1 Errata: Metamaterials and Waves in Composites

1. **Section 5.6.1: Ensemble averaging:** The definition of ensemble averaged displacements should read

$$p_j(\mathbf{x})\, \langle \mathbf{u}(\mathbf{x},t)\rangle_j = \int_{\mathcal{A}} \chi_j(\mathbf{x};\alpha)\, \mathbf{u}(\mathbf{x},t;\alpha)\, \mathrm{d}[p(\alpha)]\,.$$

2. **Section 5.6.2: The Willis equations:** The convolution operator was not clearly defined in the text. Here is an example of how the operator should be interpreted when both space and time are involved

$$\begin{aligned}\boldsymbol{G} \star (\nabla\cdot\boldsymbol{S} - \dot{\mathbf{m}}) &= \int_\Omega \mathrm{d}\mathbf{x}'\, [\boldsymbol{G}(\mathbf{x}'-\mathbf{x},t) \star (\nabla\cdot\boldsymbol{S} - \dot{\mathbf{m}})(\mathbf{x}',t)] \\ &= \int_\Omega \mathrm{d}\mathbf{x}' \left[\int_{-\infty}^{t} \mathrm{d}\tau\, \boldsymbol{G}(\mathbf{x}'-\mathbf{x},t-\tau)\cdot(\nabla\cdot\boldsymbol{S} - \dot{\mathbf{m}})(\mathbf{x}',\tau)\right].\end{aligned}$$

3. **Section 5.7: Forces** The expression for the elastic force in equation (5.111) has the wrong sign. The correct equation is

$$\mathbf{f}_4^{\text{elastic}} = hk\,[(1-A_3)\boldsymbol{1} + (A_3 - A_1)\boldsymbol{D}_{41} + (1-A_2)\boldsymbol{D}_{42}]\cdot\mathbf{u}_4\,.$$

4. **Section 5.7: A simpler model:** The dispersion relation given in the text is wrong. The correct equation is

$$\begin{aligned}&9c^2k^2 - 3k\cos(hk_2)\left[2c^2k[1+2\cos(hk_1)] + (1+c^2)^2\delta h\omega^2\sin^2\left(\frac{hk_2}{2}\right)\right] \\ &\quad + \cos^2(hk_2)\left[c^2k^2[1+2\cos(hk_1)]^2 + (1+c^2)^2\delta hk\omega^2[1+2\cos(hk_1)]\sin^2\left(\frac{hk_2}{2}\right) + (1+c^2)^2\delta^2h^2\omega^4\sin^4\left(\frac{hk_2}{2}\right)\right] \\ &\quad + (1+c^2)^2(\delta h - 2m)^2\omega^4\sin^4\left(\frac{hk_2}{2}\right)\sin^2(hk_2) = 0\,.\end{aligned}$$

When $\delta = 0$, we have

$$9c^2k^2 - 6c^2k^2\cos(hk_2)[1+2\cos(hk_1)] + c^2k^2\cos^2(hk_2)[1+2\cos(hk_1)]^2 + 4(1+c^2)^2m^2w^4\sin^4\left(\frac{hk_2}{2}\right)\sin^2(hk_2) = 0\,.$$

In the limit $h \to 0$, the above equation is degenerate.

5. **Section 5.7: A simpler model:** The inertial traction in equation (5.113) has the wrong sign. The correct expressions are
   (a) "A force $\mathbf{t}_4^{\text{inertial}} = -\mathbf{f}_4^{\text{inertial}}/(2h)$ needs to act ..."
   (b) Equation (5.113):

$$\mathbf{t}_4^{\text{inertial}} = \frac{m\omega^2}{2}\left(cu_{4y}\,\mathbf{e}_1 + \frac{u_{4x}}{c}\mathbf{e}_2\right).$$

6. **Section 5.7: A simpler model:** The expression for the elastic force at node 4 is wrong. The correct expression is

$$\begin{aligned}\mathbf{f}_4^{\text{elastic}} =& hk\left[\{1 - e^{-ihk_1}\cos(hk_2)\}u_{4x} + ie^{-ihk_1}\sin(hk_2)u_{4y}\right]\mathbf{e}_1 \\ &+ hk\left[ie^{-ihk_1}\sin(hk_2)u_{4x} + \left\{2 - e^{ihk_2} - e^{-ihk_1}\cos(hk_2)\right\}u_{4y}\right]\mathbf{e}_2\,.\end{aligned}$$

7. **Section 5.7: Momentum and stress:** The sign of the momentum density wrong. The correct expression is

$$\begin{aligned}\mathbf{p} = &-\frac{i\omega}{4ch}\left[c(e^{ihk_2}+1)\delta h\, u_{4x} + (e^{ihk_2}-1)(\delta h - 2m)u_{4y}\right]\mathbf{e}_1 \\ &-\frac{i\omega}{4h}\left[c(e^{ihk_2}-1)(\delta h - 2m)u_{4x} + (e^{ihk_2}+1)\delta h\, u_{4y}\right]\mathbf{e}_2\,.\end{aligned}$$

The same mistake is propagated in equation (5.115) and the reamining expressions for the momentum density. The correct expressions are:

(a) Equation (5.115):

$$\mathbf{p} = -\frac{i\omega}{2c}\left[(c\delta\, u_1 - ik_2 m\, u_2)\,\mathbf{e}_1 + c^2\left(-ik_2 m\, u_1 + \frac{\delta}{c}\, u_2\right)\right]\mathbf{e}_2\,.$$

(b) Expression in terms of displaceemnt gradients:

$$\mathbf{p} = -\frac{i\omega}{2c}\left[(c\delta\, u_1 - m\, u_{2,2})\,\mathbf{e}_1 + c^2\left(-m\, u_{1,2} + \frac{\delta}{c}\, u_2\right)\right]\mathbf{e}_2$$

(c) Equation (5.116):

$$\mathbf{p} = -\frac{1}{2}\left(-\frac{im\omega}{c}\, u_{2,2} - \delta\, v_1\right)\mathbf{e}_1 + \frac{1}{2}(-im\omega c\, u_{1,2} - \delta\, v_2)\,\mathbf{e}_2\,.$$

(d) Matrix form of equation (5.116):

$$\underline{\underline{\mathrm{p}}} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & im\omega c^{-1} & \delta & 0 \\ 0 & im\omega c & 0 & 0 & 0 & \delta \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \\ v_1 \\ v_2 \end{bmatrix}.$$

This error shows that quick checks of correctness of results in metamaterial calculations can be difficult because of the possibility of negative mass. In this case, such a check is possible because the momentum and the velocity should have the same sign at low frequencies.

8. **Section 5.7: Momentum and stress:** At equation (5.118) and nearby the signs are wrong.

   (a) The correct sentence is "... $\mathbf{t}_3^{\text{inertial}} = \mathbf{t}_1^{\text{inertial}} = 0$, we have $\sigma_{11}^I = 0$, $\sigma_{12}^I = 0$, $\sigma_{21}^I = m\omega^2 c u_{4y}/2$, and $\sigma_{22}^I = m\omega^2 u_{4x}/(2c)$. "

   (b) The correct form of Equation (5.118) is:

$$\boldsymbol{\sigma}^I \equiv [\sigma]^I = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ m\omega^2 c\, u_2 & m\omega^2\, u_1/c \end{bmatrix}$$

   (c) The correct form of the full matrix form of equation (5.118) is

$$[\sigma]^I = \begin{bmatrix} \sigma_{11}^I \\ \sigma_{12}^I \\ \sigma_{21}^I \\ \sigma_{22}^E \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ m\omega^2 c\, u_2 \\ m\omega^2 c^{-1}\, u_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & i\omega m\, c \\ i\omega m\, c^{-1} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

9. **Section 5.7: Momentum and stress:** At equation (5.120) and nearby the signs are wrong.

   (a) The correct form of the stress-(displacement gradient-velocity) relation is

$$[\sigma] = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} hk & 0 & 0 & hk & 0 & 0 \\ 0 & hk & hk & 0 & 0 & 0 \\ 0 & hk & hk & 0 & 0 & i\omega mc \\ hk & 0 & 0 & 3hk & i\omega mc^{-1} & 0 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \\ v_1 \\ v_2 \end{bmatrix}$$

   (b) The correct form of the (stress-momentum)-(displacement gradient-velocity) relation is

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{12} \\ \sigma_{22} \\ p_1 \\ p_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} hk & 0 & 0 & hk & 0 & 0 \\ 0 & hk & hk & 0 & 0 & i\omega mc \\ 0 & hk & hk & 0 & 0 & 0 \\ hk & 0 & 0 & 3hk & i\omega mc^{-1} & 0 \\ 0 & 0 & 0 & im\omega c^{-1} & \delta & 0 \\ 0 & im\omega c & 0 & 0 & 0 & \delta \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \\ v_1 \\ v_2 \end{bmatrix}$$

10. **Section 6.2.3: Examples of transformation-based cloaking** and **Problem 6.2**: The Greenleaf-Lassas-Uhlman map should be read in the notation of the rest of the text as:

$$\mathbf{x}(\mathbf{X}) = \begin{cases} \left(\frac{\|\mathbf{X}\|}{2} + 1\right) \frac{\mathbf{X}}{\|\mathbf{X}\|} & \text{if } \|\mathbf{X}\| < 2 \\ \mathbf{X} & \text{if } \|\mathbf{X}\| > 2\,. \end{cases}$$

11. **Section 6.3: p. 229**: The expressions for $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ in spherical coordinates should read

$$\begin{aligned} \mathbf{g}_1 &= \frac{\partial \mathbf{x}}{\partial \theta^1} = \frac{\partial \mathbf{x}}{\partial r} = \ldots \\ \mathbf{g}_2 &= \frac{\partial \mathbf{x}}{\partial \theta^2} = \frac{\partial \mathbf{x}}{\partial \theta} = \ldots \\ \mathbf{g}_3 &= \frac{\partial \mathbf{x}}{\partial \theta^3} = \frac{\partial \mathbf{x}}{\partial \phi} = \ldots\,. \end{aligned}$$

12. **Section 6.3: p. 230**: In the expression for the deformation gradient in spherical coordinates, $\mathbf{x}$ has been assumed to be a general vector rather than a position vector. That is the reason for us writing $\mathbf{x} = x_r\mathbf{e}_r + x_\theta\mathbf{e}_\theta + x_z\mathbf{e}_z$ instead of the correct form $\mathbf{x} = x_r\mathbf{e}_r$. The general vector form has been shown so that the reader can apply the same idea to other coordinate systems.

13. **Section 6.3: Equation (6.20)**: The first part of the equation should read

$$\boldsymbol{F}\cdot\boldsymbol{F}^T = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & \left(\frac{R_1 + tR}{R}\right)^2 & 0 \\ 0 & 0 & \left(\frac{R_1 + tR}{R}\right)^2 \end{bmatrix} = \ldots$$

14. **Section 6.4: Equation (6.27), p. 234** The expression for the transformed bulk modulus is confusing, i.e., is the $\kappa_x(\mathbf{x})$ on the left a tensor? The correct expression is

$$\frac{1}{\kappa_x(\mathbf{x})} = \frac{1}{\kappa(\mathbf{X})\,\det(\boldsymbol{F})}\,.$$

15. **Section 6.4 p. 240** The expression $\kappa\mathbf{x} = \kappa(r)\boldsymbol{1}$ should be replaced with $\kappa(\mathbf{x}) = \kappa(r)$. This quantity is a assumed to be a scalar in the rest of the book.

16. **Problem 6.7:** The $\cos\theta$ in the expression should be $\cos\phi$ for consistency with the rest of the book. Recall that θ is the azimuthal angle in our book and not the polar angle as implied in the paper.

17. **Problem 6.8:** The expression $\kappa(\mathbf{x}) = \kappa(\mathbf{r}) = \kappa(r)\boldsymbol{1}$ should be replaced with $\kappa(\mathbf{x}) = \kappa(r)$. The bulk modulus is a scalar quantity. Also note that $\nabla^2_\perp p$ is only the angular part of the Laplace-Beltrami operator and not the full operator.

18. **Problem 6.9:** Replace $\boldsymbol{\Sigma} = \kappa_X\boldsymbol{\nabla}_X\cdot\mathbf{U}$ with $\boldsymbol{\Sigma} = \kappa_X(\boldsymbol{\nabla}_X\cdot\mathbf{U})\boldsymbol{1}$. Note that the $\boldsymbol{V}$ in this problem is not the same as the left stretch tensor in continuum mechanics.

19. **Problem 7.2:** The problem is worded incorrectly. It should read: "Verify that the matrix $\boldsymbol{A}$ defined by the action $\boldsymbol{A}\cdot\langle\mathbf{e}_0\rangle := (\mathbf{k}\cdot\langle\mathbf{e}_0\rangle)\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\langle\mathbf{e}_0\rangle$ can be expressed as $\boldsymbol{A} = \mathbf{k}\otimes\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\boldsymbol{1}$. Show that the eigenvalues of $-\boldsymbol{A}$ are ..."

20. **Section 8.1.1, p. 285:** The solution to the Helmholtz equation given in equation (8.1) appears to be different from the one in equation (2.63). This is just a matter of convention as the solution of Problem 8.1 from the exercises indicates.

21. **Section 8.1.1, p. 285:** The requirement that if $\mathrm{Re}(k_j) \geq 0$ we must have $\mathrm{Im}(k_j) \leq 0$ is only valid if $x_3 > 0$. Treat Problem 8.1 accordingly.

22. **Problem 8.4** The properties of the medium should be changed to

$$\varepsilon(x) = \varepsilon_0\left[1 + (x - x_0)^3\right],\ \mu(x) = \mu_0\left[1 + (x - x_0)^3\right].$$

23. **Section 8.6.1 Equation (8.123):** The expression for $C^{\text{eff}}_{2222} = C^{\text{eff}}_{3333}$ is wrong. The correct expression is

$$C^{\text{eff}}_{2222} = C^{\text{eff}}_{3333} = \left\langle \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1}$$

The same applies to Problem 8.9.

24. **Problem 8.7:** Replace θ by θ_t in the expression.

25. **Problem 8.8:** The elastic property values in the problem are incorrect. The problem should read "... with Young's moduli $E_x = 9$ GPa, $E_y = E_z = 140$ GPa, Poisson's ratios $\nu_{xy} = 0.1$, $\nu_{yz} = \nu_{xz} = 0.3$, and shear moduli $G_{xz} = G_{yz} = 7$ GPa."

26. **Chapter 8, p. 309:** The quantity SfC should read $\mathbf{C}$.

27. **Chapter 8, p. 309:** The quantities $a(y), f(y)$ should read $a(z), f(z)$.