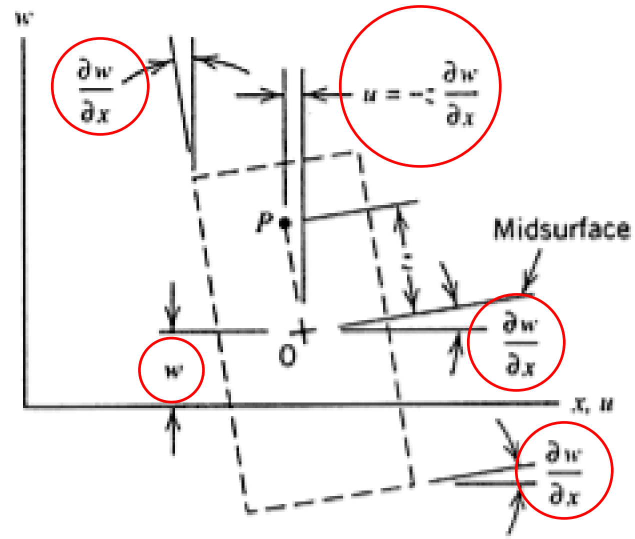



- The middle plane of the plate undergoes deflections w(x,y). The top and bottom surfaces of the plate undergo deformations almost like a rigid body along with the middle surface.

# Behavior of Plates

Thin plate theory - does not include transverse shear deformations




$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y}$$

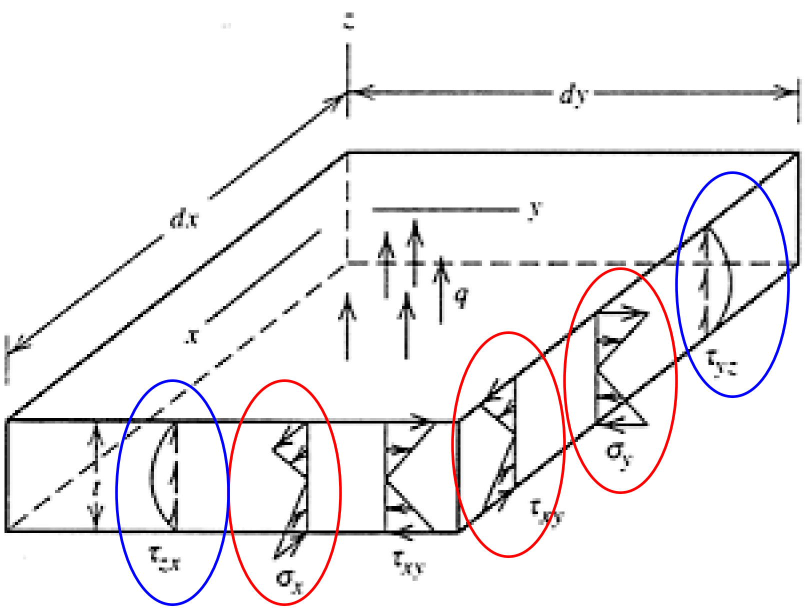
$$u = -z\frac{\partial w}{\partial x} \quad v = -z\frac{\partial w}{\partial y}$$

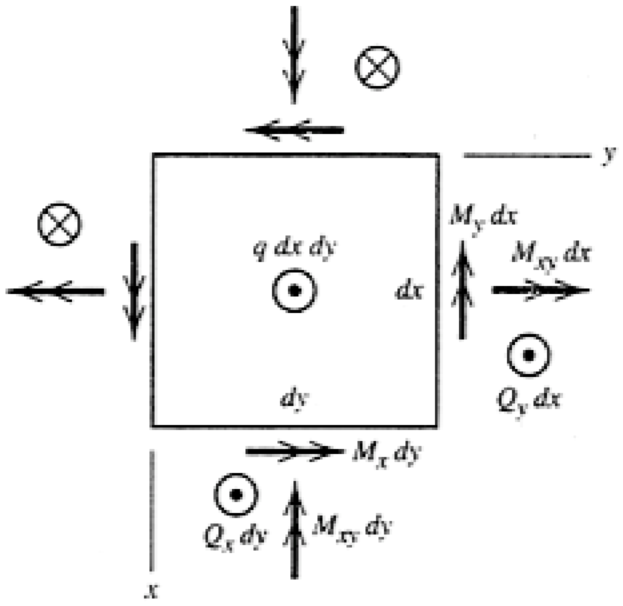
$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \quad \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\frac{\partial^2 w}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

# Behavior of Plates

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = -z\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \end{Bmatrix} \qquad \tau_{xy} = -2zG\frac{\partial^2 w}{\partial x\,\partial y} \qquad (7.1\text{-}2)$$



# Behavior of Plates

- Note that the stresses vary linearly from the middle surface. Just like bending stresses in beams.
- Also note that the shear stresses ($\tau_{xy}$) produced by bending also vary linearly from the middle surface.
- The shear stresses $\tau_{yz}$ and $\tau_{zx}$ are present and required for equilibrium, although the corresponding strains are assumed negligible. Parabolic variations of the stresses are assumed.
- The bending stresses can be simplified to resultant moments ($M_{xx}$, $M_{yy}$, $M_{xy}$). These moments are resultants of the linear stress variations through the thickness

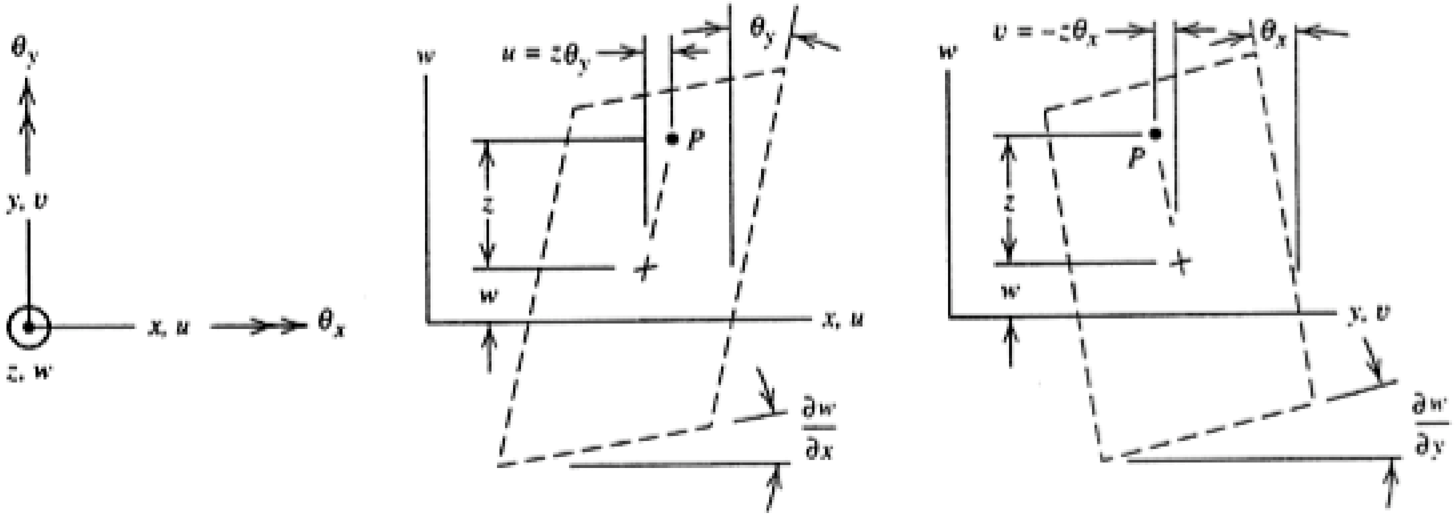
$$M_x = \int_{-t/2}^{t/2} \sigma_x z \, dz \qquad M_y = \int_{-t/2}^{t/2} \sigma_y z \, dz \qquad M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z \, dz \qquad (7.1\text{-}3)$$

# Mindlin Plate Theory

- The transverse shear deformation effects are included by relaxing the assumption that plane sections remain perpendicular to middle surface, i.e., the right angles in the BPS element are no longer preserved.
  - Planes initially normal to the middle surface may experience different rotations than the middle surface itself
  - Analogy is the Timoshenko beam theory.

# Mindlin Plate Theory

- $\theta_x$ and $\theta_y$ are rotations of lines perpendicular to the middle surface




$$\gamma_{xy} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

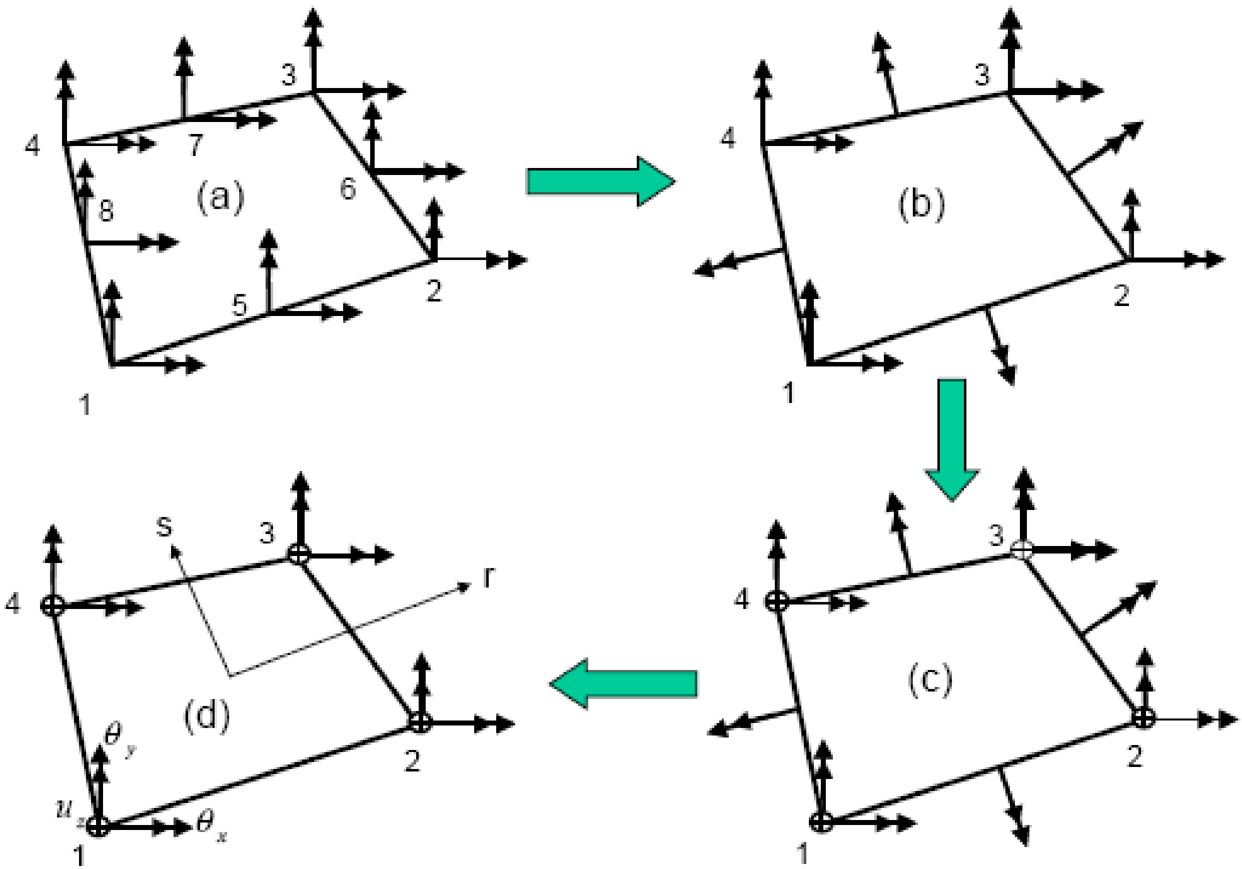
$$u = z\theta_y \qquad \varepsilon_x = z\frac{\partial \theta_y}{\partial x}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x$$

$$\upsilon = -z\theta_x \qquad \varepsilon_y = -z\frac{\partial \theta_x}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \theta_y$$

# Procedure for FEM Formulations

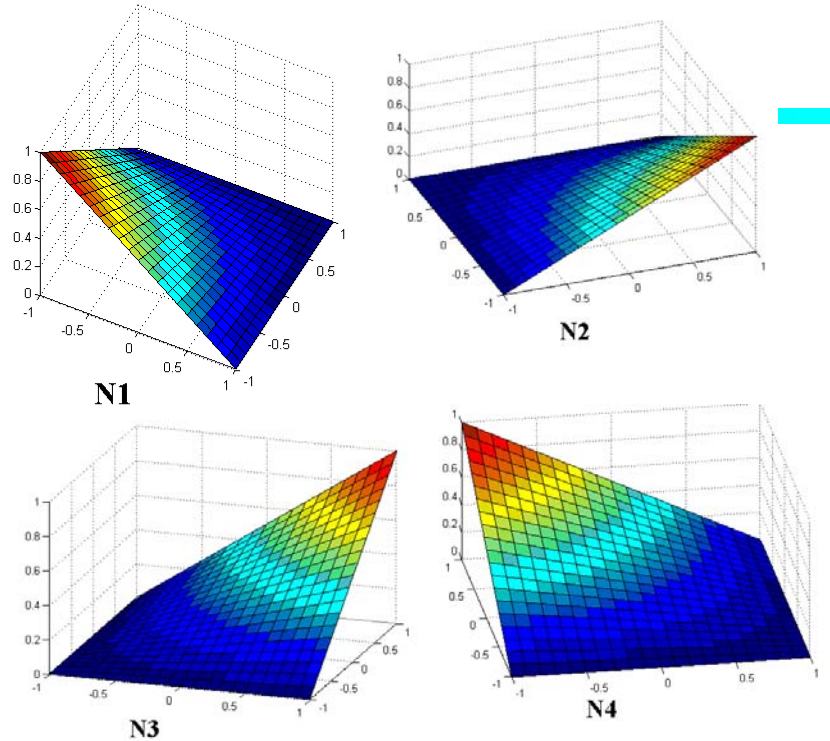



*Quadrilateral Plate Bending Element*

# Shape Functions

$$\theta_x(r,s) = \sum_{i=1}^{4} N_i(r,s)\theta_{xi} + \sum_{i=5}^{8} N_i(r,s)\,\Delta\theta_{xi}$$

$$\theta_y(r,s) = \sum_{i=1}^{8} N_i(r,s)\theta_{yi} + \sum_{i=5}^{8} N_i(r,s)\,\Delta\theta_{yi}$$




## Natural bilinear shape functions

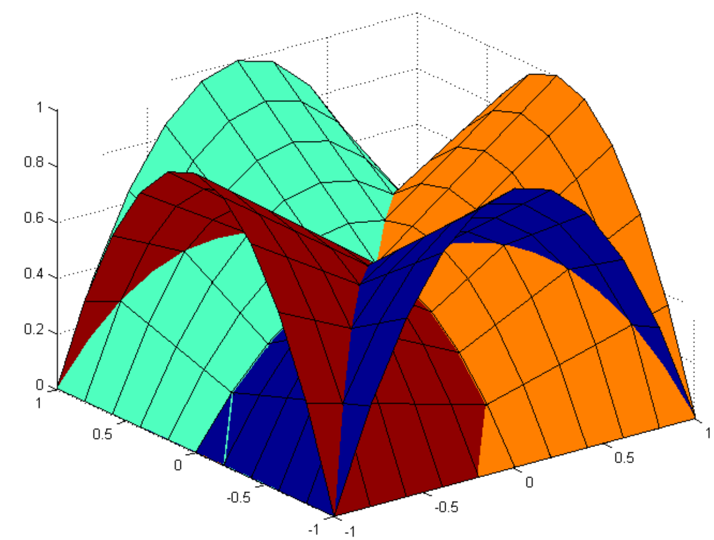
$$N_1 = (1-r)(1-s)/4 \qquad N_2 = (1+r)(1-s)/4$$

$$N_3 = (1+r)(1+s)/4 \qquad N_4 = (1-r)(1+s)/4$$

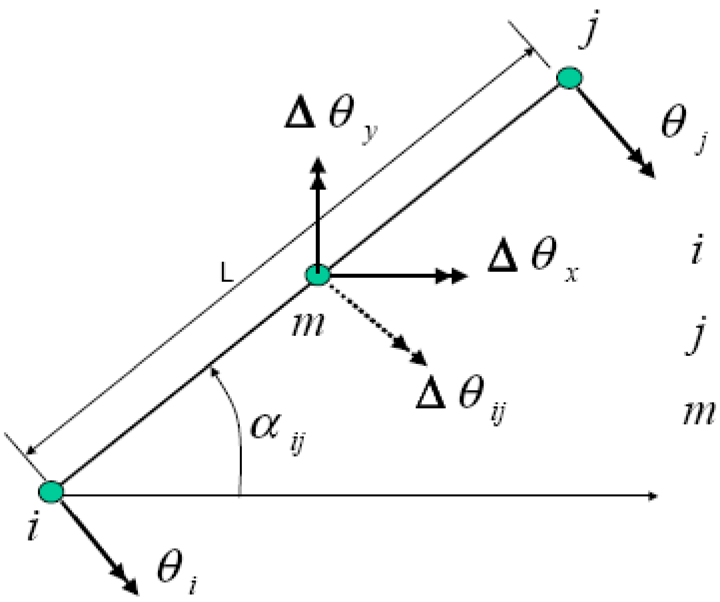
## ***Hierarchical*** functions

$$N_5 = (1-r^2)(1-s)/2 \qquad N_6 = (1+r)(1-s^2)/2$$

$$N_7 = (1-r^2)(1+s)/2 \qquad N_8 = (1-r)(1-s^2)/2$$



# Corner node displacements and mid-side rotations




$$i = 1,2,3,4$$
$$j = 2,3,4,1$$
$$m = 5,6,7,8$$

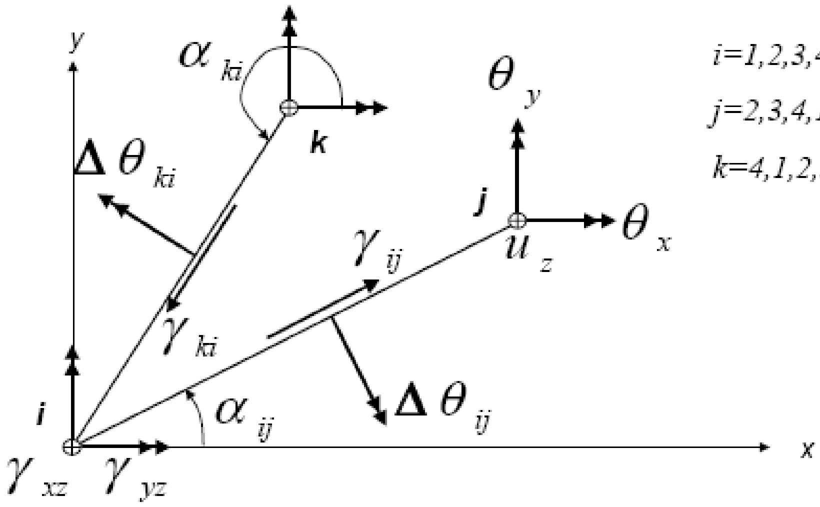
*Typical Element Side*

$$\Delta\theta_x = \sin\alpha_{ij}\ \Delta\theta_{ij}$$
$$\Delta\theta_y = -\cos\alpha_{ij}\ \Delta\theta_{ij}$$

$$u_x(r,s) = z\,\theta_y(r,s)$$
$$u_y(r,s) = -z\,\theta_x(r,s)$$

# Transverse shears Relationship

## (Mindlin Plate )




$i=1,2,3,4$

$j=2,3,4,1$

$k=4,1,2,3$

*Node Point Transverse Shears*

$$\begin{bmatrix} \gamma_{ij} \\ \gamma_{ki} \end{bmatrix} = \begin{bmatrix} \cos\alpha_{ij} & \sin\alpha_{ij} \\ \cos\alpha_{ki} & \sin\alpha_{ki} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}_i$$

$$\gamma_{ij} = \frac{1}{L}(u_{zj} - u_{zi}) - \frac{1}{2}(\theta_i + \theta_j) - \frac{2}{3}\Delta\theta_{ij}$$

$$\gamma_{ij} = \frac{1}{L}(u_{zj} - u_{zi}) - \frac{\sin\alpha_{ij}}{2}(\theta_{xi} + \theta_{xj}) + \frac{\cos\alpha_{ij}}{2}(\theta_{yi} + \theta_{yj}) - \frac{2}{3}\Delta\theta_{ij}$$

## Discrete Kirchhoff Element

$$\Delta\theta = \frac{3}{2L}(w_j - w_i) - \frac{3}{4}(\theta_i + \theta_j)$$

# STRAIN-DISPLACEMENT EQUATIONS

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = z\,\theta_y(r,s)_{,x}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = -z\,\theta_x(r,s)_{,y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = z[\theta_y(r,s)_{,y} - \theta_x(r,s)]$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{b} \begin{bmatrix} \theta_X \\ \theta_Y \\ u_Z \\ \Delta\theta \end{bmatrix} \text{ or } \mathbf{d} = \mathbf{B}\,\mathbf{u} = \mathbf{a}(z)\,\mathbf{b}(r,s)\,\mathbf{u}$$

# THE QUADRILATERAL ELEMENT STIFFNESS

$$\mathbf{k} = \int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV = \int \mathbf{b}^T \mathbf{D} \mathbf{b}\, dA$$

where $$\mathbf{D} = \int \mathbf{a}^T \mathbf{E}\, \mathbf{a}\, dz$$

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\nu^2)}$$

$$D_{12} = D_{21} = \frac{\nu Eh^3}{12(1-\nu^2)}$$

$$D_{44} = D_{55} = \frac{5Eh}{12(1+\nu)}$$

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \\ V_{xz} \\ V_{yz} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} \\ D_{31} & D_{31} & D_{33} & D_{34} & D_{35} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} \end{bmatrix} \begin{bmatrix} \psi_{xx} \\ \psi_{yy} \\ \psi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

$$\begin{bmatrix} \psi_{xx} \\ \psi_{yy} \\ \psi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ w \\ \Delta\theta \end{bmatrix}$$

node displacements $(\theta_x, \theta_y, w)$

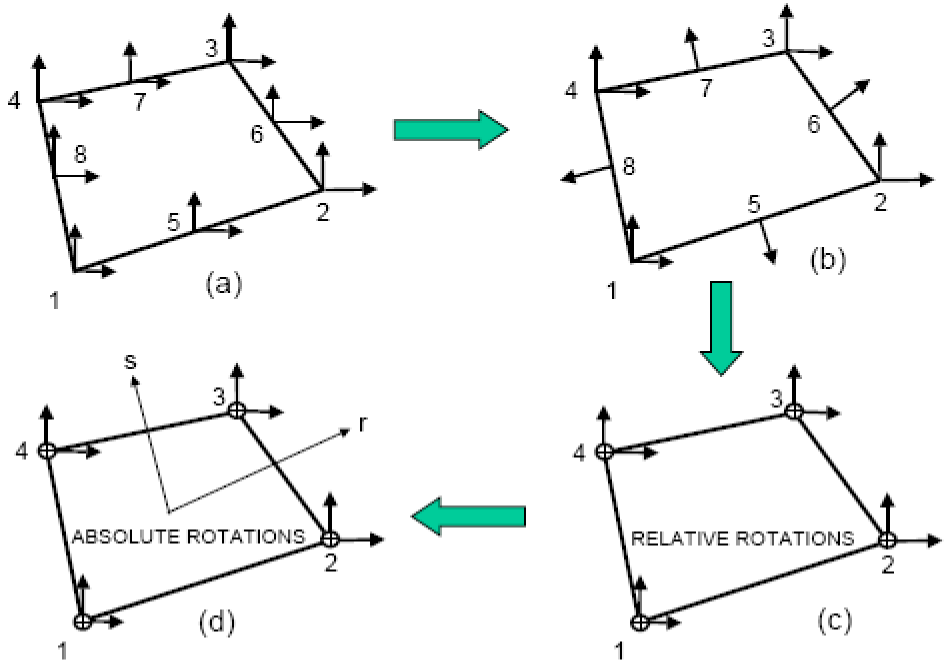
# STATIC CONDENSATION

$$\overline{\mathbf{K}} = \int \mathbf{B}^{T}\mathbf{D}\mathbf{B}\, dA = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}$$

# MEMBRANE ELEMENT WITH NORMAL ROTATIONS

# Procedure for FEM Formulations

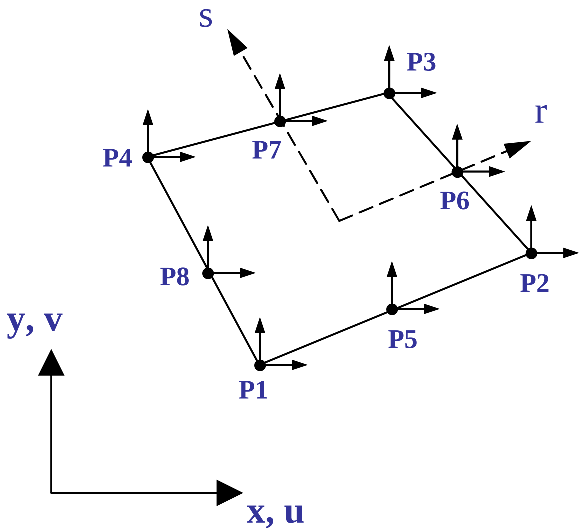



*Quadrilateral Membrane Element with Normal Rotations*

# Development of the element

1. The starting point is the nine node quadrilateral element, 16 DOF2.
2. The next step is to rotate the mid-side relative displacements to be normal and tangential to each side and the relative tangential displacement is set to zero, reducing the element to the 12 DOF
3. The third step is to introduce parabolic normal displacement constraints to eliminate the four mid-side normal displacements and to introduce four relative normal rotations at the nodes
4. The final step is to convert the relative normal rotations to absolute values and to modify the shape functions to pass the patch test. This results in the 12 by 12 element stiffness with respect to the 12 DOF

# Membrane element (4-nodes) + 4 mid-side nodes




Displacement Interpolation Extension

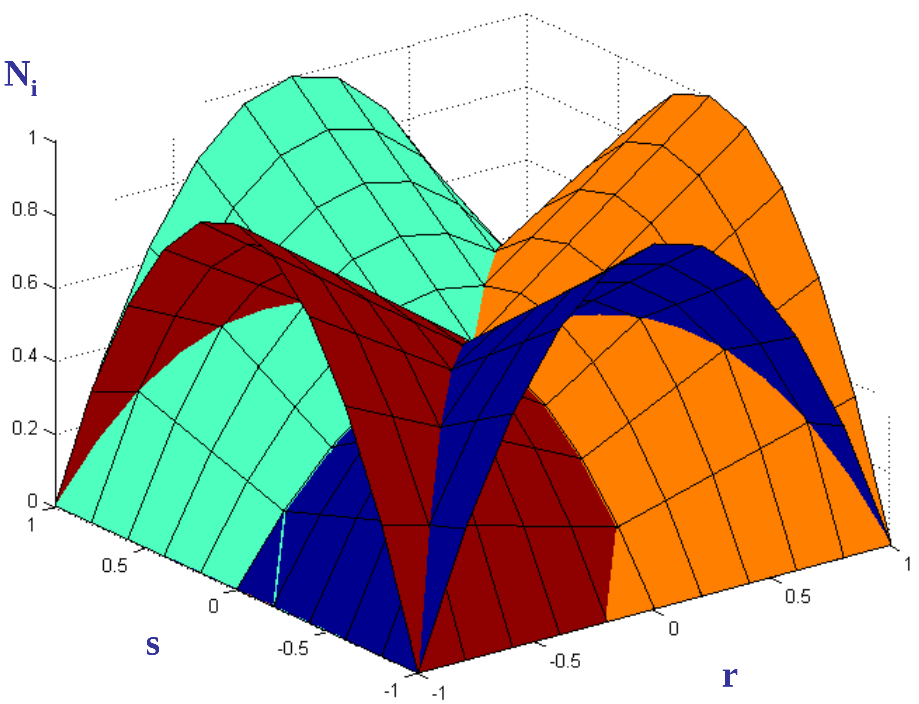
$$u(r,s) = \sum_{i=5}^{8} \mathbf{N_i}\,(r,s)\,u_i$$

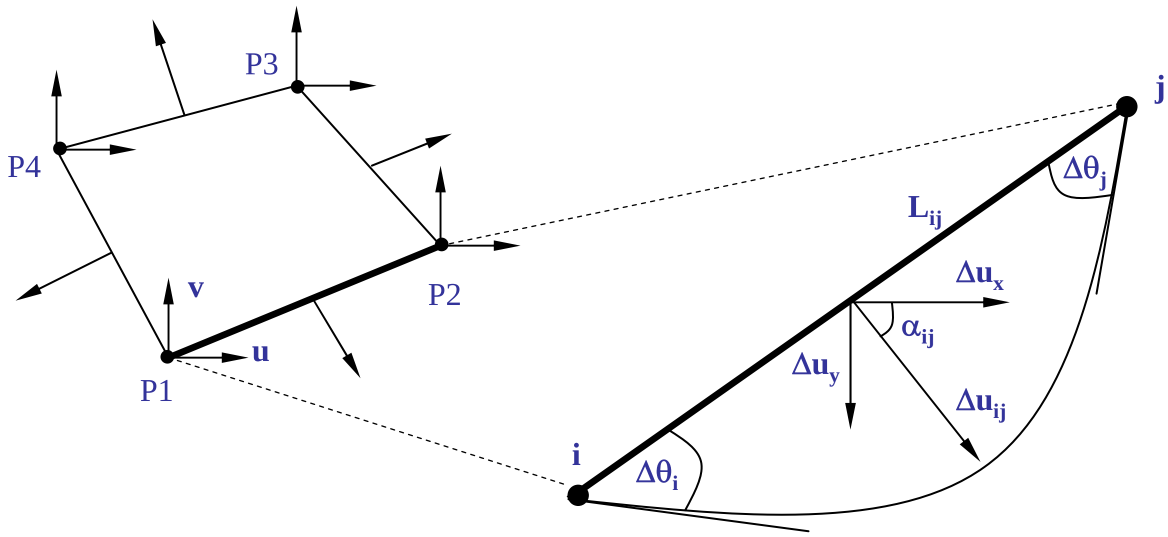
$$v(r,s) = \sum_{i=5}^{8} \mathbf{N_i}\,(r,s)\,v_i$$

Interpolation functions (i = 5:8)

$$\mathbf{N_5 = (1\text{-}r^2)\,(1\text{-}s)\,/\,2} \qquad \mathbf{N_6 = (1+r)\,(1\text{-}s^2)\,/\,2}$$

$$\mathbf{N_7 = (1\text{-}r^2)\,(1+s)\,/\,2} \qquad \mathbf{N_8 = (1\text{-}r)\,(1\text{-}s^2)\,/\,2}$$






i = 1, 2, 3, 4

j = 1, 2, 3, 4

$$\Delta u_{ij} = L_{ij}/8\,(\Delta\theta_j - \Delta\theta_i)$$

$$\Delta u_x = \cos\alpha_{ij}\,\Delta u_{ij} = \cos\alpha_{ij}\,L_{ij}/8\,(\Delta\theta_j - \Delta\theta_i)$$

$$\Delta u_y = \sin\alpha_{ij}\,\Delta u_{ij} = \sin\alpha_{ij}\,L_{ij}/8\,(\Delta\theta_j - \Delta\theta_i)$$

$$u(r,s) = \sum_{i=1}^{4} \mathbf{N_i}\,(r,s)\,u_i + \sum_{i=5}^{8} N_{\mathbf{i}}\,(r,s)\,\Delta\theta_i$$

$$v(r,s) = \sum_{i=1}^{4} \mathbf{N_i}\,(r,s)\,v_i + \sum_{i=5}^{8} N_{\mathbf{i}}\,(r,s)\,\Delta\theta_i$$

$$\delta^T = [u_1, v_1, \Delta\theta_1, u_2, v_2, \Delta\theta_2, u_3, v_3, \Delta\theta_3, u_4, v_4, \Delta\theta_4]$$

# STRAIN-DISPLACEMENT EQUATIONS

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y} \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

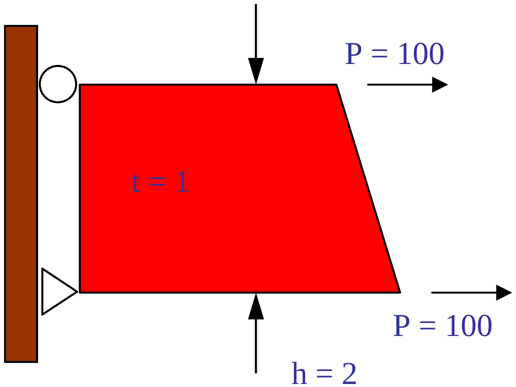
$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \Delta\theta \end{bmatrix}$$

$$\overline{\mathbf{B}}_{12} = \mathbf{B}_{12} - \frac{1}{A} \int \mathbf{B}_{12} \, dA$$

## Stiffness Matrix

$$\overline{\mathbf{K}} = \int \mathbf{B}^{\mathbf{T}} \mathbf{D} \mathbf{B} \, dV$$

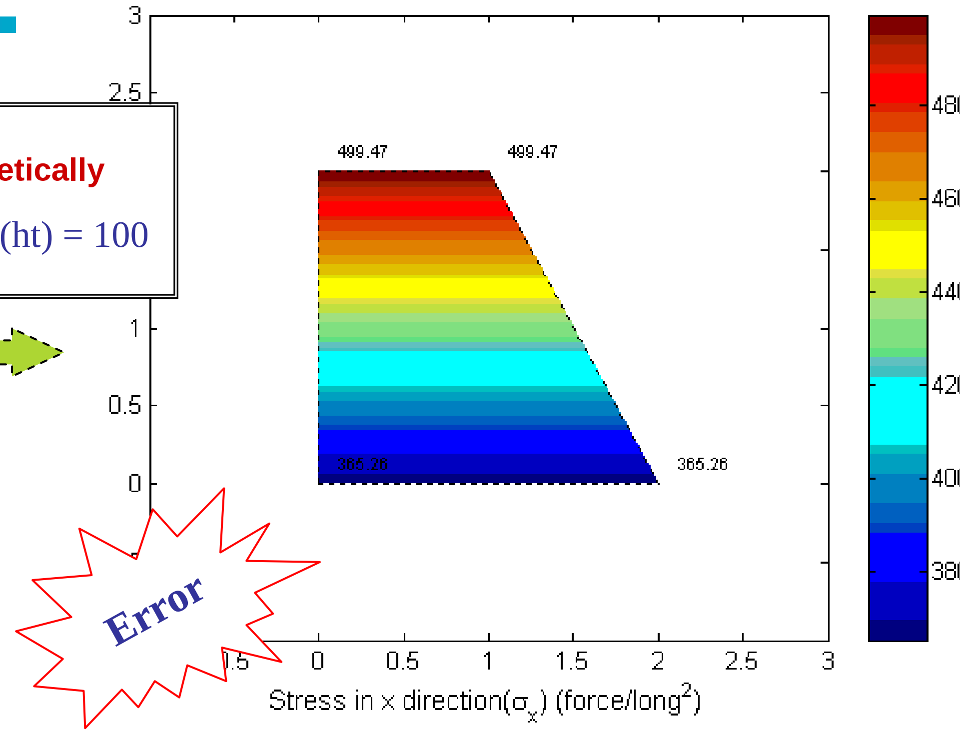
# One element test




Patch test verification
( Bruce Irons, 1972)

**Theoretically**

$\sigma_x = 2P/(ht) = 100$

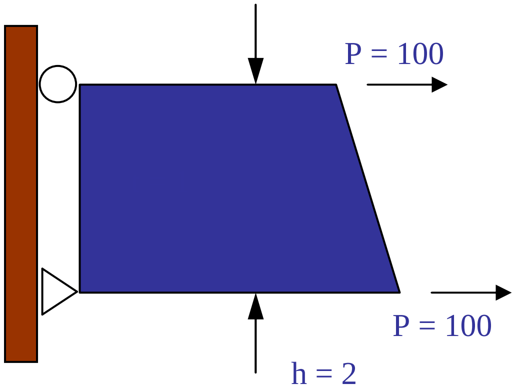



## Correction to avoid shear locking

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \mathbf{B_{11}} & \mathbf{B_{12}} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \Delta\theta \end{bmatrix}$$

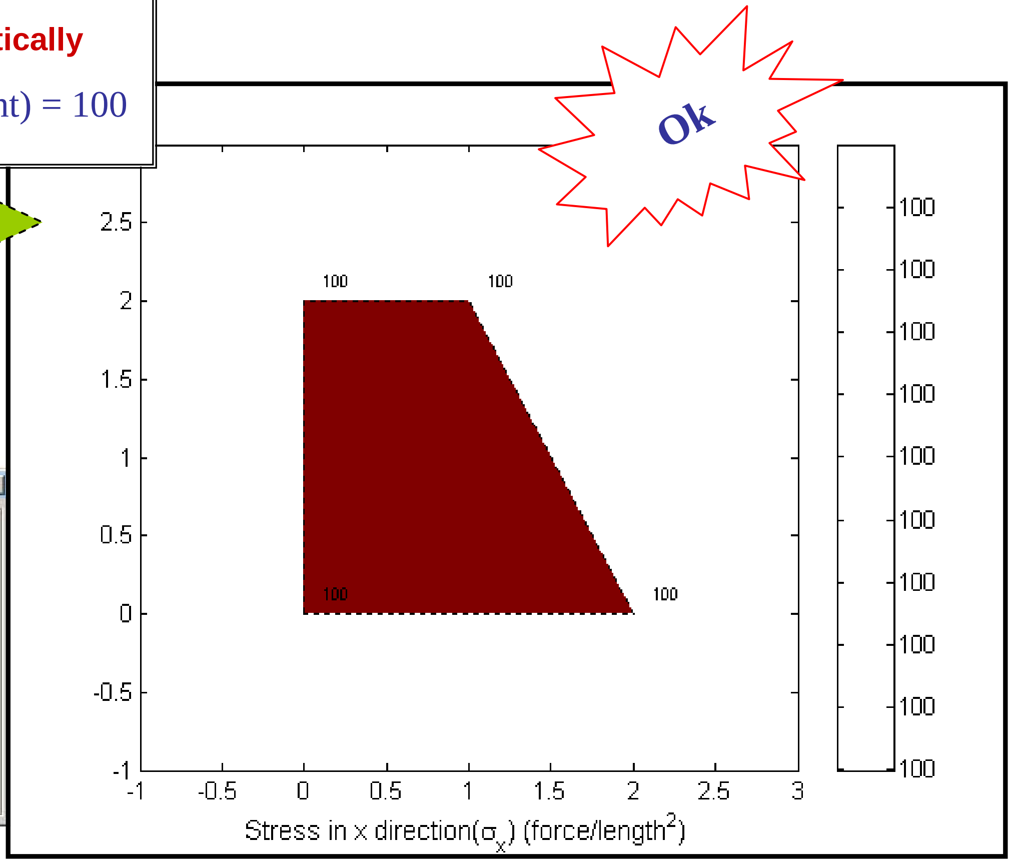
$$B_{12} = B_{12} - 1/A \int B_{12}\, dA$$

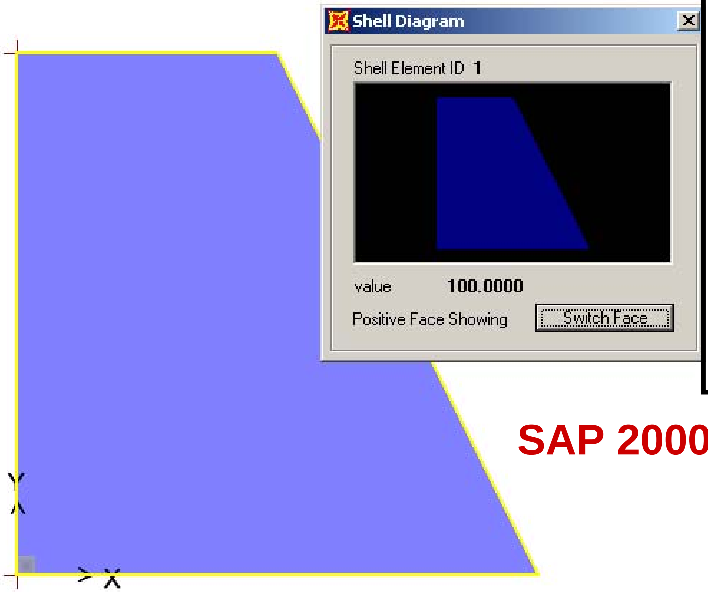
# One element test (applying correction)




**Theoretically**

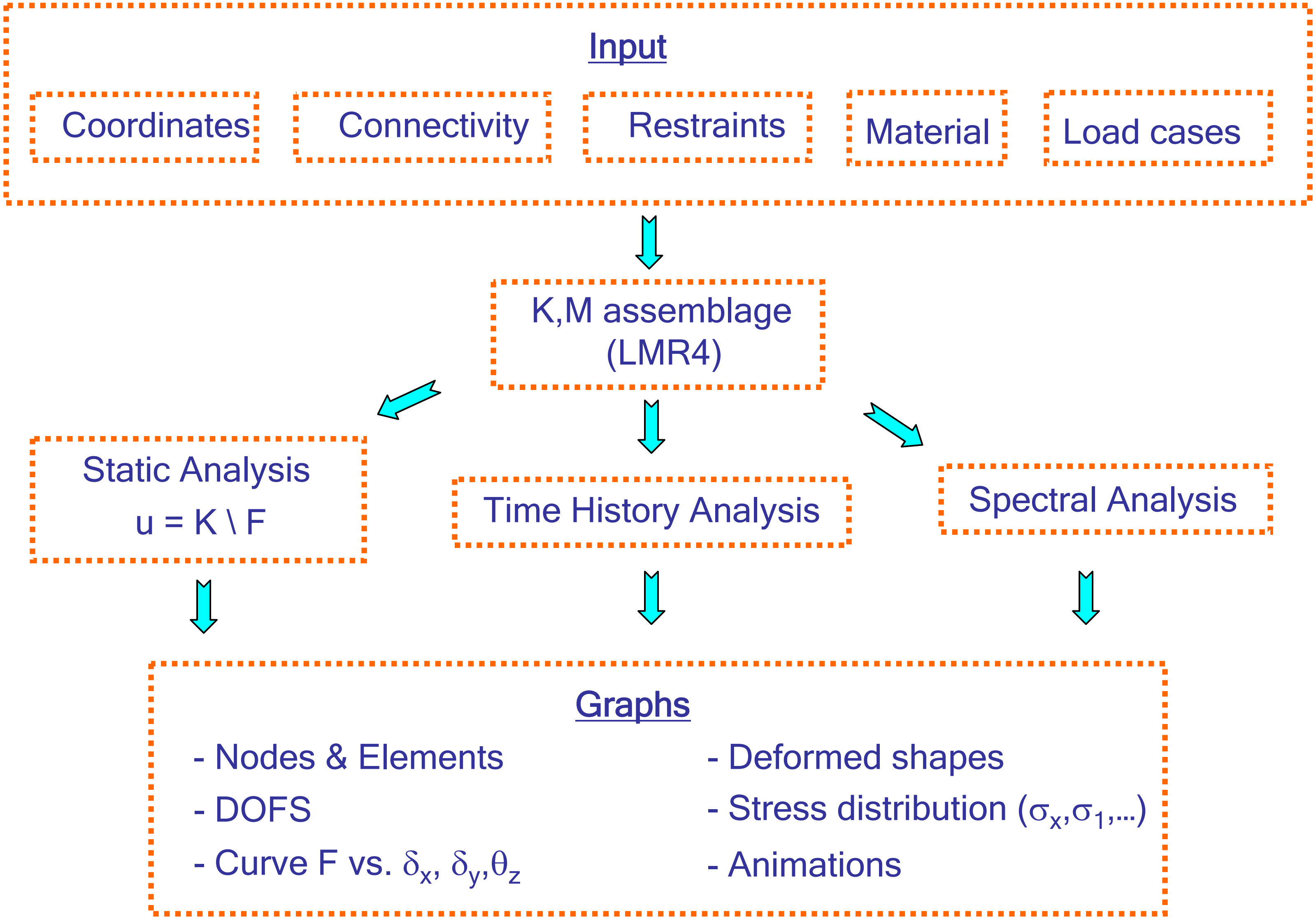
$\sigma_x = 2P/(ht) = 100$



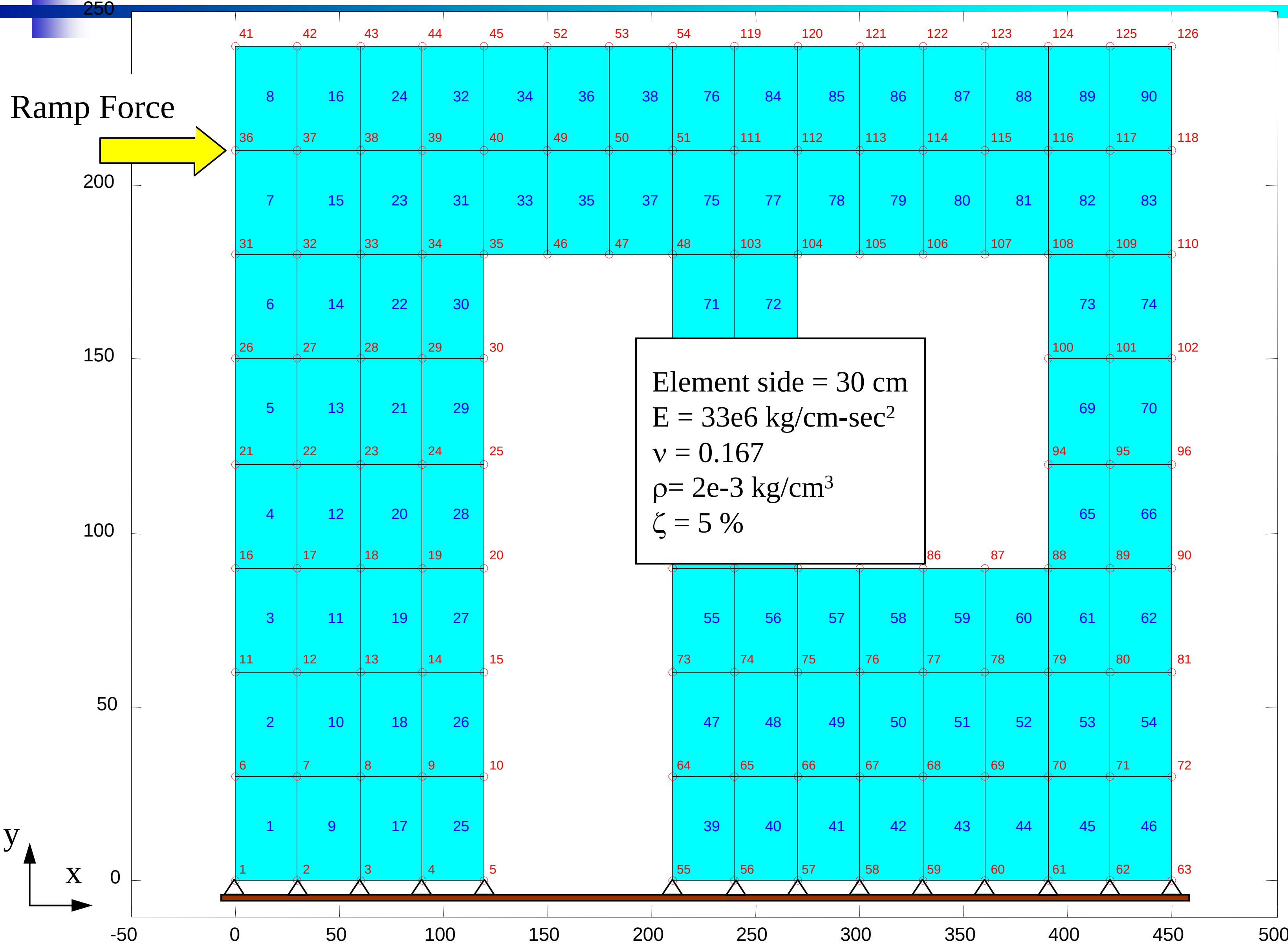




**SAP 2000**

# Implementation of Elastic Static and Dynamic Analyses

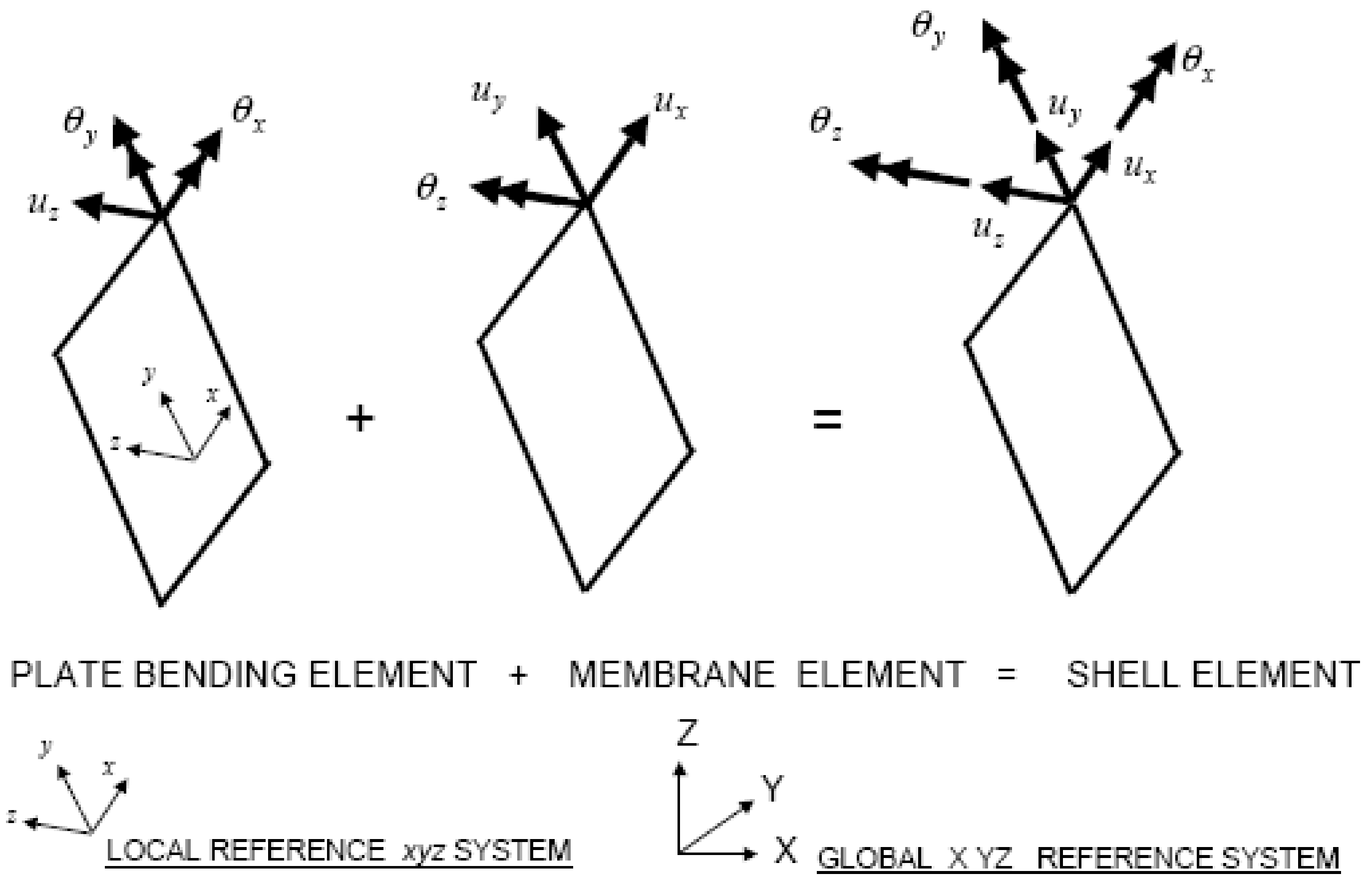


# A wall of masonry

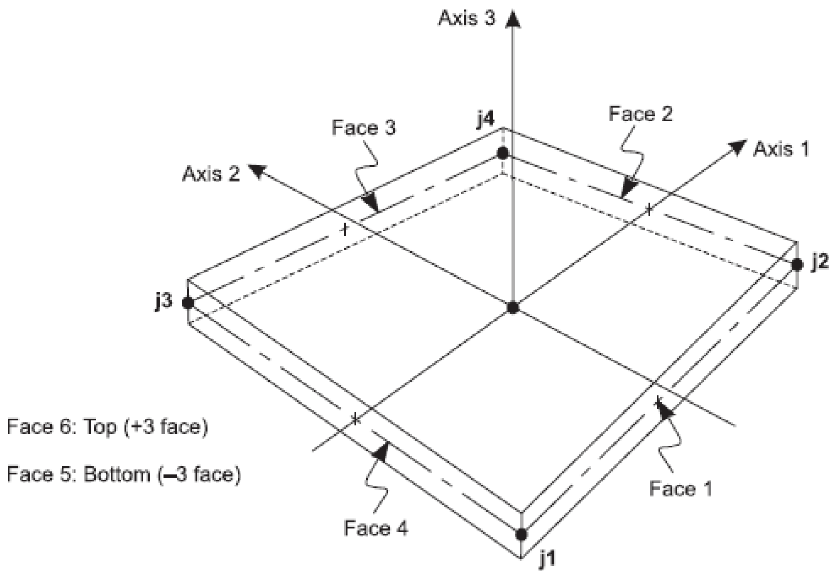


# SHELL ELEMENTS

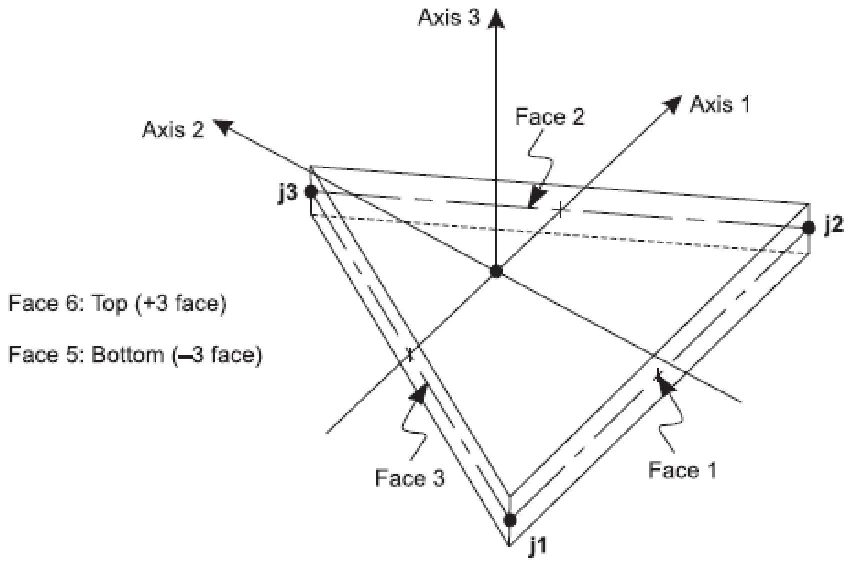
# A SIMPLE QUADRILATERAL SHELL ELEMENT



# APPLICATION CONSIDERATION



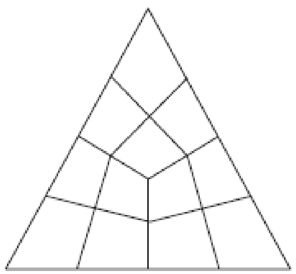

Four-node Quadrilateral Shell Element



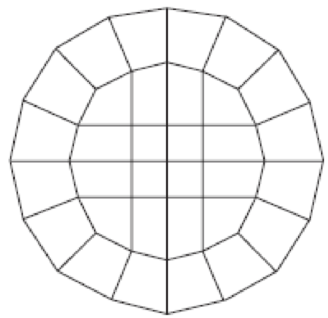

Three-node Triangular Shell Element

*Shell Element Joint Connectivity and Face Definitions*

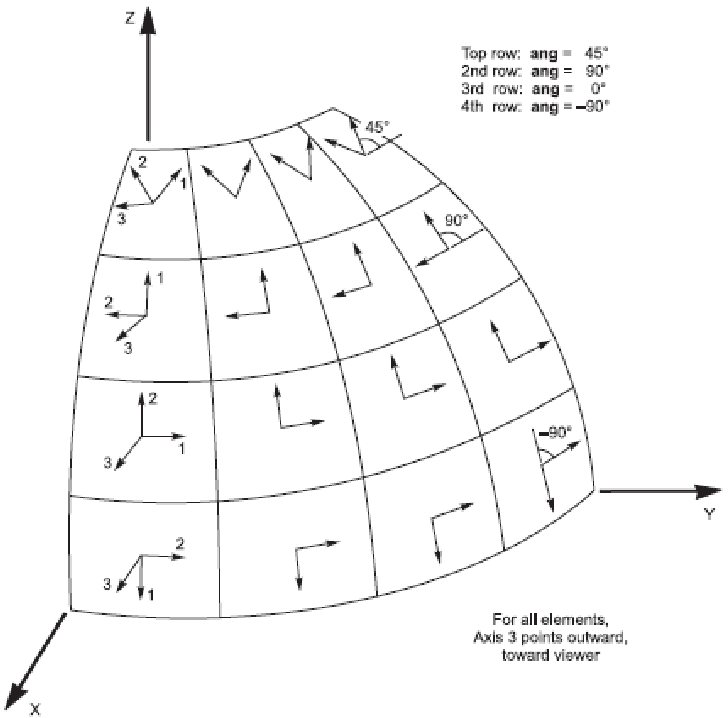
# Mesh Examples Using the Quadrilateral Shell Element



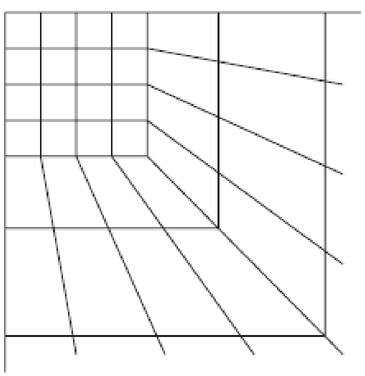
Triangular Region



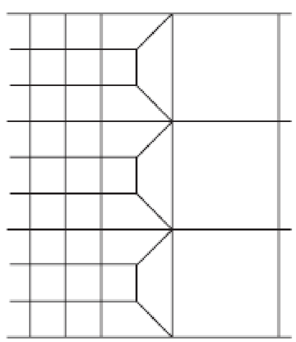
Circular Region





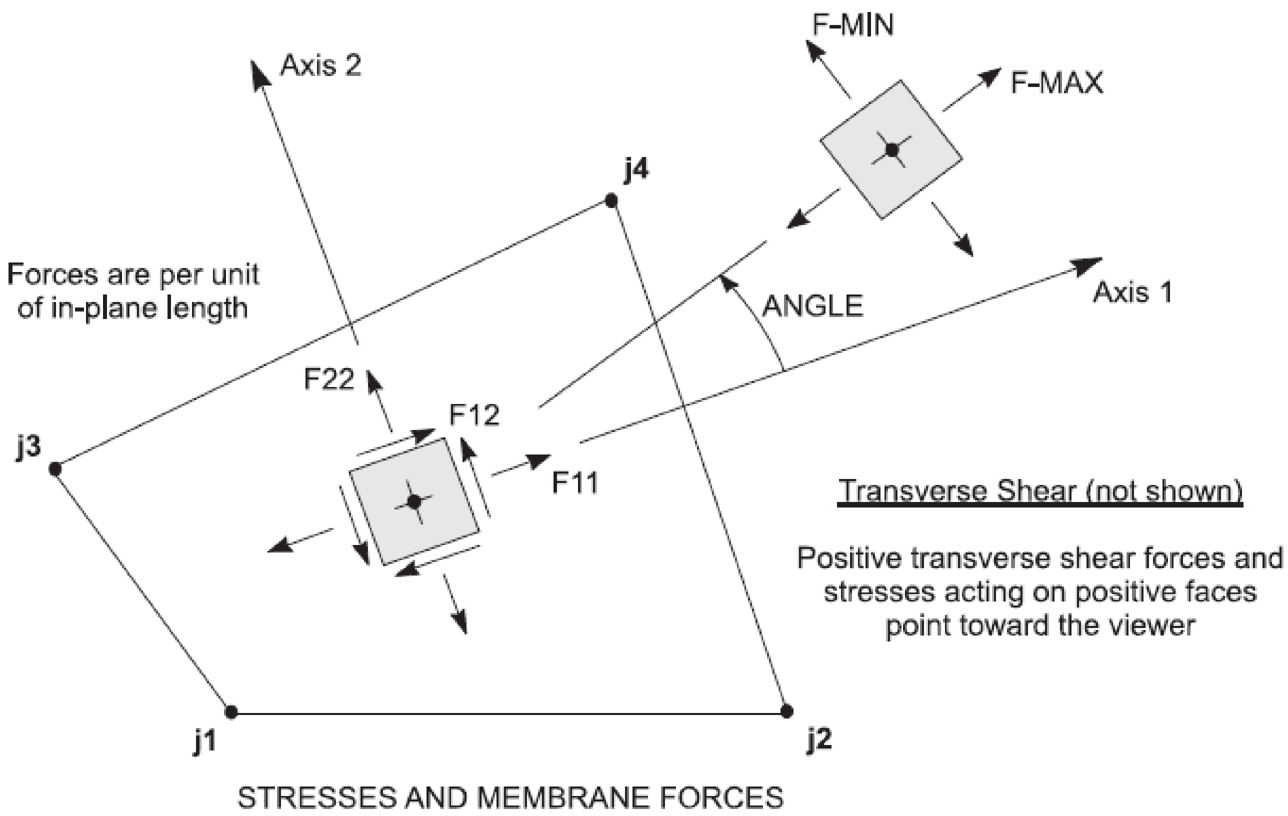


Infinite Region



Mesh Transition

# Shell Element Stresses and Internal Forces



# Shell Element Stresses and Internal Forces

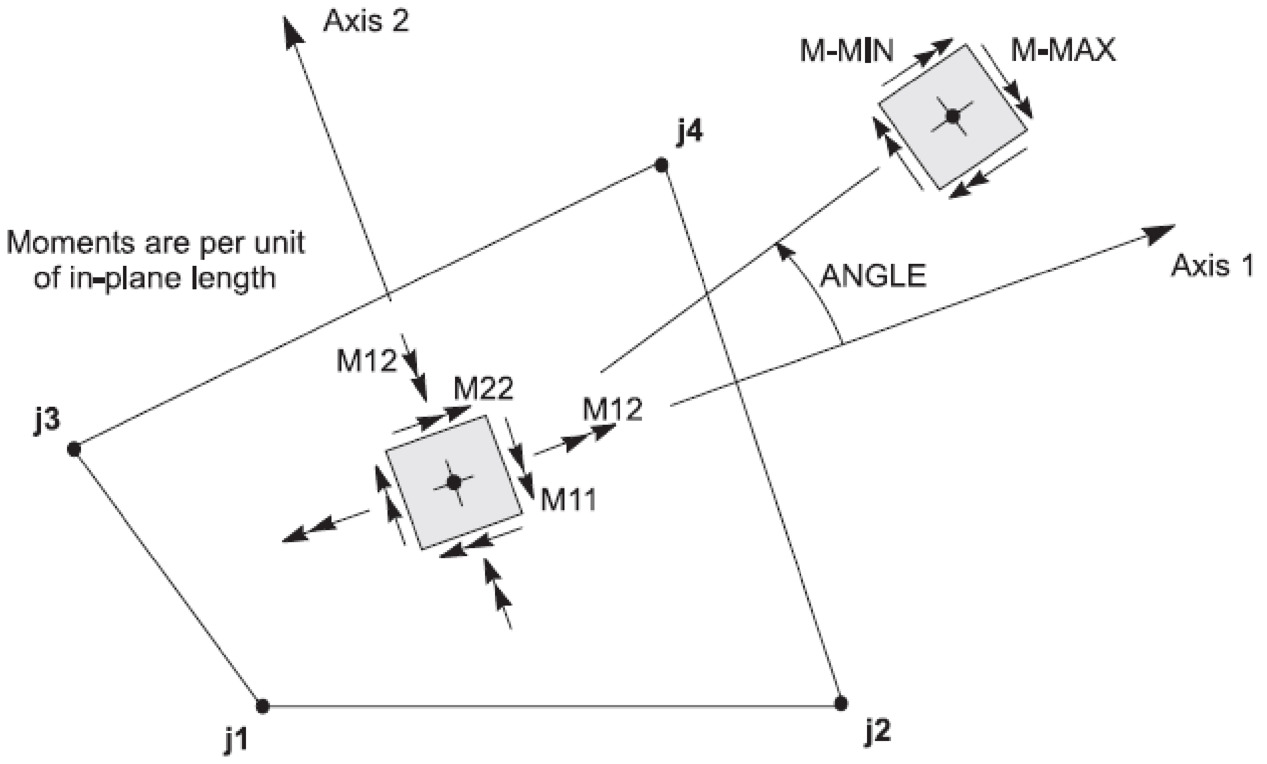



PLATE BENDING AND TWISTING MOMENTS

# Internal Force and Stress

Membrane direct forces:

$$F_{11} = \int_{-th/2}^{+th/2} \sigma_{11}\, dx_3$$

$$F_{22} = \int_{-th/2}^{+th/2} \sigma_{22}\, dx_3$$

Membrane shear force:

$$F_{12} = \int_{-th/2}^{+th/2} \sigma_{12}\, dx_3$$

Plate bending moments:

$$M_{11} = -\int_{-thb/2}^{+thb/2} t\, \sigma_{11}\, dx_3$$

$$M_{22} = -\int_{-thb/2}^{+thb/2} t\, \sigma_{22}\, dx_3$$

Plate twisting moment:

$$M_{12} = -\int_{-thb/2}^{+thb/2} t\, \sigma_{12}\, dx_3$$

Plate transverse shear forces:

$$V_{13} = \int_{-thb/2}^{+thb/2} \sigma_{13}\, dx_3$$

$$V_{23} = \int_{-thb/2}^{+thb/2} \sigma_{23}\, dx_3$$

# Internal Force-Stress Relationship

$$\sigma_{11} = \frac{F_{11}}{\mathrm{th}} - \frac{12 M_{11}}{\mathrm{thb}^3} x_3$$

$$\sigma_{22} = \frac{F_{22}}{\mathrm{th}} - \frac{12 M_{22}}{\mathrm{thb}^3} x_3$$

$$\sigma_{12} = \frac{F_{12}}{\mathrm{th}} - \frac{12 M_{12}}{\mathrm{thb}^3} x_3$$

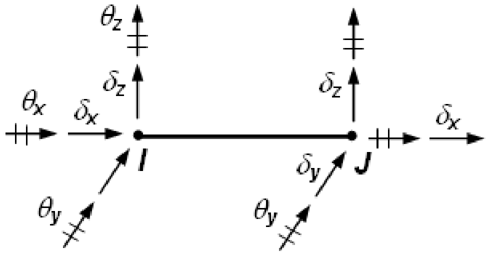
$$\sigma_{13} = \frac{V_{13}}{\mathrm{thb}}$$

$$\sigma_{23} = \frac{V_{23}}{\mathrm{thb}}$$

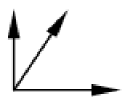
$$\sigma_{33} = 0$$

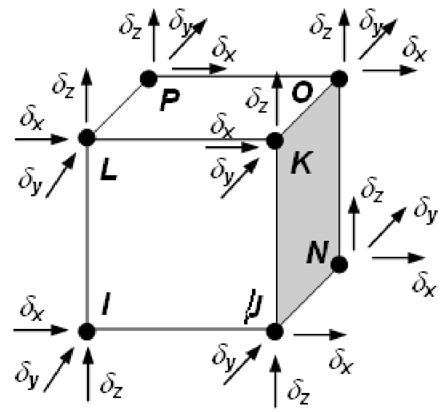
# MODELING THE GEOMETRY

# GEOMETRY AND DOFS OF NUMERICAL ANALYSIS ELEMENTS

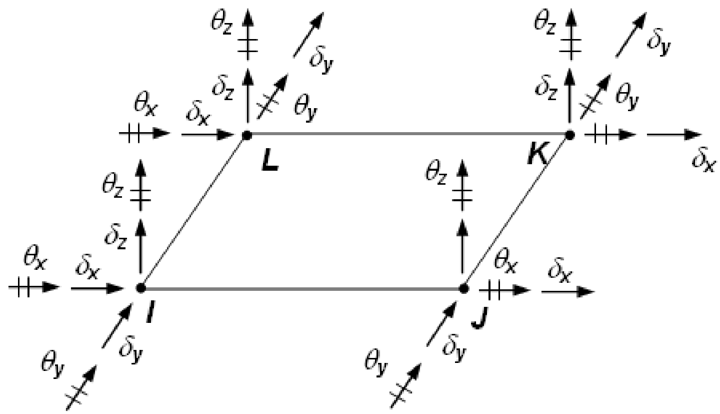



Frame (12 DOFS)



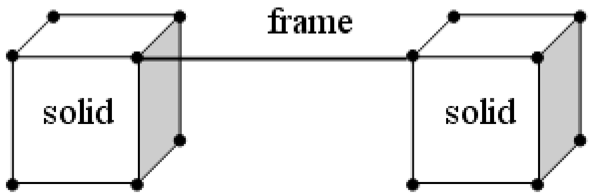


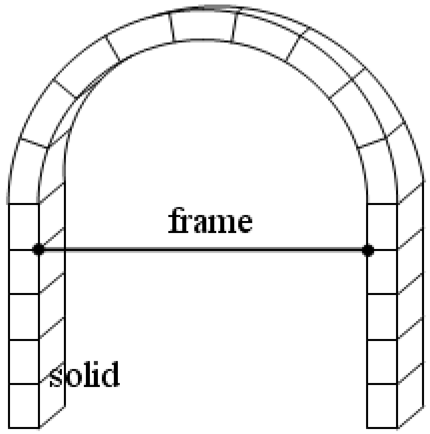

Solid (24 DOFS)



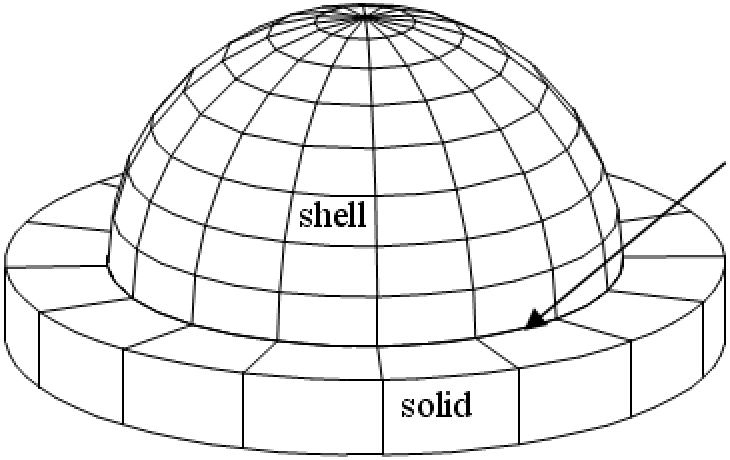

Shell (24 DOFS)

# FRAME-SOLID CONNECTION



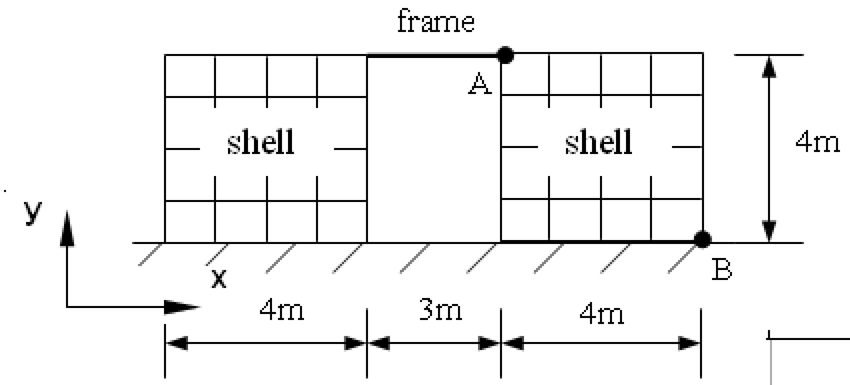



# SHELL-SOLID CONNECTION




Connectivity between shell and solid elements in typical modeling of domes

# FRAME-SHELL CONNECTION




Load case 1: Distributed horizontal and vertical loads along the upper side of every panel (total $P_v$=$P_h$=100 ton)

Shell properties:

E = 25000 ton/m$^2$

$\nu$ = 0.2

Thickness = 1 m

Frame properties:

EI = 20e5 ton-m$^2$

Area = ∞

| *Mesh* | *Load case 1* | | |
|---|---|---|---|
| | $\delta_x$*(m)* | $\delta_y$*(m)* | $\theta_z$*(rad)* |
| ***5 x 5*** | *0.280* | *0.092* | *-0.042* |
| ***10 x 10*** | *0.286* | *0.096* | *-0.005* |
| ***20 x 20*** | *0.288* | *0.099* | *0.049* |