Saint-Venant's Principe of the " Hole in Plate " Problem

Jian-zhong Zhao
Geophysics Department
College of Resource, Environment and Earth Sciences
Yunnan University
Kunming, Yunnan, China

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Abstract

The problem of the infinite plate with a central hole loaded by an equilibrium system of forces is generalized and its formulation of Special Saint-Venant's Principle is established. It is essential to develop mathematical theories of Special Saint-Venant's Principle one by one if Elasticity has to be constructed to be rational, logical, rigorous and secure mechanics.

`Keywords` : Saint-Venant's Principe, proof, provability, decay, formulation
`AMS` Subject Classifications: 74-02, 74G50

1 Introduction

Saint-Venant's Principle is essential and fundamental in Elasticity (See Ref.[1] and Ref.[2]). Boussinesq and Love announce statements of Saint-Venant's Principe (See Ref.[3] and Ref.[4]), but Mises and Sternberg argue, by citing counterexamples, that the statements are not clear, suggesting that Saint-Venant's Principle should be proved or given a mathematical formulation (See Ref.[5] and Ref.[6]). Truesdell asserts that if Saint-Venant's Principle of equipollent loads is true, it must be a mathematical consequence of the general equations of Linear Elasticity (See Ref.[7]).

There is no doubt that mathematical proof of Saint-Venant's Principle has become an academic attraction for contributors and much effort has been made for exploring its mysterious implications or deciphering its puzzle. Zanaboni "proved" a theorem trying to concern Saint-Venant's Principle in terms of work and energy (See Refs.[8],[9],[10]). However, Zhao argues that Zanaboni's theorem is false (See Ref.[11]). The work published by Toupin cites more counterexamples to explain that Love's statement is false, and then establishes a formulation of energy decay, which is considered as "a precise mathematical formulation and proof" of Saint-Venant's Principle for the elastic cylinder (See Refs.[12], [13]). Furthermore, Toupin's work seems to set up an example followed by a large number of papers to establish Toupin-type energy decay formulae for branches of continuum mechanics. Since 1965 the concept of energy decay suggested by Toupin has been widely accepted by authors, and various techniques have been developed to construct inequalities of Toupin-type decay of energy which are spread widely in continuum mechanics. Especially, the theorem given by Berdichevskii is considered as a generalization of Toupin's theorem (See Ref.[14]). Horgan and Knowles reviewed the development (See Refs.[15],[16],[17]). However, Zhao points out that Toupin's theory is not a strict mathematical proof, and Toupin's Theorem is not an exact mathematical formulation, of Saint-Venant's Principle. Interestingly and significantly, Saint-Venant's Principle stated by Love is disproved mathematically from Toupin's Theorem, so Toupin's Theorem is mathematically inconsistent with Saint-Venant's Principle (See Ref.[11]).

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Boussinesq and Love and points out that Special Saint-Venant's Principle or Modified Saint-Venant's Principle can be proved or formulated though Saint-Venant's Principle in its general form stated by Boussinesq or Love is not true (See Ref.[11]).

Saint-Venant's Principle is applied without proof here and there in the literature of Elasticity. It is essential to supplement the literature with mathematical proof or formulation of Special Saint-Venant's Principle or Modified Saint-Venant's Principle of elastic problems one by one unless Elasticity is not to be constructed to be rational, logical, rigorous and secure mechanics.

We discuss the problem of the infinite plate with a central hole loaded by an equilibrium system of forces, and prove its Special Saint-Venant's Principle in this paper.

2 Love's Statement of Saint-Venant's Principle and Its Provability

Love's Statement : "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." (See Ref.[4])

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Love, but argues by mathematical analysis that Saint-Venant's decay of strains (then stresses) described by Love's statement can be proved true by special formulating or adding supplementary conditions to the problems discussed (See Ref.[11]).

3 Saint-Venant's Principle of the Problem of the Effect of the Circular Hole in the Plate Discussed by Timoshenko and Goodier

The problem is discussed by Timoshenko and Goodier by using Saint-Venanat's Principle without proof (See Ref.[18]). Now we establish its formulation of Saint-Venanat's Principle.

Before cutting the hole, the stress distribution on the circle in the plate submitted to a uniform tension of magnitude S in the x direction is

$$(\sigma_r)_{r=a} = S\cos^2\theta \ = \ 1/2 \ S(1+\cos 2\theta), \tag{1}$$

$$(\tau_{r\theta})_{r=a} = \ -1/2 \ S\sin 2\theta. \tag{2}$$

Cutting and constructing the hole means applying an equilibrium system of forces, which is

$$r=a: \quad \sigma_r = -S\cos^2\theta = -1/2 \ S(1+\cos 2\theta), \tag{3}$$

$$\tau_{r\theta} = 1/2 \ S\sin 2\theta, \tag{4}$$

to the circle to balance the stresses expressed by Eq.(1) and Eq.(2) so that the hole becomes free from load.

On the circle $r=b$, the stresses should tend to zero, that is,

$$r=b: \quad \lim_{b\to\infty}\sigma_r = 0, \ \ \lim_{b\to\infty}\tau_{r\theta} = 0. \tag{5}$$

Let the stress function

$$\phi(r,\theta) = \varphi_0(r) + \varphi_2(r)\cos 2\theta. \tag{6}$$

Putting Eq.(6) into

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}) \ \phi(r,\theta) = 0, \tag{7}$$

we find the solution

$$\varphi_0(r) = A_0 lnr + B_0 r^2 lnr + C_0 r^2 + D_0, \tag{8}$$

$$\varphi_2(r) = A_2 r^2 + B_2 r^4 + C_2 \frac{1}{r^2} + D_2. \tag{9}$$

Then

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{A_0}{r^2} + B_0(1+2lnr) + 2C_0 - (2A_2 + \frac{6C_2}{r^4} + \frac{4D_2}{r^2})\cos 2\theta, \tag{10}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = -\frac{A_0}{r^2} + B_0(3+2lnr) + 2C_0 + (2A_2 + 12B_2 r^2 + \frac{6C_2}{r^4})\cos 2\theta, \tag{11}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \phi}{\partial \theta}) = (2A_2 + 6B_2 r^2 - \frac{6C_2}{r^4} - \frac{2D_2}{r^2})\sin 2\theta. \tag{12}$$

Considering Eq.(5),

$$B_0 = C_0 = A_2 = B_2 = 0, \tag{13}$$

then

$$\sigma_r = \frac{A_0}{r^2} - (\frac{6C_2}{r^4} + \frac{4D_2}{r^2})\cos 2\theta, \tag{14}$$

$$\sigma_\theta = -\frac{A_0}{r^2} + \frac{6C_2}{r^4}\cos 2\theta, \tag{15}$$

$$\tau_{r\theta} = -(\frac{6C_2}{r^4} + \frac{2D_2}{r^2})\sin 2\theta. \tag{16}$$

From Eq.(3), (4), (14) and (16), at $r = a$,

$$\frac{A_0}{a^2} - (\frac{6C_2}{a^4} + \frac{4D_2}{a^2})\cos 2\theta \;=\; -1/2\ S(1+\cos 2\theta) \tag{17}$$

$$-(\frac{6C_2}{a^4} + \frac{2D_2}{a^2})\sin 2\theta \;=\; 1/2\ S\sin 2\theta \tag{18}$$

Solving Eq.(17) and (18), we have

$$A_0 = -\frac{1}{2}Sa^2, \quad C_2 = -\frac{1}{4}Sa^4, \quad D_2 = \frac{1}{2}Sa^2 \tag{19}$$

Putting Eq.(19) into Eq.(14), (15) and (16), we have the final solution:

$$\sigma_r = -\frac{1}{2}S\frac{a^2}{r^2} + S(\frac{3}{2}\frac{a^4}{r^4} - 2\frac{a^2}{r^2})\cos 2\theta, \tag{20}$$

$$\sigma_\theta = \frac{1}{2}S\frac{a^2}{r^2} - \frac{3}{2}S\frac{a^4}{r^4}\cos 2\theta, \tag{21}$$

$$\tau_{r\theta} = S(\frac{3}{2}\frac{a^4}{r^4} - \frac{a2}{r^2})\sin 2\theta. \tag{22}$$

Equations (20), (21) and (22) indicate decay of stresses with r , and are the formulation of Saint-Venant's Principle of the problem of the small hole in the plate discussed by Timoshenko and Goodier.

4 Saint-Venant's Principle of Generalized " Hole in Plate " Problem

4.1 Generalized " Hole in Plate " Problem

Suppose an infinite plate with a small hole at its center loaded by an equilibrium system of forces, otherwise the plate would be free. Let the stress function be formulated as

$$\phi(r,\theta) = \sum_{n=0}^{\infty}[f_n(r)\sin(n\theta) + g_n(r)\cos(n\theta)]. \tag{23}$$

It is obvious that the stress function of Eq.(6) in the previous section is a special case of Eq.(23).

4.2 Stress Function of Generalized " Hole in Plate " Problem

Putting Eq.(23) into Eq.(7) and considering Eq.(5), we have

$$\begin{aligned}\phi(r,\theta) &= C_0 lnr + \frac{1}{r}(A_1\sin\theta + C_1\cos\theta) + rlnr(B_1\sin\theta + D_1\cos\theta) \\ &+ \sum_{n=2}^{\infty}\{\frac{1}{r^n}[A_n\sin(n\theta) + C_n\cos(n\theta)] + \frac{1}{r^{n-2}}[B_n\sin(n\theta) \\ &+ D_n\cos(n\theta)]\}\end{aligned} \tag{24}$$

4.3 Stresses of Generalized "Hole in Plate" Problem

From Eq.(24) we find that

$$\begin{aligned}\sigma_r &= \frac{C_0}{r^2} - \frac{2}{r^3}(A_1\sin\theta + C_1\cos\theta) + \frac{1}{r}(B_1\sin\theta + D_1\cos\theta) \\ &+ \sum_{n=2}^{\infty}\{\frac{-n(n+1)}{r^{n+2}}[A_n\sin(n\theta) + C_n\cos(n\theta)] \\ &- \frac{(n-1)(n+2)}{r^n}[B_n\sin(n\theta) + D_n\cos(n\theta)]\},\end{aligned} \tag{25}$$

$$\begin{aligned}\sigma_\theta &= -\frac{C_0}{r^2} + \frac{2}{r^3}(A_1\sin\theta + C_1\cos\theta) + \frac{1}{r}(B_1\sin\theta + D_1\cos\theta) \\ &+ \sum_{n=2}^{\infty}\{\frac{n(n+1)}{r^{n+2}}[A_n\sin(n\theta) + C_n\cos(n\theta)] \\ &+ \frac{(n-2)(n-1)}{r^n}[B_n\sin(n\theta) + D_n\cos(n\theta)]\},\end{aligned} \tag{26}$$

$$
\begin{aligned}
\tau_{r\theta} &= \frac{2}{r^3}(A_1\cos\theta - C_1\sin\theta) + \frac{1}{r}(-B_1\cos\theta + D_1 sin\theta) \\
&+ \sum_{n=2}^{\infty}\{\frac{n(n+1)}{r^{n+2}}[A_n\cos(n\theta) - C_n\sin(n\theta)] \\
&+ \frac{n(n-1)}{r^n}[B_n\cos(n\theta) - D_n\sin(n\theta)]\}. \qquad (27)
\end{aligned}
$$

Equations (25) and (27) satisfy Eq.(5).

4.4 Saint-Venant's Principle of Generalized "Hole in Plate" Problem

From Eq. (25) and (27), the equilibrium system of forces on the hole ($r = a$) is

$$
\begin{aligned}
\sigma_r &= \frac{C_0}{a^2} - \frac{2}{a^3}(A_1\sin\theta + C_1\cos\theta) + \frac{1}{a}(B_1\sin\theta + D_1\cos\theta) \\
&+ \sum_{n=2}^{\infty}\{\frac{-n(n+1)}{a^{n+2}}[A_n\sin(n\theta) + C_n\cos(n\theta)] \\
&- \frac{(n-1)(n+2)}{a^n}[B_n\sin(n\theta) + D_n\cos(n\theta)]\}, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{r\theta} &= \frac{2}{a^3}(A_1\cos\theta - C_1\sin\theta) + \frac{1}{a}(-B_1\cos\theta + D_1 sin\theta) \\
&+ \sum_{n=2}^{\infty}\{\frac{n(n+1)}{a^{n+2}}[A_n\cos(n\theta) - C_n\sin(n\theta)] \\
&+ \frac{n(n-1)}{a^n}[B_n\cos(n\theta) - D_n\sin(n\theta)]\}. \qquad (29)
\end{aligned}
$$

From Eq. (25), (26) and (27) we have

$$
\begin{aligned}
\lim_{r\to\infty}\sigma_r &= 0, \\
\lim_{r\to\infty}\sigma_\theta &= 0 \\
and \quad \lim_{r\to\infty}\tau_{r\theta} &= 0. \qquad (30)
\end{aligned}
$$

We prove Saint-Venant's Principle of Generalized "Hole in Plate" Problem by the end equations in terms of Eqs.(30). Equations (25), (26) and (27) are the formulation of Saint-Venant's decay of effect of the equilibrium system of forces loaded on the small hole. Equations (5), (28) and (29) are the conditions for Saint-Venant's Principle to be valid for the Problem.

5 Conclusion

Special Saint-Venant's Principle of the effect of the equilibrium system of forces loaded on the small central hole in the infinite plate is proved in this paper.

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