An Introduction to Metamaterials and Waves in Composites: Solutions Manual

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Chapter 1

Solutions for Exercises in Chapter 1

Problem 1.1 The equation of balance of the flow of a physical quantity $f(\mathbf{x}, t)$ through a body Ω with surface Γ can be expressed in the form

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\,\mathrm{dV}\right] = \int_\Gamma f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\,\mathrm{dA} + \int_\Gamma g(\mathbf{x},t)\,\mathrm{dA} + \int_\Omega h(\mathbf{x},t)\,\mathrm{dV}\,. \quad (1.1)$$

where $g(\mathbf{x}, t)$ be sources on the surface of the body, $h(\mathbf{x}, t)$ are sources inside the body, $\mathbf{n}(\mathbf{x}, t)$ is the outward unit normal to the surface Γ, $\mathbf{v}(\mathbf{x}, t)$ is the velocity of the physical particles that carry the physical quantity that is flowing, and u_n is the speed at which the bounding surface Γ is moving in the direction $\mathbf{n}$. Using the above balance relation show that the balance of linear momentum of a linear elastic body can be expressed as

$$\rho\,\dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0 \quad (1.2)$$

where $\rho(\mathbf{x}, t)$ is the mass density, $\mathbf{v}(\mathbf{x}, t)$ is the spatial velocity, $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress, and $\rho\,\mathbf{b}$ is the body force density.

Solution 1.1: The general equation for the balance of a physical quantity is

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\,\mathrm{dV}\right] = \int_\Gamma f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\,\mathrm{dA} + \int_\Gamma g(\mathbf{x},t)\,\mathrm{dA} + \int_\Omega h(\mathbf{x},t)\,\mathrm{dV}\,. \quad (1.3)$$

In this case the physical quantity of interest is the momentum density, i.e., $f(\mathbf{x}, t) = \rho(\mathbf{x}, t)\,\mathbf{v}(\mathbf{x}, t)$. The source of momentum flux at the surface is the surface traction, i.e., $g(\mathbf{x}, t) = \mathbf{t}$. The source of momentum inside the body is the body force, i.e., $h(\mathbf{x}, t) = \rho(\mathbf{x}, t)\,\mathbf{b}(\mathbf{x}, t)$. Therefore, we have

$$\frac{d}{dt}\left[\int_\Omega \rho\,\mathbf{v}\,\mathrm{dV}\right] = \int_\Gamma \rho\,\mathbf{v}[u_n - \mathbf{v}\cdot\mathbf{n}]\,\mathrm{dA} + \int_\Gamma \mathbf{t}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,. \quad (1.4)$$

The surface tractions are related to the Cauchy stress by

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}\,. \quad (1.5)$$

Therefore,

$$\frac{d}{dt}\left[\int_\Omega \rho\,\mathbf{v}\,\mathrm{dV}\right] = \int_\Gamma \rho\,\mathbf{v}[u_n - \mathbf{v}\cdot\mathbf{n}]\,\mathrm{dA} + \int_\Gamma \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,. \quad (1.6)$$

Let us assume that Ω is an arbitrary fixed control volume. Then,

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\mathrm{dV} = -\int_\Gamma \rho\,\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,\mathrm{dA} + \int_\Gamma \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,. \quad (1.7)$$

Now, from the definition of the tensor product we have (for all vectors $\mathbf{a}$)

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\,\mathbf{u}\,. \tag{1.8}$$

Therefore,

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\mathrm{dV} = -\int_\Gamma \rho\,(\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{n}\,\mathrm{dA} + \int_\Gamma \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{dA} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV}\,. \tag{1.9}$$

Using the divergence theorem

$$\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{v}\,\mathrm{dV} = \int_\Gamma \mathbf{v}\cdot\mathbf{n}\,\mathrm{dA} \tag{1.10}$$

we have

$$\int_\Omega \frac{\partial}{\partial t}(\rho\,\mathbf{v})\,\mathrm{dV} = -\int_\Omega \boldsymbol{\nabla}\cdot[\rho\,(\mathbf{v}\otimes\mathbf{v})]\,\mathrm{dV} + \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\,\mathrm{dV} + \int_\Omega \rho\,\mathbf{b}\,\mathrm{dV} \tag{1.11}$$

or,

$$\int_\Omega \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \boldsymbol{\nabla}\cdot[(\rho\,\mathbf{v})\otimes\mathbf{v}] - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right]\,\mathrm{dV} = 0\,. \tag{1.12}$$

Since Ω is arbitrary, we have

$$\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \boldsymbol{\nabla}\cdot[(\rho\,\mathbf{v})\otimes\mathbf{v}] - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,. \tag{1.13}$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{v} \tag{1.14}$$

we get

$$\frac{\partial\rho}{\partial t}\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + (\boldsymbol{\nabla}\cdot\mathbf{v})(\rho\mathbf{v}) + \boldsymbol{\nabla}(\rho\,\mathbf{v})\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0 \tag{1.15}$$

or,

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\rho\,\mathbf{v})\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0 \tag{1.16}$$

Using the identity

$$\boldsymbol{\nabla}(\varphi\,\mathbf{v}) = \varphi\,\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes(\boldsymbol{\nabla}\varphi) \tag{1.17}$$

we get

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + [\rho\,\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\otimes(\boldsymbol{\nabla}\rho)]\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0 \tag{1.18}$$

From the definition

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{v})\,\mathbf{u} \tag{1.19}$$

we have

$$[\mathbf{v}\otimes(\boldsymbol{\nabla}\rho)]\cdot\mathbf{v} = [\mathbf{v}\cdot(\boldsymbol{\nabla}\rho)]\,\mathbf{v}\,. \tag{1.20}$$

Hence,

$$\left[\frac{\partial\rho}{\partial t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} + [\mathbf{v}\cdot(\boldsymbol{\nabla}\rho)]\,\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0 \tag{1.21}$$

or,

$$\left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{v}\right]\mathbf{v} + \rho\,\frac{\partial\mathbf{v}}{\partial t} + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b} = 0\,. \tag{1.22}$$

The material time derivative of ρ is defined as

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\rho\cdot\mathbf{v}\,. \tag{1.23}$$

Therefore,

$$[\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v}]\, \mathbf{v} + \rho\, \frac{\partial \mathbf{v}}{\partial t} + \rho\, \boldsymbol{\nabla} \mathbf{v} \cdot \mathbf{v} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b} = 0\,. \tag{1.24}$$

From the balance of mass, we have

$$\dot{\rho} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} = 0\,. \tag{1.25}$$

Therefore,

$$\rho\, \frac{\partial \mathbf{v}}{\partial t} + \rho\, \boldsymbol{\nabla} \mathbf{v} \cdot \mathbf{v} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b} = 0\,. \tag{1.26}$$

The material time derivative of **v** is defined as

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \mathbf{v} \cdot \mathbf{v}\,. \tag{1.27}$$

Hence,

$$\boxed{\rho\, \dot{\mathbf{v}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho\, \mathbf{b} = 0\,.} \tag{1.28}$$

Problem 1.2 Assume that there are no surface couples on Γ or body couples in Ω. Use the angular momentum density as the conserved quantity in the general balance relation from Problem 2.1, i.e., $f = \mathbf{x} \times (\rho\,\mathbf{v})$, to show that the balance of angular momentum can be expressed as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T\,. \tag{1.29}$$

Solution 1.2: If the physical quantity to be conserved the angular momentum density, i.e., $f = \mathbf{x} \times (\rho\,\mathbf{v})$, the angular momentum source at the surface is $g = \mathbf{x} \times \mathbf{t}$ and the angular momentum source inside the body is $h = \mathbf{x} \times (\rho\,\mathbf{b})$. The angular momentum and moments are calculated with respect to a fixed origin. Hence we have

$$\frac{d}{dt}\left[\int_\Omega \mathbf{x} \times (\rho\,\mathbf{v})\,\text{dV}\right] = \int_\Gamma [\mathbf{x} \times (\rho\,\mathbf{v})][u_n - \mathbf{v}\cdot\mathbf{n}]\,\text{dA} + \int_\Gamma \mathbf{x} \times \mathbf{t}\,\text{dA} + \int_\Omega \mathbf{x} \times (\rho\,\mathbf{b})\,\text{dV}\,. \tag{1.30}$$

Assuming that Ω is a control volume, we have

$$\int_\Omega \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\,\text{dV} = -\int_\Gamma [\mathbf{x} \times (\rho\,\mathbf{v})][\mathbf{v}\cdot\mathbf{n}]\,\text{dA} + \int_\Gamma \mathbf{x} \times \mathbf{t}\,\text{dA} + \int_\Omega \mathbf{x} \times (\rho\,\mathbf{b})\,\text{dV}\,. \tag{1.31}$$

Using the definition of a tensor product we can write

$$[\mathbf{x} \times (\rho\,\mathbf{v})][\mathbf{v}\cdot\mathbf{n}] = [[\mathbf{x} \times (\rho\,\mathbf{v})] \otimes \mathbf{v}] \cdot \mathbf{n}\,. \tag{1.32}$$

Also, $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$. Therefore we have

$$\int_\Omega \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\,\text{dV} = -\int_\Gamma [[\mathbf{x} \times (\rho\,\mathbf{v})] \otimes \mathbf{v}] \cdot \mathbf{n}\,\text{dA} + \int_\Gamma \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\,\text{dA} + \int_\Omega \mathbf{x} \times (\rho\,\mathbf{b})\,\text{dV}\,. \tag{1.33}$$

Using the divergence theorem, we get

$$\int_\Omega \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\,\text{dV} = -\int_\Omega \boldsymbol{\nabla} \cdot [[\mathbf{x} \times (\rho\,\mathbf{v})] \otimes \mathbf{v}]\,\text{dV} + \int_\Gamma \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\,\text{dA} + \int_\Omega \mathbf{x} \times (\rho\,\mathbf{b})\,\text{dV}\,. \tag{1.34}$$

To convert the surface integral in the above equation into a volume integral, it is convenient to use index notation. Thus,

$$\left[\int_\Gamma \mathbf{x} \times (\boldsymbol{\sigma}\cdot\mathbf{n})\,\text{dA}\right]_i = \int_\Gamma e_{ijk}\,x_j\,\sigma_{kl}\,n_l\,\text{dA} = \int_\Gamma A_{il}\,n_l\,\text{dA} = \int_\Gamma \boldsymbol{A}\cdot\mathbf{n}\,\text{dA} \tag{1.35}$$

where $[\,]_i$ represents the i-th component of the vector. Using the divergence theorem

$$\int_\Gamma \boldsymbol{A}\cdot\mathbf{n}\,\text{dA} = \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{A}\,\text{dV} = \int_\Omega \frac{\partial A_{il}}{\partial x_l}\text{dV} = \int_\Omega \frac{\partial}{\partial x_l}(e_{ijk}\,x_j\,\sigma_{kl})\,\text{dV}\,. \tag{1.36}$$

Differentiating,

$$\begin{aligned}\int_\Gamma \boldsymbol{A}\cdot\mathbf{n}\,\text{dA} &= \int_\Omega \left[e_{ijk}\,\delta_{jl}\,\sigma_{kl} + e_{ijk}\,x_j\,\frac{\partial \sigma_{kl}}{\partial x_l}\right]\,\text{dV}\\ &= \int_\Omega \left[e_{ijk}\,\sigma_{kj} + e_{ijk}\,x_j\,\frac{\partial \sigma_{kl}}{\partial x_l}\right]\,\text{dV}\\ &= \int_\Omega \left[e_{ijk}\,\sigma_{kj} + e_{ijk}\,x_j\,[\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}]_l\right]\,\text{dV}\,.\end{aligned} \tag{1.37}$$

Expressed in direct tensor notation,

$$\int_\Gamma \boldsymbol{A}\cdot\mathbf{n}\,\text{dA} = \int_\Omega \left[[\mathcal{E} : \boldsymbol{\sigma}^T]_i + [\mathbf{x} \times (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})]_i\right]\,\text{dV} \tag{1.38}$$

where $\mathcal{E}$ is the third-order permutation tensor. Therefore,

$$\left[\int_\Gamma \mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{dA}\right]_i == \int_\Omega \left[[\mathcal{E}:\boldsymbol{\sigma}^T]_i + [\mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})]_i\right]\,\mathrm{dV} \tag{1.39}$$

or,

$$\int_\Gamma \mathbf{x}\times(\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{dA} == \int_\Omega \left[\mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right]\,\mathrm{dV}\,. \tag{1.40}$$

The balance of angular momentum can then be written as

$$\int_\Omega \mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right]\,\mathrm{dV} = -\int_\Omega \boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}]\,\mathrm{dV} + \int_\Omega \left[\mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right]\,\mathrm{dV} + \int_\Omega \mathbf{x}\times(\rho\,\mathbf{b})\,\mathrm{dV}\,. \tag{1.41}$$

Since Ω is an arbitrary volume, we have

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v})\right] = -\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] + \mathcal{E}:\boldsymbol{\sigma}^T + \mathbf{x}\times(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}) + \mathbf{x}\times(\rho\,\mathbf{b}) \tag{1.42}$$

or,

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] + \mathcal{E}:\boldsymbol{\sigma}^T\,. \tag{1.43}$$

Using the identity,

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})\cdot\mathbf{v} \tag{1.44}$$

we get

$$\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] = (\boldsymbol{\nabla}\cdot\mathbf{v})[\mathbf{x}\times(\rho\,\mathbf{v})] + (\boldsymbol{\nabla}[\mathbf{x}\times(\rho\,\mathbf{v})])\cdot\mathbf{v}\,. \tag{1.45}$$

The second term on the right can be further simplified using index notation as follows.

$$\begin{aligned}
[(\boldsymbol{\nabla}[\mathbf{x}\times(\rho\,\mathbf{v})])\cdot\mathbf{v}]_i = [(\boldsymbol{\nabla}[\rho\,(\mathbf{x}\times\mathbf{v})])\cdot\mathbf{v}]_i &= \frac{\partial}{\partial x_l}(\rho\,e_{ijk}\,x_j\,v_k)\,v_l \\
&= e_{ijk}\left[\frac{\partial\rho}{\partial x_l}x_j\,v_k\,v_l + \rho\,\frac{\partial x_j}{\partial x_l}v_k\,v_l + \rho\,x_j\,\frac{\partial v_k}{\partial x_l}v_l\right] \\
&= (e_{ijk}\,x_j\,v_k)\,\left(\frac{\partial\rho}{\partial x_l}v_l\right) + \rho\,(e_{ijk}\,\delta_{jl}\,v_k\,v_l) + e_{ijk}\,x_j\,\left(\rho\,\frac{\partial v_k}{\partial x_l}v_l\right) \\
&= [(\mathbf{x}\times\mathbf{v})(\boldsymbol{\nabla}\rho\cdot\mathbf{v}) + \rho\,\mathbf{v}\times\mathbf{v} + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]_i \\
&= [(\mathbf{x}\times\mathbf{v})(\boldsymbol{\nabla}\rho\cdot\mathbf{v}) + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})]_i\,.
\end{aligned} \tag{1.46}$$

Therefore we can write

$$\boldsymbol{\nabla}\cdot[[\mathbf{x}\times(\rho\,\mathbf{v})]\otimes\mathbf{v}] = (\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v})\,. \tag{1.47}$$

The balance of angular momentum then takes the form

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - \mathbf{x}\times(\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T \tag{1.48}$$

or,

$$\mathbf{x}\times\left[\frac{\partial}{\partial t}(\rho\,\mathbf{v}) + \rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\,\mathbf{b}\right] = -(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) - (\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v}) + \mathcal{E}:\boldsymbol{\sigma}^T \tag{1.49}$$

or,

$$\mathbf{x}\times\left[\rho\frac{\partial\mathbf{v}}{\partial t}+\frac{\partial\rho}{\partial t}\mathbf{v}+\rho\,\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}\right]=-(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\,\mathbf{v})-(\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})+\mathcal{E}:\boldsymbol{\sigma}^T \tag{1.50}$$

The material time derivative of $\mathbf{v}$ is defined as

$$\dot{\mathbf{v}}=\frac{\partial\mathbf{v}}{\partial t}+\boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{v}\,. \tag{1.51}$$

Therefore,

$$\mathbf{x}\times[\rho\,\dot{\mathbf{v}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}]=-\mathbf{x}\times\frac{\partial\rho}{\partial t}\mathbf{v}+-(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\,\mathbf{v})-(\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})+\mathcal{E}:\boldsymbol{\sigma}^T\,. \tag{1.52}$$

Also, from the conservation of linear momentum

$$\rho\,\dot{\mathbf{v}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}-\rho\,\mathbf{b}=0\,. \tag{1.53}$$

Hence,

$$\begin{aligned}0&=\mathbf{x}\times\frac{\partial\rho}{\partial t}\mathbf{v}+(\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\,\mathbf{v})+(\boldsymbol{\nabla}\rho\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})-\mathcal{E}:\boldsymbol{\sigma}^T\\&=\left(\frac{\partial\rho}{\partial t}+\rho\boldsymbol{\nabla}\cdot\mathbf{v}+\boldsymbol{\nabla}\rho\cdot\mathbf{v}\right)(\mathbf{x}\times\mathbf{v})-\mathcal{E}:\boldsymbol{\sigma}^T\,.\end{aligned} \tag{1.54}$$

The material time derivative of ρ is defined as

$$\dot{\rho}=\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\rho\cdot\mathbf{v}\,. \tag{1.55}$$

Hence,

$$(\dot{\rho}+\rho\,\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{x}\times\mathbf{v})-\mathcal{E}:\boldsymbol{\sigma}^T=0\,. \tag{1.56}$$

From the balance of mass

$$\dot{\rho}+\rho\,\boldsymbol{\nabla}\cdot\mathbf{v}=0\,. \tag{1.57}$$

Therefore,

$$\mathcal{E}:\boldsymbol{\sigma}^T=0\,. \tag{1.58}$$

In index notation,

$$e_{ijk}\,\sigma_{kj}=0\,. \tag{1.59}$$

Expanding out, we get

$$\sigma_{12}-\sigma_{21}=0\,;\;\;\sigma_{23}-\sigma_{32}=0\,;\;\;\sigma_{31}-\sigma_{13}=0\,. \tag{1.60}$$

Hence,

$$\boxed{\boldsymbol{\sigma}=\boldsymbol{\sigma}^T} \tag{1.61}$$

Problem 1.3 Show that if $\boldsymbol{\sigma} = \mathbf{C} : \varepsilon$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{u})\,. \tag{1.62}$$

Solution 1.3: Let us express the tensors in terms of components with respect to an orthonormal basis. Then we can write

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \tag{1.63}$$

The conservation of angular momentum implies that the stress tensor is symmetric,

$$\sigma_{ij} = \sigma_{ji}\,. \tag{1.64}$$

The strain tensor is, by definition, symmetric because it is a symmetrized gradient, i.e.,

$$\varepsilon_{ij} = \varepsilon_{ji}\,. \tag{1.65}$$

These imply that $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$. The strain energy density in the elastic material is given by

$$W = \tfrac{1}{2}\,\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\,C_{ijkl}\,Ve_{kl}\,Ve_{ij}\,. \tag{1.66}$$

Since the strain energy density should not change if kl and ij are interchanged, we must have $C_{ijkl} = C_{klij}$. Now,

$$[\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}]_j = \sigma_{ij,i} = [C_{ijkl}\,\varepsilon_{kl}]_{,i} = \tfrac{1}{2}\,[C_{ijkl}\,(u_{k,l} + u_{l,k})]_{,i} \tag{1.67}$$

Using the symmetries of $\mathbf{C}$, we have $C_{ijkl}\,u_{k,l} = C_{ijkl}\,u_{l,k}$. Hence

$$[\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}]_j = [C_{ijkl}\,u_{k,l}]_{,i} = [\boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{u})]_j \tag{1.68}$$

or

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{u})\,. \tag{1.69}$$

Problem 1.4 Show that for an isotropic, homogeneous, linear elastic material with stiffness tensor

$$\mathbf{C} = \lambda \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu \, \mathbf{I} \tag{1.70}$$

the divergence of the stress can be expressed as

$$\boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{u}) = (\lambda + \mu) \, \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \, \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{u}^T) \, . \tag{1.71}$$

Solution 1.4: Let us express the stiffness tensor in terms of components with respect to an orthonormal basis. Then

$$C_{ijkl} = \lambda \, \delta_{ij} \, \delta_{kl} + \mu \, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.72}$$

and

$$C_{ijkl} \, u_{k,l} = \lambda \, u_{k,k} \, \delta_{ij} + \mu \, (u_{i,j} + u_{j,i}) \, . \tag{1.73}$$

Therefore,

$$[\boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{u})]_i = [C_{ijkl} \, u_{k,l}]_{,i} = \lambda \, u_{k,ki} \, \delta_{ij} + \mu \, (u_{i,ji} + u_{j,ii}) = (\lambda + \mu) \, u_{m,mj} + \mu \, u_{j,nn} \tag{1.74}$$

or,

$$\boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{u}) = (\lambda + \mu) \, \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \, \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{u}^T) \, . \tag{1.75}$$

Problem 1.5 Verify that the volumetric strain e is given by $\boldsymbol{\nabla} \cdot \mathbf{u}$ and that the infinitesimal rotation vector is given by $\boldsymbol{\nabla} \times \mathbf{u}$.

Solution 1.5: The volumetric strain is

$$e = \text{tr}\, \boldsymbol{\varepsilon} \tag{1.76}$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor. In an orthonormal coordinate system

$$e = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = u_{1,1} + u_{2,2} + u_{3,3} \equiv \boldsymbol{\nabla} \cdot \mathbf{u} \tag{1.77}$$

where $\mathbf{u}$ is the displacement. In general,

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right] \quad \implies \quad \text{tr}\, \boldsymbol{\varepsilon} = \tfrac{1}{2}\left[\text{tr}\, \boldsymbol{\nabla}\mathbf{u} + \text{tr}\, (\boldsymbol{\nabla}\mathbf{u})^T\right] = \boldsymbol{\nabla} \cdot \mathbf{u}\,. \tag{1.78}$$

Therefore,

$$e = \boldsymbol{\nabla} \cdot \mathbf{u} \qquad \square \tag{1.79}$$

The infinitesimal rotation vector is the axial vector that of the skew-symmetric part of $\boldsymbol{\nabla}\mathbf{u}$. The axial vector $\mathbf{w}$ of a skew-symmetric tensor $\boldsymbol{W}$ satisfies the condition $\boldsymbol{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}$ for all vectors $\mathbf{a}$. In index notation (with respect to a rectangular Cartesian basis), we can write this relation as

$$W_{ip}\, a_p = e_{ijk}\, w_j\, a_k \tag{1.80}$$

Since $e_{ijk} = -e_{ikj}$, we have

$$W_{ip}\, a_p = -e_{ikj}\, w_j\, a_k \equiv -e_{ipq}\, w_q\, a_p \quad \implies \quad W_{ip} = -e_{ipq}\, w_q\,. \tag{1.81}$$

Therefore, the relation between the components of $\boldsymbol{\omega} = 1/2(\boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}^T)$ and $\boldsymbol{\theta}$ is

$$\omega_{ij} = -e_{ijk}\, \theta_k\,. \tag{1.82}$$

Multiplying both sides by e_{pij}, we get

$$e_{pij}\, \omega_{ij} = -e_{pij}\, e_{ijk}\, \theta_k = -e_{pij}\, e_{kij}\, \theta_k\,. \tag{1.83}$$

From the identity

$$e_{ijk}\, e_{pqk} = \delta_{ip}\, \delta_{jq} - \delta_{iq}\, \delta_{jp} \tag{1.84}$$

we have

$$e_{ijk}\, e_{pjk} = \delta_{ip}\, \delta_{jj} - \delta_{ij}\, \delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip} \tag{1.85}$$

Using the above identity, we get

$$e_{pij}\, \omega_{ij} = -2\delta_{pk}\, \theta_k = -2\theta_p\,. \tag{1.86}$$

Rearranging,

$$\theta_p = -\frac{1}{2}\, e_{pij}\, \omega_{ij} \tag{1.87}$$

Now, the components of the tensor $\boldsymbol{\omega}$ with respect to a Cartesian basis are given by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.88}$$

Therefore, we may write

$$\theta_p = -\frac{1}{4} e_{pij}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.89}$$

Since the curl of a vector $\mathbf{v}$ can be written in index notation as

$$\boldsymbol{\nabla} \times \mathbf{v} = e_{ijk}\, \frac{\partial u_k}{\partial x_j}\, \mathbf{e}_i \tag{1.90}$$

we have

$$e_{pij}\,\frac{\partial u_j}{\partial x_i} = [\boldsymbol{\nabla}\times\mathbf{u}]_p \qquad \text{and} \qquad e_{pij}\,\frac{\partial u_i}{\partial x_j} = -e_{pji}\frac{\partial u_i}{\partial x_j} = -[\boldsymbol{\nabla}\times\mathbf{u}]_p \tag{1.91}$$

where $[\,]_p$ indicates the p-th component of the vector inside the square brackets.

Hence,

$$\theta_p = -\frac{1}{4}\left(-[\boldsymbol{\nabla}\times\mathbf{u}]_p - [\boldsymbol{\nabla}\times\mathbf{u}]_p\right) = \frac{1}{2}\,[\boldsymbol{\nabla}\times\mathbf{u}]_p\,. \tag{1.92}$$

Therefore,

$$\boldsymbol{\theta} = \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{u} \qquad \square \tag{1.93}$$

Problem 1.6 Show that, in the absence of a body force, the displacement potentials ϕ and $\boldsymbol{\psi}$ satisfy the wave equations

$$\frac{\partial^2 \phi}{\partial t^2} = c_p^2 \,\nabla^2 \phi \,; \quad \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} = c_s^2 \,\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{\psi})^T \,. \tag{1.94}$$

Solution 1.6: A rigorous proof can be found outlined in the textbook by Aki and Richards. Here we only discuss a sufficient condition.

Consider a Helmholtz decomposition of the displacement

$$\mathbf{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\psi} \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0 \,. \tag{1.95}$$

Recall that the wave equation in a homogeneous, isotropic, and linear elastic body in the absence of body forces can be written as

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{u})^T = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \,. \tag{1.96}$$

Plugging the decomposition of $\mathbf{u}$ into the above equation gives

$$(\lambda + \mu)\boldsymbol{\nabla}[\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\psi})] + \mu \boldsymbol{\nabla} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\psi})]^T = \rho \frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\psi}) \,. \tag{1.97}$$

Expanding out,

$$(\lambda + \mu)\boldsymbol{\nabla}[\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi] + \mu \boldsymbol{\nabla} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi)]^T + \mu \boldsymbol{\nabla} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{\psi})]^T = \rho \boldsymbol{\nabla}\left(\frac{\partial^2 \phi}{\partial t^2}\right) + \rho \boldsymbol{\nabla} \times \left(\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}\right) \,. \tag{1.98}$$

where we have used the fact that the divergence of the curl of a vector field is zero and the interchangeability of the order of differentiation between space and time. Note that $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \nabla^2 \phi$,

$$\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi)]^T = \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial^2 \phi}{\partial x_j \partial x_j}\right) = \boldsymbol{\nabla}(\nabla^2 \phi) \,. \tag{1.99}$$

We can also see that

$$\begin{aligned} \boldsymbol{\nabla} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{\psi})]^T &= e_{ijk}\psi_{k,jmm} = e_{ijk}(\boldsymbol{\nabla}\boldsymbol{\psi})_{km,jm} = e_{ijk}(\boldsymbol{\nabla}\boldsymbol{\psi})^T_{mk,mj} \\ &= e_{ijk}[\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{\psi})^T]_{k,j} = \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{\psi})^T] \,. \end{aligned} \tag{1.100}$$

Therefore,

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\nabla^2 \phi) + \mu \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{\psi})^T] = \rho \boldsymbol{\nabla}\left(\frac{\partial^2 \phi}{\partial t^2}\right) + \rho \boldsymbol{\nabla} \times \left(\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}\right) \tag{1.101}$$

or,

$$\boldsymbol{\nabla}\left[(\lambda + 2\mu)\nabla^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2}\right] + \boldsymbol{\nabla} \times \left[\mu \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{\psi})^T - \rho \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}\right] = 0 \,. \tag{1.102}$$

If we take the divergence of the above equation, the curl term vanishes and if we take the curl of the equation, the gradient term vanishes. Therefore,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\left[(\lambda + 2\mu)\nabla^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2}\right] = 0 \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left[\mu \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{\psi})^T - \rho \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}\right] = 0 \,. \tag{1.103}$$

We can write the first equation as

$$(\lambda + 2\mu)\nabla^2(\nabla^2 \phi) - \rho \frac{\partial^2}{\partial t^2}(\nabla^2 \phi) = 0 \tag{1.104}$$

and the second equation as

$$\boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\left[\mu\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T\right]\right]-\rho\frac{\partial^2}{\partial t^2}[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\psi})]-\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}\left[\mu\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T\right]\right]^T+\rho\frac{\partial^2}{\partial t^2}[\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T]=0\,. \quad (1.105)$$

The first two terms above are zero because $\boldsymbol{\nabla}\cdot\boldsymbol{\psi}=0$ (we can show that the first term has this form by interchanging derivatives). Therefore, the second equation has the form

$$\boldsymbol{\nabla}\cdot\left\{\mu\left[\boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T\right]\right]^T-\rho\frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla}\boldsymbol{\psi})^T\right\}=0\,. \quad (1.106)$$

Also note that the dilatation is given by

$$e=\boldsymbol{\nabla}\cdot\mathbf{u}=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi=\nabla^2\phi \quad (1.107)$$

and the infinitesimal rotation is given by

$$\boldsymbol{\theta}=\tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{u}=\tfrac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi}=-\tfrac{1}{2}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T \quad (1.108)$$

where we have again used the identity $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})-\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{u})^T$. Plugging these into the two equations of interest gives us

$$(\lambda+2\mu)\nabla^2 e-\rho\frac{\partial^2 e}{\partial t^2}=0 \quad (1.109)$$

and

$$\mu\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\theta})^T-\rho\frac{\partial^2\boldsymbol{\theta}}{\partial t^2}=0\,. \quad (1.110)$$

If we make the new connections $\phi\leftrightarrow e$ and $\boldsymbol{\psi}\leftrightarrow\boldsymbol{\theta}$, we see that

$$(\lambda+2\mu)\nabla^2\phi-\rho\frac{\partial^2\phi}{\partial t^2}=0 \quad\text{and}\quad \mu\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T-\rho\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}=0 \quad (1.111)$$

which can also be derived as a special case without invoking any change of variables. Therefore, there exist potentials ϕ and $\boldsymbol{\psi}$ such that

$$\frac{\partial^2\phi}{\partial t^2}=c_p^2\,\nabla^2\phi \quad\text{and}\quad \frac{\partial^2\boldsymbol{\psi}}{\partial t^2}=c_s^2\,\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{\psi})^T \qquad \square \quad (1.112)$$

where $c_p^2=(\lambda+2\mu)/\rho$ and $c_s^2=\mu/\rho$.

Problem 1.7 Show, using spatial curvilinear coordinates, that the gradient of a vector field $\mathbf{v}$ can be expressed in spherical coordinates as

$$\boldsymbol{\nabla}\mathbf{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r\sin\phi}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\cot\phi}{r} v_\phi & \frac{1}{r}\frac{\partial v_\theta}{\partial \phi} \\ \frac{\partial v_\phi}{\partial r} & \frac{1}{r\sin\phi}\frac{\partial v_\phi}{\partial \theta} - \frac{\cot\phi}{r} v_\theta & \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} \end{bmatrix} \tag{1.113}$$

where $x_1 = \theta^1 \cos\theta^2 \sin\theta^3$, $x_2 = \theta^1 \sin\theta^2 \sin\theta^3$, $x_3 = \theta^1 \cos\theta^3$ and $(\theta^1, \theta^2, \theta^3) \equiv (r, \theta, \phi)$.

Solution 1.7: The relation between Cartesian and spherical coordinates is assumed to be

$$\mathbf{x} \equiv \mathbf{x}(\theta^1, \theta^2, \theta^3) \quad \text{where} \quad (\theta^1, \theta^2, \theta^3) \equiv (r, \theta, \phi)\,. \tag{1.114}$$

In explicit form,

$$x_1 = \theta^1 \cos\theta^2 \sin\theta^3\,,\ x_2 = \theta^1 \sin\theta^2 \sin\theta^3\,,\ x_3 = \theta^1 \cos\theta^3\,. \tag{1.115}$$

If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a background Cartesian frame, the orthogonal basis vectors for the spherical coordinate system are determined from

$$\begin{aligned} \mathbf{g}_1 &= \frac{\partial \mathbf{x}}{\partial \theta^1} = \cos\theta^2 \sin\theta^3\, \mathbf{e}_1 + \sin\theta^2 \sin\theta^3\, \mathbf{e}_2 + \cos\theta^3\, \mathbf{e}_3 = \mathbf{e}_r \\ \mathbf{g}_2 &= \frac{\partial \mathbf{x}}{\partial \theta^2} = -\theta^1 \sin\theta^2 \sin\theta^3\, \mathbf{e}_1 + \theta^1 \cos\theta^2 \sin\theta^3\, \mathbf{e}_2 = r\sin\phi\, \mathbf{e}_\theta \\ \mathbf{g}_3 &= \frac{\partial \mathbf{x}}{\partial \theta^3} = \theta^1 \cos\theta^2 \cos\theta^3\, \mathbf{e}_1 + \theta^1 \sin\theta^2 \cos\theta^3\, \mathbf{e}_2 - \theta^1 \sin\theta^3\, \mathbf{e}_3 = r\, \mathbf{e}_\phi\,. \end{aligned} \tag{1.116}$$

We have normalized the covariant basis vectors to get the spherical basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. Using $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$ and solving for the components of $\mathbf{g}^j$, we have

$$\begin{aligned} \mathbf{g}^1 &= \cos\theta^2 \sin\theta^3\, \mathbf{e}_1 + \sin\theta^2 \sin\theta^3\, \mathbf{e}_2 + \cos\theta^3\, \mathbf{e}_3 = \mathbf{e}_r \\ \mathbf{g}^2 &= -\frac{1}{\theta^1}\frac{\sin\theta^2}{\sin\theta^3}\mathbf{e}_1 + \frac{1}{\theta^1}\frac{\cos\theta^2}{\sin\theta^3}\mathbf{e}_2 = \frac{1}{r\sin\phi}\mathbf{e}_\theta \\ \mathbf{g}^3 &= \frac{1}{\theta^1}\cos\theta^2\cos\theta^3\,\mathbf{e}_1 + \frac{1}{\theta^1}\sin\theta^2\cos\theta^3\,\mathbf{e}_2 - \frac{1}{\theta^1}\sin\theta^3\,\mathbf{e}_3 = \frac{1}{r}\mathbf{e}_\phi\,. \end{aligned} \tag{1.117}$$

The covariant and contravariant components of the metric tensor are $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$. In matrix form

$$\underline{\underline{\mathsf{g}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\theta^1\sin\theta^3)^2 & 0 \\ 0 & 0 & (\theta^1)^2 \end{bmatrix} \quad \text{and} \quad \overline{\overline{\mathsf{g}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/(\theta^1\sin\theta^3)^2 & 0 \\ 0 & 0 & (1/\theta^1)^2 \end{bmatrix}. \tag{1.118}$$

The gradient of the vector field $\mathbf{v}$ is given by

$$\boldsymbol{\nabla}\mathbf{v} = \left[\frac{\partial v_i}{\partial \theta^j} - \Gamma^k_{ji}\, v_k\right] \mathbf{g}^i \otimes \mathbf{g}^j \quad \text{where} \quad \Gamma^k_{ji} = \tfrac{1}{2} g^{km}\left[\frac{\partial g_{mi}}{\partial \theta^j} + \frac{\partial g_{mj}}{\partial \theta^i} - \frac{\partial g_{ji}}{\partial \theta^m}\right]. \tag{1.119}$$

The non-zero Christoffel symbols are

$$\begin{aligned} &\Gamma^1_{22} = -\theta^1 \sin^2\theta^3\,,\ \Gamma^1_{33} = -\theta^1\,,\ \Gamma^2_{12} = 1/\theta^1 = \Gamma^2_{21}\,,\ \Gamma^2_{23} = \cot\theta^3 = \Gamma^2_{32} \\ &\Gamma^3_{13} = 1/\theta^1 = \Gamma^3_{31}\,,\ \Gamma^3_{22} = -\cos\theta^3 \sin\theta^3\,. \end{aligned} \tag{1.120}$$

Therefore, the gradient can be expressed in matrix form in the basis $\mathbf{g}^i \otimes \mathbf{g}^j$ as

$$\boldsymbol{\nabla}\mathbf{v} = \begin{bmatrix} \dfrac{\partial v_1}{\partial \theta^1} & \dfrac{\partial v_1}{\partial \theta^2} - \dfrac{v_2}{\theta^1} & \dfrac{\partial v_1}{\partial \theta^3} - \dfrac{v_3}{\theta^1} \\ \dfrac{\partial v_2}{\partial \theta^1} - \dfrac{v_2}{\theta^1} & \dfrac{\partial v_2}{\partial \theta^2} + \theta^1 \sin^2\theta^3\, v_1 + \cos\theta^3 \sin\theta^3\, v_3 & \dfrac{\partial v_2}{\partial \theta^3} - \cot\theta^3\, v_2 \\ \dfrac{\partial v_3}{\partial \theta^1} - \dfrac{v_3}{\theta^1} & \dfrac{\partial v_3}{\partial \theta^2} - \cot\theta^3\, v_2 & \dfrac{\partial v_3}{\partial \theta^3} + \theta^1\, v_1 \end{bmatrix} \tag{1.121}$$

In terms of (r, θ, ϕ) and in the basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$, we have $\mathbf{v} = v_j \mathbf{g}^j = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi$ where $v_r = v_1$, $v_\theta = v_2/(r\sin\phi)$, $v_\phi = v_3/r$, and $\mathbf{e}_r = \mathbf{g}^1$, $\mathbf{e}_\theta = r\sin\phi\mathbf{g}^2$, $\mathbf{e}_\phi = r\mathbf{g}^3$. Therefore we can write,

$$\boldsymbol{\nabla}\mathbf{v} = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r\sin\phi}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r} & \dfrac{1}{r}\dfrac{\partial v_r}{\partial \phi} - \dfrac{v_\phi}{r} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r\sin\phi}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r} + \dfrac{\cot\phi}{r} v_\phi & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \phi} \\ \dfrac{\partial v_\phi}{\partial r} & \dfrac{1}{r\sin\phi}\dfrac{\partial v_\phi}{\partial \theta} - \dfrac{\cot\phi}{r} v_\theta & \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{v_r}{r} \end{bmatrix} \square \tag{1.122}$$

Problem 1.8 The equation of state for an adiabatic, reversible, ideal gas is given by

$$\frac{dp}{d\rho} = \frac{\gamma\, p}{\rho}; \quad \gamma := \frac{c_p}{c_v}; \quad c^2 = \frac{\gamma\, p}{\rho} \tag{1.123}$$

where c_p is the specific heat at constant pressure, c_v is the specific heat at constant volume, and c is the wave speed. Find the value of γ in air and water. Show that for small disturbances the balance of mass in the gas can be expressed as

$$\frac{\partial p}{\partial t} + \rho_0\, c_0^2\, \nabla \cdot \mathbf{v} = 0 \tag{1.124}$$

where c_0 is the speed of sound in the medium.

Solution 1.8: The value of γ is approximately 1.4 for air. For water at room temperature, c_p = 75.3 J/mol-K and c_v = 74.5 J/mol-K. Therefore, γ for water is approximately 1.01.

For small disturbances

$$\frac{dp}{d\rho} \approx \frac{\tilde{p}}{\tilde{\rho}}; \; \frac{p}{\rho} \approx \frac{\langle p \rangle}{\langle \rho \rangle}; \; c^2 \approx c_0^2 = \frac{\gamma\, \langle p \rangle}{\langle \rho \rangle}. \tag{1.125}$$

Therefore,

$$\frac{\tilde{p}}{\tilde{\rho}} = \gamma\, \frac{\langle p \rangle}{\langle \rho \rangle} = c_0^2 \qquad \Longrightarrow \qquad \frac{\partial \tilde{p}}{\partial t} = c_0^2 \frac{\partial \tilde{\rho}}{\partial t} \tag{1.126}$$

The balance of mass can then be written as

$$\frac{1}{c_0^2} \frac{\partial \tilde{p}}{\partial t} + \langle \rho \rangle \nabla \cdot \tilde{\mathbf{v}} = 0 \tag{1.127}$$

Dropping the tildes and using $\rho_0 := \langle \rho \rangle$ leads to the required equation.

Problem 1.9 The total magnetic induction produced at a point $\mathbf{x}$ by a current density $\mathbf{J}$ located in a region Ω is given by

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_\Omega \mathbf{J}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}\mathbf{y}\,. \tag{1.128}$$

Starting from the above equation show that

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{J} + \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\left[\int_\Omega \left(\frac{\boldsymbol{\nabla}_y\cdot\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right]\,. \tag{1.129}$$

Solution 1.9: We will first show that

$$\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right) = -\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}. \tag{1.130}$$

Let

$$f(z_1,z_2,z_3) = (z_1^2+z_2^2+z_3^2)^{-1/2} = (z_q z_q)^{-1/2} \tag{1.131}$$

where $z_m = z_m(\mathbf{x})$. Then

$$\frac{\partial f}{\partial x_p} = \frac{\partial f}{\partial z_m}\frac{\partial z_m}{\partial x_p}. \tag{1.132}$$

Now,

$$\frac{\partial f}{\partial z_m} = -\tfrac{1}{2}(z_q z_q)^{-3/2}(2\delta_{mr}z_r) = -(z_q z_q)^{-3/2} z_m\,. \tag{1.133}$$

Also, if $\mathbf{z} = \mathbf{x}-\mathbf{y}$, i.e., $z_m = x_m - y_m$,

$$\frac{\partial z_m}{\partial x_p} = \delta_{mp}\,. \tag{1.134}$$

Therefore,

$$\frac{\partial f}{\partial x_p} = -(z_q z_q)^{-3/2} z_p = -\frac{x_p - y_p}{(\sqrt{z_q z_q})^3}\,. \tag{1.135}$$

Since $\boldsymbol{\nabla} f \equiv \partial f/\partial x_p$ and

$$\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{z}\| = (z_q z_q)^{1/2} \tag{1.136}$$

we have

$$\frac{\partial f}{\partial x_p} \equiv \boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right) = -\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\,. \tag{1.137}$$

Note that similar arguments can be used to show that

$$\frac{\partial f}{\partial y_p} \equiv \boldsymbol{\nabla}_y\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right) = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3} \quad\implies\quad \boldsymbol{\nabla}_y\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right) = -\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,. \tag{1.138}$$

Therefore we can express the magnetic induction as

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\int_\Omega \mathbf{J}(\mathbf{y})\times\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\,. \tag{1.139}$$

Since the integration is over $\mathbf{y}$, we will now try to take the gradient outside the integral. We can write the quantity inside the integral as

$$\mathbf{J}(\mathbf{y})\times\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}-\mathbf{y}) \equiv e_{ijk}J_j(\mathbf{y})\frac{\partial f}{\partial x_k} = e_{ijk}\frac{\partial}{\partial x_k}[J_j(\mathbf{y})f] = -e_{ikj}\frac{\partial}{\partial x_k}[J_j(\mathbf{y})f] \tag{1.140}$$

or,

$$\mathbf{J}(\mathbf{y})\times\boldsymbol{\nabla}_\mathbf{x} f(\mathbf{x}-\mathbf{y}) = -\boldsymbol{\nabla}_\mathbf{x}\times[\mathbf{J}(\mathbf{y})f(\mathbf{x}-\mathbf{y})]\,. \tag{1.141}$$

Therefore,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\,\boldsymbol{\nabla}\times\int_\Omega \frac{\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y}\,. \tag{1.142}$$

Since the divergence of the curl of a vector field is zero, we have

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,. \tag{1.143}$$

Taking the curl of **B** gives

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu_0}{4\pi}\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int_\Omega \frac{\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y}\,. \tag{1.144}$$

From the identity $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{u})^T$, we have

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu_0}{4\pi}\,\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\int_\Omega \frac{\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y}\right) - \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\int_\Omega \frac{\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y}\right)^T\right]\,. \tag{1.145}$$

Taking the gradient inside the integral, we have

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu_0}{4\pi}\,\left[\boldsymbol{\nabla}\left(\int_\Omega \mathbf{J}(\mathbf{y})\cdot\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right) - \int_\Omega \mathbf{J}(\mathbf{y})\,\nabla^2\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right]\,. \tag{1.146}$$

Using relation (1.138) gives

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu_0}{4\pi}\,\left[-\boldsymbol{\nabla}\left(\int_\Omega \mathbf{J}(\mathbf{y})\cdot\boldsymbol{\nabla}_y\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right) - \int_\Omega \mathbf{J}(\mathbf{y})\,\nabla^2\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right]\,. \tag{1.147}$$

Let us now find the Laplacian of $1/\|\mathbf{x}-\mathbf{y}\|$. Let $r = \|\mathbf{x}-\mathbf{y}\|$. If we integrate the quantity over a sphere of radius R and volume V, we have

$$\int_V \nabla^2\left(\frac{1}{r}\right)\,dV = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right)\,dV = \int_S \boldsymbol{\nabla}\left(\frac{1}{r}\right)\cdot\mathbf{n}\,dA = -\int_S \frac{1}{r^2}\mathbf{n}\cdot\mathbf{n}\,dA = -\frac{4\pi R^2}{r^2} \tag{1.148}$$

where S is the surface of the volume and **n** is the outward unit normal. This result indicates that the Laplacian has the form

$$\nabla^2\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,dV = -4\pi\delta(\mathbf{x}-\mathbf{y}) \tag{1.149}$$

where $\delta(\mathbf{x})$ is the three-dimensional Dirac delta function because of the behavior as $R\to 0$. Therefore,

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu_0}{4\pi}\,\left[-\boldsymbol{\nabla}\left(\int_\Omega \mathbf{J}(\mathbf{y})\cdot\boldsymbol{\nabla}_y\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right) + 4\pi\int_\Omega \mathbf{J}(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\right]\,. \tag{1.150}$$

Using the identity $\mathbf{J}\cdot\boldsymbol{\nabla} f = \boldsymbol{\nabla}\cdot(f\mathbf{J}) - f(\boldsymbol{\nabla}\cdot\mathbf{J})$ and the divergence theorem, we have

$$\int_\Omega \mathbf{J}(\mathbf{y})\cdot\boldsymbol{\nabla}_y\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y} = \int_\Gamma \frac{\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\cdot\mathbf{n}\,\mathrm{d}A - \int_\Omega \frac{\boldsymbol{\nabla}_y\cdot\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y}\,. \tag{1.151}$$

In the absence of current flux from the boundary, we have

$$\int_\Omega \mathbf{J}(\mathbf{y})\cdot\boldsymbol{\nabla}_y\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y} = -\int_\Omega \frac{\boldsymbol{\nabla}_y\cdot\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y}\,. \tag{1.152}$$

Therefore equation (1.150) can be written as

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mu_0}{4\pi}\,\left[\boldsymbol{\nabla}\int_\Omega \frac{\boldsymbol{\nabla}_y\cdot\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,\mathrm{d}\mathbf{y} + 4\pi\mathbf{J}(\mathbf{x})\right] = \mu_0\,\mathbf{J} + \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\left[\int_\Omega\left(\frac{\boldsymbol{\nabla}_y\cdot\mathbf{J}(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\right)\,\mathrm{d}\mathbf{y}\right]\quad\square \tag{1.153}$$

Problem 1.10 The relation between the electromotive force (emf) around a loop and the magnetic induction is given by the integral form of Faraday's law:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -k \, \frac{D}{Dt} \left[\int_\Gamma \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}A \right] . \tag{1.154}$$

Derive the differential form of Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \,. \tag{1.155}$$

Solution 1.10: The total derivative of a scalar field, φ, is defined as

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\varphi \tag{1.156}$$

where $\mathbf{v}$ is the velocity with which the scalar is being convected. The equivalent relation for a vector field $\mathbf{u}$ is

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\mathbf{u} \cdot \mathbf{v} \,. \tag{1.157}$$

Therefore,

$$\frac{D\mathbf{B}}{Dt} = \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla}\mathbf{B} \cdot \mathbf{v} \,. \tag{1.158}$$

Using the identity $\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = (\boldsymbol{\nabla} \cdot \mathbf{b})\mathbf{a} - (\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b} + (\boldsymbol{\nabla}\mathbf{a}) \cdot \mathbf{b} - (\boldsymbol{\nabla}\mathbf{b}) \cdot \mathbf{a}$, we have, for constant $\mathbf{v}$,

$$\boldsymbol{\nabla}\mathbf{B} \cdot \mathbf{v} = \boldsymbol{\nabla} \times (\mathbf{B} \times \mathbf{v}) + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{v} \,. \tag{1.159}$$

Hence,

$$\frac{D\mathbf{B}}{Dt} = \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times (\mathbf{B} \times \mathbf{v}) + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{v} \,. \tag{1.160}$$

Therefore, using $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and Stokes' theorem, we have

$$\begin{aligned} \frac{D}{Dt} \left[\int_\Gamma \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}A \right] &= \int_\Gamma \left[\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times (\mathbf{B} \times \mathbf{v}) + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{v} \right] \cdot \mathbf{n} \, \mathrm{d}A \\ &= \int_\Gamma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, \mathrm{d}A + \oint_C (\mathbf{B} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{l} \,. \end{aligned} \tag{1.161}$$

Plugging these into Faraday's law, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -k \left[\int_\Gamma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, \mathrm{d}A + \oint_C (\mathbf{B} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{l} \right] \tag{1.162}$$

or

$$\oint_C [\mathbf{E} - k \, \mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l} = -k \int_\Gamma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, \mathrm{d}A \,. \tag{1.163}$$

For a body that is moving at small speeds relative to the speed of light in vacuum, we can use Galilean covariance to work in a moving frame to eliminate $\mathbf{v}$. We can also use the fact that k equals 1 under these conditions in SI units, to get the relation

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_\Gamma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, \mathrm{d}A \quad \Longrightarrow \quad \int_\Gamma (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{n} \, \mathrm{d}A = -\int_\Gamma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, \mathrm{d}A \tag{1.164}$$

where $\mathbf{E} \leftarrow \mathbf{E} - \mathbf{v} \times \mathbf{B}$ and Stokes' theorem has been used. Invoking the arbitrariness of Γ, we then get the differential form of Faraday's equation

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad \square \tag{1.165}$$

Problem 1.11 Assume that the electric field, electric displacement, magnetic field, and magnetic inductions depend harmonically of time, i.e.,

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \text{Re}\{\widehat{\mathbf{E}}(\mathbf{x})\, e^{-i\omega t}\}\,; \quad \mathbf{B}(\mathbf{x},t) = \text{Re}\{\widehat{\mathbf{B}}(\mathbf{x})\, e^{-i\omega t}\} \\ \widetilde{\mathbf{D}}(\mathbf{x},t) &= \text{Re}\{\widehat{\mathbf{D}}(\mathbf{x})\, e^{-i\omega t}\}\,; \quad \mathbf{H}(\mathbf{x},t) = \text{Re}\{\widehat{\mathbf{H}}(\mathbf{x})\, e^{-i\omega t}\} \end{aligned} \tag{1.166}$$

where ω has an infinitesimally small imaginary part. Show that Maxwell's equations under these conditions can be written as

$$\begin{aligned} &\boldsymbol{\nabla} \times \mathbf{E} = i\omega\, \boldsymbol{\mu}(\mathbf{x},\omega) \cdot \mathbf{H}(\mathbf{x})\,; \quad \boldsymbol{\nabla} \times \mathbf{H} = -i\omega\, \boldsymbol{\varepsilon}(\mathbf{x},\omega) \cdot \mathbf{E}(\mathbf{x}) \\ &\boldsymbol{\nabla} \cdot \mathbf{B} = 0\,; \quad \boldsymbol{\nabla} \cdot \mathbf{D} = 0\,. \end{aligned} \tag{1.167}$$

Why does ω need a small imaginary part? Compare these equations with the Fourier transformed form of Maxwell's equations.

Solution 1.11: If we plug in the assumed harmonic solutions into Maxwell's equations we get

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad &\implies \quad \boldsymbol{\nabla} \times \widehat{\mathbf{E}}\, e^{-i\omega t} = i\omega\, \widehat{\mathbf{B}}(\mathbf{x})\, e^{-i\omega t} \quad &\implies \quad \boldsymbol{\nabla} \times \widehat{\mathbf{E}} = i\omega\, \widehat{\mathbf{B}}(\mathbf{x}) \\ \boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \widetilde{\mathbf{D}}}{\partial t} \quad &\implies \quad \boldsymbol{\nabla} \times \widehat{\mathbf{H}}\, e^{-i\omega t} = -i\omega\, \widehat{\mathbf{D}}(\mathbf{x})\, e^{-i\omega t} \quad &\implies \quad \boldsymbol{\nabla} \times \widehat{\mathbf{H}} = -i\omega\, \widehat{\mathbf{D}}(\mathbf{x}) \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad &\implies \quad \boldsymbol{\nabla} \cdot \widehat{\mathbf{B}}\, e^{-i\omega t} = 0 \quad &\implies \quad \boldsymbol{\nabla} \cdot \widehat{\mathbf{B}} = 0 \\ \boldsymbol{\nabla} \cdot \widetilde{\mathbf{D}} = 0 \quad &\implies \quad \boldsymbol{\nabla} \cdot \widehat{\mathbf{D}}\, e^{-i\omega t} = 0 \quad &\implies \quad \boldsymbol{\nabla} \cdot \widehat{\mathbf{D}} = 0 \end{aligned} \tag{1.168}$$

or,

$$\boldsymbol{\nabla} \times \widehat{\mathbf{E}} = i\omega\, \widehat{\mathbf{B}}(\mathbf{x})\,;\; \boldsymbol{\nabla} \times \widehat{\mathbf{H}} = -i\omega\, \widehat{\mathbf{D}}(\mathbf{x})\,;\; \boldsymbol{\nabla} \cdot \widehat{\mathbf{B}} = 0\,;\; \boldsymbol{\nabla} \cdot \widehat{\mathbf{D}} = 0\,. \tag{1.169}$$

Similarly, plugging the harmonic solutions into the constitutive equations we get (using $\tau = t - t'$)

$$\begin{aligned} \widehat{\mathbf{H}}(\mathbf{x})\, e^{-i\omega t} &= \int_{-\infty}^{\infty} \bar{\boldsymbol{K}}_B(\mathbf{x}, t-t') \cdot \left[\widehat{\mathbf{B}}(\mathbf{x})\, e^{-i\omega t'}\right]\, \mathrm{d}t' = \left[\int_{-\infty}^{\infty} \bar{\boldsymbol{K}}_B(\mathbf{x},\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau\right] \cdot \widehat{\mathbf{B}}(\mathbf{x})\, e^{-i\omega t} \\ \widehat{\mathbf{D}}(\mathbf{x})\, e^{-i\omega t} &= \int_{-\infty}^{\infty} \bar{\boldsymbol{K}}_E(\mathbf{x}, t'-t) \cdot \left[\widehat{\mathbf{E}}(\mathbf{x})\, e^{-i\omega t'}\right]\, \mathrm{d}t' = \left[\int_{-\infty}^{\infty} \bar{\boldsymbol{K}}_E(\mathbf{x},\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau\right] \cdot \widehat{\mathbf{E}}(\mathbf{x})\, e^{-i\omega t} \end{aligned} \tag{1.170}$$

or,

$$\widehat{\mathbf{H}}(\mathbf{x}) = [\boldsymbol{\mu}(\mathbf{x},\omega)]^{-1} \cdot \widehat{\mathbf{B}}(\mathbf{x})\,;\; \widehat{\mathbf{D}}(\mathbf{x}) = \boldsymbol{\varepsilon}(\mathbf{x},\omega) \cdot \widehat{\mathbf{E}}(\mathbf{x}) \tag{1.171}$$

where

$$[\boldsymbol{\mu}(\mathbf{x},\omega)]^{-1} = \int_{-\infty}^{\infty} \bar{\boldsymbol{K}}_B(\mathbf{x},\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau\,;\; \boldsymbol{\varepsilon}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} \bar{\boldsymbol{K}}_E(\mathbf{x},\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau\,. \tag{1.172}$$

In general $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ are complex, rank-2 tensor quantities. The integrals in the above equations converge when the imaginary part of ω is positive (since $\bar{\boldsymbol{K}}_B = \bar{\boldsymbol{K}}_E = 0$ when $\tau > 0$). To see this, observe that $\exp(-i\omega\tau) = \exp(\text{Im}(\omega)\,\tau)[\cos(\text{Re}(\omega)\,\tau) - i\sin(\text{Re}(\omega)\,\tau)]$. Now, $\exp(-i\omega\tau)$ is an analytic function of ω. Since a sum of analytic functions is analytic and a convergent integral of analytic functions is also analytic, the functions $\boldsymbol{\varepsilon}(\mathbf{x},\omega)$ and $\boldsymbol{\mu}(\mathbf{x},\omega)$ are analytic functions of ω in the upper half ω-plane, $\text{Im}(\omega) > 0$.

Substituting the constitutive equations into Maxwell's equations and dropping the hats gives us

$$\boldsymbol{\nabla} \times \mathbf{E} = i\omega\, \boldsymbol{\mu}(\mathbf{x},\omega) \cdot \mathbf{H}(\mathbf{x})\,;\; \boldsymbol{\nabla} \times \mathbf{H} = -i\omega\, \boldsymbol{\varepsilon}(\mathbf{x},\omega) \cdot \mathbf{E}(\mathbf{x})\,;\; \boldsymbol{\nabla} \cdot \mathbf{B} = 0\,;\; \boldsymbol{\nabla} \cdot \mathbf{D} = 0\,. \tag{1.173}$$

The frequency ω needs a small imaginary part so that the fields are zero at $t \to -\infty$. To see this express ω in terms of its real and imaginary parts to get

$$\begin{aligned} \text{Re}\left[Ae^{-i\omega t}\right] &= \text{Re}\left[Ae^{-i(\omega_r + i\omega_i)t}\right] = \text{Re}\left[Ae^{-i\omega_r t}\, e^{\omega_i t}\right] \\ &= \text{Re}\left[A(\cos\omega_r t - i\sin\omega_r t)\, e^{\omega_i t}\right] = A\, \cos(\omega_r t)\, e^{\omega_i t}\,. \end{aligned} \tag{1.174}$$

As $t \to -\infty$ the cosine term lies between 0 and 1. Hence for the fields to go to zero we need a non-zero value of ω_i.

In the Fourier transformed Maxwell equations the fields depend both on $\mathbf{x}$ and ω whereas in the time harmonic case they depend only on $\mathbf{x}$.

Problem 1.12 Show that the transverse electric (TE) wave equation for a material with anisotropic permeability and permittivity

$$\boldsymbol{\mu} = \boldsymbol{\mu}(x_2, x_3) \equiv \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & \mu_{23} \\ 0 & \mu_{23} & \mu_{33} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x_2, x_3) \equiv \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \tag{1.175}$$

is given by

$$\overline{\boldsymbol{\nabla}} \cdot \left[\left(\boldsymbol{R}_\perp \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{R}_\perp^T\right) \cdot \overline{\boldsymbol{\nabla}} E_1\right] + \omega^2 \, \varepsilon_{11} \, E_1 = 0 \tag{1.176}$$

where $\overline{\boldsymbol{\nabla}}$ indicates the two-dimensional gradient, and

$$\boldsymbol{M} \equiv \begin{bmatrix} \mu_{22} & \mu_{23} \\ \mu_{23} & \mu_{33} \end{bmatrix} \; ; \; \boldsymbol{R}_\perp \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \,. \tag{1.177}$$

Solution 1.12: We know that

$$\boldsymbol{\nabla} \times \mathbf{E} = E_{1,3} \, \mathbf{e}_2 - E_{1,2} \, \mathbf{e}_3 = i\omega \boldsymbol{\mu} \cdot \mathbf{H} \,. \tag{1.178}$$

Now,

$$\boldsymbol{\mu} \cdot \mathbf{H} = \mu_{ij} H_j \mathbf{e}_i = \mu_{11} H_1 \mathbf{e}_1 + (\mu_{22} H_2 + \mu_{23} H_3)\mathbf{e}_2 + (\mu_{32} H_2 + \mu_{33} H_3)\mathbf{e}_3 \,. \tag{1.179}$$

Therefore,

$$H_1 = 0 \, ; \;\; \mu_{22} H_2 + \mu_{23} H_3 = (i\omega)^{-1} E_{1,3} \, ; \;\; \mu_{32} H_2 + \mu_{33} H_3 = -(i\omega)^{-1} E_{1,2} \tag{1.180}$$

or,

$$\begin{bmatrix} \mu_{22} & \mu_{23} \\ \mu_{32} & \mu_{33} \end{bmatrix} \begin{bmatrix} H_2 \\ H_3 \end{bmatrix} = (i\omega)^{-1} \begin{bmatrix} E_{1,3} \\ -E_{1,2} \end{bmatrix} \,. \tag{1.181}$$

Solving for H_2 and H_3, and using the notation $\boldsymbol{M}$ for the two-dimensional permeability tensor, leads to

$$H_2 = (i\omega)^{-1} \left(M_{11}^{-1} E_{1,3} - M_{12}^{-1} E_{1,2}\right) \; ; \;\; H_3 = (i\omega)^{-1} \left(M_{21}^{-1} E_{1,3} - M_{22}^{-1} E_{1,2}\right) \,. \tag{1.182}$$

At this stage recall that

$$\boldsymbol{\nabla} \times \mathbf{H} = (H_{3,2} - H_{2,3})\mathbf{e}_1 = -i\omega \boldsymbol{\varepsilon} \cdot \mathbf{E} \,. \tag{1.183}$$

Since

$$\boldsymbol{\varepsilon} \cdot \mathbf{E} = \varepsilon_{ij} E_j \mathbf{e}_i = \varepsilon_{11} E_1 \mathbf{e}_1 \tag{1.184}$$

we have

$$H_{3,2} - H_{2,3} = -i\omega\varepsilon_{11} E_1 \,. \tag{1.185}$$

Plugging in the expressions for H_2 and H_3 results in

$$\left(M_{21}^{-1} E_{1,3} - M_{22}^{-1} E_{1,2}\right)_{,2} - \left(M_{11}^{-1} E_{1,3} - M_{12}^{-1} E_{1,2}\right)_{,3} = \omega^2 \varepsilon_{11} E_1 \tag{1.186}$$

or

$$\left(M_{22}^{-1} E_{1,2} - M_{21}^{-1} E_{1,3}\right)_{,2} + \left(-M_{12}^{-1} E_{1,2} + M_{11}^{-1} E_{1,3}\right)_{,3} = -\omega^2 \varepsilon_{11} E_1 \,. \tag{1.187}$$

Notice that the quantities inside the bracket can be obtained by rotating the original tensor $\boldsymbol{M}$ by 90 degrees, i.e.,

$$\boldsymbol{R} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.188}$$

in which case

$$\widehat{\boldsymbol{M}} := \boldsymbol{R}^T \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{R} \equiv \begin{bmatrix} M_{22}^{-1} & -M_{21}^{-1} \\ -M_{12}^{-1} & M_{11}^{-1} \end{bmatrix} \,. \tag{1.189}$$

Therefore,

$$\widehat{\boldsymbol{M}} \cdot \overline{\boldsymbol{\nabla}} E_1 = \widehat{M}_{ij} E_{1,j} \mathbf{e}_i = (M_{22}^{-1} E_{1,2} - M_{21}^{-1} E_{1,3}) \mathbf{e}_2 + (-M_{12}^{-1} E_{1,2} + M_{11}^{-1} E_{1,3}) \mathbf{e}_3 \tag{1.190}$$

and

$$\overline{\boldsymbol{\nabla}} \cdot (\widehat{\boldsymbol{M}} \cdot \overline{\boldsymbol{\nabla}} E_1) = (\widehat{M}_{ij} E_{1,j})_{,i} = (M_{22}^{-1} E_{1,2} - M_{21}^{-1} E_{1,3})_{,2} + (-M_{12}^{-1} E_{1,2} + M_{11}^{-1} E_{1,3})_{,3} \,. \tag{1.191}$$

Comparison with equation (1.187) gives us

$$\overline{\boldsymbol{\nabla}} \cdot (\widehat{\boldsymbol{M}} \cdot \overline{\boldsymbol{\nabla}} E_1) + \omega^2 \varepsilon_{11} E_1 = 0 \quad \square \tag{1.192}$$

Problem 1.13 Maxwell's equations can be expressed in a form similar to the equations of elastodynamics at a fixed frequency, i.e.,

$$\omega^2 \boldsymbol{\varepsilon} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot (\mathbf{C} : \boldsymbol{\nabla}\mathbf{E}) \,. \tag{1.193}$$

Show that the tensor $\mathbf{C}$ has the symmetries

$$C_{ijk\ell} = -C_{jik\ell} = -C_{ij\ell k} = C_{k\ell ij} \,. \tag{1.194}$$

Solution 1.13: The components of the tensor $\mathbf{C}$ are given by

$$C_{ijk\ell} = e_{inj} e_{kp\ell} [\boldsymbol{\mu}^{-1}]_{np} \,. \tag{1.195}$$

If we switch ij and $k\ell$ we have

$$C_{k\ell ij} = e_{kn\ell} e_{ipj} [\boldsymbol{\mu}^{-1}]_{np} \,. \tag{1.196}$$

Since n and p are dummy indices, we can switch them around to get

$$C_{k\ell ij} = e_{kp\ell} e_{inj} [\boldsymbol{\mu}^{-1}]_{pn} \,. \tag{1.197}$$

The energy density of a magnetic field is

$$W = \tfrac{1}{2}\mathbf{B} \cdot \mathbf{H} = \tfrac{1}{2}(\boldsymbol{\mu} \cdot \mathbf{H}) \cdot \mathbf{H} = \tfrac{1}{2}\mu_{ij} H_j H_i \,. \tag{1.198}$$

Since the energy density must remain unchanged if we switch the indices i and j, we must have

$$\mu_{ij} = \mu_{ji} \quad \implies \quad [\boldsymbol{\mu}^{-1}]_{ij} = [\boldsymbol{\mu}^{-1}]_{ji} \,. \tag{1.199}$$

Therefore, equation (1.197) can be written as

$$C_{k\ell ij} = e_{kp\ell} e_{inj} [\boldsymbol{\mu}^{-1}]_{np} = C_{ijkl} \,. \tag{1.200}$$

Let us next switch the indices ij to ji in equation (1.195). Then we have

$$C_{jik\ell} = e_{jni} e_{kp\ell} [\boldsymbol{\mu}^{-1}]_{np} \,. \tag{1.201}$$

Now, $e_{nji} = -e_{ijn}$ and hence $C_{jik\ell} = -C_{ijk\ell}$. The same argument can be used to show that $C_{ijk\ell} = -C_{ij\ell k}$. $\square$

Chapter 2

Solutions for Exercises in Chapter 2

Problem 2.1 Show that for time harmonic plane waves of the form

$$\mathbf{u}(\mathbf{x},t) = \widehat{\mathbf{u}}\, e^{i\omega(\mathbf{s}\cdot\mathbf{x}-t)} \; . \tag{2.1}$$

the governing equations for the dynamics of an isotropic and homogeneous linear elastic body decouple into the equations

$$\left(\rho - \frac{\mu}{c^2}\right)(\widehat{\mathbf{u}} \times \mathbf{s}) = \mathbf{0} \quad \text{and} \quad \left(\rho - \frac{\lambda+2\mu}{c^2}\right)(\widehat{\mathbf{u}} \cdot \mathbf{s}) = 0 \; . \tag{2.2}$$

Solution 2.1: The elastodynamic wave equation for an isotropic, homogeneous body is

$$(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu\,\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{u})^T = \rho\,\ddot{\mathbf{u}} \, . \tag{2.3}$$

Let us solve this problem in rectangular Cartesian coordinates and note that the results translate over to other coordinate systems. For a plane wave solution

$$u_i = \widehat{u}_i e^{i\omega(s_m x_m - t)} \tag{2.4}$$

we have

$$\boldsymbol{\nabla}\mathbf{u} \equiv u_{i,j} = i\omega s_m \delta_{mj} \widehat{u}_i e^{i\omega(s_m x_m - t)} = i\omega s_j u_i \tag{2.5}$$

and

$$(\boldsymbol{\nabla}\mathbf{u})^T \equiv u_{j,i} = i\omega s_i u_j \; . \tag{2.6}$$

Therefore,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{u})^T \equiv u_{j,ii} = i\omega s_i u_{j,i} = (i\omega)^2 s_i s_i u_j = -\omega^2 s_i s_i u_j \equiv -\omega^2(\mathbf{s}\cdot\mathbf{s})\mathbf{u} \; . \tag{2.7}$$

Also,

$$\boldsymbol{\nabla}\cdot\mathbf{u} \equiv u_{i,i} = i\omega s_i u_i \; . \tag{2.8}$$

Hence,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) \equiv u_{i,ij} = i\omega s_i u_{i,j} = (i\omega)^2 s_i s_j u_i \equiv -\omega^2(\mathbf{s}\otimes\mathbf{s})\cdot\mathbf{u} \; . \tag{2.9}$$

Finally

$$\dot{\mathbf{u}} \equiv \dot{u}_i = -i\omega\widehat{u}_i e^{i\omega(s_m x_m - t)} = -i\omega u_i \tag{2.10}$$

and

$$\ddot{\mathbf{u}} \equiv \ddot{u}_i = -i\omega\dot{u}_i = (i\omega)^2 u_i \equiv -\omega^2\mathbf{u} \; . \tag{2.11}$$

Therefore, if we plug in the plane wave solution into the governing equations we get

$$(\lambda+\mu)\,(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}+\mu\,(\mathbf{s}\cdot\mathbf{s})\,\widehat{\mathbf{u}}=\rho\,\widehat{\mathbf{u}}\,. \tag{2.12}$$

Now

$$[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\times\mathbf{s}\equiv e_{kij}s_i s_p\widehat{u}_p s_j\equiv(\mathbf{s}\times\mathbf{s})(\mathbf{s}\cdot\widehat{\mathbf{u}})=\mathbf{0} \tag{2.13}$$

and

$$[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\cdot\mathbf{s}\equiv s_i s_j\widehat{u}_j s_i\equiv(\mathbf{s}\cdot\mathbf{s})(\mathbf{u}\cdot\mathbf{s})\,. \tag{2.14}$$

Therefore, taking the cross product of the governing equation with $\mathbf{s}$ gives

$$(\lambda+\mu)\,[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\times\mathbf{s}+\mu\,(\mathbf{s}\cdot\mathbf{s})\,(\widehat{\mathbf{u}}\times\mathbf{s})=\rho\,(\widehat{\mathbf{u}}\times\mathbf{s}) \tag{2.15}$$

or,

$$[\rho-\mu\,(\mathbf{s}\cdot\mathbf{s})]\,(\widehat{\mathbf{u}}\times\mathbf{s})=\mathbf{0}\,. \tag{2.16}$$

Using the definition of $\mathbf{s}$ we have $\mathbf{s}\cdot\mathbf{s}=1/c^2$, which leads to the first decoupled equation

$$\left[\rho-\frac{\mu}{c^2}\right](\widehat{\mathbf{u}}\times\mathbf{s})=\mathbf{0}\,. \qquad \square \tag{2.17}$$

Similarly, taking the dot product of the governing equation with $\mathbf{s}$, we have

$$(\lambda+\mu)\,[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\cdot\mathbf{s}+\mu\,(\mathbf{s}\cdot\mathbf{s})\,(\widehat{\mathbf{u}}\cdot\mathbf{s})=\rho\,(\widehat{\mathbf{u}}\cdot\mathbf{s}) \tag{2.18}$$

or,

$$(\lambda+\mu)\,(\mathbf{s}\dot{\mathbf{s}})(\widehat{\mathbf{u}}\cdot\mathbf{s})+\mu\,(\mathbf{s}\cdot\mathbf{s})\,(\widehat{\mathbf{u}}\cdot\mathbf{s})=\rho\,(\widehat{\mathbf{u}}\cdot\mathbf{s}) \tag{2.19}$$

Replacing $\mathbf{s}\cdot\mathbf{s}$ with $1/c^2$ gives us our second decoupled equation

$$\left[\rho-\frac{\lambda+2\mu}{c^2}\right](\widehat{\mathbf{u}}\cdot\mathbf{s})=0\,. \qquad \square \tag{2.20}$$

Problem 2.2 Derive the relations

$$R_{pp} = \frac{4c_s^4\alpha^2\rho^2 - Z_pZ_s(1-2c_s^2\alpha^2)^2}{4c_s^4\alpha^2\rho^2 + Z_pZ_S(1-2c_s^2\alpha^2)^2}\,;\;\; R_{ps} = \frac{4c_s^2\alpha\rho Z_s(1-2c_s^2\alpha^2)}{4c_s^4\alpha^2\rho^2 + Z_pZ_S(1-2c_s^2\alpha^2)^2}\,. \tag{2.21}$$

starting from

$$\begin{aligned} \kappa_p^2\sin(2\theta_{ip})(1-R_{pp}) &= \kappa_s^2\cos(2\theta_{rs})R_{ps} \\ (\kappa_s^2 - 2\kappa_p^2\sin^2\theta_{ip})(1+R_{pp}) &= \kappa_s^2\sin(2\theta_{rs})R_{ps}\,. \end{aligned} \tag{2.22}$$

Solution 2.2: From the first equation we have

$$2\frac{\kappa_p^2}{\kappa_s^2}\sin\theta_{ip}\cos\theta_{ip}(1-R_{pp}) = (1-2\sin^2\theta_{rs})R_{ps} \tag{2.23}$$

or,

$$2\frac{c_s^2}{c_p^2}\sin\theta_{ip}\cos\theta_{ip}(1-R_{pp}) = (1-2\sin^2\theta_{rs})R_{ps} \tag{2.24}$$

or,

$$2c_s^2\alpha\frac{\rho}{Z_p}(1-R_{pp}) = (1-2c_s^2\alpha^2)R_{ps}\,. \tag{2.25}$$

or,

$$2c_s^2\alpha\rho(1-R_{pp}) = Z_p(1-2c_s^2\alpha^2)R_{ps}\,. \tag{2.26}$$

From the second equation,

$$\left(1-2\frac{\kappa_p^2}{\kappa_s^2}\sin^2\theta_{ip}\right)(1+R_{pp}) = 2\sin\theta_{rs}\cos\theta_{rs}R_{ps} \tag{2.27}$$

or,

$$\left(1-2\frac{c_s^2}{c_p^2}\sin^2\theta_{ip}\right)(1+R_{pp}) = 2c_s^2\frac{\sin\theta_{rs}}{c_s}\frac{\cos\theta_{rs}}{c_s}R_{ps} \tag{2.28}$$

or,

$$\left(1-2c_s^2\alpha^2\right)(1+R_{pp}) = 2c_s^2\alpha\frac{\rho}{Z_s}R_{ps} \tag{2.29}$$

or,

$$Z_s\left(1-2c_s^2\alpha^2\right)(1+R_{pp}) = 2c_s^2\alpha\rho R_{ps}\,. \tag{2.30}$$

The required relations are obtained in a straightforward manner by solving equations (2.26) and (2.30) for R_{pp} and R_{ps}.

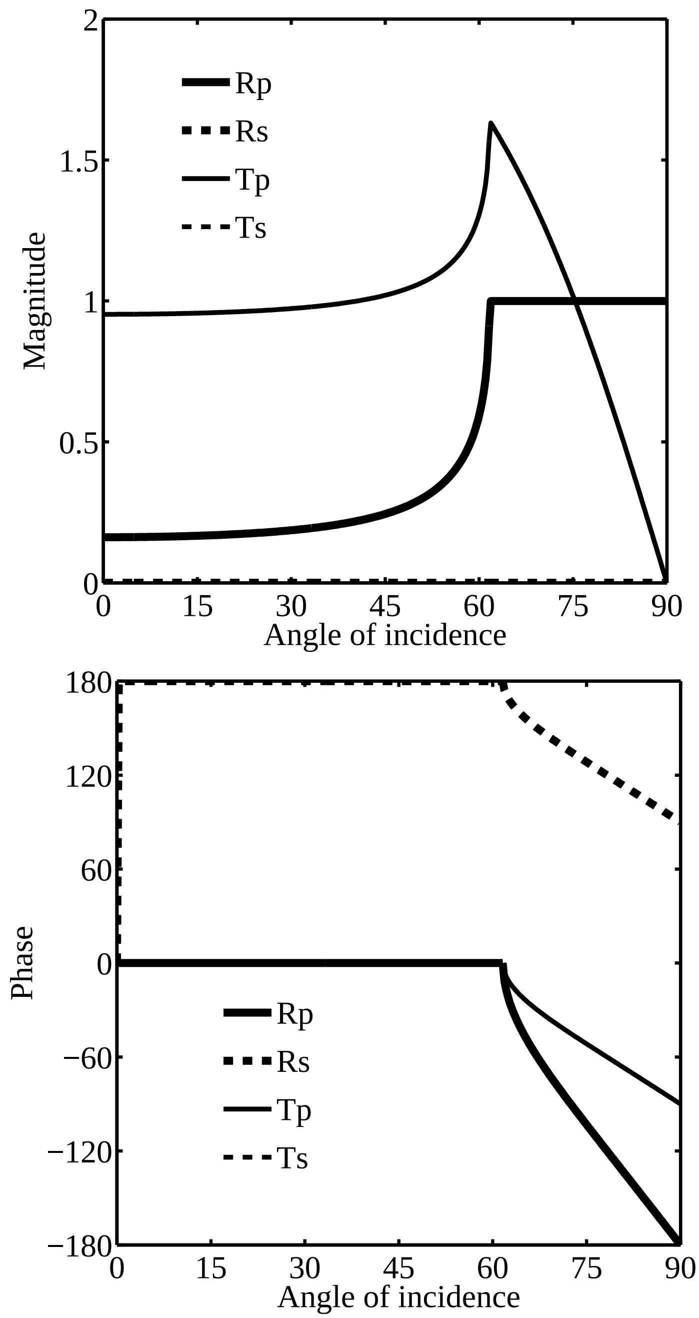
Problem 2.3 Verify the reflection coefficient relations given in the equations for a P-wave incident upond th einterface between two solids using a symbolic computation tool if needed. Solve these equations numerically and plot the magnitude and phase of the reflection and transmission coefficients as a function of the angle of incidence for the interface between two materials with $\rho_1 = 820$, $c_{p1} = 1320$, $c_{s1} = 1.0e-4$, $\rho_2 =$1000, $c_{p2} =$1500, $c_{s2} = 1.0e-4$.

Solution 2.3: A Mathematica script that shows that the calculation is given below.

```
In[1]:= Phr := Rp*Phi
Pht := Tp*Phi
Psr := Rs*Phi
Pst := Ts*Phi
pi = Phi*Exp[I* kp1 *(x1*sti - x2*cti)]
pr = Phr*Exp[I* kp1 *(x1*str + x2*ctr)]
pt = Pht*Exp[I* kp2 *(x1*stt - x2*ctt)]
sr = Psr*Exp[I* ks1 *(x1*strs + x2*ctrs)]
st = Pst*Exp[I* ks2 *(x1*stts - x2*ctts)]
ui1 := D[pi,x1]
ui2 := D[pi,x2]
ur1 := D[pr,x1]
ur2 := D[pr,x2]
ut1 := D[pt,x1]
ut2 := D[pt,x2]
urs1 := D[sr,x2]
urs2 := -D[sr,x1]
uts1 := D[st,x2]
uts2 := -D[st,x1]
s12i := 2*mu1*D[D[pi,x1],x2]
s12r := 2*mu1*D[D[pr,x1],x2]
s12t := 2*mu2*D[D[pt,x1],x2]
s12rs := mu1*(D[D[sr,x2],x2]-D[D[sr,x1],x1])
s12ts := mu2*(D[D[st,x2],x2]-D[D[st,x1],x1])
s22i := lambda1*(D[D[pi,x1],x1]+D[D[pi,x2],x2])+2*mu1*D[D[pi,x2],x2]
s22r := lambda1*(D[D[pr,x1],x1]+D[D[pr,x2],x2])+2*mu1*D[D[pr,x2],x2]
s22t := lambda2*(D[D[pt,x1],x1]+D[D[pt,x2],x2])+2*mu2*D[D[pt,x2],x2]
s22rs := -2*mu1*D[D[sr,x1],x2]
s22ts := -2*mu2*D[D[st,x1],x2]
s12top := FullSimplify[s12i + s12r + s12rs/. x2->0]
s12bot := FullSimplify[s12t + s12ts/. x2->0]
s12diff = Collect[FullSimplify[(s12bot - s12top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
s22top := FullSimplify[s22i + s22r + s22rs/. x2->0]
s22bot := FullSimplify[s22t + s22ts/. x2->0]
s22diff = Collect[FullSimplify[(s22bot - s22top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
u1top := FullSimplify[ui1 + ur1 + urs1/. x2->0]
u1bot := FullSimplify[ut1 + uts1/. x2->0]
u1diff = Collect[FullSimplify[(u1bot - u1top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
u2top := FullSimplify[ui2 + ur2 + urs2/. x2->0]
u2bot := FullSimplify[ut2 + uts2/. x2->0]
u2diff = Collect[FullSimplify[(u2bot - u2top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
s12diffa = s12diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
s22diffa = s22diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
u1diffa = u1diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
u2diffa = u2diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
s12diffb := s12diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
s22diffb := s22diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
u1diffb := u1diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
u2diffb := u2diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
s12diffc = s12diffb /. {sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
s22diffc = s22diffb /. { sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
u1diffc = u1diffb /. {sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
u2diffc = u2diffb /. {sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
s12diffd := -s12diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
s22diffd := -s22diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
u1diffd := -u1diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
u2diffd := -u2diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
s12diffe = Collect[FullSimplify[s12diffd*(Zp1*Zp2*Zs1^2*Zs2^2/omega^2)],
        {Rp, Rs, Tp, Ts}, Simplify]
s22diffe = Collect[FullSimplify[s22diffd*(Zp1^2*Zp2^2*Zs1*Zs2/omega^2)],
        {Rp, Rs, Tp, Ts}, Simplify]
u1diffe = Collect[FullSimplify[u1diffd*(Zs1*Zs2/(I*omega))],
        {Rp, Rs, Tp, Ts}, Simplify]
u2diffe = Collect[FullSimplify[u2diffd*(Zp1*Zp2/(I*omega))],
        {Rp, Rs, Tp, Ts}, Simplify]
Out[5]= E^(I kp1 (sti x1-cti x2)) Phi
```

```
Out[6]= E^(I kp1 (str x1+ctr x2)) Phi Rp
Out[7]= E^(I kp2 (stt x1-ctt x2)) Phi Tp
Out[8]= E^(I ks1 (strs x1+ctrs x2)) Phi Rs
Out[9]= E^(I ks2 (stts x1-ctts x2)) Phi Ts
Out[32]= -2 cti kp1^2 mu1 sti+2 ctr E^(-I kp1 sti x1+I kp1 str x1) kp1^2 mu1 Rp str
        +E^(-I kp1 sti x1) Rs (ctrs^2 E^(I ks1 strs x1) ks1^2 mu1
        -E^(I ks1 strs x1) ks1^2 mu1 strs^2)
        +2 ctt E^(-I kp1 sti x1+I kp2 stt x1) kp2^2 mu2 stt Tp
        +E^(-I kp1 sti x1+I ks2 stts x1) ks2^2 mu2 (-ctts^2+stts^2) Ts
Out[35]= kp1^2 (cti^2 (lambda1+2 mu1)+lambda1 sti^2)+
        E^(-I kp1 sti x1+I kp1 str x1) kp1^2 Rp (ctr^2 (lambda1+2 mu1)+lambda1 str^2)
        -2 ctrs E^(-I kp1 sti x1+I ks1 strs x1) ks1^2 mu1 Rs strs
        -E^(-I kp1 sti x1+I kp2 stt x1) kp2^2 (ctt^2 (lambda2+2 mu2)+lambda2 stt^2) Tp
        -2 ctts E^(-I kp1 sti x1+I ks2 stts x1) ks2^2 mu2 stts Ts
Out[38]= -I ctrs E^(-I kp1 sti x1+I ks1 strs x1) ks1 Rs
        -I kp1 sti-I E^(-I kp1 sti x1+I kp1 str x1) kp1 Rp str
        +I E^(-I kp1 sti x1+I kp2 stt x1) kp2 stt Tp
        -I ctts E^(-I kp1 sti x1+I ks2 stts x1) ks2 Ts
Out[41]= I cti kp1-I ctr E^(-I kp1 sti x1+I kp1 str x1) kp1 Rp
        +I E^(-I kp1 sti x1+I ks1 strs x1) ks1 Rs strs
        -I ctt E^(-I kp1 sti x1+I kp2 stt x1) kp2 Tp
        -I E^(-I kp1 sti x1+I ks2 stts x1) ks2 stts Ts
Out[42]= -2 cti kp1^2 mu1 sti+2 cti kp1^2 mu1 Rp sti
        +E^(-I kp1 sti x1) Rs (ctrs^2 E^(I kp1 sti x1) ks1^2 mu1
        -E^(I kp1 sti x1) kp1^2 mu1 sti^2)
        +2 ctt kp1 kp2 mu2 sti Tp+ks2^2 mu2 (-ctts^2+(kp1^2 sti^2)/ks2^2) Ts
Out[43]= -2 ctrs kp1 ks1 mu1 Rs sti+kp1^2 (cti^2 (lambda1+2 mu1)+lambda1 sti^2)
        +kp1^2 Rp (cti^2 (lambda1+2 mu1)+lambda1 sti^2)
        -kp2^2 (ctt^2 (lambda2+2 mu2)+(kp1^2 lambda2 sti^2)/kp2^2) Tp
        -2 ctts kp1 ks2 mu2 sti Ts
Out[44]= -I ctrs ks1 Rs-I kp1 sti-I kp1 Rp sti+I kp1 sti Tp-I ctts ks2 Ts
Out[45]= I cti kp1-I cti kp1 Rp+I kp1 Rs sti-I ctt kp2 Tp-I kp1 sti Ts
Out[50]= -((2 alpha cs1^2 omega^2 rho1^2)/Zp1)+(2 alpha cs1^2 omega^2 rho1^2 Rp)/Zp1
        +(2 alpha cs2^2 omega^2 rho2^2 Tp)/Zp2
        +E^(-I alpha omega x1) Rs (-alpha^2 cs1^2 E^(I alpha omega x1) omega^2 rho1
        +(cs1^2 E^(I alpha omega x1) omega^2 rho1^3)/Zs1^2)
        +omega^2 rho2 Ts (alpha^2 cs2^2-(cs2^2 rho2^2)/Zs2^2)
Out[51]= (omega^2 (alpha^2 cp1^2 (-2 mu1+cp1^2 rho1)
        +(cp1^2 rho1^2 (-2 mu1+cp1^2 rho1+2 cs1^2 rho1))/Zp1^2))/cp1^2
        +(1/(cp1^2))omega^2 Rp (alpha^2 cp1^2 (-2 mu1+cp1^2 rho1)
        +(cp1^2 rho1^2 (-2 mu1+cp1^2 rho1+2 cs1^2 rho1))/Zp1^2)
        -(1/(cp2^2))omega^2 Tp (alpha^2 cp2^2 (-2 mu2+cp2^2 rho2)
        +(cp2^2 rho2^2 (-2 mu2+cp2^2 rho2+2 cs2^2 rho2))/Zp2^2)
        -(2 alpha cs1^2 omega^2 rho1^2 Rs)/Zs1-(2 alpha cs2^2 omega^2 rho2^2 Ts)/Zs2
Out[52]= -I alpha omega-I alpha omega Rp+I alpha omega Tp
        -(I omega rho1 Rs)/Zs1-(I omega rho2 Ts)/Zs2
Out[53]= I alpha omega Rs-I alpha omega Ts+(I omega rho1)/Zp1
        -(I omega rho1 Rp)/Zp1-(I omega rho2 Tp)/Zp2
Out[58]= -2 alpha cs2^2 rho2^2 Tp Zp1 Zs1^2 Zs2^2
        +2 alpha cs1^2 rho1^2 Zp2 Zs1^2 Zs2^2-2 alpha cs1^2 rho1^2 Rp Zp2 Zs1^2 Zs2^2
        +cs1^2 rho1 Rs Zp1 Zp2 (-rho1^2+alpha^2 Zs1^2) Zs2^2
        +cs2^2 rho2 Ts Zp1 Zp2 Zs1^2 (rho2^2-alpha^2 Zs2^2)
Out[59]= 2 alpha cs2^2 rho2^2 Ts Zp1^2 Zp2^2 Zs1
        +2 alpha cs1^2 rho1^2 Rs Zp1^2 Zp2^2 Zs2
        +rho1 (2 alpha^2 cs1^2 Zp1^2-cp1^2 (rho1^2+alpha^2 Zp1^2)) Zp2^2 Zs1 Zs2
        +rho1 Rp (2 alpha^2 cs1^2 Zp1^2-cp1^2 (rho1^2+alpha^2 Zp1^2)) Zp2^2 Zs1 Zs2
        +rho2 Tp Zp1^2 (cp2^2 rho2^2+alpha^2 (cp2^2-2 cs2^2) Zp2^2) Zs1 Zs2
Out[60]= rho2 Ts Zs1+rho1 Rs Zs2+alpha Zs1 Zs2+alpha Rp Zs1 Zs2-alpha Tp Zs1 Zs2
Out[61]= rho2 Tp Zp1-rho1 Zp2+rho1 Rp Zp2-alpha Rs Zp1 Zp2+alpha Ts Zp1 Zp2
```

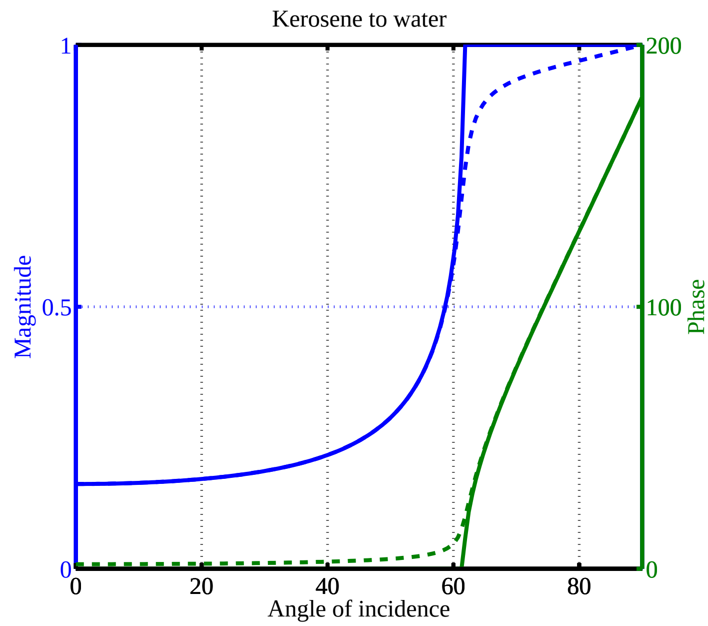
Plots of the magnitude and phase for the reflection and transmission coefficients are given below.


Rp
Rs
Tp
Ts
Magnitude
Angle of incidence
0
0.5
1
1.5
2
0
15
30
45
60
75
90
Rp
Rs
Tp
Ts
Phase
Angle of incidence
180
120
60
0
−60
−120
−180

Problem 2.4 Consider a plane acoustic wave propagating from kerosene into water (at room temperature). Kerosene has a density of 820 kg/m^3 and a sound speed of 1320 m/s while water has a density of 1000 kg/m^3 and s sound speed of 1500 m/s. Plot the magnitude and phase of the reflection coefficient (R) as a function of the angle of incidence. Is there any angle at which the entire energy of the wave is transmitted through the interface? At what angle does total internal reflection occur (i.e., the transmission coefficient becomes zero)? What happens as the angle of incidence is increased beyond the angle at which total internal reflection first occurs?

Now consider the case where the materials absorb a small fraction of the energy of the acoustic wave. In that case we can add a damping factor (α) to the refractive index n, i.e, $n \to n(1 + i\alpha)$. Plot the magnitude and phase and a function of incidence angle for α = 0.01. Is there total internal reflection in this situation?

Solution 2.4: See the plot below (the phases have been multiplied by -1).



There is no angle at which there is no reflection. Total internal reflection occurs at approximately 62°. At incidence angles greater than 62° $R = 1$ but the phase of the reflected wave changes. There is no total internal reflection for absorbing media and there is a phase lag between incident and reflected waves.

Problem 2.5 We have defined the refractive index for acoustic waves propagating from a medium with phase velocity c_1 into a medium with phase velocity c_2 as $n = c_1/c_2$. If we choose a reference medium, e.g., air with a sound speed of c_0, the we can have an alternative definition of the refractive indexes n_1 and n_2 of the two media given by $n_1 = c_0/c_1$ and $n_2 = c_0/c_2$ in which case $n = n_2/n_1$. We have mentioned earlier that waves cannot propagate in the medium if the phase velocity is imaginary. How then can waves propagate in a medium with a complex refractive index?

Solution 2.5: The phase velocity is related to the wave number ($k = \|\mathbf{k}\|$) by $k = \omega/c$. Therefore, for a medium with phase velocity c_1, we have $k = \omega/c_1 = \omega n_1/c_0$. So we can write a plane wave solution in the form

$$p = p_0 e^{i(\omega n_1 x/c_0 - \omega t)} \, . \tag{2.31}$$

If $n_1 = n_1(1 + i\alpha)$ we have

$$p = p_0 e^{i[\omega n_1(1+i\alpha)x/c_0 - \omega t]} = p_0 e^{i[\omega n_1 x/c_0 - \omega t] - \omega n_1 \alpha x/c_0} \tag{2.32}$$

or

$$p = p_0 e^{-\omega n_1 \alpha x/c_0} e^{i(\omega n_1 x/c_0 - \omega t)} \, . \tag{2.33}$$

Hence the phase speed remains real and only the amplitude decreases because of the imaginary part of the refractive index.

Problem 2.6 Maxwell's equations for an isotropic material at fixed frequency may be expressed as

$$\boldsymbol{\nabla}\times\mathbf{E} = i\omega\mu\mathbf{H}\,;\quad \boldsymbol{\nabla}\cdot\mathbf{H} = 0\,;\quad \boldsymbol{\nabla}\cdot\mathbf{E} = 0\,. \tag{2.34}$$

Show that for a plane wave electric field $\mathbf{E}(\mathbf{x}) = \mathbf{E}_0\exp(i\mathbf{k}\cdot\mathbf{x})$ the wavenumber vector is perpendicular to the fields, i.e., $\mathbf{k}\cdot\mathbf{E}_0 = 0$. Then show that this implies that the magnetic field is also a plane wave of the form $\mathbf{H}(\mathbf{x}) = \mathbf{H}_0\exp(i\mathbf{k}\cdot\mathbf{x})$ where

$$\mathbf{H}_0 = -\frac{1}{\omega\mu}(\mathbf{k}\times\mathbf{E}_0)\quad\text{and}\quad \mathbf{k}\cdot\mathbf{H}_0 = 0\,. \tag{2.35}$$

Recall also that for fixed frequency

$$\nabla^2\mathbf{H} + \frac{\omega^2}{c^2}\mathbf{H} = \mathbf{0}\,. \tag{2.36}$$

Show that the above equation implies that for a plane wave

$$(\|\mathbf{k}\|)^2 = \frac{\omega^2}{c^2}\,. \tag{2.37}$$

Solution 2.6: It is convenient to work out this exercise in rectangular Cartesian coordinates.

First, from the relation $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ we have

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\partial}{\partial x_r}\left(E_{0r}e^{ik_m x_m}\right) = iE_{0r}k_m\delta_{mr}e^{i\mathbf{k}\cdot\mathbf{x}} = iE_{0r}k_r e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.38}$$

or,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = i(\mathbf{k}\cdot\mathbf{E}_0)e^{i\mathbf{k}\cot\mathbf{x}} = 0\,. \tag{2.39}$$

Therefore,

$$\mathbf{k}\cdot\mathbf{E}_0 = 0\,.\qquad\square \tag{2.40}$$

Recall that for a vector field $\mathbf{v}(\mathbf{x})$

$$\boldsymbol{\nabla}\times\mathbf{v} = e_{pqr}v_{r,q}\mathbf{e}_p\,. \tag{2.41}$$

So we have

$$\boldsymbol{\nabla}\times\mathbf{E} = e_{pqr}E_{r,q}\mathbf{e}_p = e_{pqr}E_{0r}\frac{\partial}{\partial x_q}\left(e^{ik_m x_m}\right)\mathbf{e}_p = ie_{pqr}E_{0r}k_m\delta_{mq}e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_p = i(e_{pqr}k_qE_{0r}\mathbf{e}_p)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.42}$$

or

$$\boldsymbol{\nabla}\times\mathbf{E} = (\mathbf{k}\times\mathbf{E}_0)e^{i\mathbf{k}\cdot\mathbf{x}} = i\omega\mu\mathbf{H}\,. \tag{2.43}$$

Hence,

$$\mathbf{H} = -\frac{i}{\omega\mu}(\mathbf{k}\times\mathbf{E}_0)e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{H}_0e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.44}$$

where

$$\mathbf{H}_0 := \frac{1}{i\omega\mu}(\mathbf{k}\times\mathbf{E}_0)\,. \tag{2.45}$$

Hence the magnetic field also has the form of a plane wave. This field has to satisfy the relation $\boldsymbol{\nabla}\cdot\mathbf{H} = 0$, i.e,

$$\boldsymbol{\nabla}\cdot\mathbf{H} = \frac{\partial}{\partial x_r}\left(H_{0r}e^{ik_m x_m}\right) = iH_{0r}k_m\delta_{mr}e^{i\mathbf{k}\cdot\mathbf{x}} = iH_{0r}k_re^{i\mathbf{k}\cdot\mathbf{x}} = i(\mathbf{k}\cdot\mathbf{H}_0)e^{\mathbf{k}\cdot\mathbf{x}} = 0\,. \tag{2.46}$$

Therefore,

$$\mathbf{k}\cdot\mathbf{H}_0 = 0\,.\qquad\square \tag{2.47}$$

The Laplacian of the magnetic field is

$$\begin{aligned}\nabla^2\mathbf{H} &= H_{i,jj}\mathbf{e}_i = \frac{\partial^2 H_i}{\partial x_j \partial x_j}\mathbf{e}_i \\ &= \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\left(H_{0i}\, e^{ik_m x_m}\right)\right]\mathbf{e}_i = \frac{\partial}{\partial x_j}\left[iH_{0i}k_m\delta_{mj}e^{ik_m x_m}\right]\mathbf{e}_i = \frac{\partial}{\partial x_j}\left[iH_{0i}k_j e^{ik_m x_m}\right]\mathbf{e}_i \\ &= -H_{0i}k_j k_m \delta_{mj} e^{ik_m x_m}\mathbf{e}_i = -H_{0i}k_j k_j e^{ik_m x_m}\mathbf{e}_i = -(\mathbf{k}\cdot\mathbf{k})\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}}\,.\end{aligned} \tag{2.48}$$

Therefore

$$\nabla^2\mathbf{H} + \frac{\omega^2}{c^2}\,\mathbf{H} = -(\mathbf{k}\cdot\mathbf{k})\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{\omega^2}{c^2}\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{0} \tag{2.49}$$

which implies that

$$\mathbf{k}\cdot\mathbf{k} = (\|\mathbf{k}\|)^2 = \frac{\omega^2}{c^2}\,. \qquad \square \tag{2.50}$$

Problem 2.7 Express Fresnel's equations for perpendicular incidence in terms of electromagnetic impedances and then calculate the reflection and transmission coefficients for a medium that is impedance matched with a silicone rubber dielectric material.

Solution 2.7: Fresnel's equations, for a **E** wave polarized perpendicular to the plane of incidence, are

$$R = \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t} \tag{2.51}$$

and

$$T = \frac{2\frac{n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}\,. \tag{2.52}$$

For the situation where the incident **E** wave is polarized perpendicular to the plane of incidence, the electrical impedance is defined as

$$Z_i = \frac{1}{\cos\theta_i}\sqrt{\frac{\mu_i}{\varepsilon_i}} = \frac{c_0\mu_i}{n_i\cos\theta_i}\,. \tag{2.53}$$

Since the polarization does not change when we move from medium 1 to medium 2 or vice versa, we can write

$$Z_1 = \frac{c_0\mu_1}{n_1\cos\theta_i} \quad \text{and} \quad Z_2 = \frac{c_0\mu_2}{n_2\cos\theta_t}\,. \tag{2.54}$$

Plugging these into the expressions for R and T gives us

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{and} \quad T = \frac{2Z_2}{Z_2 + Z_1} \qquad \square \tag{2.55}$$

Let us assume that medium 2 is a silicone rubber. We want to find a medium 1 that is impedance matched with medium 2, i.e., $Z_1 = Z_2$, $R = 0$, and $T = 1$. Then we must find a medium that satisfies

$$\frac{\mu_1}{n_1\cos\theta_i} = \frac{\mu_2}{n_2\cos\theta_t}\,. \tag{2.56}$$

From Snell's law,

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \qquad \Longrightarrow \qquad \cos\theta_t = \frac{1}{n_2}\sqrt{n_2^2 - n_1^2\sin^2\theta_i}\,. \tag{2.57}$$

Plugging the first form of Snell's law into (2.56) gives us

$$\frac{\tan\theta_t}{\tan\theta_i} = \frac{\mu_1}{\mu_2}\,. \tag{2.58}$$

Since $\theta_i = \theta_t$ in impedance-matched media, we must have $\mu_1 = \mu_2$. If we plug the second form of Snell's law into (2.56) and rearrange, we have

$$\frac{n_2^2}{\mu_2^2} = \frac{n_1^2}{\mu_2^2}\sin^2\theta_i + \frac{n_1^2}{\mu_1^2}\cos^2\theta_i\,. \tag{2.59}$$

Since impedance matching requires that $\mu_1 = \mu_2$, we must have $n_1 = n_2$ (for ordinary materials) and therefore $\varepsilon_1 = \varepsilon_2$. Therefore, we will have to find a material that has exactly the same electrical properties as silicone rubber for impedance matching.

Problem 2.8 Consider a slab of material in an impedance tube. The bulk modulus and density of air on both sides of the slab are 1.42×10^5 Pa and 1.20 kg/m^3, respectively. The Young's modulus (E), Poisson's ratio (ν), and density (ρ) of aluminum are 70 GPa, 0.33, and 2700 kg/m^3, respectively. Assume that the phase velocity in aluminum can be obtained from the relation $c = \sqrt{\kappa/\rho}$ where $\kappa = E/(3(1-2\nu))$ is the bulk modulus.

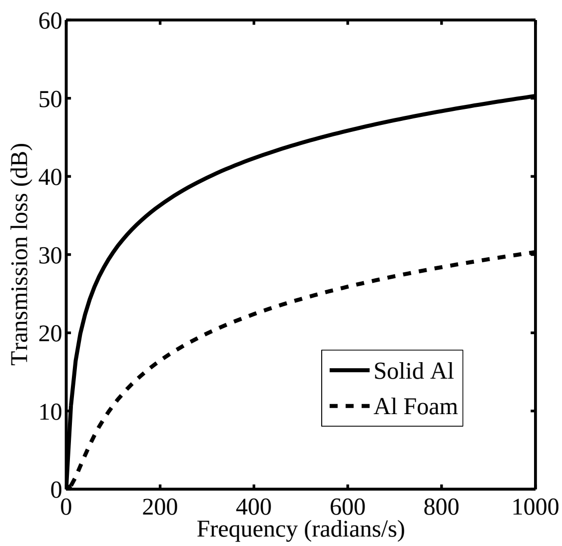
1. The transmission loss due to the slab is calculated using the relation

$$\text{TL (dB)} = 10\log_{10}\left(\frac{1}{T^2}\right) \tag{2.60}$$

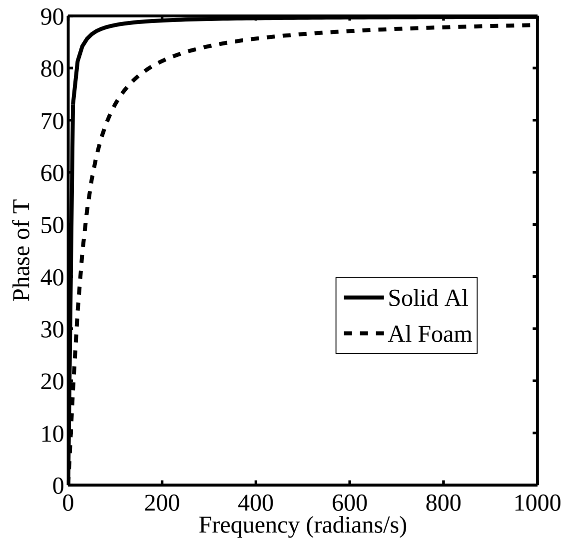
where T is the transmission coefficient. Plot the transmission loss for a 10 cm thick aluminum slab. Compare the transmission loss due to the solid slab with that for a similar slab made of aluminum foam with an aluminum volume fraction (f) of 10%. Assume that the effective foam density is given by $\rho_{\text{eff}} = f\rho_1 + (1-f)\rho_2$ and that the effective foam Young's modulus is given by $E_{\text{eff}} = E(\rho_{\text{eff}}/rho)^2$. The Poisson's ratio of the foam is 0.33. What does the imaginary part of the transmission coefficient indicate? What is the effect of slab density on the transmission loss?

2. Next plot the transmission losses for aluminum and aluminum slabs for a fixed frequency as a function of slab thickness. Assume a frequency of 100 Hz and keep in mind that ω has units of radians/s and not cycles/s. Such a plot is called a mass law plot in acoustics. What would the mass law effect be if ρ_{eff} were a function of frequency and the system had a resonance frequency of 100 Hz?

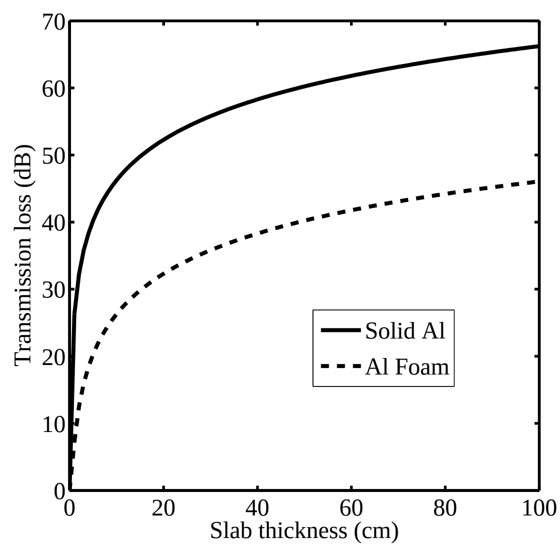
Solution 2.8: 1) Assume normal incidence. The quantities that we need to calculate the transmission coefficient for the foam are E = 0.7 GPa, ρ = 271 kg/m^3. The transmission loss plots is below.



A plot of the phase of the transmission loss is shown below.



2) For a frequency of $\omega = (2\pi)(100)$ radians/s the transmission loss plot is as shown in the plot below.



Problem 2.9 Show that the transmission coefficient for an slab with incident TE-waves can be expressed as

$$T_{13} = \frac{T_{12}T_{23}\, e^{ik_{z2}(d_2-d_1)}}{1 - R_{21}R_{23}\, e^{2ik_{z2}(d_2-d_1)}}. \tag{2.61}$$

Also verify that the series expansion of the above equation is

$$T_{13} = T_{12}T_{23}\, e^{i\theta} + T_{12}T_{23}R_{21}R_{23}\, e^{3i\theta} + T_{12}T_{23}R_{21}^2R_{23}^2\, e^{5i\theta} + \dots . \tag{2.62}$$

Solution 2.9: The process of finding the transmission coefficient is identical to that used to find the generalized reflection coefficient.

As before, we superpose solutions of the form

$$E_y(Z) = E_0 \exp(\pm ik_z Z) \tag{2.63}$$

where $Z = 0$ at the interface. To make sure that the above form can be used in all the layers, we will express all equations in a single coordinate system with $z = -d_1$ at interface $1-2$ and $z = -d_2$ at interface $2-3$.

In medium 1, the electric field consists of a incident part and a reflected part,

$$\begin{aligned} E_{y1}(Z) = E_i + E_r &= E_0 \exp(-ik_{z1}Z) + \widetilde{R}_{12}E_0 \exp(ik_{z1}Z) \\ &= E_0 \exp(-ik_{z1}Z)\left[1 + \widetilde{R}_{12}\exp(2ik_{z1}Z)\right] \end{aligned} \tag{2.64}$$

where $\widetilde{R}_{23}$ is the generalized reflection coefficient at the interface $1-2$. Let us now change the variable so that interface $1-2$ with $Z = 0$ is at $z = -d_1$, i.e, we set $Z = z + d_1$. Then we can write the above equation as

$$\begin{aligned} E_{y1}(z) &= E_0 \exp[-ik_{z1}(z+d_1)]\left[1 + \widetilde{R}_{12}\exp[2ik_{z1}(z+d_1)]\right] \\ &= E_0 \exp(-ik_{z1}d_1)\left[\exp(-ik_{z1}z) + \widetilde{R}_{12}\exp[ik_{z1}(z+2d_1)]\right] \\ &= A_1\left[\exp(-ik_{z1}z) + \widetilde{R}_{12}\exp[ik_{z1}(z+2d_1)]\right] \end{aligned} \tag{2.65}$$

where $A_1 := E_0 \exp(-ik_{z1}d_1)$. Similarly, in medium 2, we have

$$E_{y2}(Z) = E_i + E_r = A\exp(-ik_{z2}Z) + \widetilde{R}_{23}A\exp(ik_{z2}Z) \tag{2.66}$$

where A is the amplitude of the incident wave in medium 2 and $\widetilde{R}_{23}$ is the generalized reflection coefficient at interface $2-3$. A change of variables, $Z = z + d_2$, gives us

$$\begin{aligned} E_{y2}(z) &= A\exp[-ik_{z2}(z+d_2)] + \widetilde{R}_{23}A\exp[ik_{z2}(z+d_2)] \\ &= A\exp(-ik_{z2}d_2)\left[\exp(-ik_{z2}z) + \widetilde{R}_{23}\exp[ik_{z2}(z+2d_2)]\right] \\ &= A_2\left[\exp(-ik_{z2}z) + \widetilde{R}_{23}\exp[ik_{z2}(z+2d_2)]\right] \end{aligned} \tag{2.67}$$

where $A_2 := A\exp(-ik_{z2}d_2)$. There is no reflected wave in medium 3 and we have

$$E_{y3}(Z) = E_i = B\exp(-ik_{z3}Z) \tag{2.68}$$

where B is the amplitude of the transmitted wave in medium 3. With a change of variables $Z = z + d_2$ we have

$$E_{y3}(z) = B\exp[-ik_{z3}(z+d_2)] = A_3\exp(-ik_{z3}z) \tag{2.69}$$

where $A_3 := B\exp(-ik_{z3}d_2)$.

The electric field in the three layers, expressed in a single coordinate system with $z = -d_1$ at interface $1-2$ and $z = -d_2$ at interface $2-3$, are therefore

$$\begin{aligned}
E_{y1}(z) &= A_1 \left[\exp(-ik_{z1}z) + \widetilde{R}_{12}\exp[ik_{z1}(z+2d_1)]\right] \\
E_{y2}(z) &= A_2 \left[\exp(-ik_{z2}z) + \widetilde{R}_{23}\exp[ik_{z2}(z+2d_2)]\right] \\
E_{y3}(z) &= A_3 \exp(-ik_{z3}z)\,.
\end{aligned} \tag{2.70}$$

Let us now examine the fields above and below interface $1-2$ at $z = -d_1$. From the above equations we have

$$\begin{aligned}
E_{y1}(-d_1) &= A_1 \left[\exp(ik_{z1}d_1) + \widetilde{R}_{12}\exp[ik_{z1}d_1]\right] \\
E_{y2}(-d_1) &= A_2 \left[\exp(ik_{z2}d_1) + \widetilde{R}_{23}\exp[ik_{z2}(-d_1+2d_2)]\right]\,.
\end{aligned} \tag{2.71}$$

Similarly, at interface $2-3$, we have

$$\begin{aligned}
E_{y2}(-d_2) &= A_2 \left[\exp(ik_{z2}d_2) + \widetilde{R}_{23}\exp[ik_{z2}d_2]\right] \\
E_{y3}(-d_2) &= A_3 \exp(ik_{z3}d_2)\,.
\end{aligned} \tag{2.72}$$

Consider medium 2 below interface $1-2$. Then the transmitted wave from medium 1 has the form

$$E_t = T_{12}A_1\exp(ik_{z1}d_1) \tag{2.73}$$

where T_{12} is the transmission coefficient going from medium 1 to medium 2. The reflected wave from interface $2-3$ is also reflected at interface $1-2$ and adds to the transmitted wave from medium 1 to medium 2. This reflected wave has the form

$$E_r = R_{21}A_2\widetilde{R}_{23}\exp[ik_{z2}(-d_1+2d_2)] \tag{2.74}$$

where R_{21} is the reflection coefficient going from medium 2 to medium 1. These two waves sum to the downgoing wave in medium 2,

$$E_t + E_r = E_i = A_2\exp(ik_{z2}d_1)\,. \tag{2.75}$$

Plugging in the expressions for E_t and E_r,

$$A_2\exp(ik_{z2}d_1) = T_{12}A_1\exp(ik_{z1}d_1) + R_{21}A_2\widetilde{R}_{23}\exp[ik_{z2}(-d_1+2d_2)] \tag{2.76}$$

or

$$A_2 = T_{12}A_1\exp[i(k_{z1}-k_{z2})d_1] + R_{21}A_2\widetilde{R}_{23}\exp[2ik_{z2}(-d_1+d_2)] \tag{2.77}$$

or

$$\frac{A_2}{A_1} = \frac{T_{12}\exp[i(k_{z1}-k_{z2})d_1]}{1 - R_{21}\widetilde{R}_{23}\exp[2ik_{z2}(d_2-d_1)]}\,. \tag{2.78}$$

Now consider the transmitted wave going from medium 1 to medium 3. In medium 1, the downgoing wave at interface $1-2$ is,

$$E_i = A_1\exp(ik_{z1}d_1)\,. \tag{2.79}$$

In medium 3, the downgoing wave at interface $2-3$ is

$$E_t = A_3\exp(ik_{z3}d_2)\,. \tag{2.80}$$

If T_{13} is the transmission coefficient for waves going from medium 1 to medium 3, we have

$$E_t = T_{13}E_i \quad \Longrightarrow \quad A_3\exp(ik_{z3}d_2) = T_{13}A_1\exp(ik_{z1}d_1)\,. \tag{2.81}$$

Therefore,

$$\frac{A_3}{A_1} = T_{13}\exp(ik_{z1}d_1 - ik_{z3}d_2)\,. \tag{2.82}$$

Similarly, if we consider only the downgoing wave is from medium 2 to medium 3, we have

$$E_i = A_2\exp(ik_{z2}d_2)\,. \tag{2.83}$$

If T_{23} is the transmission coefficient for waves going from medium 2 to medium 3, we have

$$E_t = T_{23}E_i \quad \Longrightarrow \quad A_3\exp(ik_{z3}d_2) = T_{23}A_2\exp(ik_{z2}d_2)\,. \tag{2.84}$$

Therefore,

$$T_{23}\frac{A_2}{A_3} = \exp(-ik_{z2}d_2 + ik_{z3}d_2)\,. \tag{2.85}$$

Multiply (2.82) and (2.85) to get

$$T_{23}\frac{A_2}{A_1} = T_{13}\exp(ik_{z1}d_1 - ik_{z2}d_2)\,. \tag{2.86}$$

Substitute (2.78) to get

$$T_{13}\exp(ik_{z1}d_1 - ik_{z2}d_2) = \frac{T_{23}T_{12}\exp[i(k_{z1}-k_{z2})d_1]}{1 - R_{21}\widetilde{R}_{23}\exp[2ik_{z2}(d_2-d_1)]} \tag{2.87}$$

or,

$$T_{13} = \frac{T_{23}T_{12}\exp[ik_{z2}(d_2-d_1)]}{1 - R_{21}\widetilde{R}_{23}\exp[2ik_{z2}(d_2-d_1)]} \qquad \square \tag{2.88}$$

If we consider only one reflection at interface $2-3$, we have $\widetilde{R}_{23} = R_{23}$.
To expand in series, we observe that $0 \leq 1 - R_{21}R_{23}\exp[2ik_{z2}(d_2-d_1)] < 1$ and recall that

$$\frac{1}{1-f}\Big|_{f=a} = \frac{1}{1-a}\sum_{n=0}^{\infty}\left(\frac{f-a}{1-a}\right)^n\,. \tag{2.89}$$

If we expand around $f=0$, we have

$$\frac{1}{1-f}\Big|_{f=0} = 1 + f + f^2 + f^3 + \ldots \tag{2.90}$$

If $f = R_{21}R_{23}\exp[2ik_{z2}(d_2-d_1)] = R_{21}R_{23}\exp(2i\theta)$, we can write (2.88) as

$$T_{13} = T_{23}T_{12}\exp(i\theta)\left[1 + R_{21}R_{23}\exp(2i\theta) + R_{21}^2R_{23}^2\exp(4i\theta) + \ldots\right]\,. \tag{2.91}$$

Expanded out,

$$T_{13} = T_{23}T_{12}\exp(i\theta) + T_{23}T_{12}R_{21}R_{23}\exp(3i\theta) + T_{23}T_{12}R_{21}^2R_{23}^2\exp(5i\theta) + \ldots \qquad \square \tag{2.92}$$

Chapter 3

Solutions for Exercises in Chapter 3

Problem 3.1 The acoustic potential satisfies Helmholtz equation

$$\nabla^2\varphi + k^2\varphi = 0 \tag{3.1}$$

Show that for the problem of the acoustic spherical lens the solutions of this equation for outgoing waves can be expressed in the form

$$\varphi = \sum_{n=0}^{\infty} A_n h_n(kr) P_n(\cos\theta) \tag{3.2}$$

and those for incoming waves can be expressed in the form

$$\varphi = \sum_{n=0}^{\infty} B_n j_n(kr) P_n(\cos\theta)\ . \tag{3.3}$$

Solution 3.1: Because of the symmetry in ϕ we can express the Helmholtz equation in in spherical coordinates as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + k^2\varphi = 0 \tag{3.4}$$

We use separation of variables to solve this equation,

$$\varphi = R(r)T(\theta) \tag{3.5}$$

Plug this solution into the equation to get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)T + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)R + k^2RT = 0 \tag{3.6}$$

Divide by RT and multiply by r^2 to get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + k^2r^2 = 0 \tag{3.7}$$

Therefore,

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R &= \frac{CR}{r^2}\\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) &= -CT\end{aligned} \tag{3.8}$$

Let us first solve the θ equation. Make the substitution $\eta = \cos\theta$, which means that $d\eta = -\sin\theta d\theta$ and $\sin\theta = \sqrt{1-\eta^2}$. Then

$$\frac{d()}{d\theta} = \frac{d()}{d\eta}\frac{d\eta}{d\theta} = -\sin\theta\frac{d()}{d\eta} \tag{3.9}$$

which leads to

$$-\frac{d}{d\eta}\left(-\sin^2\theta\frac{dT}{d\eta}\right) = -CT \tag{3.10}$$

or

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dT}{d\eta}\right] + CT = 0 \tag{3.11}$$

or,

$$(1-\eta^2)\frac{d^2T}{d\eta^2} - 2\eta\frac{dT}{d\eta} + CT = 0\ . \tag{3.12}$$

This equation has solutions for $C = m(m+1)$ where $m = 0, 1, 2, 3, \ldots$ and its solutions are the Legendre functions, that is

$$T = P_m(\cos\theta)\ . \tag{3.13}$$

For the radial the part of the equation we now have

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2R - m(m+1)R = 0 \tag{3.14}$$

Let $\rho = kr$. Then $d\rho/dr = k$ and

$$\frac{d()}{dr} = \frac{d()}{d\rho}\frac{d\rho}{dr} = k\frac{d()}{d\rho} \tag{3.15}$$

which leads to

$$k\frac{d}{d\rho}\left(r^2k\frac{dR}{d\rho}\right) + k^2r^2R - m(m+1)R = 0 \tag{3.16}$$

or

$$k^2r^2\frac{d^2R}{d\rho^2} + 2rk\frac{dR}{d\rho} + k^2r^2R - m(m+1)R = 0 \tag{3.17}$$

or

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + [\rho^2R - m(m+1)]\frac{R}{\rho^2} = 0 \tag{3.18}$$

This is similar to Bessel's equations and has solutions

$$R = 1/\sqrt{\rho}J_{m+1/2}(\rho) = j_m(\rho) \tag{3.19}$$

and

$$R = 1/\sqrt{\rho}Y_{m+1/2}(\rho) = y_m(\rho) \tag{3.20}$$

Therefore, the solution of the problem of the form

$$\varphi = \sum_{m=0}^{\infty} a_m j_m(kr)P_m(\cos\theta) \qquad \square \tag{3.21}$$

is bounded at $r = 0$ and hence is suitable for the interior of the sphere.

For outgoing waves we are not concerned with boundedness at $r = 0$ and we recall that spherical waves from a source have the form $\exp(ikr)/r$. We know that $h_m^{(1)} = j_m + iy_m$ has this behavior and hence the solution suitable for outgoing waves is

$$\varphi = \sum_{m=0}^{\infty} a_m h_m(kr)P_m(\cos\theta) \qquad \square \tag{3.22}$$

Problem 3.2 A classic problem in scattering is that of acoustic waves from an infinitely long fluid cylinder of circular cross-section. Show that, for harmonic waves, the solutions of the governing equation $\nabla^2 p + k^2 p = 0$ have the form

$$p = \sum_{n=0}^{\infty} A_n \cos(n\theta) \begin{bmatrix} J_n(kr) \\ H_n^{(1)}(kr) \end{bmatrix} \tag{3.23}$$

where $J_n(z)$ is a Bessel function of the first kind and $H_n^{(1)}$ is a Hankel function of the first kind. Then follow the standard procedure of matching boundary conditions at the surface of the cylinder to show for an incident plane wave of unit amplitude $p^i = \exp(ik_1 x)$ and a scattered wave given by

$$p^s = \sum_{n=0}^{\infty} B_n \cos(n\theta) H_n^{(1)}(k_1 r) \tag{3.24}$$

that coefficient B_n has the form

$$B_n = -\frac{e_n i^n}{1 + iC_n} \quad \text{with} \quad C_n := \cfrac{\cfrac{{J_n}'(k_2 a) Y_n(k_1 a)}{J_n(k_2 a){J_n}'(k_1 a)} - \xi\eta \cfrac{{Y_n}'(k_1 a)}{{J_n}'(k_1 a)}}{\cfrac{{J_n}'(k_2 a) J_n(k_1 a)}{J_n(k_2 a){J_n}'(k_1 a)} - \xi\eta} \tag{3.25}$$

where $Y_n(z)$ is a Bessel function of the second kind, $\xi = \rho_2/\rho_1$, and $\eta = c_2/c_1$.

Solution 3.2: Since the cylinder is infinitely long, we can ignore dependence on the z coordinate and express the governing equation in cylindrical coordinates, (r, θ, z), as

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + k^2 p = 0\,. \tag{3.26}$$

Using separation of variables, $p(r,\theta) = R(r)T(\theta)$, we get

$$\frac{r^2}{R}\frac{d^2 R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + k^2 r^2 = -\frac{1}{T}\frac{d^2 T}{d\theta^2} = m^2\,. \tag{3.27}$$

Solving

$$\frac{d^2 T}{d\theta^2} + m^2 T = 0 \tag{3.28}$$

gives us solutions of the form

$$T(\theta) = C_1 \exp(im\theta) + C_2 \exp(-im\theta)\,. \tag{3.29}$$

Periodicity in θ, i.e., the requirement that $T(\theta) = T(2\pi + \theta)$, means that m is an integer. Since only the real part of the pressure is of interest, we can write the solution as

$$T(\theta) = C_m \cos(m\theta)\,. \tag{3.30}$$

Also, the equation for $R(r)$ is

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} + (k^2 r^2 - m^2)R = 0 \tag{3.31}$$

If we use a change of variables, $x \to kr$, we have $dR/dr = dR/dx\, dx/dr = k\, dR/dx$ and $d^2R/dr^2 = d/dr(dR/dr) = d/dx(dR/dr)\, dx/dr = k^2 d^2R/dx^2$. Therefore we have

$$x^2 \frac{d^2 R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2)R = 0 \quad \text{with} \quad x = kr \tag{3.32}$$

which is Bessel's equations with solutions of the form

$$R(r) = C_3 J_m(kr) + C_4 Y_m(kr)\,. \tag{3.33}$$

For the incident wave, we take only the regular part of the solution (which is not singular at $r = 0$), and we have

$$R_i(r) = C_{3m} J_m(kr)\,. \tag{3.34}$$

For the scattered wave, we also include the irregular part of the solution because it has the correct asymptotic behavior as $r \to \infty$, and we can write

$$R_s(r) = C_{4m} H_m^{(1)}(kr) \tag{3.35}$$

where $H_m^{(1)}$ is a Hankel function of the first kind. Therefore, a solution of the wave equation has the form

$$p(r,\theta) = A_m \cos(m\theta) \begin{bmatrix} J_m(kr) \\ H_m^{(1)}(kr) \end{bmatrix}\,. \tag{3.36}$$

To get the general solution, we superpose the solutions for particular values of m to get

$$p(r,\theta) = \sum_{m=0}^{\infty} A_m \cos(m\theta) \begin{bmatrix} J_m(kr) \\ H_m^{(1)}(kr) \end{bmatrix} \quad \square \tag{3.37}$$

Now let us consider an incident plane wave of the form

$$p_i(r,\theta) = \exp(ik_1 x) = \exp(ik_1 r\cos\theta) = \sum_{m=0}^{\infty} e_m i^m \cos(m\theta) J_m(k_1 r) \tag{3.38}$$

where

$$e_m := \begin{cases} 1 & \text{for } m = 0 \\ 2 & \text{for } m > 0 \end{cases}\,. \tag{3.39}$$

The scattered wave has the form

$$p_s(r,\theta) = \sum_{m=0}^{\infty} B_m \cos(m\theta) H_m^{(1)}(k_1 r)\,. \tag{3.40}$$

The waves reflected and refracted inside the sphere have the form

$$p_q(r,\theta) = \sum_{m=0}^{\infty} D_m \cos(m\theta) J_m(k_2 r)\,. \tag{3.41}$$

Let the density and sound speed of the incident medium be (ρ_1, c_1) and that of the fluid cylinder be (ρ_2, c_2). The boundary conditions at the surface of the cylinder, $r = a$, are the continuity of pressure, p, and the radial component of the displacement, $[\mathbf{u}]_r = 1/(\rho\omega^2)[\boldsymbol{\nabla} p]_r = (\rho\omega^2)\partial p/\partial r$ with $\omega = kc$. Taking derivatives with respect to r of (3.38), (3.40), and (3.41), we have

$$\begin{aligned} \frac{\partial p_i}{\partial r} &= k_1 \sum_{m=0}^{\infty} e_m i^m \cos(m\theta) J_m^{'}(k_1 r) \\ \frac{\partial p_s}{\partial r} &= k_1 \sum_{m=0}^{\infty} B_m \cos(m\theta) \left[J_m^{'}(k_1 r) + iY_m^{'}(k_1 r)\right] \\ \frac{\partial p_q}{\partial r} &= k_2 \sum_{m=0}^{\infty} D_m \cos(m\theta) J_m^{'}(k_2 r)\,. \end{aligned} \tag{3.42}$$

At $r = a$,

$$p_i(a,\theta) = \sum_{m=0}^{\infty} e_m i^m \cos(m\theta) J_m(k_1 a)\,, \quad p_s(a,\theta) = \sum_{m=0}^{\infty} B_m \cos(m\theta)\left[J_m(k_1 a) + iY_m(k_1 a)\right]$$
$$p_q(a,\theta) = \sum_{m=0}^{\infty} D_m \cos(m\theta) J_m(k_2 a)\,. \tag{3.43}$$

and

$$\begin{aligned} u_{ri}(a,\theta) &= \frac{k_1}{\rho_1\omega^2}\sum_{m=0}^{\infty} e_m i^m \cos(m\theta) J_m^{'}(k_1 a) \\ u_{rs}(a,\theta) &= \frac{k_1}{\rho_1\omega^2}\sum_{m=0}^{\infty} B_m \cos(m\theta)\left[J_m^{'}(k_1 a) + iY_m^{'}(k_1 a)\right] \\ u_{rq}(a,\theta) &= \frac{k_2}{\rho_2\omega^2}\sum_{m=0}^{\infty} D_m \cos(m\theta) J_m^{'}(k_2 a)\,. \end{aligned} \tag{3.44}$$

Superposing the solutions, using the boundary conditions at $r = a$, and equating terms we have

$$\begin{aligned} e_m i^m \cos(m\theta) J_m(k_1 a) + B_m \cos(m\theta)\left[J_m(k_1 a) + iY_m(k_1 a)\right] &= D_m \cos(m\theta) J_m(k_2 a) \\ e_m i^m \cos(m\theta) J_m^{'}(k_1 a) + B_m \cos(m\theta)\left[J_m^{'}(k_1 a) + iY_m^{'}(k_1 a)\right] &= \frac{\rho_1 k_2}{\rho_2 k_1} D_m \cos(m\theta) J_m^{'}(k_2 a)\,. \end{aligned} \tag{3.45}$$

Eliminating the dependence on θ and using $\xi = \rho_2/\rho_1$,$\eta = c_2/c_1 = k_1/k_2$ gives us

$$\begin{aligned} e_m i^m J_m(k_1 a) + B_m\left[J_m(k_1 a) + iY_m(k_1 a)\right] &= D_m J_m(k_2 a) \\ e_m i^m J_m^{'}(k_1 a) + B_m\left[J_m^{'}(k_1 a) + iY_m^{'}(k_1 a)\right] &= \frac{1}{\xi\eta} D_m J_m^{'}(k_2 a)\,. \end{aligned} \tag{3.46}$$

Solving for B_m, we get

$$B_m = -\frac{e_m i^m\left[-J_m(k_1 a) J_m^{'}(k_2 a) + J_m(k_2 a) J_m^{'}(k_1 a)\xi\eta\right]}{-J_m(k_1 a) J_m'(k_2 a) + J_m(k_2 a) J_m'(k_1 a)\xi\eta - iJ_m'(k_2 a) Y_m(k_1 a) + iJ_m(k_2 a) Y_m'(k_1 a)\xi\eta} \tag{3.47}$$

or

$$B_m = -\frac{e_m i^m}{1 + i\left[\dfrac{-J_m^{'}(k_2 a) Y_m(k_1 a) + J_m(k_2 a) Y_m^{'}(k_1 a)\xi\eta}{-J_m(k_1 a) J_m^{'}(k_2 a) + J_m(k_2 a) J_m^{'}(k_1 a)\xi\eta}\right]} = -\frac{e_m i^m}{1 + iC_m} \quad \square \tag{3.48}$$

where

$$C_m := \frac{-J_m^{'}(k_2 a) Y_m(k_1 a) + J_m(k_2 a) Y_m^{'}(k_1 a)\xi\eta}{-J_m(k_1 a) J_m^{'}(k_2 a) + J_m(k_2 a) J_m^{'}(k_1 a)\xi\eta}\,. \tag{3.49}$$

Divide the numerator and denominator by $J_m(k_2 a) J_m^{'}(k_1 a)$ to get

$$C_m := \frac{\dfrac{J_m^{'}(k_2 a) Y_m(k_1 a)}{J_m(k_2 a) J_m^{'}(k_1 a)} - \dfrac{Y_m^{'}(k_1 a)}{J_m^{'}(k_1 a)}\xi\eta}{\dfrac{J_m(k_1 a) J_m^{'}(k_2 a)}{J_m(k_2 a) J_m^{'}(k_1 a)} - \xi\eta} \quad \square\,. \tag{3.50}$$

Problem 3.3 If a is the radius of the sphere and the wavelength of the incident plane wave is much larger than a we have the condition $ka \ll 1$. Find the asymptotic solutions for the acoustic lens problem for this case. What are the asymptotic solutions when $ka \gg 1$?

Solution 3.3: The scattered and internal pressure wave fields for a spherical acoustic lens are

$$p_s = i\omega\rho_1 \sum_{n=0}^{\infty} \frac{i^n(2n+1)h_n(k_1 r)P_n(\cos\phi)[\eta j_n(k_1 a){j_n}'(k_2 a) - \xi {j_n}'(k_1 a)j_n(k_2 a)]}{\eta h_n(k_1 a){j_n}'(k_2 a) - \xi {h_n}'(k_1 a)j_n(k_2 a)} \tag{3.51}$$

$$p_q = -\frac{\omega\rho_2}{k_1^2 a^2} \sum_{n=0}^{\infty} \frac{i^n(2n+1)j_n(k_2 r)P_n(\cos\phi)}{\eta h_n(k_1 a){j_n}'(k_2 a) - \xi {h_n}'(k_1 a)j_n(k_2 a)} \tag{3.52}$$

where $h_n(kr) = j_n(kr) + iy_n(kr)$. For small arguments, $ka \ll 1$,

$$\begin{aligned} j_n(ka) &\to \frac{2^n n!}{(2n+1)!}(ka)^n = \frac{(ka)^n}{(2n+1)!!} \\ h_n(ka) &\to (i)^{-1}\frac{(2n)!}{2^n n!}\,\frac{1}{(ka)^{n+1}} = -i\frac{(2n-1)!!}{(ka)^{n+1}}\,. \end{aligned} \tag{3.53}$$

Therefore,

$$\begin{aligned} j_n^{'}(ka) &= -j_{n+1}(ka) + \frac{n}{ka}\,j_n(ka) \to -\frac{(ka)^{n+1}}{(2n+3)!!} + \frac{n(ka)^{n-1}}{(2n+1)!!} \\ h_n^{'}(ka) &= -h_{n+1}(ka) + \frac{n}{ka}\,h_n(ka) \to i\frac{(2n+1)!!}{(ka)^{n+2}} - in\frac{(2n-1)!!}{(ka)^{n+2}}\,. \end{aligned} \tag{3.54}$$

We can now calculate the products,

$$\begin{aligned} j_n(k_1 a){j_n}'(k_2 a) &= \frac{(k_1 a)^n}{(2n+1)!!}\left[-\frac{(k_2 a)^{n+1}}{(2n+3)!!} + \frac{n(k_2 a)^{n-1}}{(2n+1)!!}\right] \\ {j_n}'(k_1 a)j_n(k_2 a) &= \left[-\frac{(k_1 a)^{n+1}}{(2n+3)!!} + \frac{n(k_1 a)^{n-1}}{(2n+1)!!}\right]\frac{(k_2 a)^n}{(2n+1)!!} \\ h_n(k_1 a){j_n}'(k_2 a) &= -i\frac{(2n-1)!!}{(k_1 a)^{n+1}}\left[-\frac{(k_2 a)^{n+1}}{(2n+3)!!} + \frac{n(k_2 a)^{n-1}}{(2n+1)!!}\right] \\ {h_n}'(k_1 a)j_n(k_2 a) &= \left[i\frac{(2n+1)!!}{(k_1 a)^{n+2}} - in\frac{(2n-1)!!}{(k_1 a)^{n+2}}\right]\frac{(k_2 a)^n}{(2n+1)!!} \end{aligned} \tag{3.55}$$

Therefore, using Mathematica, we can calculate the ratio needed to find the scattered field:

$$\begin{aligned} &\frac{\eta j_n(k_1 a){j_n}'(k_2 a) - \xi {j_n}'(k_1 a)j_n(k_2 a)}{\eta h_n(k_1 a){j_n}'(k_2 a) - \xi {h_n}'(k_1 a)j_n(k_2 a)} \\ &\quad = \frac{i(k_1 a)^{2n+1}(2n+1)(2n+3)\left[\dfrac{k_1 k_2 a^2(\xi k_1 - \eta k_2)}{(2n+1)!!(2n+3)!!} + \dfrac{n(\eta k_1 - \xi k_2)}{(2n+1)!!(2n+1)!!}\right]}{\eta k_1 k_2^2 a^2 - (2n+3)[n\eta k_1 + (1+n)\xi k_2]} \end{aligned} \tag{3.56}$$

We can verify that this ratio rapidly drops off as n increases. We therefore retain only the first term, $n = 0$, and have

$$\sum_{i=0}^{\infty} \frac{\eta j_n(k_1 a){j_n}'(k_2 a) - \xi {j_n}'(k_1 a)j_n(k_2 a)}{\eta h_n(k_1 a){j_n}'(k_2 a) - \xi {h_n}'(k_1 a)j_n(k_2 a)} \approx \frac{ik_1^2 a^3(\eta k_2 - \xi k_1)}{\eta k_1 k_2 a^2 - 3\xi}\,. \tag{3.57}$$

Using the definitions $\eta = k_2/k_1$ and $\xi = \rho_2/\rho 1$, the scattered field may then be written as

$$p_s \approx -\omega\rho_1 h_0(k_1 r)P_0(\cos\phi)\,\frac{k_1 a^3(\rho_1 k_2^2 - \rho_2 k_1^2)}{\rho_1 k_2^2 a^2 - 3\rho_2}\,. \tag{3.58}$$

Now,

$$\omega = k_1 c_1\,, \quad h_0(k_1 r) = -\frac{i}{k_1 r} e^{ik_1 r} \quad \text{and} \quad P_0(\cos\phi) = 1\,. \tag{3.59}$$

Hence,

$$p_s \approx -\frac{ik_1\rho_1 c_1}{4\pi r}(4\pi a^2) e^{ik_1 r}\left[\frac{a(\rho_2 k_1^2 - \rho_1 k_2^2)}{\rho_1 k_2^2 a^2 - 3\rho_2}\right] \qquad \square \tag{3.60}$$

Further simplification is possible, but the above result shows that the solution is equivalent to that for a simple point source.

For the field inside the sphere, we calculate the quantity in the denominator after substituting for η and xi to get

$$\eta h_n(k_1 a) j_n{}'(k_2 a) - \xi h_n{}'(k_1 a) j_n(k_2 a) = \frac{i(k_1 a)^{-2-n}(k_2 a)^n [\rho_1 k_2^2 a^2 - (2n+3)(\rho_2 + (\rho_1+\rho_2)n)]}{\rho_1[3 + 4n(n+2)]}\,. \tag{3.61}$$

Compared to $n = 0$, the contribution of the inverse of this quantity is quite small an can be ignored. Therefore we have

$$\sum_{i=0}^{\infty} \frac{1}{\eta h_n(k_1 a) j_n{}'(k_2 a) - \xi h_n{}'(k_1 a) j_n(k_2 a)} \approx \frac{3i\rho_1 k_1^2 a^2}{3\rho_2 - \rho_1 k_2^2 a^2} \tag{3.62}$$

and we can express the internal pressure field as

$$p_q \approx -ik_2 c_2 \rho_2\, j_0(k_2 r) P_0(\cos\phi)\ \left[\frac{3\rho_1}{3\rho_2 - \rho_1 k_2^2 a^2}\right]\,. \tag{3.63}$$

Now,

$$j_0(k_2 r) = \frac{\sin k_2 r}{k_2 r} \quad \text{and} \quad P_0(\cos\phi) = 1\,. \tag{3.64}$$

Hence,

$$p_q \approx -ik_2 c_2 \rho_2 \frac{\sin k_2 r}{k_2 r}\left[\frac{3\rho_1}{3\rho_2 - \rho_1 k_2^2 a^2}\right] \quad \square \tag{3.65}$$

Once again, further simplification is possible.

For large arguments, $ka \gg 1$,

$$\begin{aligned} j_n(ka) &\approx \frac{1}{ka}\sin\left(ka - \frac{n\pi}{2}\right) \\ h_n(ka) &\approx i^{-n-1}\frac{1}{ka}\exp(ika) = -\frac{i}{ka}\ \exp\left[i\left(ka - \frac{n\pi}{2}\right)\right]\,. \end{aligned} \tag{3.66}$$

Therefore, ignoring terms containing $1/(ka)^2$,

$$\begin{aligned} j_n^{'}(ka) &= -j_{n+1}(ka) + \frac{n}{ka}\, j_n(ka) \approx -\frac{1}{ka}\sin\left(ka - \frac{(n+1)\pi}{2}\right) \\ h_n^{'}(ka) &= -h_{n+1}(ka) + \frac{n}{ka}\, h_n(ka) \approx \frac{i}{ka}\ \exp\left[i\left(ka - \frac{(n+1)\pi}{2}\right)\right]\,. \end{aligned} \tag{3.67}$$

The products that we need are,

$$\begin{aligned} \eta j_n(k_1 a) j_n{}'(k_2 a) &= -\frac{\eta}{k_1 k_2 a^2}\sin\left(k_1 a - \frac{n\pi}{2}\right)\,\sin\left(k_2 a - \frac{(n+1)\pi}{2}\right) \\ \xi j_n{}'(k_1 a) j_n(k_2 a) &= -\frac{\xi}{k_1 k_2 a^2}\sin\left(k_2 a - \frac{n\pi}{2}\right)\,\sin\left(k_1 a - \frac{(n+1)\pi}{2}\right) \\ \eta h_n(k_1 a) j_n{}'(k_2 a) &= \frac{i\eta}{k_1 k_2 a^2}\ \exp\left[i\left(k_1 a - \frac{n\pi}{2}\right)\right]\,\sin\left(k_2 a - \frac{(n+1)\pi}{2}\right) \\ \xi h_n{}'(k_1 a) j_n(k_2 a) &= \frac{i\xi}{k_1 k_2 a^2}\sin\left(k_2 a - \frac{n\pi}{2}\right)\ \exp\left[i\left(k_1 a - \frac{(n+1)\pi}{2}\right)\right]\,. \end{aligned} \tag{3.68}$$

Therefore,

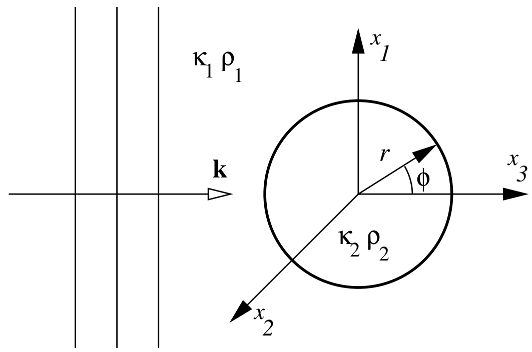
$$
\begin{aligned}
&\frac{\eta j_n(k_1a){j_n}'(k_2a) - \xi {j_n}'(k_1a)j_n(k_2a)}{\eta h_n(k_1a){j_n}'(k_2a) - \xi {h_n}'(k_1a)j_n(k_2a)} \\
&\quad = \frac{\xi \sin\left(k_2a - \frac{n\pi}{2}\right)\, \sin\left(k_1a - \frac{(n+1)\pi}{2}\right) - \eta \sin\left(k_1a - \frac{n\pi}{2}\right)\, \sin\left(k_2a - \frac{(n+1)\pi}{2}\right)}{i\eta\, \exp\left[i\left(k_1a - \frac{n\pi}{2}\right)\right]\, \sin\left(k_2a - \frac{(n+1)\pi}{2}\right) - i\xi \sin\left(k_2a - \frac{n\pi}{2}\right)\, \exp\left[i\left(k_1a - \frac{(n+1)\pi}{2}\right)\right]} \\
&\quad = -e^{\left[i\left(-k_1a + \frac{n\pi}{2}\right)\right]} \frac{\eta \cos\left(k_2a - \frac{n\pi}{2}\right) \sin\left(k_1a - \frac{n\pi}{2}\right) - \xi \cos\left(k_1a - \frac{n\pi}{2}\right) \sin\left(k_2a - \frac{n\pi}{2}\right)}{i\eta \cos\left(k_1a - \frac{n\pi}{2}\right) + \xi \sin\left(k_2a - \frac{n\pi}{2}\right)}.
\end{aligned}
\tag{3.69}
$$

This series is divergent, but more can be said about the behavior of acoustic waves in this "geometrical acoustics" limit. See Morse and Ingard, p. 340, p. 426.

Problem 3.4 From the following equations

$$
\begin{aligned}
p_s &= i\omega\rho_1 \sum_{n=0}^{\infty} \frac{i^n(2n+1)h_n(k_1 r)P_n(\cos\phi)[\eta j_n(k_1 a)j_n{}'(k_2 a) - \xi j_n{}'(k_1 a)j_n(k_2 a)]}{\eta h_n(k_1 a)j_n{}'(k_2 a) - \xi h_n{}'(k_1 a)j_n(k_2 a)} \\
p_q &= -\frac{\omega\rho_2}{k_1^2 a^2} \sum_{n=0}^{\infty} \frac{i^n(2n+1)j_n(k_2 r)P_n(\cos\phi)}{\eta h_n(k_1 a)j_n{}'(k_2 a) - \xi h_n{}'(k_1 a)j_n(k_2 a)}\,.
\end{aligned}
\tag{3.70}
$$

identify the conditions under which there can be resonance within the system. Plot the scattered pressure as a function of the frequency when the sphere in the figure below is made of air.



Solution 3.4: We can express the above equations in terms of the frequency ω by using the relations $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$. Then we have

$$
\begin{aligned}
p_s &= i\omega\rho_1 \sum_{n=0}^{\infty} \frac{i^n(2n+1)h_n(\frac{\omega r}{c_1})P_n(\cos\phi)[\eta j_n(\frac{\omega a}{c_1})j_n{}'(\frac{\omega a}{c_2}) - \xi j_n{}'(\frac{\omega a}{c_1})j_n(\frac{\omega a}{c_2})]}{\eta h_n(\frac{\omega a}{c_1})j_n{}'(\frac{\omega a}{c_2}) - \xi h_n{}'(\frac{\omega a}{c_1})j_n(\frac{\omega a}{c_2})} \\
p_q &= -\frac{\rho_2 c_1^2}{\omega a^2} \sum_{n=0}^{\infty} \frac{i^n(2n+1)j_n(\frac{\omega r}{c_2})P_n(\cos\phi)}{\eta h_n(\frac{\omega a}{c_1})j_n{}'(\frac{\omega a}{c_2}) - \xi h_n{}'(\frac{\omega a}{c_1})j_n(\frac{\omega a}{c_2})}\,.
\end{aligned}
\tag{3.71}
$$

Clearly, we will have resonances when

$$
\eta h_n\left(\frac{\omega a}{c_1}\right) j_n{}'\left(\frac{\omega a}{c_2}\right) - \xi h_n{}'\left(\frac{\omega a}{c_1}\right) j_n\left(\frac{\omega a}{c_2}\right) = 0 \quad \text{for } n = 0\ldots\infty \qquad \square \tag{3.72}
$$

For $n = 0$, the condition for resonance is

$$
\rho_1 c_1 \left(\text{sinc}\frac{\omega a}{c_2} - \cos\frac{\omega a}{c_2}\right) - \rho_2(c_1 - i\omega a)\text{sinc}\frac{\omega a}{c_2} = 0 \tag{3.73}
$$

where $\text{sinc}(x) = \sin(\pi x)/(\pi x)$ is the sinc function. For $\omega a/c_2 \ll 1$, a series expansion around $\omega a/c_2 = 0$ gives

$$
\rho_1 c_1 \left(\frac{\omega a}{c_2}\right)^2 - 3\rho_2(c_1 - i\omega a) = 0 \tag{3.74}
$$

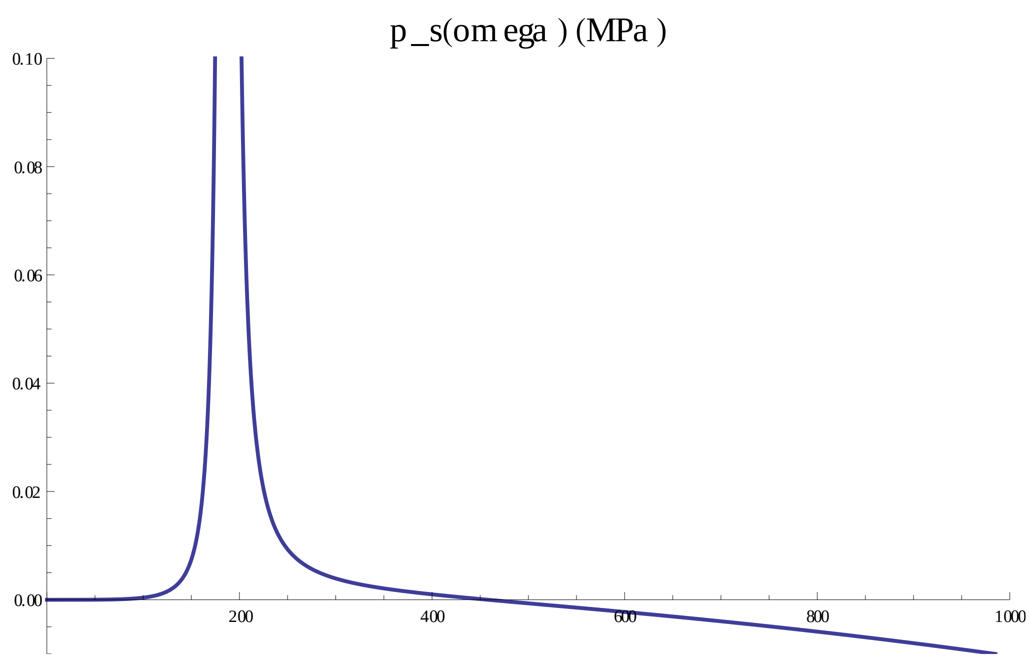
which is a quadratic equation that can be solved for ω.

We can plot the scattered wave at $r = 1.2a$ and $\phi = \pi/2$ using Mathematica. A version of the script is given below.

```
hn = SphericalHankelH1[n, x]
hnp = D[hn, x]
jn = SphericalBesselJ[n, x]
jnp = D[jn, x]
```

```
hnc1 = hn /. (x -> omega a/c1)
hnpc1 = hnp /. (x -> omega a/c1)
jnc1 = jn /. (x -> omega a/c1)
jnpc1 = jnp /. (x -> omega a/c1)
jnc2 = jn /. (x -> omega a/c2)
jnpc2 = jnp /. (x -> omega a/c2
hnr = hn /. (x -> omega r /c1);
Pn = LegendreP[n, Cos[phi]];
tt1 = eta jnc1 jnpc2 - xi jnpc1 jnc2 /. {eta -> c1/c2,
    xi -> rho2/rho1};
tt2 = eta hnc1 jnpc2 - xi hnpc1 jnc2 /. {eta -> c1/c2,
    xi -> rho2/rho1};
val = I^n (2 n + 1) hnr Pn tt1/tt2;
ps = Sum[I omega rho1 val, {n, 0, 10}];
pswaterair = ps /. {c1 -> 1500, c2 -> 346, rho1 -> 1000, rho2 -> 1.18};
psairphi = pswaterair /. {phi -> Pi/2, r -> 1.2 a};
pssimple = psairphi /. {a -> 0.11};
Plot[Re[pssimple]/10^6, {omega, 0, 1000},
 PlotRange -> {{0, 1000}, {-0.01, 0.1}}, PlotStyle -> Thick,
 PlotLabel -> Style["p_s(omega) (MPa)", FontSize -> 18]]
```

A plot of p_s vs. ω is shown below.



Problem 3.5 Consider a plane harmonic acoustic wave incident from a medium with bulk modulus κ_1 and density ρ_1 upon a sphere with bulk modulus κ_2 and density ρ_2. Let the radius of the sphere be a. The sphere is coated with a fluid layer of density $\rho_3 = \rho_2$, radius b, and a bulk modulus given by

$$\frac{b^3}{\kappa_3} = \frac{a^3}{\kappa_1} + \frac{b^3 - a^3}{\kappa_2} \,. \tag{3.75}$$

Find the scattering cross-section of the coated sphere.

Solution 3.5: Let the incident pressure wave have unit amplitude. Then,

$$p_i = \exp(ik_1 r \cos\phi) = \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_1 r) P_n(\cos\phi) \,. \tag{3.76}$$

The scattered pressure field in medium 1 has the form

$$p_s = \sum_{n=0}^{\infty} i^n (2n+1) A_n h_n(k_1 r) P_n(\cos\phi) \,. \tag{3.77}$$

Therefore the total field in medium 1 is $p_1 = p_i + p_s$. The pressure field in the core medium 2 has the form

$$p_2 = \sum_{n=0}^{\infty} i^n (2n+1) D_n j_n(k_2 r) P_n(\cos\phi) \,. \tag{3.78}$$

The pressure field in the coating medium 3 has the form

$$p_3 = \sum_{n=0}^{\infty} i^n (2n+1) \left[B_n j_n(k_3 r) + C_n y_n(k_3 r)\right] P_n(\cos\phi) \,. \tag{3.79}$$

The corresponding radial displacements are given by $u = \rho\omega^2 \partial p/\partial r$. Hence,

$$\begin{aligned}
u_1 &= \rho_1\omega^2 \sum_{n=0}^{\infty} i^n (2n+1) k_1 \left[j_n^{'}(k_1 r) + A_n h_n^{'}(k_1 r)\right] P_n(\cos\phi) \\
u_2 &= \rho_2\omega^2 \sum_{n=0}^{\infty} i^n (2n+1) k_2 D_n j_n^{'}(k_2 r) P_n(\cos\phi) \\
u_3 &= \rho_3\omega^2 \sum_{n=0}^{\infty} i^n (2n+1) k_3 \left[B_n j_n^{'}(k_3 r) + C_n y_n^{'}(k_3 r)\right] P_n(\cos\phi) \,.
\end{aligned} \tag{3.80}$$

The boundary conditions at $r = a$ are $p_2 = p_3$ and $u_2 = u_3$. At $r = b$, $p_1 = p_3$, $u_1 = u_3$. Applying these BCs we have, at $r = a$,

$$\begin{aligned}
D_n j_n(k_2 a) &= B_n j_n(k_3 a) + C_n y_n(k_3 a) \\
\rho_2 k_2 D_n j_n^{'}(k_2 a) &= \rho_3 k_3 \left[B_n j_n^{'}(k_3 a) + C_n y_n^{'}(k_3 a)\right]
\end{aligned} \tag{3.81}$$

and at $r = b$,

$$\begin{aligned}
j_n(k_1 b) + A_n h_n(k_1 b) &= B_n j_n(k_3 b) + C_n y_n(k_3 b) \\
\rho_1 k_1 \left[j_n^{'}(k_1 b) + A_n h_n^{'}(k_1 b)\right] &= \rho_3 k_3 \left[B_n j_n^{'}(k_3 b) + C_n y_n^{'}(k_3 b)\right] \,.
\end{aligned} \tag{3.82}$$

Solving for A_n, B_n, C_n, and D_n gives

$$A_n = -\frac{\alpha_n}{\Delta_n},\; B_n = \frac{\beta_n}{\Delta_n},\; C_n = \frac{\gamma_n}{\Delta_n} D_n = \frac{\delta_n}{\Delta_n} \tag{3.83}$$

where the numerators are

$$
\begin{aligned}
\alpha_n &= \xi_1\xi_2 j_n^{'}(k_1b)j_n^{'}(k_2a)\left[j_n(k_3b)y_n(k_3a) - j_n(k_3a)y_n(k_3b)\right] \\
&\quad + \xi_1\eta_2 j_n^{'}(k_1b)j_n(k_2a)\left[j_n^{'}(k_3a)y_n(k_3b) - j_n(k_3b)y_n^{'}(k_3a)\right] \\
&\quad + \xi_2\eta_1 j_n(k_1b)j_n^{'}(k_2a)\left[j_n(k_3a)y_n^{'}(k_3b) - j_n^{'}(k_3b)y_n(k_3a)\right] \\
&\quad + \eta_1\eta_2 j_n(k_1b)j_n(k_2a)\left[j_n^{'}(k_3b)y_n^{'}(k_3a) - j_n^{'}(k_3a)y_n^{'}(k_3b)\right] , \\
\beta_n &= \frac{i\xi_1}{k_1^2b^2}\left[\xi_2 j_n^{'}(k_2a)y_n(k_3a) - \eta_2 j_n(k_2a)y_n^{'}(k_3a)\right] , \\
\gamma_n &= \frac{i\xi_1}{k_1^2b^2}\left[\eta_2 j_n(k_2a)j_n^{'}(k_3a) - \xi_2 j_n^{'}(k_2a)j_n(k_3a)\right] , \\
\delta_n &= -\frac{i\xi_1\eta_2}{k_1^2k_3^2a^2b^2},
\end{aligned}
\tag{3.84}
$$

the denominator is

$$
\begin{aligned}
\Delta_n &= \xi_1\xi_2 h_n^{'}(k_1b)j_n^{'}(k_2a)\left[j_n(k_3b)y_n(k_3a) - j_n(k_3a)y_n(k_3b)\right] \\
&\quad + \xi_1\eta_2 h_n^{'}(k_1b)j_n(k_2a)\left[j_n^{'}(k_3a)y_n(k_3b) - j_n(k_3b)y_n^{'}(k_3a)\right] \\
&\quad + \xi_2\eta_1 h_n(k_1b)j_n^{'}(k_2a)\left[j_n(k_3a)y_n^{'}(k_3b) - j_n^{'}(k_3b)y_n(k_3a)\right] \\
&\quad + \eta_1\eta_2 h_n(k_1b)j_n(k_2a)\left[j_n^{'}(k_3b)y_n^{'}(k_3a) - j_n^{'}(k_3a)y_n^{'}(k_3b)\right] ,
\end{aligned}
\tag{3.85}
$$

and

$$
\xi_1 = \frac{\rho_1}{\rho_3}\,, \;\; \xi_2 = \frac{\rho_2}{\rho_3}\,, \;\; \eta_1 = \frac{k_3}{k_1} = \frac{c_1}{c_3}\,, \;\; \eta_2 = \frac{k_3}{k_2} = \frac{c_2}{c_3}. \tag{3.86}
$$

If $\rho_3 = \rho_2$, we have $\xi_2 = 1$. Also, since

$$
\frac{b^3}{\kappa_3} = \frac{a^3}{\kappa_1} + \frac{b^3 - a^3}{\kappa_2} \tag{3.87}
$$

and

$$
c_3 = \sqrt{\frac{\kappa_3}{\rho_3}}, \quad c_2 = \sqrt{\frac{\kappa_2}{\rho_2}}, \quad \text{and} \quad c_1 = \sqrt{\frac{\kappa_1}{\rho_1}} \qquad \implies \qquad \frac{\eta_1}{\eta_2} = \sqrt{\frac{\kappa_1\rho_2}{\kappa_2\rho_1}} \tag{3.88}
$$

we have

$$
\eta_1 = \frac{c_1}{c_3} = \sqrt{\frac{[b^3\kappa_1 + a^3(\kappa_2-\kappa_1)]}{b^3\kappa_2}\,\frac{\rho_2}{\rho_1}} \quad \text{and} \quad \eta_2 = \frac{c_2}{c_3} = \sqrt{\frac{[b^3\kappa_1 + a^3(\kappa_2-\kappa_1)]}{b^3\kappa_1}}. \tag{3.89}
$$

Therefore,

$$
\begin{aligned}
&\xi_1\xi_2 = \xi_1 = \frac{\rho_1}{\rho_2}\,, \;\; \xi_1\eta_2 = \sqrt{\frac{[b^3\kappa_1 + a^3(\kappa_2-\kappa_1)]}{b^3\kappa_1}\,\frac{\rho_1^2}{\rho_2^2}} \\
&\xi_2\eta_1 = \eta_1 = \sqrt{\frac{[b^3\kappa_1 + a^3(\kappa_2-\kappa_1)]}{b^3\kappa_2}\,\frac{\rho_2}{\rho_1}}\,, \;\; \eta_1\eta_2 = \frac{[b^3\kappa_1 + a^3(\kappa_2-\kappa_1)]}{b^3}\sqrt{\frac{1}{\kappa_1\kappa_2}\,\frac{\rho_2}{\rho_1}}.
\end{aligned}
\tag{3.90}
$$

Also,

$$
k_3 = \frac{\omega}{c_3} = \frac{\omega\sqrt{\rho_3}}{\sqrt{\kappa_3}}\;;\;\; k_2 = \frac{\omega}{c_2} = \frac{\omega\sqrt{\rho_2}}{\sqrt{\kappa_2}}\;;\;\; k_1 = \frac{\omega}{c_1} = \frac{\omega\sqrt{\rho_1}}{\sqrt{\kappa_1}}\,. \tag{3.91}
$$

In terms of media 1 and 2,

$$k_3 = \omega\sqrt{\rho_2}\sqrt{\frac{a^3}{b^3\kappa_1} + \frac{b^3 - a^3}{b^3\kappa_2}}. \tag{3.92}$$

The scattered wave therefore has the form

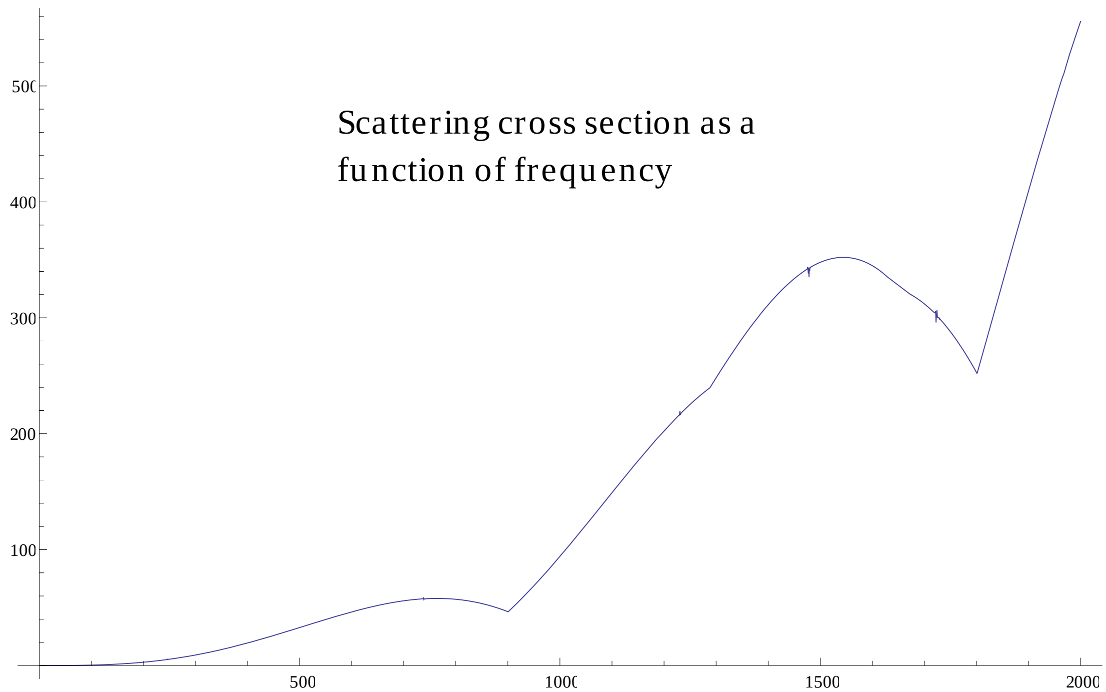
$$p_s = \sum_{n=0}^{\infty} \frac{i^n(2n+1)\alpha_n h_n(k_1 r) P_n(\cos\phi)}{\Delta_n} \tag{3.93}$$

where all quantities have been expressed in terms of $\rho_1, \rho_2, \kappa_1, \kappa_2$.

Following the approach discussed in the text, the scattering cross-section is given by

$$\gamma_{cs} = \frac{4\pi}{k_1^2}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left\|\frac{\alpha_n}{\Delta_n}\right\|^2 . \tag{3.94}$$

We can plot the scattering cross-section for particular choices of geometry and material properties. For example, if $a = 1$ m, $b = 1.2$ m, $\rho_1 = 1.2$ kg/m^3, $\rho_2 = 1000$ kg/m^3, $\kappa_1 = 1.42 \times 10^5$ Pa, $\kappa_2 = 2.2$ GPa, we get the cross-section as a function of frequency shown in the plot below.



Problem 3.6 A vector displacement potential for shear waves can be expressed in terms of scalar Debye potentials (Π, χ) as

$$\boldsymbol{\psi} = \boldsymbol{\nabla} \times (\Pi r \mathbf{e}_r) + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\chi r \mathbf{e}_r)) \tag{3.95}$$

Express the displacement and stress components for an elastic material in terms of these potentials. What are the forms of the potential that can be used to represent an incident plane SV-wave and a plane SH-wave? What are the appropriate forms of the expansions for the scattered fields.

Solution 3.6: The Debye potentials are applicable to problems involving spherical symmetry. The displacement field in an isotropic and linear elastic material can be expressed as

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} \times \boldsymbol{\psi} \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0 \tag{3.96}$$

where $\varphi \equiv \varphi(r, \theta, \phi)$ and $\boldsymbol{\psi} \equiv \boldsymbol{\psi}(r, \theta, \phi)$. The elastic wave equation

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) = -\omega^2 \rho \mathbf{u} \tag{3.97}$$

is satisfied if the potentials satisfy the Helmholtz equations

$$\nabla^2\varphi + \frac{\omega^2}{c_p^2}\varphi = 0\,, \quad \nabla^2\Pi + \frac{\omega^2}{c_s^2}\Pi = 0\,, \quad \nabla^2\chi + \frac{\omega^2}{c_s^2}\chi = 0 \tag{3.98}$$

where $\Pi \equiv \Pi(r, \theta, \phi)$ and $\chi \equiv \chi(r, \theta, \phi)$.

Let us find the components of the displacement field term by term. In spherical coordinates

$$\boldsymbol{\nabla} \times (\Pi r \mathbf{e}_r) = \frac{1}{\sin\phi}\frac{\partial \Pi}{\partial \theta}\mathbf{e}_\phi - \frac{\partial \Pi}{\partial \phi}\mathbf{e}_\theta \quad \text{and} \quad \boldsymbol{\nabla} \times (\chi r \mathbf{e}_r) = \frac{1}{\sin\phi}\frac{\partial \chi}{\partial \theta}\mathbf{e}_\phi - \frac{\partial \chi}{\partial \phi}\mathbf{e}_\theta\,. \tag{3.99}$$

Also,

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\Pi r \mathbf{e}_r) &= -\frac{1}{r}\left(\frac{1}{\sin^2\phi}\frac{\partial^2 \Pi}{\partial \theta^2} + \cot\phi\frac{\partial \Pi}{\partial \phi} + \frac{\partial^2 \Pi}{\partial \phi^2}\right)\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial \Pi}{\partial \phi} + \frac{\partial^2 \Pi}{\partial r \partial \phi}\right)\mathbf{e}_\phi \\
&\quad + \frac{1}{r\sin\phi}\left(\frac{\partial \Pi}{\partial \theta} + r\frac{\partial^2 \Pi}{\partial r \partial \theta}\right)\mathbf{e}_\theta \\
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\chi r \mathbf{e}_r) &= -\frac{1}{r}\left(\frac{1}{\sin^2\phi}\frac{\partial^2 \chi}{\partial \theta^2} + \cot\phi\frac{\partial \chi}{\partial \phi} + \frac{\partial^2 \chi}{\partial \phi^2}\right)\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial \chi}{\partial \phi} + \frac{\partial^2 \chi}{\partial r \partial \phi}\right)\mathbf{e}_\phi \\
&\quad + \frac{1}{r\sin\phi}\left(\frac{\partial \chi}{\partial \theta} + r\frac{\partial^2 \chi}{\partial r \partial \theta}\right)\mathbf{e}_\theta\,,
\end{aligned} \tag{3.100}$$

and

$$\begin{aligned}
&\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\chi r \mathbf{e}_r) = \\
&-\frac{1}{r^2\sin\phi}\left(\cot\phi\frac{\partial^2 \chi}{\partial \theta \partial \phi} + \frac{\partial^3 \chi}{\partial \theta \partial \phi^2} + \frac{1}{\sin^2\phi}\frac{\partial^3 \chi}{\partial \theta^3} + 2r\frac{\partial^2 \chi}{\partial r \partial \theta} + r^2\frac{\partial^3 \chi}{\partial r^2 \partial \theta}\right)\mathbf{e}_\phi \\
&+ \frac{1}{r^2}\left[\cot\phi\frac{\partial^2 \chi}{\partial \phi^2} + \frac{\partial^3 \chi}{\partial \phi^3} + \frac{1}{\sin^2\phi}\left(-\frac{\partial \chi}{\partial \phi} - 2\cot\phi\frac{\partial^2 \chi}{\partial \theta^2} + \frac{\partial^3 \chi}{\partial \theta^2 \partial \phi}\right) + r\left(2\frac{\partial^2 \chi}{\partial r \partial \phi} + r\frac{\partial^3 \chi}{\partial r^2 \partial \phi}\right)\right]\mathbf{e}_\theta\,.
\end{aligned} \tag{3.101}$$

Now,

$$\mathbf{u}_p = \boldsymbol{\nabla}\varphi = \frac{\partial \varphi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \varphi}{\partial \phi}\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial \varphi}{\partial \theta}\mathbf{e}_\theta \tag{3.102}$$

and

$$\mathbf{u}_s = \boldsymbol{\nabla} \times \boldsymbol{\psi} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\Pi r \mathbf{e}_r) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\chi r \mathbf{e}_r)) \tag{3.103}$$

Plugging in the expressions for the two terms, we have

$$\begin{aligned}
\mathbf{u}_s = &-\frac{1}{r}\left(\frac{1}{\sin^2\phi}\frac{\partial^2\Pi}{\partial\theta^2} + \cot\phi\frac{\partial\Pi}{\partial\phi} + \frac{\partial^2\Pi}{\partial\phi^2}\right)\mathbf{e}_r \\
&+ \left[\left(\frac{1}{r}\frac{\partial\Pi}{\partial\phi} + \frac{\partial^2\Pi}{\partial r\partial\phi}\right) - \frac{1}{r^2\sin\phi}\left(\cot\phi\frac{\partial^2\chi}{\partial\theta\partial\phi} + \frac{\partial^3\chi}{\partial\theta\partial\phi^2} + \frac{1}{\sin^2\phi}\frac{\partial^3\chi}{\partial\theta^3} + 2r\frac{\partial^2\chi}{\partial r\partial\theta} + r^2\frac{\partial^3\chi}{\partial r^2\partial\theta}\right)\right]\mathbf{e}_\phi \\
&+ \left\{\frac{1}{r\sin\phi}\left(\frac{\partial\Pi}{\partial\theta} + r\frac{\partial^2\Pi}{\partial r\partial\theta}\right) + \frac{1}{r^2}\left[\cot\phi\frac{\partial^2\chi}{\partial\phi^2} + \frac{\partial^3\chi}{\partial\phi^3} + \frac{1}{\sin^2\phi}\left(-\frac{\partial\chi}{\partial\phi} - 2\cot\phi\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial^3\chi}{\partial\theta^2\partial\phi}\right)\right.\right. \\
&\qquad \left.\left. + r\left(2\frac{\partial^2\chi}{\partial r\partial\phi} + r\frac{\partial^3\chi}{\partial r^2\partial\phi}\right)\right]\right\}\mathbf{e}_\theta\,.
\end{aligned} \tag{3.104}$$

Combining u_p and u_s to get $\mathbf{u} = u_r\mathbf{e}_r + u_\phi\mathbf{e}_\phi + u_\theta\mathbf{e}_\theta$, we have

$$\begin{aligned}
u_r &= \frac{\partial\varphi}{\partial r} - \frac{1}{r}\left(\frac{1}{\sin^2\phi}\frac{\partial^2\Pi}{\partial\theta^2} + \cot\phi\frac{\partial\Pi}{\partial\phi} + \frac{\partial^2\Pi}{\partial\phi^2}\right) \\
u_\phi &= \frac{1}{r}\frac{\partial\varphi}{\partial\phi} + \left(\frac{1}{r}\frac{\partial\Pi}{\partial\phi} + \frac{\partial^2\Pi}{\partial r\partial\phi}\right) \\
&\qquad - \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^2\phi}\frac{\partial^2\chi}{\partial\theta^2} + \cot\phi\frac{\partial\chi}{\partial\phi} + \frac{\partial^2\chi}{\partial\phi^2} + 2r\frac{\partial\chi}{\partial r} + r^2\frac{\partial^2\chi}{\partial r^2}\right) \\
u_\theta &= \frac{1}{r\sin\phi}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r\sin\phi}\left(\frac{\partial\Pi}{\partial\theta} + r\frac{\partial^2\Pi}{\partial r\partial\theta}\right) \\
&\quad + \frac{1}{r^2}\left[\cot\phi\frac{\partial^2\chi}{\partial\phi^2} + \frac{\partial^3\chi}{\partial\phi^3} + \frac{1}{\sin^2\phi}\left(-\frac{\partial\chi}{\partial\phi} - 2\cot\phi\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial^3\chi}{\partial\theta^2\partial\phi}\right)\right. \\
&\qquad \left. + r\left(2\frac{\partial^2\chi}{\partial r\partial\phi} + r\frac{\partial^3\chi}{\partial r^2\partial\phi}\right)\right]
\end{aligned} \tag{3.105}$$

We can simplify these further using the Helmholtz equations satisfied by the potentials. Thus,

$$\nabla^2\Pi = \frac{\partial^2\Pi}{\partial r^2} + \frac{2}{r}\frac{\partial\Pi}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2\Pi}{\partial\theta^2} + \frac{\cot\phi}{r^2}\frac{\partial\Pi}{\partial\phi} + \frac{1}{r^2}\frac{\partial^2\Pi}{\partial\phi^2} = -\frac{\omega^2}{c_s^2}\Pi \tag{3.106}$$

or

$$r\left(\frac{\partial^2\Pi}{\partial r^2} + \frac{2}{r}\frac{\partial\Pi}{\partial r}\right) + \frac{r\omega^2}{c_s^2}\Pi = -\frac{1}{r}\left(\frac{1}{\sin^2\phi}\frac{\partial^2\Pi}{\partial\theta^2} + \cot\phi\frac{\partial\Pi}{\partial\phi} + \frac{\partial^2\Pi}{\partial\phi^2}\right)\,. \tag{3.107}$$

Therefore we can express the radial displacement as

$$u_r(r,\theta,\phi) = \frac{\partial\varphi}{\partial r} + \frac{\partial^2}{\partial r^2}(r\Pi) + \frac{\omega^2}{c_s^2}(r\Pi) \qquad \square \tag{3.108}$$

Similarly, using

$$\frac{1}{\sin^2\phi}\frac{\partial^2\chi}{\partial\theta^2} + \cot\phi\frac{\partial\chi}{\partial\phi} + \frac{\partial^2\chi}{\partial\phi^2} = -r^2\left(\frac{\partial^2\chi}{\partial r^2} + \frac{2}{r}\frac{\partial\chi}{\partial r}\right) - \frac{r^2\omega^2}{c_s^2}\chi \tag{3.109}$$

we have

$$u_\phi = \frac{1}{r}\frac{\partial \varphi}{\partial \phi} + \left(\frac{1}{r}\frac{\partial \Pi}{\partial \phi} + \frac{\partial^2 \Pi}{\partial r \partial \phi}\right) - \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \theta}\left(-r^2\frac{\partial^2 \chi}{\partial r^2} - 2r\frac{\partial \chi}{\partial r} - \frac{r^2\omega^2}{c_s^2}\chi + 2r\frac{\partial \chi}{\partial r} + r^2\frac{\partial^2 \chi}{\partial r^2}\right) \tag{3.110}$$

or

$$u_\phi = \frac{1}{r}\frac{\partial \varphi}{\partial \phi} + \frac{1}{r}\frac{\partial \Pi}{\partial \phi} + \frac{\partial^2 \Pi}{\partial r \partial \phi} + \frac{1}{\sin\phi}\frac{\omega^2}{c_s^2}\frac{\partial \chi}{\partial \theta} \tag{3.111}$$

or

$$u_\phi = \frac{1}{r}\frac{\partial}{\partial \phi}\left[\varphi + \frac{\partial}{\partial r}(r\Pi)\right] + \frac{1}{\sin\phi}\frac{\omega^2}{c_s^2}\frac{\partial \chi}{\partial \theta} \qquad \square \tag{3.112}$$

For u_θ, we reorganize the expression to get

$$u_\theta = \frac{1}{r\sin\phi}\frac{\partial \varphi}{\partial \theta} + \frac{1}{r\sin\phi}\left(\frac{\partial \Pi}{\partial \theta} + r\frac{\partial^2 \Pi}{\partial r \partial \theta}\right) + \frac{1}{r^2}\left[\frac{\partial}{\partial \phi}\left(\frac{1}{\sin^2\phi}\frac{\partial^2 \chi}{\partial \theta^2} + \cot\phi\frac{\partial \chi}{\partial \phi} + \frac{\partial^2 \chi}{\partial \phi^2}\right) + \frac{2\cot\phi}{\sin^2\phi}\frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{\sin^2\phi}\frac{\partial \chi}{\partial \phi} + \frac{1}{\sin^2\phi}\left(-\frac{\partial \chi}{\partial \phi} - 2\cot\phi\frac{\partial^2 \chi}{\partial \theta^2}\right) + r\left(2\frac{\partial^2 \chi}{\partial r \partial \phi} + r\frac{\partial^3 \chi}{\partial r^2 \partial \phi}\right)\right] \tag{3.113}$$

or

$$u_\theta = \frac{1}{r\sin\phi}\frac{\partial \varphi}{\partial \theta} + \frac{1}{r\sin\phi}\left(\frac{\partial \Pi}{\partial \theta} + r\frac{\partial^2 \Pi}{\partial r \partial \theta}\right) - \frac{\partial}{\partial \phi}\left(\frac{\partial^2 \chi}{\partial r^2} + \frac{2}{r}\frac{\partial \chi}{\partial r} + \frac{\omega^2}{c_s^2}\chi\right) + \frac{\partial}{\partial \phi}\left(\frac{2}{r}\frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial r^2}\right) \tag{3.114}$$

or

$$u_\theta = \frac{1}{r\sin\phi}\left(\frac{\partial \varphi}{\partial \theta} + \frac{\partial \Pi}{\partial \theta} + r\frac{\partial^2 \Pi}{\partial r \partial \theta}\right) - \frac{\omega^2}{c_s^2}\frac{\partial \chi}{\partial \phi} \tag{3.115}$$

or

$$u_\theta = \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\left[\varphi + \frac{\partial}{\partial r}(r\Pi)\right] - \frac{\omega^2}{c_s^2}\frac{\partial \chi}{\partial \phi} \qquad \square \tag{3.116}$$

Now that we have expressions for the displacements, we can find the stresses using the stress-strain and strain-displacement relations. Using

$$\boldsymbol{\sigma} = \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{1} + \mu[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T] \tag{3.117}$$

with

$$\boldsymbol{\nabla}\mathbf{u} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \phi} & \frac{1}{r\sin\phi}\frac{\partial u_r}{\partial \theta} \\ \frac{\partial u_\phi}{\partial r} & \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} & \frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial \theta} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \phi} & \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta} \end{bmatrix}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta} \tag{3.118}$$

gives

$$\begin{aligned}\sigma_{rr} &= \frac{\lambda}{r}\left(\cot\phi\, u_\phi + 2u_r + \frac{\partial u_\phi}{\partial \phi} + \frac{1}{\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + (\lambda+2\mu)\frac{\partial u_r}{\partial r} \\ \sigma_{r\phi} &= \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r}\right) \\ \sigma_{r\theta} &= \mu\left(\frac{1}{r\sin\phi}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right) .\end{aligned} \tag{3.119}$$

Plugging in the relations for the displacements, and following a similar procedure as before, gives the expressions for the stresses:

$$\sigma_{rr} = -\lambda\frac{\omega^2}{c_p^2}\varphi + 2\mu\left[\frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial}{\partial r}(r\Pi)\right) + \frac{\omega^2}{c_s^2}\frac{\partial}{\partial r}(r\Pi)\right] \qquad \square \tag{3.120}$$

$$\sigma_{r\phi} = \mu\left[2\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial \phi}\left(\varphi + \frac{\partial}{\partial r}(r\Pi)\right)\right\} + \frac{\omega^2}{c_s^2}\left\{\frac{\partial \Pi}{\partial \phi} + \frac{1}{\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{\partial \chi}{\partial r} - \frac{\chi}{r}\right)\right\}\right] \qquad \square \tag{3.121}$$

and

$$\sigma_{r\theta} = \mu\left[\frac{2}{\sin\phi}\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial \theta}\left(\varphi + \frac{\partial}{\partial r}(r\Pi)\right)\right\} + \frac{\omega^2}{c_s^2}\left\{\frac{1}{\sin\phi}\frac{\partial \Pi}{\partial \theta} - \frac{\partial}{\partial \phi}\left(\frac{\partial \chi}{\partial r} - \frac{\chi}{r}\right)\right\}\right] \qquad \square \tag{3.122}$$

For an SV-wave, the direction of polarization can be taken to be the x-direction and the direction of propagation the z-direction. The out-of-plane displacements are zero, i.e., $u_y = 0$ and the displacement in the direction of propagation is also zero, i.e., $u_z = 0$. Therefore the displacement vector in Cartesian coordinates has the form $\mathbf{u} = u_x\mathbf{e}_x$. This form of the displacement vector can be derived from a vector potential of the form

$$\boldsymbol{\psi} = F(x,y,z)\mathbf{e}_y \quad \Longrightarrow \quad \mathbf{u} = \boldsymbol{\nabla}\times\boldsymbol{\psi} = -\frac{\partial F}{\partial z}\mathbf{e}_x + \frac{\partial F}{\partial x}\mathbf{e}_z \,. \tag{3.123}$$

Therefore, the potential $\boldsymbol{\psi}$ for an incident SV-wave of unit amplitude will have the form

$$\boldsymbol{\psi}_{SV} = \exp(ik_s z)\mathbf{e}_y \qquad \square \tag{3.124}$$

where $k_s = \omega/c_s$. Now, the transformation of coordinates from a Cartesian to a spherical basis is

$$\begin{bmatrix}\mathbf{e}_r\\ \mathbf{e}_\phi\\ \mathbf{e}_\theta\end{bmatrix} = \begin{bmatrix}\cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi\\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi\\ -\sin\theta & \cos\theta & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{bmatrix} . \tag{3.125}$$

The inverse relationship is

$$\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{bmatrix} = \begin{bmatrix}\cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta\\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta\\ \cos\phi & -\sin\phi & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_r\\ \mathbf{e}_\phi\\ \mathbf{e}_\theta\end{bmatrix} . \tag{3.126}$$

We can use $z = r\cos\phi$ and the above relationship to write

$$\boldsymbol{\psi}_{SV} = \exp(ik_s r\cos\phi)\left[\sin\theta\sin\phi\,\mathbf{e}_r + \sin\theta\cos\phi\,\mathbf{e}_\phi + \cos\theta\,\mathbf{e}_\theta\right] . \tag{3.127}$$

Using the series expansion

$$\exp(iz\cos\alpha) = \sum_{n=0}^{\infty} i^n(2n+1)j_n(z)P_n(\cos\alpha) \tag{3.128}$$

we can write

$$\boldsymbol{\psi}_{SV} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_s r) P_n(\cos\phi) \left[\sin\theta \sin\phi \, \mathbf{e}_r + \sin\theta \cos\phi \, \mathbf{e}_\phi + \cos\theta \, \mathbf{e}_\theta\right] \tag{3.129}$$

or,

$$\begin{aligned} \boldsymbol{\psi}_{SV} &= \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_s r) P_n(\cos\phi) \cos\theta \, \mathbf{e}_\theta \\ &\quad + \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_s r) P_n(\cos\phi) \left[\sin\phi \, \mathbf{e}_r + \cos\phi \, \mathbf{e}_\phi\right] \sin\theta \,. \end{aligned} \tag{3.130}$$

In the case of an SV-wave the incident wave has the form

$$\phi_i = 0 \,, \;\; \boldsymbol{\psi}_i = \boldsymbol{\nabla}\times(\Pi_i r \mathbf{e}_r) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\chi_i r \mathbf{e}_r)) \equiv (F_\pi \mathbf{e}_\theta + G_\pi \mathbf{e}_\phi) + (F_\chi \mathbf{e}_r + G_\chi \mathbf{e}_\theta + H_\chi \mathbf{e}_\phi) \tag{3.131}$$

For equations (3.130) and (3.131) to match, the potentials Π_i and χ_i may have the forms

$$\begin{aligned} \Pi_i &= -\Pi_0 \sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} j_n(k_s r) P_n^1(\cos\phi) \cos\theta \\ \chi_i &= -\chi_0 \frac{i}{k_s} \sum_{n=1}^{\infty} (-i)^n \frac{(2n+1)}{n(n+1)} j_n(k_s r) P_n^1(\cos\phi) \sin\theta \end{aligned} \qquad \square \tag{3.132}$$

Other forms can also be found.

For an SH-wave, the only allowable displacements are out-of-plane, i.e., $u_x = u_z = 0$ and the displacement vector has the form $\mathbf{u} = u_y \mathbf{e}_y$. In this case, we can choose a potential of the form

$$\boldsymbol{\psi} = G(x,z)\mathbf{e}_x \quad \implies \quad \mathbf{u} = \boldsymbol{\nabla}\times\boldsymbol{\psi} = \frac{\partial G}{\partial z}\mathbf{e}_y \,. \tag{3.133}$$

Therefore the potential $\boldsymbol{\psi}$ for an incident SH-wave of unit amplitude will have the form

$$\boldsymbol{\psi}_{SH} = \exp(i k_s z)\mathbf{e}_x \qquad \square \tag{3.134}$$

We can follow the procedure used for SV-waves to find expressions for the potentials for SH-waves.

Problem 3.7 Consider three identical infinitely long rigid circular cylinders arranged in an equilateral triangle. Assume that a plane wave with wavenumber k is incident on the system along the line joining two of the cylinders and that $ka = 2$ where a is the radius of the cylinder. Assume that the cylinders are separated by a distance $b = \pi a$. Calculate the effective potential field (ϕ) around a cylinder at a distance $r = 0.5b$ as a function of angle.

Solution 3.7: Let us define the origin of the global coordinate system at the center of one of the cylinders. Then the centers of the three cylinders are, in cylindrical coordinates, at

$$\mathbf{b}_1 = (0, 0)\,, \quad \mathbf{b}_2 = (\pi a, 0)\,, \quad \mathbf{b}_3 = (\pi a, 2\pi/3)\,. \tag{3.135}$$

Let as also define local coordinate systems, $(r_i, \theta_i), i = 1, 2, 3$, at the center of each cylinder. The the coordinates of a point in the plane with respect to the origin is

$$\mathbf{r} = \mathbf{b}_1 + \mathbf{r}_1 = \mathbf{b}_2 + \mathbf{r}_2 = \mathbf{b}_3 + \mathbf{r}_3\,. \tag{3.136}$$

Let the regular part of the solution be composed of terms $\psi_m(\mathbf{r}_j)$ and let the general solution be composed of terms $\chi_m(\mathbf{r}_j)$. The total field outside the first cylinder is

$$\phi_1 = \sum_{m=-\infty}^{\infty} A_{1m}\psi_m(\mathbf{r}_1) + \sum_{m=-\infty}^{\infty} B_{1m}\chi_m(\mathbf{r}_1) + \sum_{m=-\infty}^{\infty} B_{2m}\chi_m(\mathbf{r}_2) + \sum_{m=-\infty}^{\infty} B_{3m}\chi_m(\mathbf{r}_3)\,. \tag{3.137}$$

The solutions $\chi_m(\mathbf{r}_j)$ may be singular at the location $\mathbf{r}_j = (0, \theta)$ but not at locations $\mathbf{r}_p = (0, \theta)$ where $p \neq j$. Then, Graf's addition theorem implies that

$$\begin{aligned} \chi_m(\mathbf{r}_2) &= \sum_{n=-\infty}^{\infty} S_{mn}(\mathbf{b}_2 - \mathbf{b}_1)\psi_n(\mathbf{r}_1) \quad \text{for} \quad r_1 < \|\mathbf{b}_2 - \mathbf{b}_1\| \\ \chi_m(\mathbf{r}_3) &= \sum_{n=-\infty}^{\infty} S_{mn}(\mathbf{b}_3 - \mathbf{b}_1)\psi_n(\mathbf{r}_1) \quad \text{for} \quad r_1 < \|\mathbf{b}_3 - \mathbf{b}_1\|\;. \end{aligned} \tag{3.138}$$

where, for cylindrical scatterers,

$$\psi_n(\mathbf{r}) = J_n(k\,\|\mathbf{r}\|)\,e^{in\theta}\;; \;\; \chi_m(\mathbf{r}) = H_m^{(1)}(k\,\|\mathbf{r}\|)\,e^{im\theta}\;; \;\; S_{nm}(\mathbf{b}) = H_{n-m}^{(1)}(k\,\|\mathbf{b}\|)\,e^{i(n-m)\beta} \tag{3.139}$$

and

$$\cos\theta = \frac{\mathbf{r}\cdot\mathbf{e}_x}{\|\mathbf{r}\|} \quad \cos\beta = \frac{\mathbf{b}\cdot\mathbf{e}_x}{\|\mathbf{b}\|}\,. \tag{3.140}$$

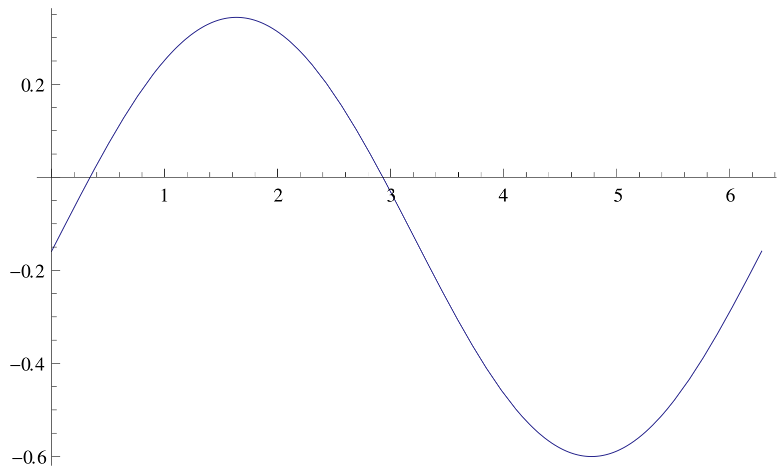
Therefore we can write

$$\phi_1 = \sum_{m=-\infty}^{\infty}\left[B_{1m}\chi_m(\mathbf{r}_1) + \left(A_{1m} + \sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_2 - \mathbf{b}_1)B_{2n} + \sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_3 - \mathbf{b}_1)B_{3n}\right)\psi_m(\mathbf{r}_1)\right]\,. \tag{3.141}$$

Applying boundary conditions $\partial\phi_j/\partial r_j = 0$ and $r_j = a$ gives us

$$\begin{aligned} B_{1m}H_m^{'}(ka) + \left[\sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_1 - \mathbf{b}_2)B_{2n} + \sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_1 - \mathbf{b}_3)B_{3n}\right] J_m^{'}(ka) = -A_{1m}J_m^{'}(ka) \\ B_{2m}H_m^{'}(ka) + \left[\sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_2 - \mathbf{b}_1)B_{1n} + \sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_2 - \mathbf{b}_3)B_{3n}\right] J_m^{'}(ka) = -A_{2m}J_m^{'}(ka) \\ B_{3m}H_m^{'}(ka) + \left[\sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_3 - \mathbf{b}_1)B_{1n} + \sum_{n=-\infty}^{\infty} S_{nm}(\mathbf{b}_3 - \mathbf{b}_2)B_{2n}\right] J_m^{'}(ka) = -A_{3m}J_m^{'}(ka) \end{aligned}$$

An example solution for incident waves of unit amplitude with $m, n, -1, 0, 1$ is shown below. Better solutions can be obtained for larger ranges of m, n.



psin = BesselJ[n, kr]/k;
chim = HankelH1[m, kr]/k;
Smn = HankelH1[$n-m$, kb]Exp[$I(n-m)$beta];
chimr1 = chim/.{$r \to$ r1};
psimr1 = psin/.{$n \to m, r \to$ r1};
Smnb12 = Smn/.{$b \to$ b12, beta $\to$ beta12};
Smnb13 = Smn/.{$b \to$ b13, beta $\to$ beta13};
SumB2 = Sum[Smnb12B2[n], {$n, -p, p$}];
SumB3 = Sum[Smnb13B3[n], {$n, -p, p$}];
phi1 = B1[m]chimr1 + (A1[m] + SumB2 + SumB3)psimr1;
Dphi1Dr = D[phi1, r1];
Dphi1Dra =
Dphi1Dr/.{r1 $\to a$, b12 $\to$ Pia, b13 $\to$ Pia, beta13 $\to$ 2Pi/3, beta12 $\to$ 0};
Dphi1Drb = Dphi1Dra/.{$k \to 2/a$};
Dphi1Drc = Dphi1Drb/.{$p \to 1$};
eq11 = Dphi1Drc/.{m-> − 1};
eq12 = Dphi1Drc/.{m->0};
eq13 = Dphi1Drc/.{m->1};

chimr2 = chim/.{$r \to$ r2};
psimr2 = psin/.{$n \to m, r \to$ r2};
Smnb21 = Smn/.{$b \to$ b21, beta $\to$ beta21};
Smnb23 = Smn/.{$b \to$ b23, beta $\to$ beta23};
SumB21 = Sum[Smnb21B1[n], {$n, -p, p$}];
SumB23 = Sum[Smnb23B3[n], {$n, -p, p$}];
phi2 = B2[m]chimr2 + (A2[m] + SumB21 + SumB23)psimr2;
Dphi2Dr = D[phi2, r2];
Dphi2Dra =
Dphi2Dr/.{r2 $\to a$, b21 $\to$ Pia, b23 $\to$ Pia, beta21 $\to$ −Pi, beta23 $\to$ 5Pi/6};
Dphi2Drb = Dphi2Dra/.{$k \to 2/a$};
Dphi2Drc = Dphi2Drb/.{$p \to 1$};
eq21 = Dphi2Drc/.{m-> − 1};
eq22 = Dphi2Drc/.{m->0};
eq23 = Dphi2Drc/.{m->1};

chimr3 = chim/.{$r \to$ r3};
psimr3 = psin/.{$n \to m, r \to$ r3};
Smnb31 = Smn/.{$b \to$ b31, beta $\to$ beta31};
Smnb32 = Smn/.{$b \to$ b32, beta $\to$ beta32};
SumB31 = Sum[Smnb31B1[n], {$n, -p, p$}];

```
SumB32 = Sum[Smnb32B2[n], {n, -p, p}];
phi3 = B3[m]chimr3 + (A3[m] + SumB31 + SumB32)psimr3;
Dphi3Dr = D[phi3, r3];
Dphi3Dra =
Dphi3Dr/.{r3 -> a, b31 -> Pia, b32 -> Pia, beta31 -> 5Pi/3, beta32 -> -2Pi/3};
Dphi3Drb = Dphi3Dra/.{k -> 2/a};
Dphi3Drc = Dphi3Drb/.{p -> 1};
eq31 = Dphi3Drc/.{m-> - 1};
eq32 = Dphi3Drc/.{m->0};
eq33 = Dphi3Drc/.{m->1};


neq11 =
eq11/.{A1[0] -> 1, A1[-1] -> 1, A1[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq12 =
eq12/.{A1[0] -> 1, A1[-1] -> 1, A1[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq13 =
eq13/.{A1[0] -> 1, A1[-1] -> 1, A1[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq21 =
eq21/.{A2[0] -> 1, A2[-1] -> 1, A2[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq22 =
eq22/.{A2[0] -> 1, A2[-1] -> 1, A2[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq23 =
eq23/.{A2[0] -> 1, A2[-1] -> 1, A2[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq31 =
eq31/.{A3[0] -> 1, A3[-1] -> 1, A3[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq32 =
eq32/.{A3[0] -> 1, A3[-1] -> 1, A3[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
neq33 =
eq33/.{A3[0] -> 1, A3[-1] -> 1, A3[1] -> 1, B1[-1] -> B111, B2[-1] -> B211,
B3[-1] -> B311, B1[0] -> B10, B2[0] -> B20, B3[0] -> B30, B1[1] -> B11,
B2[1] -> B21, B3[1] -> B31}
```

$$\frac{1}{2}\mathrm{B111}(-\mathrm{HankelH1}[0,2]+\mathrm{HankelH1}[2,2])+\frac{1}{2}(-\mathrm{BesselJ}[0,2]+\mathrm{BesselJ}[2,2])$$
$$(1+\mathrm{B211HankelH1}[0,2\pi]+\mathrm{B311HankelH1}[0,2\pi]+\mathrm{B20HankelH1}[1,2\pi]+$$
$$\mathrm{B30}e^{\frac{2i\pi}{3}}\mathrm{HankelH1}[1,2\pi]+\mathrm{B21HankelH1}[2,2\pi]+\mathrm{B31}e^{-\frac{2i\pi}{3}}\mathrm{HankelH1}[2,2\pi]\Big)$$

$$-\mathrm{B10HankelH1}[1,2]-$$
$$\mathrm{BesselJ}[1,2](1+\mathrm{B20HankelH1}[0,2\pi]+\mathrm{B30HankelH1}[0,2\pi]+\mathrm{B21HankelH1}[1,2\pi]-$$
$$\mathrm{B211HankelH1}[1,2\pi]-\mathrm{B311}e^{-\frac{2i\pi}{3}}\mathrm{HankelH1}[1,2\pi]+\mathrm{B31}e^{\frac{2i\pi}{3}}\mathrm{HankelH1}[1,2\pi]\Big)$$

$$\frac{1}{2}\text{B11}(\text{HankelH1}[0,2]-\text{HankelH1}[2,2])+\frac{1}{2}(\text{BesselJ}[0,2]-\text{BesselJ}[2,2])$$
$$(1+\text{B21HankelH1}[0,2\pi]+\text{B31HankelH1}[0,2\pi]-\text{B20HankelH1}[1,2\pi]-$$
$$\text{B30}e^{-\frac{2i\pi}{3}}\text{HankelH1}[1,2\pi]+\text{B211HankelH1}[2,2\pi]+\text{B311}e^{\frac{2i\pi}{3}}\text{HankelH1}[2,2\pi]\Big)$$

$$\frac{1}{2}\text{B211}(-\text{HankelH1}[0,2]+\text{HankelH1}[2,2])+\frac{1}{2}(-\text{BesselJ}[0,2]+\text{BesselJ}[2,2])$$
$$(1+\text{B111HankelH1}[0,2\pi]+\text{B311HankelH1}[0,2\pi]-\text{B10HankelH1}[1,2\pi]+$$
$$\text{B30}e^{\frac{5i\pi}{6}}\text{HankelH1}[1,2\pi]+\text{B11HankelH1}[2,2\pi]+\text{B31}e^{-\frac{i\pi}{3}}\text{HankelH1}[2,2\pi]\Big)$$

$$-\text{B20HankelH1}[1,2]-$$
$$\text{BesselJ}[1,2](1+\text{B10HankelH1}[0,2\pi]+\text{B30HankelH1}[0,2\pi]-\text{B11HankelH1}[1,2\pi]+$$
$$\text{B111HankelH1}[1,2\pi]-\text{B311}e^{-\frac{5i\pi}{6}}\text{HankelH1}[1,2\pi]+\text{B31}e^{\frac{5i\pi}{6}}\text{HankelH1}[1,2\pi]\Big)$$

$$\frac{1}{2}\text{B21}(\text{HankelH1}[0,2]-\text{HankelH1}[2,2])+\frac{1}{2}(\text{BesselJ}[0,2]-\text{BesselJ}[2,2])$$
$$(1+\text{B11HankelH1}[0,2\pi]+\text{B31HankelH1}[0,2\pi]+\text{B10HankelH1}[1,2\pi]-$$
$$\text{B30}e^{-\frac{5i\pi}{6}}\text{HankelH1}[1,2\pi]+\text{B111HankelH1}[2,2\pi]+\text{B311}e^{\frac{i\pi}{3}}\text{HankelH1}[2,2\pi]\Big)$$

$$\frac{1}{2}\text{B311}(-\text{HankelH1}[0,2]+\text{HankelH1}[2,2])+$$
$$\frac{1}{2}(-\text{BesselJ}[0,2]+\text{BesselJ}[2,2])(1+\text{B111HankelH1}[0,2\pi]+$$
$$\text{B211HankelH1}[0,2\pi]+\text{B10}e^{-\frac{i\pi}{3}}\text{HankelH1}[1,2\pi]+\text{B20}e^{-\frac{2i\pi}{3}}\text{HankelH1}[1,2\pi]+$$
$$\text{B11}e^{-\frac{2i\pi}{3}}\text{HankelH1}[2,2\pi]+\text{B21}e^{\frac{2i\pi}{3}}\text{HankelH1}[2,2\pi]\Big)$$

$$-\text{B30HankelH1}[1,2]-\text{BesselJ}[1,2]$$
$$\Big(1+\text{B10HankelH1}[0,2\pi]+\text{B20HankelH1}[0,2\pi]+\text{B11}e^{-\frac{i\pi}{3}}\text{HankelH1}[1,2\pi]-\text{B111}$$
$$e^{\frac{i\pi}{3}}\text{HankelH1}[1,2\pi]+\text{B21}e^{-\frac{2i\pi}{3}}\text{HankelH1}[1,2\pi]-\text{B211}e^{\frac{2i\pi}{3}}\text{HankelH1}[1,2\pi]\Big)$$

$$\frac{1}{2}\text{B31}(\text{HankelH1}[0,2]-\text{HankelH1}[2,2])+$$
$$\frac{1}{2}(\text{BesselJ}[0,2]-\text{BesselJ}[2,2])(1+\text{B11HankelH1}[0,2\pi]+$$
$$\text{B21HankelH1}[0,2\pi]-\text{B10}e^{\frac{i\pi}{3}}\text{HankelH1}[1,2\pi]-\text{B20}e^{\frac{2i\pi}{3}}\text{HankelH1}[1,2\pi]+$$
$$\text{B211}e^{-\frac{2i\pi}{3}}\text{HankelH1}[2,2\pi]+\text{B111}e^{\frac{2i\pi}{3}}\text{HankelH1}[2,2\pi]\Big)$$

```
Neq11 = N[neq11];
Neq12 = N[neq12];
Neq13 = N[neq13];
Neq21 = N[neq21];
Neq22 = N[neq22];
Neq23 = N[neq23];
Neq31 = N[neq31];
Neq32 = N[neq32];
Neq33 = N[neq33];
```

```
sol = Solve[{Neq11 == 0, Neq12 == 0, Neq13 == 0, Neq21 == 0, Neq22 == 0,
Neq23 == 0, Neq31 == 0, Neq32 == 0, Neq33 == 0},
{B111, B10, B11, B211, B20, B21, B311, B30, B31}]
```

$$\{\{\text{B111} \to -0.00104653 - 0.0877363i,$$
$$\text{B10} \to -0.471547 - 0.187902i, \text{B11} \to -0.011804 - 0.0776243i,$$
$$\text{B211} \to -0.0503947 - 0.0819051i, \text{B20} \to -0.535029 - 0.225689i,$$
$$\text{B21} \to 0.0362463@ - 0.110834i, \text{B311} \to -0.0351225 - 0.133928i,$$
$$\text{B30} \to -0.70556 - 0.347127i, \text{B31} \to -0.0345999 - 0.135592i\}\}$$

B111a = (B111/.sol)[[1]];
B10a = (B10/.sol)[[1]];
B11a = (B11/.sol)[[1]];
B211a = (B211/.sol)[[1]];
B20a = (B20/.sol)[[1]];
B21a = (B21/.sol)[[1]];
B311a = (B311/.sol)[[1]];
B30a = (B30/.sol)[[1]];
B31a = (B31/.sol)[[1]];

fieldPhi1 = Sum[phi1(*k*Exp[*Im*theta]), {*m*, −*p*, *p*}];
fPhi1a =
fieldPhi1/.{r1 → 0.5Pi*a*, b12 → Pi*a*, b13 → Pi*a*, beta13 → 2Pi/3,
beta12 → 0, *k* → 2/*a*, *p* → 1};
fPhi1b =
fPhi1a/.{A1[0] → 1, A1[−1] → 1, A1[1] → 1, B1[−1] → B111a, B2[−1] → B211a,
B3[−1] → B311a, B1[0] → B10a, B2[0] → B20a, B3[0] → B30a, B1[1] → B11a,
B2[1] → B21a, B3[1] → B31a}

$$(0.051091@ - 0.128215i) - (0.249892@ - 0.00298075i)e^{-i\text{theta}} + (0.221091@ - 0.0336205i)e^{i\text{theta}}$$

fPhi1c = Re[fPhi1b];
Plot[Re[−*I*fPhi1b], {theta, 0, 2Pi}]

Chapter 4

Solutions for Exercises in Chapter 4

Problem 4.1 Show that the effective permittivity of an array of thin wires with finite conductivity can be expressed in the form

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\Omega_p^2}{\omega^2 + i\Gamma\omega}\right) \qquad \text{with} \qquad \Gamma := \frac{\varepsilon_0 b^2 \Omega_p^2}{\pi a^2 \sigma}. \tag{4.1}$$

Solution 4.1 The Drude model for the effective permittivity has the form

$$\varepsilon(\omega) = \varepsilon_0 + \sum_j \frac{a_j}{\omega_0^2 - \omega^2 - i\omega\gamma}. \tag{4.2}$$

For high frequencies, $\omega \gg \omega_0$, we can write the above equation as

$$\varepsilon(\omega) = \varepsilon_0 - \sum_j \frac{a_j}{\omega^2 + i\omega\gamma}. \tag{4.3}$$

Using the plasma frequency $\omega_p^2 := \frac{1}{\varepsilon_0}\sum_j a_j$, we can write

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma}\right). \tag{4.4}$$

For an array of thin wires, we seek a similar equation for the *effective* permittivity $\varepsilon(\omega)$ of the form

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\Omega_p^2}{\omega^2 + i\omega\Gamma}\right). \tag{4.5}$$

where the *effective* plasma frequency is ω_p and the *effective* damping coefficient is Γ.

We have already see that the effective plasma frequency for an square array of thin wires is given by

$$\Omega_p^2 = \frac{2\pi c^2}{b^2 \ln(b/a)} \tag{4.6}$$

where c the speed of light in vacuum, a is the radius of the wires, and b is the spacing between the wires. Alternative expressions are possible.

We now want to find the effective damping constant Γ for wires with finite conductivity.

The electric field, E_z, along the wires is related to the current induced, I, and the total inductance, L, per unit length by the relation

$$E_z = L\frac{dI}{dt} = -i\omega L I. \tag{4.7}$$

If the wires have finite conductivity, σ, we have

$$E_z = -i\omega L I + \frac{I}{\sigma\pi a^2} = -i\omega\left(L - \frac{1}{\omega\sigma\pi a^2}\right) I = -i\omega L^{'} I \quad \text{where} \;\; L^{'} := \left(L - \frac{1}{i\omega\sigma\pi a^2}\right) I\,. \tag{4.8}$$

The axial component of the magnetic potential, **A**, around each wire is given by

$$A_z(r) \approx \frac{I\mu_0}{2\pi}\ln\left(\frac{b}{r}\right) \quad \text{where} \quad I = \pi a^2 N v e\,. \tag{4.9}$$

We can estimate the inductance, L, from the magnetic flux per unit length passing though a representative volume element of the composite,

$$\Phi = LI = \mu_0\int_a^{r_c} H_\theta(r)\,dr = A_z(a) \quad \Longrightarrow \quad L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad \Longrightarrow \quad \ln(b/a) = 2\pi L/\mu_0\,. \tag{4.10}$$

Therefore,

$$\Omega_p^2 = \frac{\mu_0 c^2}{b^2 L} = \frac{1}{\varepsilon_0 b^2 L} \quad \Longrightarrow \quad L = \frac{1}{\varepsilon_0 b^2 \Omega_p^2}\,. \tag{4.11}$$

Now, the polarization per unit volume in the homogenized medium, with wires of finite conductivity, is

$$P = -\frac{E_z}{\omega^2 b^2 L^{'}} = (\varepsilon - \varepsilon_0)\,E_z \tag{4.12}$$

or

$$\varepsilon = \varepsilon_0 - \frac{1}{\omega^2 b^2\left[\dfrac{1}{\varepsilon_0 b^2\Omega_p^2} - \dfrac{1}{i\omega\sigma\pi a^2}\right]}\,. \tag{4.13}$$

Therefore,

$$\varepsilon(\omega) = \varepsilon_0\left[1 - \frac{\Omega_p^2}{\omega^2 + i\omega\Gamma}\right] \tag{4.14}$$

where

$$\Gamma := \frac{b^2\varepsilon_0\Omega_p^2}{\pi a^2\sigma} \qquad \square \tag{4.15}$$

Problem 4.2 Derive the effective permittivity relation

$$\varepsilon_{\text{eff}}(\omega) = \varepsilon_0 \left[1 - \frac{2\pi f c^2}{\omega^2 + 2i\omega\pi f c^2 \varepsilon_0 \langle\sigma\rangle^{-1}}\right] \tag{4.16}$$

using the approach in Felbacq and Bouchitté (2005).

Solution 4.2: The composite studied by Felbacq and Bouchitté in their paper on left-handed media is a volume of isotropic and homogeneous electromagnetic material containing a periodic array of thin wires. The region outside the volume is vacuum.

The starting point of the approach is the determination of a polarization field due to the current induced in each wire by an external TE-wave (s-polarized) field. If the wires are oriented in the z-direction, then the incident wave has a non-zero component $E_z(x, y)$.

We assume the correctness of the scattering calculations of Felbacq and Bouchitté that lead to a polarization, in the volume element around the j-th wire, of

$$P \approx 4i \left[-\frac{\pi}{2i}\ln(ka)\right](1 - S)E_z = -2\pi \ln\left(\frac{2\pi a}{\lambda}\right)(1 - S)E_z \tag{4.17}$$

where $k = 2\pi/\lambda = \omega/c$ is the wave number, S is the scattering amplitude inside the unit cell (which can be imagined as a cylinder of diameter b in the $x - y$ plane with a circular wire of radius a at the center) due to the wire. For a unit cell that is small relative to the wavelength, i.e., $a \ll b \ll \lambda$, we can write

$$P \approx -2\pi \ln\left(\frac{2\pi a}{b}\right)(1 - S)E_z\,. \tag{4.18}$$

Define

$$\gamma := -\frac{1}{\eta^2}\ln\left(\frac{2\pi a}{b}\right) \quad \text{where} \quad \eta := \frac{b}{\lambda}\,. \tag{4.19}$$

Then we can write the polarization as

$$P \approx 2\pi\gamma\eta^2(E_z - SE_z)\,. \tag{4.20}$$

Let us define the average electric field in the unit cell, Ω, as

$$\langle E_z\rangle := \frac{1}{V}\int_\Omega E_z\,\text{dV} \tag{4.21}$$

where V is the volume of the unit cell. Only the polarization, SE_z, contributes a non-zero amount to the volume average and we can write

$$P \approx 2\pi\gamma\eta^2\left(E_z - \langle E_z\rangle\right)\,. \tag{4.22}$$

The wave equation for a TE wave in an isotropic and homogeneous medium in the absence of sources is

$$\nabla^2 E_z + k^2 E_z = 0 \qquad \text{where} \quad k^2 = \omega^2\varepsilon\mu = \frac{\omega^2}{c^2} \tag{4.23}$$

and ε and μ are the permittivity and magnetic permeability of the medium. If we add a source term due to the polarization P, we can write, in the limit $\eta \to 0$,

$$\nabla^2 E_z + k^2 E_z = \frac{P}{b^2} = \frac{2\pi\gamma}{\lambda^2}\left(E_z - \langle E_z\rangle\right) \tag{4.24}$$

We want to express $\langle E_z\rangle$ in terms of E_z so that we can get an equation of the form

$$\nabla^2 E_z + k_{\text{eff}}^2 E_z = 0\,. \tag{4.25}$$

We can calculate the required effective properties from k_{eff}.

To find the average field in the unit cell, we can examine the fields in the individual components. In the "matrix" outside the wires, the electric field satisfies

$$\nabla^2 E_z + k_m^2 E_z = 0 \qquad \text{with} \quad k_m^2 = \omega^2 \varepsilon_m \mu_m \tag{4.26}$$

where ε_m, μ_m are the permittivity and magnetic permeability of the medium. The wave equation in the wire is

$$\nabla^2 E_z + k_w^2 E_z = 0 \qquad \text{with} \quad k_w^2 = \omega^2 \varepsilon_w \mu_w \tag{4.27}$$

where ε_w, μ_w are the permittivity and magnetic permeability of the wire. We can write the above equation as

$$\nabla^2 E_z + \omega^2 \varepsilon_0 \left(1 + \frac{i\sigma}{\varepsilon_0 \omega}\right) \mu_w E_z = 0 \tag{4.28}$$

where ε_0 is a reference permittivity. Define the relative electric displacement inside the wires as

$$D_z = \frac{i\sigma}{\varepsilon_0 \omega} E_z \,. \tag{4.29}$$

Then,

$$\nabla^2 E_z + \omega^2 \varepsilon_0 \mu_w E_z = -\omega^2 \varepsilon_0 \mu_w D_z \tag{4.30}$$

The average electrical conductivity of the wires is assumed to be

$$\langle \sigma \rangle = \frac{1}{V} \int_\Omega \sigma dV = \frac{\pi a^2}{b^2}\, \sigma =: \theta\sigma \,. \tag{4.31}$$

Then we can write,

$$D_z = \frac{i \langle \sigma \rangle}{\varepsilon_0 \omega} \frac{E_z}{\theta} \,. \tag{4.32}$$

Felbacq and Bouchitté assume that $\langle E_z \rangle = E_z/\theta$, and express the wave equation in the form

$$\nabla^2 E_z + \omega^2 \varepsilon_0 \mu_w E_z + \omega^2 \varepsilon_0 \mu_w \frac{i \langle \sigma \rangle}{\varepsilon_0 \omega} \langle E_z \rangle = 0 \,. \tag{4.33}$$

Defining $k_0^2 := \omega^2 \varepsilon_0 \mu_w$, we have

$$\nabla^2 E_z + k_0^2 E_z + k_0^2 \frac{i \langle \sigma \rangle}{\varepsilon_0 \omega} \langle E_z \rangle = 0 \,. \tag{4.34}$$

The authors combine equations (4.26) and (4.34) into a single equation of the form

$$\nabla^2 E_z + k_n^2 E_z = -\chi_n k_0^2 \frac{i \langle \sigma \rangle}{\varepsilon_0 \omega} \langle E_z \rangle \tag{4.35}$$

where χ_n is an indicator function (1 inside the wires and 0 outside) with $n = 0$ inside the wires and $n = m$ in the matrix. Comparing equations (4.24) and (4.35), we have

$$\frac{2\pi\gamma}{\lambda^2} \left(E_z - \langle E_z \rangle\right) = -k_0^2 \frac{i \langle \sigma \rangle}{\varepsilon_0 \omega} \langle E_z \rangle \,. \tag{4.36}$$

Solving for $\langle E_z \rangle$ gives

$$\langle E_z \rangle = \frac{2\pi\varepsilon_0 \omega \gamma}{-ik_0^2 \lambda^2 \langle \sigma \rangle + 2\pi \varepsilon_0 \omega \gamma} Ez \,. \tag{4.37}$$

Plugging the above expression into equation (4.24) gives us

$$\nabla^2 E_z + \left(k_n^2 - \frac{2\pi \langle \sigma \rangle \gamma k_0^2}{2i\varepsilon_0 \gamma \omega + \langle \sigma \rangle k_0^2 \lambda^2}\right) E_z = 0 \,. \tag{4.38}$$

Therefore, the effective wave vector is

$$k_{\text{eff}}^2 = k_n^2 - \frac{2\pi \langle \sigma \rangle \gamma k_0^2}{2i\varepsilon_0 \gamma \omega + \langle \sigma \rangle k_0^2 \lambda^2} \,. \tag{4.39}$$

Substituting $k_{\text{eff}}^2 = \omega^2 \varepsilon_{\text{eff}} \mu_0$, $k_n^2 = \omega^2 \varepsilon_m \mu_0$, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, and $\mu_0 = 1/(\varepsilon_0 c^2)$, we get

$$\omega^2 \varepsilon_{\text{eff}} \mu_0 = \omega^2 \mu_0 \left[\varepsilon_m - \frac{2\pi\varepsilon_0 c^2 \gamma \langle \sigma \rangle}{\omega^2 \langle \sigma \rangle \lambda^2 + 2i\omega\pi\varepsilon_0 c^2 \gamma}\right] \tag{4.40}$$

or

$$\varepsilon_{\text{eff}} = \varepsilon_m - \frac{\dfrac{2\pi\gamma\varepsilon_0 c^2}{\lambda^2}}{\omega^2 + \dfrac{2i\omega\pi\varepsilon_0 c^2 \gamma}{\lambda^2 \langle \sigma \rangle}} \,. \tag{4.41}$$

Define

$$f := \frac{\gamma}{\lambda^2} \,. \tag{4.42}$$

Then

$$\varepsilon_{\text{eff}} = \varepsilon_m - \frac{2\pi f c^2 \varepsilon_0}{\omega^2 + 2i\omega\pi f \varepsilon_0 c^2 \langle \sigma \rangle^{-1}} \,. \tag{4.43}$$

Let us now take the reference permittivity to be $\varepsilon_0 = \varepsilon_m$. Then,

$$\varepsilon_{\text{eff}} = \varepsilon_0 \left[1 - \frac{2\pi f c^2}{\omega^2 + 2i\omega\pi f \varepsilon_0 c^2 \langle \sigma \rangle^{-1}}\right] \qquad \square \tag{4.44}$$

Problem 4.3 Derive the effective magnetic permeability for the single split-ring resonator array using the approach used for the array of cylinders.

Solution 4.3: For an array of single split-ring resonators, we have

$$2\pi a\left(-i\omega L - \frac{1}{i\omega C} + R\right) I = i\omega\mu_0\pi a^2\left(H_0 - fI\right) . \tag{4.45}$$

Using

$$L = 4\pi a\mu_0 \ln\left(\frac{2\pi a}{w}\right) , \quad C = \frac{4w^2}{gc^2\mu_0} , \quad R = \frac{4\pi a}{cw\sqrt{\varepsilon_m}} , \;\; \varepsilon_m = \varepsilon_{\text{ring}}/\varepsilon_0 \tag{4.46}$$

and solving for I gives,

$$I = -\frac{2\omega^2 aw^2 c\sqrt{\varepsilon_m}\mu_0 H_0}{16i\pi\omega aw - c\sqrt{\varepsilon_m}\mu_0(gc^2 + 2aw^2 f\omega^2) + 16\pi\omega^2 aw^2 c\sqrt{\varepsilon_m}\mu_0 \ln(2\pi a/w)} \tag{4.47}$$

Using the relation

$$\mu_{\text{eff}} = \frac{\mu_0 H_0}{H_0 - fI} \tag{4.48}$$

and plugging in the expression for I leads to

$$\mu_{\text{eff}} = \mu_0\left[1 - \frac{2\omega^2 aw^2 c\sqrt{\varepsilon_m}\mu_0 f}{\omega^2[16\pi aw^2\ln(2\pi a/w)](c\sqrt{\varepsilon_m}\mu_0) + i\omega(16\pi aw) - gc^3\sqrt{\varepsilon_m}\mu_0}\right] \tag{4.49}$$

or

$$\mu_{\text{eff}} = \mu_0\left[1 - \frac{\omega^2 f}{\omega^2\, 8\pi\ln\left(\dfrac{2\pi a}{w}\right) + i\omega\,\dfrac{8\pi}{wc\sqrt{\varepsilon_m}\mu_0} - \dfrac{gc^2}{2aw^2}}\right] \tag{4.50}$$

or

$$\mu_{\text{eff}} = \mu_0\left[1 + \frac{\omega^2 F}{\omega_0^2 - \omega^2 - i\omega\Gamma}\right] \tag{4.51}$$

where

$$F := \frac{f}{8\pi\ln(2\pi a/w)} , \;\; \omega_0^2 := \frac{1}{8\pi\ln(2\pi a/w)}\,\frac{gc^2}{2aw^2} , \;\; \Gamma := \frac{1}{\ln(2\pi a/w)}\,\frac{1}{wc\sqrt{\varepsilon_m}\mu_0} . \tag{4.52}$$

Problem 4.4 Show that

$$\mu^{\text{eff}} = \mu_0 \left[1 + \frac{f\omega^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}\right] \tag{4.53}$$

for the double split cylinder geometry using the assumptions in Pendry et al. (1999).

Solution 4.4: In Pendry's approach, the equation for the balance of emf takes the form

$$-i\omega L I - \frac{I}{i\omega C} + 2\pi a R I = i\omega\mu_0(H_0 - fI)(\pi a^2)\,. \tag{4.54}$$

Following the procedure from the previous problem we get

$$\mu^{\text{eff}} = \mu_0 \left[1 + \frac{f\omega^2}{\dfrac{1}{\pi a^2\mu_0 C} - \dfrac{\omega}{\pi a^2\mu_0}(i2\pi a R + \omega L)}\right] \tag{4.55}$$

or,

$$\mu^{\text{eff}} = \mu_0 \left[1 + \frac{f\omega^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}\right] \tag{4.56}$$

where

$$\omega_0^2 = \frac{1}{\pi a^2\mu_0 C}\ ,\ \ \Gamma = \frac{2R}{a\mu_0}\ ,\ \ L = \pi a^2\mu_0\,. \tag{4.57}$$

The gap capacitance is assumed to be approximately

$$C = \frac{\pi a\varepsilon}{3d} = \frac{\pi a}{3d\mu_0 c^2}\,. \tag{4.58}$$

Then, we have

$$\omega_0^2 = \frac{3d}{\pi^2 a^3\mu_0\varepsilon}\ ,\ \ \Gamma = \frac{2}{a\mu_0\sigma}\ ,\ \ L = \pi a^2\mu_0\,. \tag{4.59}$$

Problem 4.5 Show that for evanescent TM-waves incident on a slab lens in vacuum with axis along the z-direction, the effective transmision coefficient is

$$\lim_{\substack{\mu_r \to -1 \\ \varepsilon_r \to -1}} T = e^{-ik_{zi}d} \tag{4.60}$$

where d is the thickness of the slab and k_{zi} is the the z-component of the wave vector in the medium of incidence.

Solution 4.5: The procedure is identical to that for TE waves.

The TM wave equation has the form

$$\overline{\boldsymbol{\nabla}} \cdot \left(\frac{1}{\varepsilon(z)} \overline{\boldsymbol{\nabla}} H_y \right) + \omega^2 \mu(z) H_y = 0\,. \tag{4.61}$$

Now,

$$\overline{\boldsymbol{\nabla}} H_y = \frac{\partial H_y}{\partial x} \mathbf{e}_x + \frac{\partial H_y}{\partial z} \mathbf{e}_z \tag{4.62}$$

and

$$\overline{\boldsymbol{\nabla}} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_z}{\partial z} \tag{4.63}$$

Therefore, the TM wave equation can be written as

$$\overline{\boldsymbol{\nabla}} \cdot \left(\frac{1}{\varepsilon(z)} \frac{\partial H_y}{\partial x} \mathbf{e}_x + \frac{1}{\varepsilon(z)} \frac{\partial H_y}{\partial z} \mathbf{e}_z \right) + \omega^2 \mu(z) H_y = 0 \tag{4.64}$$

or

$$\frac{\partial}{\partial x} \left(\frac{1}{\varepsilon(z)} \frac{\partial H_y}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\varepsilon(z)} \frac{\partial H_y}{\partial z} \right) + \omega^2 \mu(z) H_y = 0 \tag{4.65}$$

or

$$\frac{1}{\varepsilon(z)} \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial}{\partial z} \left(\frac{1}{\varepsilon(z)} \frac{\partial H_y}{\partial z} \right) + \omega^2 \mu(z) H_y = 0 \tag{4.66}$$

or

$$\frac{\partial^2 H_y}{\partial x^2} + \varepsilon(z) \frac{\partial}{\partial z} \left(\frac{1}{\varepsilon(z)} \frac{\partial H_y}{\partial z} \right) + \omega^2 \varepsilon(z) \mu(z) H_y = 0\,. \tag{4.67}$$

With solutions of the form

$$H_y(x, z) = \widetilde{H}_y(z) \exp(\pm i k_x x) \tag{4.68}$$

we can write the wave equation as

$$\frac{d^2 \widetilde{H}_y}{dz^2} + k_z^2 \widetilde{H}_y = 0 \quad \text{where} \quad k_z^2 = \omega^2 \varepsilon(z) \mu(z) - k_x^2\,. \tag{4.69}$$

Continuity of the tangential components of the magnetic and electric field at the interface can then be used to find equations that are identical in form to the TE fields with

$$R = \frac{\mu_1 k_{z1} - \mu_2 k_{z2}}{\mu_1 k_{z1} + \mu_2 k_{z2}} \quad \text{and} \quad T = \frac{2\mu_1 k_{z1}}{\mu_1 k_{z1} + \mu_2 k_{z2}}\,. \tag{4.70}$$

The result follows in a manner identical to that for TE waves.

Problem 4.6 Show that evanescent pressure waves in an acoustic medium can also be amplified by a negative refractive index acoustic slab lens.

Solution 4.6: The procedure is identical to that for TE and TM waves once we see that the wave equation can be expressed as

$$\overline{\boldsymbol{\nabla}} \cdot \left(\frac{1}{\rho(z)} \overline{\boldsymbol{\nabla}} p \right) + \frac{\omega^2}{\kappa(z)}\, p = 0\,. \tag{4.71}$$

Expanded out,

$$\frac{\partial^2 p}{\partial x^2} + \rho(z) \frac{\partial}{\partial z} \left(\frac{1}{\rho(z)} \frac{\partial p}{\partial z} \right) + \omega^2 \frac{\rho(z)}{\kappa(z)} p = 0\,. \tag{4.72}$$

With solutions of the form

$$p(x,z) = \widetilde{p}(z) \exp(\pm i k_x x) \tag{4.73}$$

we can write the acoustic wave equation for each layer as

$$\frac{d^2 \widetilde{p}}{dz^2} + k_z^2 \widetilde{p} = 0 \quad \text{where} \quad k_z^2 = \omega^2 \frac{\rho(z)}{\kappa(z)} - k_x^2\,. \tag{4.74}$$

Continuity of the pressure and the normal component of the displacement can then be used to find the refelection and transmission coefficients is terms of k_z for each layer. Using the evanescent solutions

$$k_z = i \sqrt{k_x^2 - \omega^2 \rho / \kappa} \tag{4.75}$$

in the reflection and transmission coefficients and taking the limit as $\rho \to -1$ and $\kappa \to -1$ gives us the required amplification behavior.

Chapter 5

Solutions for Exercises in Chapter 5

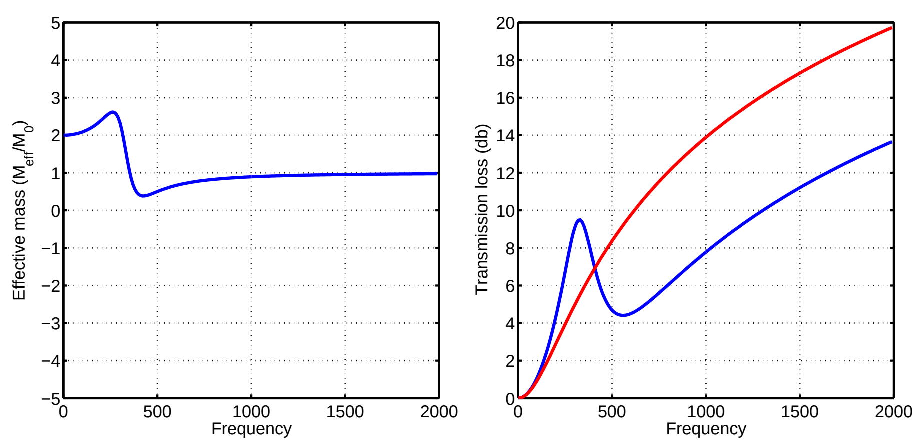
Problem 5.1 Plot the effective mass and transmission loss for a bar of mass $M_0 = 2$ kg with a cavity containing a mass $m_1 = 2$ kg and two springs with $k_1 = k_2 = 10^5(1 + 0.5i)$ N/m. Compare the transmission loss with the mass law for the system. The impedance used to calculate the mass law is $Z = i\,\omega\,(M_0 + m_1)$.

What happens when the mass m_1 in the system is also frequency-dependent and has the form

$$m_1 = m_0 + \frac{\omega_0^2\, m_2}{\omega_0^2 - \omega^2}\,;\;\; \omega_0 := \sqrt{\frac{2\,G}{m_2}} \tag{5.1}$$

with $m_1 = m_2 = 1$ kg and $G = 10^4$ N/m?

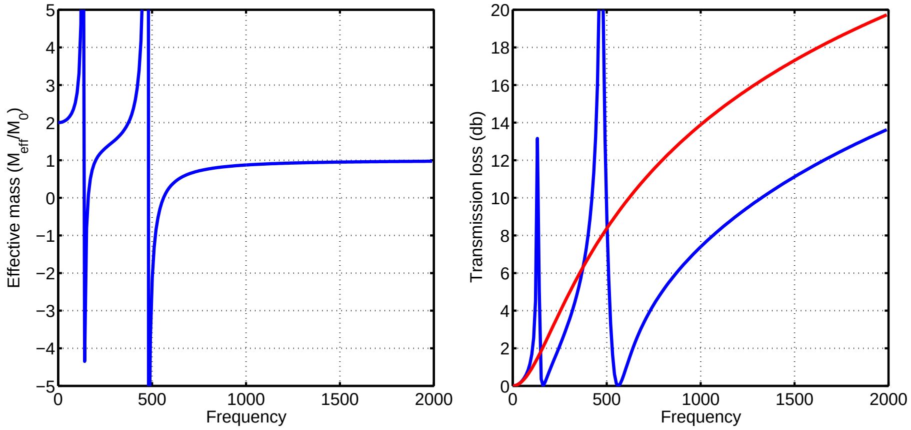
Solution 5.1 For the first part of the problem, we get the plots shown below. These are typical of what we observe in impedance tubes.



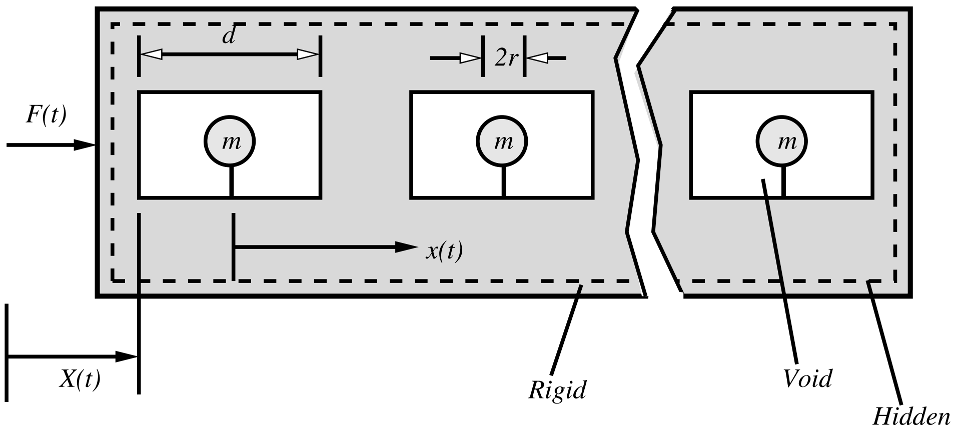
When the mass m_1 in the system is also frequency-dependent and has the form

$$m_1 = m_0 + \frac{\omega_0^2\, m_2}{\omega_0^2 - \omega^2}\,;\;\; \omega_0 := \sqrt{\frac{2\,G}{m_2}} \tag{5.2}$$

and $m_1 = m_2 = 1$ kg and $G = 10^4$ N/m we get the curves shown below.


Effective mass (M_{eff}/M_0)
5
4
3
2
1
0
−1
−2
−3
−4
−5
0
500
1000
1500
2000
Frequency
Transmission loss (db)
20
18
16
14
12
10
8
6
4
2
0
0
500
1000
1500
2000
Frequency

Problem 5.2 Consider the modified rigid bar with voids shown in the figure below.



Each ball is attached to the bar by a massless beam with a circular cross section and radius h. Calculate the effective dynamic mass of the system. Plot the effective mass as a function of frequency. Are there regions where the effective mass is negative?

Solution 5.2: The equation of motion for each isotropic, homogeneous, linear elastic beam of constant cross-section is

$$EI\frac{\partial^4 u(y,t)}{\partial y^4} + \rho A \frac{\partial^2 u(y,t)}{\partial t^2} = 0 \tag{5.3}$$

where E is the Young's modulus, ρ is the mass density, I is the area moment of inertia, A is the cross-sectional area, and $u(y)$ is the displacement. For a beam with negligible mass compared to the mass at the end, the governing equation reduces to

$$EI\frac{\partial^4 u(y,t)}{\partial y^4} = 0\,. \tag{5.4}$$

The boundary conditions at the point of attachment to the rigid bar are

$$u(0) = U(X) \quad \text{and} \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0\,. \tag{5.5}$$

At the free end of the beam, $y = L$, the bending moment and shear force is equal to the inertial load due to the mass m, i.e.,

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=L} = 0 \quad \text{and} \quad EI\left.\frac{\partial^3 u}{\partial y^3}\right|_{y=L} = m\left.\frac{\partial^2 u}{\partial t^2}\right|_{y=L}\,. \tag{5.6}$$

If we assume a time harmonic displacement

$$u(X,y,t) = \text{Re}[u(X,y)\exp(-i\omega t)] \tag{5.7}$$

then we need to find $u(X,y)$ such that

$$EI\frac{\partial^4 u(X,y)}{\partial y^4} = 0 \tag{5.8}$$

with

$$u(X,0) = U(X)\,, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0\,, \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=L} = 0 \quad \text{and} \quad EI\left.\frac{\partial^3 u}{\partial y^3}\right|_{y=L} = -m\omega^2 u(L)\,. \tag{5.9}$$

The solution to the above equation can be found by direct integration and has the form

$$u(X, y) = U(X)\frac{6EI - m\omega^2(2L^3 - 3Ly^2 + y^3)}{6EI - 2m\omega^2 L^3}. \tag{5.10}$$

The displacement of the beam-tip mass is

$$u(X, L) = \frac{6EI}{6EI - 2m\omega^2 L^3} U(X). \tag{5.11}$$

To find the effective mass of the system we can equate the total momentum to the effective momentum,

$$M_0 U(X) + m u(X, L) = M_{\text{eff}} U(X). \tag{5.12}$$

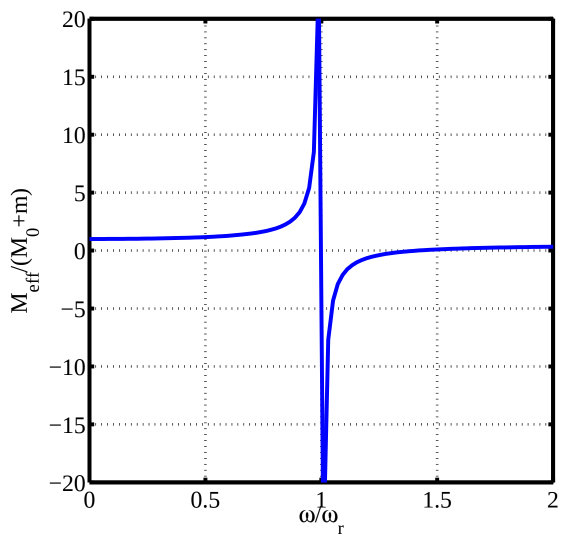
Therefore,

$$M_{\text{eff}} = M_0 + \frac{6EI}{6EI - 2m\omega^2 L^3} \quad \text{where} \quad I = \frac{\pi h^4}{4} \qquad \square \tag{5.13}$$

The resonance frequency is

$$\omega_r = \sqrt{\frac{3EI}{mL^3}}. \tag{5.14}$$

A plot of the effective mass as a function of frequency is shown below. The effective mass is negative for a small range of frequencies close to resonance.



Problem 5.3 Consider the model of an elastic bar containing hidden springs and masses. Plot the effective mass of the bar as a function of frequency and compare the result to the rigid case. Can you think of other representations of an elastic bar with hidden springs and masses?

Extend the the model of the elastic bar containing springs and masses to two-dimensions and plot the effective mass of the system in polar coordinates as a function of angle.

Solution 5.3: The equation of motion of an elastic bar with internal mass-spring systems can be written as

$$(\mathbf{K} - \omega^2 \mathbf{M})\,\mathbf{u} = \mathbf{f}\,. \tag{5.15}$$

This system of equations can be solved for **u**. The effective mass of the system may then be calculated using

$$M_{\text{eff}} = M_0 + \sum_{i=1}^{n} m_i u_i / U \tag{5.16}$$

where $U = u_{n+1} - u_0$ is the displacement of the elastic body. For the model in Figure 5.9, if we look at the situation where there is only one internal mass, we have

$$\mathbf{K} = \begin{bmatrix} 2K_0 + k & -k & -K_0 \\ -k & 2k & -k \\ -K_0 & -k & 2K_0 + k \end{bmatrix} \; ; \; \mathbf{M} = \begin{bmatrix} M_0/2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M_0/2 \end{bmatrix} \; ; \; \mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \; ; \; \mathbf{f} = \begin{bmatrix} K_0 u_a \\ 0 \\ K_0 u_b \end{bmatrix} \tag{5.17}$$

where u_a and u_b are the displacements at locations where physical measurements can be made. Solving the system of equations gives us an effective mass

$$M_{\text{eff}} = \frac{M_0}{2}\,\frac{u_0 + u_2}{U} + m\frac{u_1}{U} = \frac{2K_0[2k(m + M_0) - mM_0\omega^2]}{4kK_0 - 2[K_0 m + k(m + M_0)]\omega^2 + mM_0\omega^4}\,\frac{u_a + u_b}{U} \tag{5.18}$$

or

$$M_{\text{eff}} = \frac{4km + 2M_0(2k - m\omega^2)}{4k - 2\left[m + \dfrac{k(m + M_0)}{K_0}\right]\omega^2 + \dfrac{mM_0\omega^4}{K_0}}\,\frac{u_a + u_b}{U}\,. \tag{5.19}$$

Note that if we do not include the spatial variation of the phase of the wave, $M_{\text{eff}} \to 0$ as $\omega \to \infty$, which is unphysical. Also note that in the limit $K_0 \to \infty$, $u_a = u_b = U$ and we have

$$M_{\text{eff}} = \frac{4km + 2M_0(2k - m\omega^2)}{4k - 2m\omega^2} = M_0 + \frac{2km}{2k - m\omega^2}\,. \tag{5.20}$$

This is identical to the expression that we had derived for the equivalent rigid bar,

$$M_{\text{eff}} = M_0 + \frac{2km}{2k - m\omega^2}\,. \tag{5.21}$$

Going back to the unphysical nature of the expression for M_{eff} for the elastic bar, we are clearly missing an important factor. We have tacitly assumed that the phase velocity of the system remains unchanged as the frequency changes. This assumption is not valid for elastic bodies.

For a elastic body with a periodic distribution of hidden masses, we can use Bloch wave solutions of the form

$$U = u_b = u_a \exp(i\kappa L) \tag{5.22}$$

where $\kappa = f(\omega)$ is the phase speed in the elastic bar and L is the distance between points a and b. Then we have

$$M_{\text{eff}} = \frac{4km + 2M_0(2k - m\omega^2)}{4k - 2\left[m + \dfrac{k(m + M_0)}{K_0}\right]\omega^2 + \dfrac{mM_0\omega^4}{K_0}}\,[\cos\kappa L + 1]\,. \tag{5.23}$$

We now have to find the function $\kappa(\omega)$. If the internal mass is hidden, the effective dispersion relation for the elastic bar is

$$\omega^2 = \frac{2K_0}{M_{\text{eff}}}(1 - \cos\kappa L)\,. \tag{5.24}$$

Expressing $\cos\kappa L$ in terms of ω and M_{eff}, plugging into the expression for M_{eff}, and solving for M_{eff} gives

$$M_{\text{eff}} = \frac{8km + 4M_0(2k - m\omega^2)}{8k - 2\left[2m + \dfrac{k(m+M_0)}{K_0}\right]\omega^2 + \dfrac{mM_0\omega^4}{K_0}}\,. \tag{5.25}$$

This expression has the same form as (5.23) and hence also the same unphysical behavior as $\omega \to \infty$.

An alternative approach is to match the actual and effective dispersion relations directly. The equations of motion of the two masses can be expressed as

$$\begin{aligned} &K_0(2U_0 - U_{-L} - U_L) + k(2U_0 - u_{-L} - u_0) - \omega^2 M_0 U_0 = 0 \\ &k(2u_0 - U_0 - U_L) - \omega^2 m u_0 = 0 \end{aligned} \tag{5.26}$$

where

$$U_\alpha := U\exp[i\kappa(x+\alpha)] \quad \text{and} \quad u_\alpha = u\exp[i\kappa(x+\alpha)]\,. \tag{5.27}$$

If we match the dispersion relations from the above equations with those of an effective medium with mass M_{eff} and stiffness $K_{\text{eff}} = K_0 + k/2$, we get

$$M_{\text{eff}} = \frac{2km + M_0(2k - m\omega^2)}{2k - \frac{2K_0}{2K_0+k}\,m\omega^2}\,. \tag{5.28}$$

This model reduces to the rigid bar model as $K_0 \to \infty$. If the limit $\omega \to 0$, we have

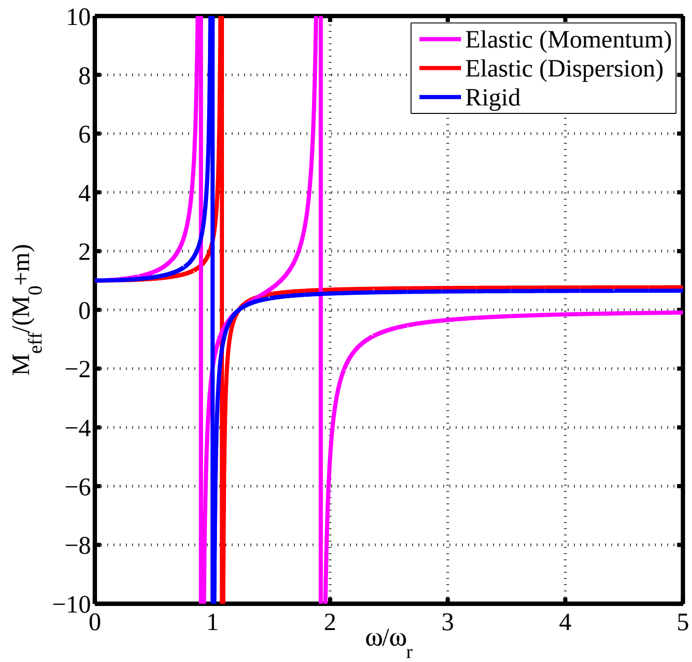
$$M_{\text{eff}} = m + M0\,. \tag{5.29}$$

In the limit $\omega \to \infty$, we have $M_{\text{eff}} = M_0(1 + k/(2K_0))$. This expression using a dispersion-based approach for the effective mass, while not ideal, is better than that from the previous momentum-based approach. The dependence on k and K_0 as $\omega \to \infty$ indicates that the model we have used may not be an accurate representation of an elastic bar with hidden masses. A script showing the calculation is given below.

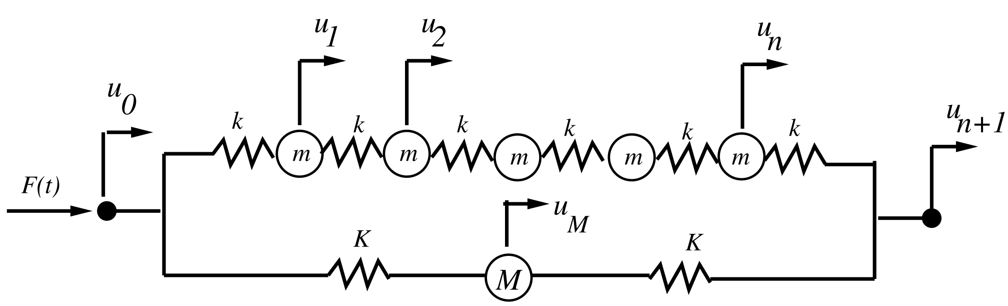
```
UUa[n_] := BB Exp[I (kappa  (x + n  L ) - omega t)]
uua[n_] := bb Exp[I (kappa  (x + n  L ) - omega t)]
Uaeq1 = M0 D[D[UUa[0], t], t] + K0 (2 UUa[0] - UUa[ -1] - UUa[1]) +
   k (2 UUa[0] - uua[-1] - uua[0])
Uaeq2 = m D[D[uua[0], t], t] + k (2 uua[0] - UUa[0] - UUa[1])
Ba11 = Coefficient[Uaeq1, BB]
Ba12 = Coefficient[Uaeq1, bb]
Ba21 = Coefficient[Uaeq2, BB]
Ba22 = Coefficient[Uaeq2, bb]
Bamat = {{Ba11, Ba12}, {Ba21, Ba22}}
detBa = Det[Bamat] /E^(-2 I (omega t - kappa  x)) // FullSimplify
detBaa = ComplexExpand[Re[detBa]] // FullSimplify
Collect[detBaa, {omega, k, K0, m}]
Keff = K0 + k/2
Bacosql = (2 Keff - meff omega^2)/(2 Keff)
Bacosql2 = 2 Bacosql^2 - 1
Bameffeq =
 detBaa /. {Cos[kappa L] -> acosql, Cos[2 kappa L] ->  Bacosql2} //
  FullSimplify
```

```
Basol = Solve[Bameffeq == 0, meff] // FullSimplify
BaMeff = (meff /. Basol)[[1]]
Limit[BaMeff, omega -> 0]
Limit[BaMeff, omega -> Infinity]
Limit[BaMeff, K0 -> Infinity] // FullSimplify
```

A representative plot of the effective mass versus frequency is shown below. Note that the two resonance peaks due to the two spring stiffnesses is observed in an actual experiments. However, the effective behavior at large enough wavelengths identifies only the first (local) resonance peak.



Another possible representation is shown below.



A two dimensional model can be created by superposing the x- and y-direction responses, assuming they are uncoupled. In matrix form,

$$\mathbf{M}_{\text{eff}}\mathbf{U} = \begin{bmatrix} M_x^{\text{eff}} & 0 \\ 0 & M_y^{\text{eff}} \end{bmatrix} \begin{bmatrix} U_x \\ U_y \end{bmatrix} . \tag{5.30}$$

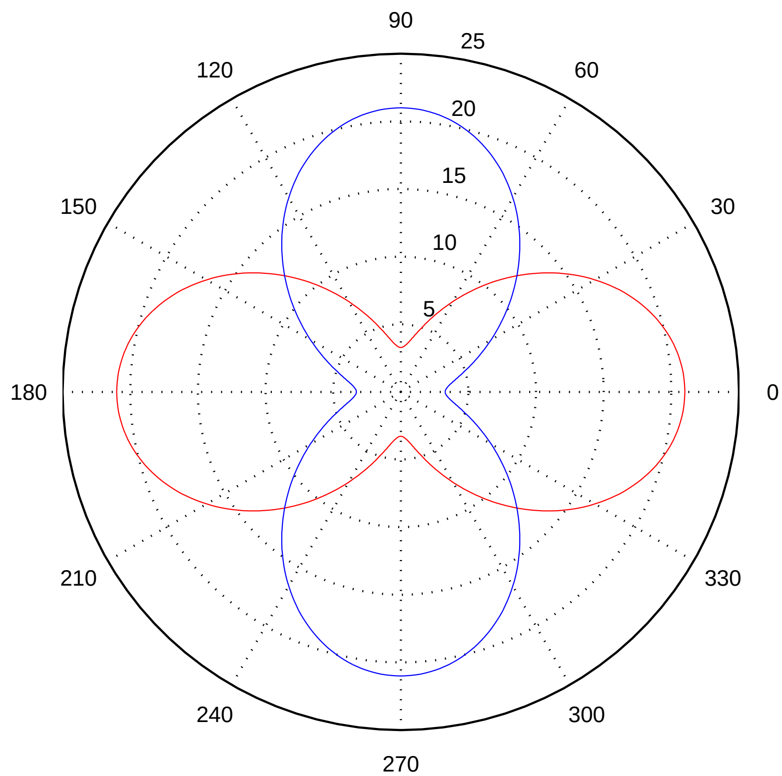
where

$$M_x^{\text{eff}} = \frac{2k_x m + M_0(2k_x - m\omega^2)}{2k_x + \frac{k_x^2}{K_0} - m\omega^2}$$
$$M_y^{\text{eff}} = \frac{2k_y m + M_0(2k_y - m\omega^2)}{2k_y + \frac{k_y^2}{K_0} - m\omega^2}\,. \tag{5.31}$$

We can rotate the mass matrix using

$$\mathbf{M}_{\text{eff}} = \mathbf{R}\begin{bmatrix} M_x^{\text{eff}} & 0 \\ 0 & M_y^{\text{eff}} \end{bmatrix}\mathbf{R}^T \quad \text{where} \quad \mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \square\,. \tag{5.32}$$

A plot of the components M_{xx} (red) and M_{yy} (blue) of the effective mass matrix is shown in the figure below.



Problem 5.4 Show that the effective Young's modulus function for a one-dimensional lossy material is analytic in the entire complex plane except for isolated singularities in the negative Im(ω) part of the plane. Draw a schematic of the locations of the singularities. Also show that $\overline{E}(\omega) = E(-\overline{\omega})$ for this model, where $\overline{(\bullet)}$ indicates the complex conjugate.

Solution 5.4: The effective Young's modulus is given by

$$E(\omega) = E_0 \sum_j \left(1 - \frac{\alpha_j \omega_j^2}{\omega^2 - \omega_j^2 + i\omega\gamma_j}\right)^{-1} . \tag{5.33}$$

If we expand this function in terms of $\omega = \omega_r + i\omega_i$, we get

$$E(\omega) = E_0 \left[1 - \sum_j \frac{\alpha_j \omega_j}{(\alpha_j + \omega_j)\omega_j + (\omega_i - i\omega_r)(\gamma_j + \omega_i - i\omega_r)}\right] . \tag{5.34}$$

If $\overline{\omega} = \omega_r - i\omega_i$, we have

$$E(-\overline{\omega}) = E_0 \left[\sum_j \frac{\omega_j^2 + (\omega_i + I\omega_r)(\gamma_j + \omega_i + I\omega_r)}{(\alpha_j + \omega_j)\omega_j + (\omega_i + I\omega_r)(\gamma_j + \omega_i + I\omega_r)}\right] . \tag{5.35}$$

The real and imaginary parts of $E(\omega) = E_r(\omega) + iE_i(\omega)$ are

$$\begin{aligned}
\frac{E_r}{E_0} &= \frac{[\omega_i(\gamma_j + \omega_i) + \omega_j^2][\omega_i(\gamma_j + \omega_i) + \omega_j(\alpha_j + \omega_j)] + [\gamma_j^2 + 2\gamma_j\omega_i + 2\omega_i^2 - \omega_j(\alpha_j + 2\omega_j)]\omega_r^2 + \omega_r^4}{[\omega_i(\gamma_j + \omega_i) + \omega_j(\alpha_j + \omega_j)]^2 + [\gamma_j^2 + 2\gamma_j\omega_i + 2\omega_i^2 - 2\omega_j(\alpha_j + \omega_j)]\omega_r^2 + \omega_r^4} \\
\frac{E_i}{E_0} &= -\frac{\alpha_j(\gamma_j + 2\omega_i)\omega_j\omega_r}{[\omega_i(\gamma_j + \omega_i) + \omega_j(\alpha_j + \omega_j)]^2 + [\gamma_j^2 + 2\gamma_j\omega_i + 2\omega_i^2 - 2\omega_j(\alpha_j + \omega_j)]\omega_r^2 + \omega_r^4}
\end{aligned} \tag{5.36}$$

Therefore,

$$\overline{E}(\omega) = E_r(\omega) - iE_i(\omega) = E_0 \left[\sum_j \frac{\omega_j^2 + (\omega_i + i\omega_r)(\gamma_j + \omega_i + i\omega_r)}{(\alpha_j + \omega_j)\omega_j + (\omega_i + i\omega_r)(\gamma_j + \omega_i + i\omega_r)}\right] = E(-\overline{\omega}) . \tag{5.37}$$

This shows that $\overline{E}(\omega) = E(-\overline{\omega})$ for this model.

To show analyticity, we can use the Cauchy-Riemann equations,

$$\frac{\partial E_r}{\partial \omega_r} - \frac{\partial E_i}{\partial \omega_i} = 0 \quad \text{and} \quad \frac{\partial E_r}{\partial \omega_i} + \frac{\partial E_i}{\partial \omega_r} = 0 . \tag{5.38}$$

Detailed calculations show that these equations indeed hold and hence the function is analytic in the complex ω-plane. To locate the poles, we try to find the conditions under which the denominator in the expression for $E(\omega)$ is zero, i.e.,

$$\gamma_j\omega_i + \omega_i^2 + \alpha_j\omega_j + \omega_j^2 - \omega_r = 0 \quad \text{and} \quad -\gamma_j\omega_r - 2\omega_i\omega_r = 0 . \tag{5.39}$$

From the second equation, we have

$$\omega_i = -\frac{\gamma_j}{2} . \tag{5.40}$$

Plugging this into the first equation gives,

$$\omega_r = \pm\sqrt{-\gamma_j^2/4 + \alpha_j\omega_j + \omega_j^2} . \tag{5.41}$$

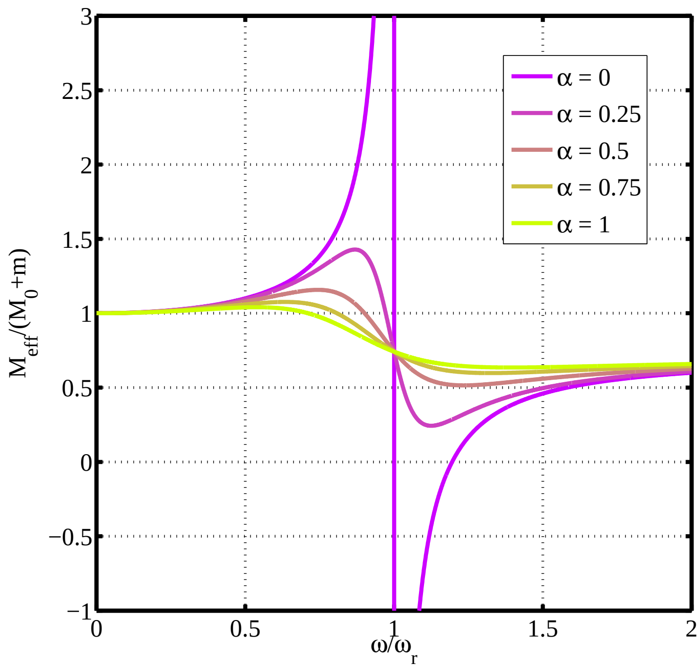
The poles are therefore on the negative part of the ω plane below the real ω axis and are spaced equally on both sides of the imaginary ω axis.

Problem 5.5 Examine the effective mass as a function of frequency when there is dissipation in the system. What is the predominant effect that you observe as the amount of dissipation increases?

Solution 5.5: We can include dissipation in a spring-mass system by adding an imaginary part to the spring stiffness. For the elastic bar, if we define

$$k \to k + i\alpha \quad \text{and} \quad K_0 \to K_0 + i\alpha \tag{5.42}$$

we get a plot of the form shown below. The effect of damping is to smooth out the resonance peak.



Problem 5.6 Starting from

$$\mathbf{p}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}(t-\tau) \cdot \mathbf{v}(\tau) \, \mathrm{d}\tau \,. \tag{5.43}$$

show that

$$\widehat{\mathbf{P}}(\omega) = \boldsymbol{M}(\omega) \cdot \widehat{\mathbf{V}}(\omega) \tag{5.44}$$

where

$$\boldsymbol{M}(\omega) := \widehat{\boldsymbol{H}}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{H}(s) \, e^{i\omega s} \, \mathrm{d}s \,. \tag{5.45}$$

Solution 5.6: Assume

$$\mathbf{p}(t) = \widehat{\mathbf{P}} \, \exp(-i\omega t) \quad \text{and} \quad \mathbf{v}(t) = \widehat{\mathbf{V}} \, \exp(-i\omega t) \,. \tag{5.46}$$

Then,

$$\widehat{\mathbf{P}} \exp(-i\omega t) = \int_{-\infty}^{\infty} \boldsymbol{H}(t-\tau) \cdot \widehat{\mathbf{V}} \, \exp(-i\omega\tau) \, \mathrm{d}\tau \,. \tag{5.47}$$

Define

$$s := t - \tau \qquad \Longrightarrow \qquad \mathrm{d}\tau = -\mathrm{d}s \tag{5.48}$$

which gives

$$\widehat{\mathbf{P}} \exp(-i\omega t) = \int_{-\infty}^{\infty} \boldsymbol{H}(s) \cdot \widehat{\mathbf{V}} \, \exp[-i\omega(t-s)] \, \mathrm{d}s = \left[\int_{-\infty}^{\infty} \boldsymbol{H}(s) \, \exp(i\omega s) \, \mathrm{d}s \right] \cdot \widehat{\mathbf{V}} \, \exp(-i\omega t) \tag{5.49}$$

or

$$\widehat{\mathbf{P}} = \boldsymbol{M}(\omega) \cdot \widehat{\mathbf{V}} \tag{5.50}$$

where

$$\boldsymbol{M}(\omega) := \int_{-\infty}^{\infty} \boldsymbol{H}(s) \, \exp(i\omega s) \, \mathrm{d}s \qquad \square \tag{5.51}$$

Problem 5.7 Starting from the relation

$$\mathbf{q}_\mathrm{r}(t) = \int_0^{2\pi} \overline{\mathbf{x}}_\mathrm{r}(\varphi, t) \times \mathrm{d}\mathbf{p}_\mathrm{r}(\varphi, t) \tag{5.52}$$

show that the angular momentum of the ring can be expressed in the form

$$\mathbf{q}_\mathrm{r}(t) = \mathbf{q}_0 + \mathrm{Re}(\widehat{\mathbf{q}}_\mathrm{r}\, e^{-i\omega t}) \tag{5.53}$$

where

$$\mathbf{q}_0 = -m r_0^2 \omega_r \mathbf{e}_3 \quad \text{and} \quad \widehat{\mathbf{q}}_\mathrm{r} = \frac{\varepsilon m r_0^2}{2}\left[-\left(\frac{2i\omega_r\widehat{\Omega}_2}{\omega} + 3\widehat{\Omega}_1\right)\mathbf{e}_1 + \left(\frac{2i\omega_r\widehat{\Omega}_1}{\omega} - 3\widehat{\Omega}_2\right)\mathbf{e}_2\right]. \tag{5.54}$$

Solution 5.7: The angular momentum of the ring is given by

$$\mathbf{q}_\mathrm{r}(t) = \int_0^{2\pi} \overline{\mathbf{x}}_\mathrm{r}(\varphi, t) \times \mathrm{d}\mathbf{p}_\mathrm{r}(\varphi, t) \tag{5.55}$$

where

$$\overline{\mathbf{x}}_\mathrm{r}(\varphi, t) = \mathbf{u}_\mathrm{o}(t) + \boldsymbol{R}(t) \cdot \mathbf{X}_\mathrm{r}(\varphi) \tag{5.56}$$

and

$$\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi, t) = \left(\frac{m\,\mathrm{d}\varphi}{2\pi}\right)\frac{d}{dt}[\overline{\mathbf{x}}_\mathrm{r}(\varphi, t)]\ . \tag{5.57}$$

We know that

$$\mathbf{u}_\mathrm{o}(t) = \varepsilon\,\mathrm{Re}(\widehat{\mathbf{u}}_\mathrm{o}\, e^{-i\omega t}) \quad \text{and} \quad \boldsymbol{R}(t) = \boldsymbol{Q}(t) + \boldsymbol{B}(t)\cdot\boldsymbol{Q}(t) \quad \text{where} \quad \boldsymbol{B}(t) = \varepsilon\,\mathrm{Re}(\widehat{\boldsymbol{B}}\, e^{-i\omega t})\,. \tag{5.58}$$

Therefore,

$$\frac{d\overline{\mathbf{x}}_\mathrm{r}}{dt} = \frac{d\mathbf{u}_\mathrm{o}}{dt} + \frac{d\boldsymbol{R}}{dt}\cdot\mathbf{X}_\mathrm{r}(\varphi) = \frac{d\mathbf{u}_\mathrm{o}}{dt} + \left(\frac{d\boldsymbol{Q}}{dt} + \frac{d\boldsymbol{B}}{dt}\cdot\boldsymbol{Q}(t) + \boldsymbol{B}(t)\cdot\frac{d\boldsymbol{Q}}{dt}\right)\cdot\mathbf{X}_\mathrm{r}(\varphi)\,. \tag{5.59}$$

Now,

$$\frac{d\mathbf{u}_\mathrm{o}}{dt} = \varepsilon\,\mathrm{Re}(-i\omega\widehat{\mathbf{u}}_\mathrm{o}\, e^{-i\omega t}) = \mathbf{v}_\mathrm{o}(t) \quad \text{and} \quad \frac{d\boldsymbol{B}}{dt} = \varepsilon\,\mathrm{Re}(-i\omega\widehat{\boldsymbol{B}}\, e^{-i\omega t})\,. \tag{5.60}$$

Also, since

$$\boldsymbol{Q}(t) = \begin{bmatrix} \cos(\omega_r t) & \sin(\omega_r t) & 0 \\ -\sin(\omega_r t) & \cos(\omega_r t) & 0 \\ 0 & 0 & \cos(0) \end{bmatrix} \tag{5.61}$$

we have

$$\frac{d\boldsymbol{Q}}{dt} = \omega_r \begin{bmatrix} -\sin(\omega_r t) & \cos(\omega_r t) & 0 \\ -\cos(\omega_r t) & -\sin(\omega_r t) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \omega_r\,\boldsymbol{Q}(\omega_r + \pi/2) =: \omega_r\,\widetilde{\boldsymbol{Q}}(t)\,. \tag{5.62}$$

Therefore we can write

$$\frac{d\overline{\mathbf{x}}_\mathrm{r}}{dt} = \mathbf{v}_\mathrm{o}(t) + \left(\omega_r\widetilde{\boldsymbol{Q}}(t) + \frac{d\boldsymbol{B}}{dt}\cdot\boldsymbol{Q}(t) + \omega_r\boldsymbol{B}(t)\cdot\widetilde{\boldsymbol{Q}}(t)\right)\cdot\mathbf{X}_\mathrm{r}(\varphi)\,. \tag{5.63}$$

So, the cross product inside the integral can be expressed as

$$\begin{aligned} \overline{\mathbf{x}}_\mathrm{r}(\varphi, t) \times \mathrm{d}\mathbf{p}_\mathrm{r}(\varphi, t) = d\alpha\,&[\mathbf{u}_\mathrm{o}(t) + \boldsymbol{Q}(t)\cdot\mathbf{X}_\mathrm{r}(\varphi) + \{\boldsymbol{B}(t)\cdot\boldsymbol{Q}(t)\}\cdot\mathbf{X}_\mathrm{r}(\varphi)] \\ &\times\left[\mathbf{v}_\mathrm{o}(t) + \left\{\omega_r[\boldsymbol{1} + \boldsymbol{B}(t)]\cdot\widetilde{\boldsymbol{Q}}(t)\right\}\cdot\mathbf{X}_\mathrm{r}(\varphi) + \left\{\frac{d\boldsymbol{B}}{dt}\cdot\boldsymbol{Q}(t)\right\}\cdot\mathbf{X}_\mathrm{r}(\varphi)\right] \end{aligned} \tag{5.64}$$

where $d\alpha := (m/2\pi)d\varphi$. Multiplying terms by term and eliminating terms that are $O(\varepsilon^2)$, we get

$$\begin{aligned}\overline{\mathbf{x}}_\mathrm{r}(\varphi,t)\times\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi,t) = d\alpha\Big[&\{\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)\}\times\mathbf{v}_\mathrm{o} + \omega_r\mathbf{u}_\mathrm{o}\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\\ &+\omega_r\{\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)\}\times[\{(\boldsymbol{1}+\boldsymbol{B})\cdot\widetilde{\boldsymbol{Q}}\}\cdot\mathbf{X}_\mathrm{r}(\varphi)] + \omega_r[(\boldsymbol{B}\cdot\boldsymbol{Q})\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\\ &+\{\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)\}\times[(\boldsymbol{B}'\cdot\boldsymbol{Q})\cdot\mathbf{X}_\mathrm{r}(\varphi)]\Big]\end{aligned} \tag{5.65}$$

Integrating term by term gives

$$\begin{aligned}\int_0^{2\pi}\overline{\mathbf{x}}_\mathrm{r}(\varphi,t)\times\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi,t)d\varphi = \frac{m}{2\pi}\Big[&\left(\boldsymbol{Q}\cdot\int_0^{2\pi}\mathbf{X}_\mathrm{r}(\varphi)\mathrm{d}\varphi\right)\times\mathbf{v}_\mathrm{o} + \omega_r\mathbf{u}_\mathrm{o}\times\left(\widetilde{\boldsymbol{Q}}\cdot\int_0^{2\pi}\mathbf{X}_\mathrm{r}(\varphi)\mathrm{d}\varphi\right)\\ &+\omega_r\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi + \omega_r\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[(\boldsymbol{B}\cdot\widetilde{\boldsymbol{Q}})\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi\\ &+\omega_r\int_0^{2\pi}[(\boldsymbol{B}\cdot\boldsymbol{Q})\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi + \int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times\left[(\boldsymbol{B}'\cdot\boldsymbol{Q})\cdot\mathbf{X}_\mathrm{r}(\varphi)\right]\mathrm{d}\varphi\Big]\,.\end{aligned} \tag{5.66}$$

Since $\mathbf{X}_\mathrm{r}(\varphi) = r_0\cos\varphi\mathbf{e}_1 + r_0\sin\varphi\mathbf{e}_2$, the first two terms are zero. Also,

$$\boldsymbol{B}' = \frac{d\boldsymbol{B}}{dt} = \varepsilon\,\mathrm{Re}(-i\omega\widehat{\boldsymbol{B}}\,e^{-i\omega t}) = \mathrm{Re}(-i\omega\boldsymbol{B})\,. \tag{5.67}$$

If we define $\boldsymbol{A} := \boldsymbol{B}\cdot\boldsymbol{Q}$ and $\widetilde{\boldsymbol{A}} := \boldsymbol{B}\cdot\widetilde{\boldsymbol{Q}}$, we have

$$\begin{aligned}\int_0^{2\pi}\overline{\mathbf{x}}_\mathrm{r}(\varphi,t)\times\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi,t)d\varphi = \frac{m}{2\pi}\Big[&\\ &\omega_r\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi + \omega_r\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{A}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi\\ &+\omega_r\int_0^{2\pi}[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi - i\omega\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)\mathrm{d}\varphi\Big]\,.\end{aligned} \tag{5.68}$$

Let us look at each term in the above expression separately. The first term is

$$\begin{aligned}\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \int_0^{2\pi} e_{ijk}Q_{jp}X_p(\varphi)\widetilde{Q}_{kq}X_q(\varphi)\mathbf{e}_i\mathrm{d}\varphi\\ &= e_{ijk}Q_{jp}\widetilde{Q}_{kq}\left[\int_0^{2\pi}X_p(\varphi)X_q(\varphi)\mathrm{d}\varphi\right]\mathbf{e}_i\,.\end{aligned} \tag{5.69}$$

Similarly, the other three terms can be written as,

$$\begin{aligned}\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{A}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= e_{ijk}Q_{jp}\widetilde{A}_{kq}\left[\int_0^{2\pi}X_p(\varphi)X_q(\varphi)\mathrm{d}\varphi\right]\mathbf{e}_i\\ \int_0^{2\pi}[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= e_{ijk}A_{jp}\widetilde{Q}_{kq}\left[\int_0^{2\pi}X_p(\varphi)X_q(\varphi)\mathrm{d}\varphi\right]\mathbf{e}_i\\ \int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= e_{ijk}Q_{jp}A_{kq}\left[\int_0^{2\pi}X_p(\varphi)X_q(\varphi)\mathrm{d}\varphi\right]\mathbf{e}_i\,.\end{aligned} \tag{5.70}$$

Now,

$$\int_0^{2\pi}X_p(\varphi)X_q(\varphi)\mathrm{d}\varphi \equiv r_0^2\begin{bmatrix}\pi & 0 & 0\\ 0 & \pi & 0\\ 0 & 0 & 0\end{bmatrix} \equiv r_0^2\pi(\delta_{pq}-\delta_{p3}\delta_{q3})\,. \tag{5.71}$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}(Q_{jp}\widetilde{Q}_{kp} - Q_{j3}\widetilde{Q}_{k3})\mathbf{e}_i \\
\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{A}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}(Q_{jp}\widetilde{A}_{kp} - Q_{j3}\widetilde{A}_{k3})\mathbf{e}_i \\
\int_0^{2\pi}[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}(A_{jp}\widetilde{Q}_{kp} - A_{j3}\widetilde{Q}_{k3})\mathbf{e}_i \\
\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}(Q_{jp}A_{kp} - Q_{j3}A_{k3})\mathbf{e}_i\,.
\end{aligned} \tag{5.72}$$

Because $Q_{33} = 1$ and all other $Q_{i3} = \widetilde{Q}_{i3} = 0$, the above relations simplify to

$$\begin{aligned}
\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}Q_{jp}\widetilde{Q}_{kp}\mathbf{e}_i \\
\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{A}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}(Q_{jp}\widetilde{A}_{kp} - \delta_{j3}\widetilde{A}_{k3})\mathbf{e}_i \\
\int_0^{2\pi}[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\widetilde{\boldsymbol{Q}}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}A_{jp}\widetilde{Q}_{kp}\mathbf{e}_i \\
\int_0^{2\pi}[\boldsymbol{Q}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\times[\boldsymbol{A}\cdot\mathbf{X}_\mathrm{r}(\varphi)]\mathrm{d}\varphi &= \pi r_0^2 e_{ijk}(Q_{jp}A_{kp} - \delta_{j3}A_{k3})\mathbf{e}_i\,.
\end{aligned} \tag{5.73}$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi}\overline{\mathbf{x}}_\mathrm{r}(\varphi,t)\times\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi,t)d\varphi =& \frac{mr_0^2}{2}e_{ijk}\Big[\omega_r\left(Q_{jp}\widetilde{Q}_{kp} + (Q_{jp}\widetilde{A}_{kp} - \delta_{j3}\widetilde{A}_{k3}) + A_{jp}\widetilde{Q}_{kp}\right) \\
& - i\omega(Q_{jp}A_{kp} - \delta_{j3}A_{k3})\Big]\mathbf{e}_i\,.
\end{aligned} \tag{5.74}$$

Recall that $\boldsymbol{A} = \boldsymbol{B}\cdot\boldsymbol{Q}$ and $\widetilde{\boldsymbol{A}} = \boldsymbol{B}\cdot\widetilde{\boldsymbol{Q}}$. Substituting these into the above expression gives

$$\begin{aligned}
\int_0^{2\pi}\overline{\mathbf{x}}_\mathrm{r}(\varphi,t)\times\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi,t)d\varphi =& \frac{mr_0^2}{2}e_{ijk}\Big[\omega_r\big(Q_{jp}\widetilde{Q}_{kp} \\
& + (Q_{jp}B_{km}\widetilde{Q}_{mp} - \delta_{j3}B_{km}\widetilde{Q}_{m3}) + B_{jm}Q_{mp}\widetilde{Q}_{kp}\big) \\
& - i\omega(Q_{jp}B_{km}Q_{mp} - \delta_{j3}B_{kp}Q_{p3})\Big]\mathbf{e}_i\,.
\end{aligned} \tag{5.75}$$

Now $\widetilde{Q}_{m3} = 0$ and $Q_{p3} = \delta_{p3}$. Therefore,

$$\begin{aligned}
\int_0^{2\pi}\overline{\mathbf{x}}_\mathrm{r}(\varphi,t)\times\mathrm{d}\mathbf{p}_\mathrm{r}(\varphi,t)d\varphi =& \frac{mr_0^2}{2}e_{ijk}\Big[\omega_r\big(Q_{jp}\widetilde{Q}_{kp} \\
& + B_{km}Q_{jp}\widetilde{Q}_{mp} + B_{jm}Q_{mp}\widetilde{Q}_{kp}\big) \\
& - i\omega(B_{km}Q_{jp}Q_{mp} - \delta_{j3}B_{k3})\Big]\mathbf{e}_i\,.
\end{aligned} \tag{5.76}$$

We also have,

$$\boldsymbol{Q}\cdot\widetilde{\boldsymbol{Q}}^T \equiv Q_{jp}\widetilde{Q}_{kp} \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{Q}\cdot\boldsymbol{Q}^T \equiv Q_{jp}Q_{kp} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.77}$$

and

$$\boldsymbol{B} \equiv B_{ij} \equiv \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} . \tag{5.78}$$

Therefore,

$$B_{km}Q_{jp}\widetilde{Q}_{mp} \equiv (\boldsymbol{Q}\cdot\widetilde{\boldsymbol{Q}}^T)\cdot\boldsymbol{B}^T \equiv \begin{bmatrix} -b_3 & 0 & b_1 \\ 0 & -b_3 & b_2 \\ 0 & 0 & 0 \end{bmatrix} , \tag{5.79}$$

$$B_{jm}Q_{mp}\widetilde{Q}_{kp} \equiv \boldsymbol{B}\cdot(\boldsymbol{Q}\cdot\widetilde{\boldsymbol{Q}}^T) \equiv \begin{bmatrix} b_3 & 0 & 0 \\ 0 & b_3 & 0 \\ -b_1 & -b_2 & 0 \end{bmatrix} , \tag{5.80}$$

$$B_{km}Q_{jp}Q_{mp} \equiv (\boldsymbol{Q}\cdot\boldsymbol{Q}^T)\cdot\boldsymbol{B}^T \equiv \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} , \tag{5.81}$$

and

$$\delta_{j3}B_{k3} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_2 & -b_1 & 0 \end{bmatrix} . \tag{5.82}$$

Hence,

$$\omega_r\big(Q_{jp}\widetilde{Q}_{kp} + B_{km}Q_{jp}\widetilde{Q}_{mp} + B_{jm}Q_{mp}\widetilde{Q}_{kp}\big) \equiv \omega_r \begin{bmatrix} 0 & -1 & b_1 \\ 1 & 0 & b_2 \\ -b_1 & -b_2 & 0 \end{bmatrix} \tag{5.83}$$

and

$$i\omega(B_{km}Q_{jp}Q_{mp} - \delta_{j3}B_{k3}) \equiv i\omega \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -2b_2 & 2b_1 & 0 \end{bmatrix} \tag{5.84}$$

Therefore,

$$\omega_r e_{ijk}\big(Q_{jp}\widetilde{Q}_{kp} + B_{km}Q_{jp}\widetilde{Q}_{mp} + B_{jm}Q_{mp}\widetilde{Q}_{kp}\big) \equiv \omega_r \begin{bmatrix} 2b_2 \\ -2b_1 \\ -2 \end{bmatrix} \tag{5.85}$$

and

$$i\omega e_{ijk}(B_{km}Q_{jp}Q_{mp} - \delta_{j3}B_{k3}) \equiv i\omega \begin{bmatrix} -3b_1 \\ -3b_2 \\ -2b_3 \end{bmatrix} \tag{5.86}$$

Hence the angular momentum vector has the form

$$\begin{aligned} \int_0^{2\pi} \bar{\mathbf{x}}_\mathrm{r}(\varphi,t)\times d\mathbf{p}_\mathrm{r}(\varphi,t)d\varphi =& \frac{mr_0^2}{2}e_{ijk}\Big[\omega_r\big(Q_{jp}\widetilde{Q}_{kp} + B_{km}Q_{jp}\widetilde{Q}_{mp} + B_{jm}Q_{mp}\widetilde{Q}_{kp}\big) \\ & - i\omega(B_{km}Q_{jp}Q_{mp} - \delta_{j3}B_{k3})\Big]\mathbf{e}_i \\ =& mr_0^2\omega_r \begin{bmatrix} b_2 \\ -b_1 \\ -1 \end{bmatrix} + mr_0^2 \begin{bmatrix} 3i\omega b_1/2 \\ 3i\omega b_2/2 \\ i\omega b_3 \end{bmatrix} \end{aligned} \tag{5.87}$$

Recall that

$$\mathbf{b} = \varepsilon\,\mathrm{Re}(\widehat{\mathbf{b}}\,e^{-i\omega t}) = \boldsymbol{\theta}_\mathrm{o} = \varepsilon\,\mathrm{Re}(\widehat{\boldsymbol{\theta}}_\mathrm{o}\,e^{-i\omega t}) = \varepsilon\,\mathrm{Re}(i\widehat{\boldsymbol{\Omega}}_\mathrm{o}/\omega\,e^{-i\omega t})\,. \tag{5.88}$$

Therefore, if there is no friction between the spindle and the cavity,

$$\widehat{b}_1 = \frac{i\widehat{\Omega}_1}{\omega}\,;\;\; \widehat{b}_2 = \frac{i\widehat{\Omega}_2}{\omega}\,;\;\; \widehat{b}_3 = 0 \quad \text{and} \quad \omega\widehat{b}_1 = i\widehat{\Omega}_1\,;\;\; \omega\widehat{b}_2 = i\widehat{\Omega}_2\,; \tag{5.89}$$

We can then write

$$\int_0^{2\pi} \overline{\mathbf{x}}_{\rm r}(\varphi,t) \times {\rm d}\mathbf{p}_{\rm r}(\varphi,t) d\varphi = \varepsilon m r_0^2 \omega_r \begin{bmatrix} {\rm Re}(\widehat{b}_2\, e^{-i\omega t}) \\ -{\rm Re}(\widehat{b}_1\, e^{-i\omega t}) \\ -1/\varepsilon \end{bmatrix} + \frac{\varepsilon m r_0^2}{2} \begin{bmatrix} 3{\rm Re}(i\omega \widehat{b}_1\, e^{-i\omega t}) \\ 3{\rm Re}(i\omega \widehat{b}_2\, e^{-i\omega t}) \\ 0 \end{bmatrix} \tag{5.90}$$

or,

$$\int_0^{2\pi} \overline{\mathbf{x}}_{\rm r}(\varphi,t) \times {\rm d}\mathbf{p}_{\rm r}(\varphi,t) d\varphi = -\varepsilon m r_0^2 \omega_r \begin{bmatrix} {\rm Re}(-i\widehat{\Omega}_2/\omega\, e^{-i\omega t}) \\ {\rm Re}(i\widehat{\Omega}_1/\omega\, e^{-i\omega t}) \\ 1 \end{bmatrix} + \frac{\varepsilon m r_0^2}{2} \begin{bmatrix} 3{\rm Re}(-\widehat{\Omega}_1\, e^{-i\omega t}) \\ 3{\rm Re}(-\widehat{\Omega}_2\, e^{-i\omega t}) \\ 0 \end{bmatrix}. \tag{5.91}$$

Reorganization leads to

$$\int_0^{2\pi} \overline{\mathbf{x}}_{\rm r}(\varphi,t) \times {\rm d}\mathbf{p}_{\rm r}(\varphi,t) d\varphi = -m r_0^2 \omega_r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{\varepsilon m r_0^2}{2} {\rm Re}\left(\begin{bmatrix} -2i\omega_r \widehat{\Omega}_2/\omega - 3\widehat{\Omega}_1 \\ 2i\omega_r \widehat{\Omega}_1/\omega - 3\widehat{\Omega}_2 \\ 0 \end{bmatrix} e^{-i\omega t} \right). \tag{5.92}$$

Therefore, the angular momentum is given by

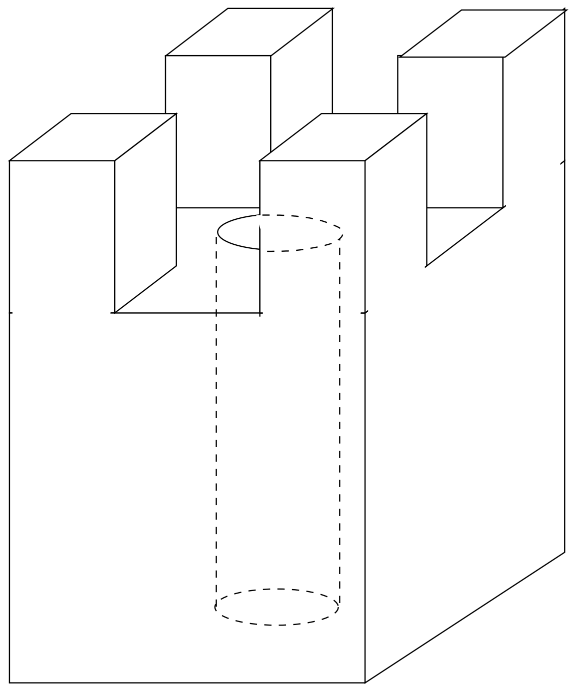
$$\mathbf{q}_{\rm r}(t) = \mathbf{q}_0 + {\rm Re}(\widehat{\mathbf{q}}_{\rm r}\, e^{-i\omega t}) \tag{5.93}$$

where

$$\mathbf{q}_0 = -m r_0^2 \omega_r \mathbf{e}_3 \quad \text{and} \quad \widehat{\mathbf{q}}_{\rm r} = \frac{\varepsilon m r_0^2}{2} \left[-\left(\frac{2i\omega_r \widehat{\Omega}_2}{\omega} + 3\widehat{\Omega}_1 \right) \mathbf{e}_1 + \left(\frac{2i\omega_r \widehat{\Omega}_1}{\omega} - 3\widehat{\Omega}_2 \right) \mathbf{e}_2 \right] .\square \tag{5.94}$$

Problem 5.8 Using the Helmholtz model as an analogy, come up with another model system that can have a negative effective elastic modulus.

Solution 5.8: Several examples can be found in Shu Zhang's doctoral dissertation. The model below is the unit cell of a periodic structure has a negative effective elastic bulk modulus for acoustic waves.



Problem 5.9 Verify that the ensemble averaged equations

$$\begin{aligned} \langle \varepsilon \rangle &= \varepsilon_0 - \mathsf{S}_x \star \langle S \rangle - \boldsymbol{\mathcal{M}}_x \star \langle \mathbf{m} \rangle \\ \langle \dot{\mathbf{u}} \rangle &= \dot{\mathbf{u}}_0 - \boldsymbol{\mathcal{S}}_t \star \langle S \rangle - \boldsymbol{M}_t \star \langle \mathbf{m} \rangle \ . \end{aligned} \tag{5.95}$$

are correct, i.e., the quantities C_x, $\boldsymbol{\mathcal{M}}_x$, $\boldsymbol{\mathcal{S}}_t$, and $\boldsymbol{M}_t$ can be taken outside the integrals during averaging.

Solution 5.9: Let us start with the solution for the displacement

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}, t) + [\boldsymbol{G} \star (\boldsymbol{\nabla} \cdot \boldsymbol{S} - \dot{\mathbf{m}})](\mathbf{x}, t) \,. \tag{5.96}$$

The first thing to note is that the convolution is over both space and time, i.e., the $\star$ notation indicates both convolution in space-time and convolution in space,

$$\begin{aligned} \boldsymbol{G} \star (\boldsymbol{\nabla} \cdot \boldsymbol{S} - \dot{\mathbf{m}}) &= \int_\Omega \mathrm{d}\mathbf{x}' \left[\boldsymbol{G}(\mathbf{x}' - \mathbf{x}, t) \star (\boldsymbol{\nabla} \cdot \boldsymbol{S} - \dot{\mathbf{m}})(\mathbf{x}', t)\right] \\ &= \int_\Omega \mathrm{d}\mathbf{x}' \left[\int_{-\infty}^t \mathrm{d}\tau \, \boldsymbol{G}(\mathbf{x}' - \mathbf{x}, t - \tau) \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{S} - \dot{\mathbf{m}})(\mathbf{x}', \tau)\right] \,. \end{aligned} \tag{5.97}$$

Consider the term containing the divergence of $\boldsymbol{S}$. We have

$$\boldsymbol{G} \star (\boldsymbol{\nabla} \cdot \boldsymbol{S}) = \int_\Omega \mathrm{d}\mathbf{x}' \left[\int_{-\infty}^t \boldsymbol{G} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{S}) \, \mathrm{d}\tau\right] \,. \tag{5.98}$$

Switching the order of integration gives

$$\boldsymbol{G} \star (\boldsymbol{\nabla} \cdot \boldsymbol{S}) = \int_{-\infty}^t \mathrm{d}\tau \left[\int_\Omega \boldsymbol{G} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{S}) \, \mathrm{d}\mathbf{x}'\right] \,. \tag{5.99}$$

The formula for integration by parts is

$$\int_\Omega \boldsymbol{F} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{G}) \, \mathrm{d}\Omega = \int_\Gamma \mathbf{n} \cdot (\boldsymbol{G} \cdot \boldsymbol{F}^T) \, \mathrm{d}\Gamma - \int_\Omega \boldsymbol{\nabla} \boldsymbol{F} : \boldsymbol{G}^T \, \mathrm{d}\Omega \,. \tag{5.100}$$

With the appropriate choice of boundary conditions, we have

$$\int_\Omega \boldsymbol{F} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{G}) \, \mathrm{d}\Omega = -\int_\Omega \boldsymbol{\nabla} \boldsymbol{F} : \boldsymbol{G}^T \, \mathrm{d}\Omega \,. \tag{5.101}$$

Therefore, from (5.99) and (5.101), and using the symmetry of $\boldsymbol{S}$, we have

$$\boldsymbol{G} \star (\boldsymbol{\nabla} \cdot \boldsymbol{S}) = -\int_{-\infty}^t \mathrm{d}\tau \left[\int_\Omega \boldsymbol{\nabla} \boldsymbol{G} : \boldsymbol{S} \, \mathrm{d}\mathbf{x}'\right] \,. \tag{5.102}$$

Since we are dealing with time derivatives of displacement, we need the quantity

$$\frac{d}{dt}[\boldsymbol{G} \star (\boldsymbol{\nabla} \cdot \boldsymbol{S})] = -\frac{d}{dt}\left[\int_{-\infty}^t \mathrm{d}\tau \left(\int_\Omega \boldsymbol{\nabla} \boldsymbol{G}(\mathbf{x}' - \mathbf{x}, t - \tau) : \boldsymbol{S}(\mathbf{x}', \tau) \, \mathrm{d}\mathbf{x}'\right)\right] \,. \tag{5.103}$$

Ignoring the spatial dependence for now, define

$$F(t) := \int_{-\infty}^t \mathrm{d}\tau \left(\int_\Omega \boldsymbol{\nabla} \boldsymbol{G}(\mathbf{x}' - \mathbf{x}, t - \tau) : \boldsymbol{S}(\mathbf{x}', \tau) \, \mathrm{d}\mathbf{x}'\right) =: \int_{-\infty}^t \mathrm{d}\tau \, f(t, \tau) \tag{5.104}$$

where

$$f(t, \tau) := \int_\Omega \boldsymbol{\nabla} \boldsymbol{G}(\mathbf{x}' - \mathbf{x}, t - \tau) : \boldsymbol{S}(\mathbf{x}', \tau) \, \mathrm{d}\mathbf{x}' \,. \tag{5.105}$$

Then from the generalized Leibniz rule, we have

$$\frac{d}{dt}[F(t)] = f(t,t) + \int_{-\infty}^{t} \mathrm{d}\tau \, \frac{\partial}{\partial t}[f(t,\tau)] \,. \tag{5.106}$$

With the assumption that $\boldsymbol{\nabla}\boldsymbol{G}(\mathbf{x}'-\mathbf{x},0) = 0$, we then have

$$\frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})] = -\int_{-\infty}^{t} \mathrm{d}\tau \frac{\partial}{\partial t}\left(\int_\Omega \boldsymbol{\nabla}\boldsymbol{G}(\mathbf{x}'-\mathbf{x},t-\tau) : \boldsymbol{S}(\mathbf{x}',\tau)\,\mathrm{d}\mathbf{x}'\right) \,. \tag{5.107}$$

Now we can use Reynold's transport theorem to take the time derivative inside the integral over space (noting that the boundary term is zero for linear elastic deformations). Then we have

$$\frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})] = -\int_{-\infty}^{t} \mathrm{d}\tau \left[\int_\Omega \frac{\partial}{\partial t}(\boldsymbol{\nabla}\boldsymbol{G}(\mathbf{x}'-\mathbf{x},t-\tau) : \boldsymbol{S}(\mathbf{x}',\tau)\,\mathrm{d}\mathbf{x}')\right] \,. \tag{5.108}$$

Since $\boldsymbol{S}$ is not a function of t, we have

$$\frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})] = -\int_{-\infty}^{t} \mathrm{d}\tau \left[\int_\Omega \frac{\partial}{\partial t}[\boldsymbol{\nabla}\boldsymbol{G}(\mathbf{x}'-\mathbf{x},t-\tau)] : \boldsymbol{S}(\mathbf{x}',\tau)\,\mathrm{d}\mathbf{x}'\right] \,. \tag{5.109}$$

Switching back the order of integration,

$$\frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})] = -\int_\Omega \mathrm{d}\mathbf{x}' \left[\int_{-\infty}^{t} \frac{\partial}{\partial t}(\boldsymbol{\nabla}\boldsymbol{G}) : \boldsymbol{S}\,\mathrm{d}\tau\right] \,. \tag{5.110}$$

We can define

$$\boldsymbol{\mathcal{S}}_t := \frac{\partial}{\partial t}(\boldsymbol{\nabla}\boldsymbol{G}) \,. \tag{5.111}$$

Alternatively, to make sure that the correct gradient of $\boldsymbol{G}$ is used when (5.110) is written in index notation, it is useful to take advantage of the symmetry of $\boldsymbol{S}$ to define

$$\boldsymbol{\mathcal{S}}_t := \tfrac{1}{2}\left[\frac{\partial}{\partial t}(\boldsymbol{\nabla}\boldsymbol{G} + \boldsymbol{\nabla}\boldsymbol{G}^T)\right] \tag{5.112}$$

where some care is needed in the definition of the transpose of a third order tensor. Then can write (5.110) as

$$\frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})] = -\int_\Omega \mathrm{d}\mathbf{x}' \left[\int_{-\infty}^{t} \boldsymbol{\mathcal{S}}_t(\mathbf{x}'-\mathbf{x},t-\tau) : \boldsymbol{S}(\mathbf{x}',\tau)\,\mathrm{d}\tau\right] \,. \tag{5.113}$$

Reverting back to convolution notation, with the slight inconsistency of notation mentioned above,

$$\frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})] = -\int_\Omega \boldsymbol{\mathcal{S}}_t(\mathbf{x}'-\mathbf{x},t)\star\boldsymbol{S}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}' = -\boldsymbol{\mathcal{S}}_t\star\boldsymbol{S} \,. \tag{5.114}$$

Recall that $\boldsymbol{G}$ is the Green's function for a domain with reference stiffness tensor C_0 and reference density ρ_0. Therefore, the ensemble average of $\boldsymbol{\mathcal{S}}_t$ is independent of the material properties of the domain and we have

$$\left\langle \frac{d}{dt}[\boldsymbol{G}\star(\boldsymbol{\nabla}\cdot\boldsymbol{S})]\right\rangle = -\boldsymbol{\mathcal{S}}_t\star\langle\boldsymbol{S}\rangle \qquad \square$$

A similar process can be used to show that the quantities C_x, $\boldsymbol{\mathcal{M}}_x$, and $\boldsymbol{M}_t$ can be also taken outside the integrals during averaging.

Problem 5.10 Given what you know about the Willis equations, develop a simple model that exhibits coupling between the elastic response and density.

Solution 5.10: Clearly, any translation-invariant periodic structure will show the required coupling. See, for example, Ankit Srivastava and Sia Nemat-Nasser, (2011), "Overall Dynamic Properties of 3-D periodic elastic composites" and Shuvalov, AL and Kutsenko, AA and Norris, AN and Poncelet, O. (2011), "Effective Willis constitutive equations for periodically stratified anisotropic elastic media".

Problem 5.11 Find the relation between the stress-momentum vector and the displacement gradient-velocity vector for the Milton model for the case where $c = 1/2$ and $d = 1/4$. Comment on your findings.

Solution 5.11: Recall that

$$\underline{\underline{\mathsf{u}}}_5 = \frac{1}{2c}\begin{bmatrix} c(1+d) & -(1-d^2) & c(1-d) & (1-d^2) \\ -c^2 & c(1-d) & c^2 & c(1+d) \end{bmatrix} \begin{bmatrix} \underline{\underline{\mathsf{u}}}_2 \\ \underline{\underline{\mathsf{u}}}_4 \end{bmatrix} =: \begin{bmatrix} \underline{\underline{\mathsf{B}}}_1 & \underline{\underline{\mathsf{B}}}_2 \end{bmatrix} \begin{bmatrix} \underline{\underline{\mathsf{u}}}_2 \\ \underline{\underline{\mathsf{u}}}_4 \end{bmatrix} \tag{5.115}$$

Recall also that

$$\begin{aligned} \alpha &= h\left(A_1 + A_1^{-1} + A_2 + A_2^{-1} + A_3 + A_3^{-1} - 6\right) \\ \boldsymbol{\beta}_1 &= A_2\boldsymbol{B}_1 + \boldsymbol{B}_2 - \boldsymbol{1}, \quad \boldsymbol{\beta}_2 = A_2^{-1}\boldsymbol{B}_2 + \boldsymbol{B}_1 - \boldsymbol{1} \\ \boldsymbol{\beta}_3 &= A_2\boldsymbol{B}_2 + \boldsymbol{B}_1 - \boldsymbol{1}, \quad \boldsymbol{\beta}_4 = A_2^{-1}\boldsymbol{B}_1 + \boldsymbol{B}_2 - \boldsymbol{1} \end{aligned} \tag{5.116}$$

where

$$A_1 = e^{-ih(k_1+k_2)}, \quad A_2 = e^{ihk_2}, \quad A_3 = e^{-ih(k_1-k_2)}\,. \tag{5.117}$$

The dispersion relation for the system is

$$\det\left[k\alpha\boldsymbol{1} - \omega^2\left\{m_5\left(\boldsymbol{\beta}_1^T\cdot\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2^T\cdot\boldsymbol{\beta}_2\right) + m_6\left(\boldsymbol{\beta}_3^T\cdot\boldsymbol{\beta}_3 + \boldsymbol{\beta}_4^T\cdot\boldsymbol{\beta}_4\right)\right\}\right] = 0\,. \tag{5.118}$$

We can use Mathematica to compute the determinant above. The resulting expression, for $m_5 = hm$ and $m_6 = -hm + \delta h^2$, $c = 1/2$, and $d = 1/4$ is

$$\begin{aligned} &1280k^2 + \cos(hk_2)\Big[256k^2\Big\{-3 + 2\cos(hk_1)\left[-3 + \{1+\cos(hk_1)\}\cos(hk_2)\right]\Big\} \\ &\quad - 377\delta h\omega^2\Big\{6k - \left[2k + \delta h\omega^2 + 4k\cos(hk_1)\right]\cos(hk_2) + \delta h\omega^2\cos^2(hk_2)\Big\}\sin^2\left(\frac{hk_2}{2}\right)\Big] \\ &\quad + 2\left[-64k^2 + 377(\delta h - 2m)^2\omega^4\sin^4\left(\frac{hk_2}{2}\right)\right]\sin^2(hk_2) = 0 \end{aligned} \tag{5.119}$$

The inertial force at node 4 is

$$\begin{aligned} \mathbf{f}_4^{\text{inertial}} &= \frac{m_5\omega^2}{4c^2h}\left(A_2C_1\,\mathbf{x}_5\otimes\mathbf{c}_1 + C_2\,\mathbf{x}_5\otimes\mathbf{c}_2\right)\cdot\mathbf{u}_4 \\ &\quad + \frac{m_6\omega^2}{4c^2h}\left(C_3\,\mathbf{x}_6\otimes\mathbf{c}_1 + A_2C_1\,\mathbf{x}_6\otimes\mathbf{c}_2\right)\cdot\mathbf{u}_4\,. \end{aligned} \tag{5.120}$$

where

$$C_1 = c^2 + d^2 - 1\,,\;\; C_2 = c^2 + (1-d)^2\,,\;\; C_3 = c^2 + (1+d)^2\,, \tag{5.121}$$

and

$$\begin{aligned} \mathbf{x}_5 &= -ch\,\mathbf{e}_1 - (1+d)h\,\mathbf{e}_2\,,\;\; \mathbf{x}_6 = ch\,\mathbf{e}_1 - (1-d)h\,\mathbf{e}_2\,, \\ \mathbf{c}_1 &= -c\,\mathbf{e}_1 + (1-d)\,\mathbf{e}_2\,,\;\; \mathbf{c}_2 = c\,\mathbf{e}_1 + (1+d)\,\mathbf{e}_2\,. \end{aligned} \tag{5.122}$$

Therefore,

$$\mathbf{x}_5\otimes\mathbf{c}_1 = \begin{bmatrix} c^2h & -c(1-d)h \\ c(1+d)h & -(1-d^2)h \end{bmatrix}, \qquad \mathbf{x}_5\otimes\mathbf{c}_2 = \begin{bmatrix} -c^2h & -c(1+d)h \\ -c(1+d)h & -(1+d)^2h \end{bmatrix} \tag{5.123}$$

$$\mathbf{x}_6\otimes\mathbf{c}_1 = \begin{bmatrix} -c^2h & -c(-1+d)h \\ -c(-1+d)h & -(-1+d)^2h \end{bmatrix}, \qquad \mathbf{x}_6\otimes\mathbf{c}_2 = \begin{bmatrix} c^2h & c(1+d)h \\ c(-1+d)h & (-1+d^2)h \end{bmatrix} \tag{5.124}$$

Plugging these, and the previously defined value of A_2, into the expression for the inertial force gives us

$$\mathbf{f}_4^{\text{inertial}} = \frac{\omega^2}{4c}\begin{bmatrix} (c^2+d^2-1)e^{ihk_2}m_5[cu_{4x}-(1-d)u_{4y}] \\ -[c^2+(1+d)^2]m_6[cu_{4x}-(1-d)u_{4y}] \\ -[c^2+(1-d)^2]m_5[cu_{4x}+(1+d)u_{4y}] \\ +(c^2+d^2-1)e^{ihk_2}m_6[cu_{4x}+(1+d)u_{4y}] \\ \\ c^3(e^{ihk_2}-1)[(1+d)m_5-(1-d)m_6]u_{4x} \\ +c(d^2-1)\left[(1-d)m_5-(1+d)m_6+e^{ihk_2}[(1+d)m_5-(1-d)m_6]\right]u_{4x} \\ +(1-d^2)^2(-1+e^{ihk_2})(m_5+m_6)u_{4y} \\ +c^2\left[-(1+d)^2m_5-(-1+d)^2m_6+(-1+d^2)e^{ihk_2}(m_5+m_6)\right]u_{4y} \end{bmatrix}. \tag{5.125}$$

In the limit $h \to 0$, with $u_1 = u_{4x}$, $u_2 = u_{4y}$, $m_5 = hm$, and $m_6 = -hm + \delta h^2$, we have

$$\mathbf{t}_4^{\text{inertial}} = -\frac{1}{2h}\mathbf{f}_4^{\text{inertial}} = \frac{m\omega^2}{2}\begin{bmatrix} cu_2 - du_1 \\ \dfrac{(1-d^2)u_1 + cdu_2}{c} \end{bmatrix}. \tag{5.126}$$

The effective stress in the structure due to these tractions is related to the traction components by

$$\mathbf{t}_4^{\text{inertial}} = \begin{bmatrix}\sigma_{21}\\ \sigma_{22}\end{bmatrix} = \frac{m\omega^2}{2}\begin{bmatrix} -du_1 + cu_2 \\ \dfrac{1-d^2}{c}u_1 + du_2 \end{bmatrix}. \tag{5.127}$$

The components σ_{11} and σ_{12} are zero. Then, we can write the stress tensor as

$$\boldsymbol{\sigma}^I = \begin{bmatrix}\sigma_{11}^I\\ \sigma_{21}^I\\ \sigma_{12}^I\\ \sigma_{22}^I\end{bmatrix} = \frac{m\omega^2}{2c}\begin{bmatrix} 0 \\ -cdu_1 + c^2u_2 \\ 0 \\ (1-d^2)u_1 + cdu_2 \end{bmatrix} = \frac{m\omega^2}{2c}\begin{bmatrix} 0 & 0 \\ -cd & c^2 \\ 0 & 0 \\ (1-d^2) & cd \end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}. \tag{5.128}$$

Noting that $v_1 = -i\omega u_1$ and $v_2 = -i\omega u_2$, we have

$$\boldsymbol{\sigma}^I = \begin{bmatrix}\sigma_{11}^I\\ \sigma_{21}^I\\ \sigma_{12}^I\\ \sigma_{22}^I\end{bmatrix} = \frac{i\omega m}{2c}\begin{bmatrix} 0 & 0 \\ -cd & c^2 \\ 0 & 0 \\ (1-d^2) & cd \end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}. \tag{5.129}$$

Let us now examine the force at node 4 due to the elastic springs, given by

$$\mathbf{f}_4^{\text{elastic}} = hk\left[(1-A_3)\boldsymbol{1} + (A_3 - A_1)\boldsymbol{D}_{41} + (1-A_2)\boldsymbol{D}_{42}\right]\cdot\mathbf{u}_4 \tag{5.130}$$

where, with $\theta_{41} = 5\pi/4$ and $\theta_{42} = 3\pi/2$, we have

$$\begin{aligned} \boldsymbol{D}_{41} &= \begin{bmatrix}\cos^2\theta_{41} & \cos\theta_{41}\sin\theta_{41}\\ \cos\theta_{41}\sin\theta_{41} & \sin^2\theta_{41}\end{bmatrix} = \begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix} \\ \boldsymbol{D}_{42} &= \begin{bmatrix}\cos^2\theta_{42} & \cos\theta_{42}\sin\theta_{42}\\ \cos\theta_{42}\sin\theta_{42} & \sin^2\theta_{42}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}. \end{aligned} \tag{5.131}$$

The elastic force is independent of c and d and has the form

$$\mathbf{f}_4^{\text{elastic}} = hk\begin{bmatrix} \{1 - e^{-ihk_1}\cos(hk_2)\}u_{4x} + i\,e^{-ihk_1}\sin(hk_2)u_{4y} \\ ie^{-ihk_1}\sin(hk_2)u_{4x} + \left\{2 - e^{ihk_2} - e^{-ihk_1}\cos(hk_2)\right\}u_{4y} \end{bmatrix}. \tag{5.132}$$

The traction due to the elastic force is

$$\mathbf{t}_4^{\text{elastic}} = \frac{1}{2h}\mathbf{f}_4^{\text{elastic}} = \frac{k}{2}\begin{bmatrix} \{1 - e^{-ihk_1}\cos(hk_2)\}u_{4x} + i\, e^{-ihk_1}\sin(hk_2)u_{4y} \\ ie^{-ihk_1}\sin(hk_2)u_{4x} + \left\{2 - e^{ihk_2} - e^{-ihk_1}\cos(hk_2)\right\}u_{4y} \end{bmatrix} . \tag{5.133}$$

Following the approach in the text, we have

$$\boldsymbol{\sigma}^E = \begin{bmatrix} \sigma_{11}^E \\ \sigma_{21}^E \\ \sigma_{12}^E \\ \sigma_{22}^E \end{bmatrix} = \frac{hk}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{bmatrix} . \tag{5.134}$$

Combining the inertial and elastic components of the stress,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} hk & 0 & 0 & hk & 0 & 0 \\ 0 & hk & hk & 0 & -i\omega md & i\omega mc \\ 0 & hk & hk & 0 & 0 & 0 \\ hk & 0 & 0 & 3hk & \dfrac{i\omega m(1-d^2)}{c} & i\omega md \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \\ v_1 \\ v_2 \end{bmatrix} . \tag{5.135}$$

The linear momentum density in the unit cell is

$$\mathbf{p} = -\frac{i\omega}{2h^2}\left[m_5(A_2\boldsymbol{B}_1 + \boldsymbol{B}_2)\cdot\mathbf{u}_4 + m_6(A_2\boldsymbol{B}_2 + \boldsymbol{B}_1)\cdot\mathbf{u}_4\right] . \tag{5.136}$$

In matrix form, with $m_5 = hm$ and $m_6 = -hm + \delta h^2$,

$$\mathbf{p} = -\frac{i\omega}{4h}\begin{bmatrix} \left[(1+d)\delta h - 2dm + e^{ihk_2}\left\{\delta(1-d)h + 2dm\right\}\right]u_{4x} + \frac{1-d^2}{c}(e^{ihk_2}-1)(\delta h - 2m)\,u_{4y} \\ c(e^{ihk_2}-1)(\delta h - 2m)\,u_{4x} + \left[(1-d)\delta h + 2dm + e^{ihk_2}\left\{\delta(1+d)h - 2dm\right\}\right]u_{4y} \end{bmatrix} . \tag{5.137}$$

In the limit $h \to 0$, with $\{u_{4x}, u_{4y}\} = \{u_1, u_2\}$, we have

$$\mathbf{p} = \frac{-i\omega}{2c}\begin{bmatrix} c(\delta + idk_2m)u_1 - i(1-d^2)k_2mu_2 \\ -c\left[ick_2mu_1 + (-\delta + idk_2m)u_2\right] \end{bmatrix} . \tag{5.138}$$

Recall that for plane waves, $u_{1,2} = ik_2u_1$ and $u_{2,2} = ik_2u_2$. Then we can write,

$$\mathbf{p} = \frac{-i\omega}{2c}\begin{bmatrix} c\delta u_1 + dmu_{1,2} - (1-d^2)mu_{2,2} \\ -c\left[cmu_{1,2} - \delta u_2 + dmu_{2,2}\right] \end{bmatrix} . \tag{5.139}$$

Now, $v_1 = -i\omega u_1$ and $v_2 = -i\omega u_2$. Therefore

$$\mathbf{p} = \frac{1}{2c}\begin{bmatrix} c\delta v_1 - i\omega cdmu_{1,2} + i\omega(1-d^2)mu_{2,2} \\ c\left[i\omega cmu_{1,2} + \delta v_2 + i\omega dmu_{2,2}\right] \end{bmatrix} . \tag{5.140}$$

In the matrix form we seek,

$$\mathbf{p} = \frac{1}{2c}\begin{bmatrix} 0 & -i\omega cdm & 0 & i\omega(1-d^2)m & c\delta & 0 \\ 0 & i\omega c^2m & 0 & i\omega cdm & 0 & c\delta \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \\ v_1 \\ v_2 \end{bmatrix} . \tag{5.141}$$

Combining (5.135) and (5.141), we have

$$\begin{bmatrix}\boldsymbol{\sigma}\\ \mathbf{p}\end{bmatrix}=\begin{bmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{12}\\ \sigma_{22}\\ p_1\\ p_2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}hk & 0 & 0 & hk & 0 & 0\\ 0 & hk & hk & 0 & -i\omega md & i\omega mc\\ 0 & hk & hk & 0 & 0 & 0\\ hk & 0 & 0 & 3hk & \dfrac{i\omega m(1-d^2)}{c} & i\omega md\\ 0 & -i\omega md & 0 & \dfrac{i\omega m(1-d^2)}{c} & \delta & 0\\ 0 & i\omega mc & 0 & i\omega md & 0 & \delta\end{bmatrix}\begin{bmatrix}u_{1,1}\\ u_{1,2}\\ u_{2,1}\\ u_{2,2}\\ v_1\\ v_2\end{bmatrix}. \quad (5.142)$$

For the situation where $c = 1/2$ and $d = 1/4$,

$$\begin{bmatrix}\boldsymbol{\sigma}\\ \mathbf{p}\end{bmatrix}=\begin{bmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{12}\\ \sigma_{22}\\ p_1\\ p_2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}hk & 0 & 0 & hk & 0 & 0\\ 0 & hk & hk & 0 & -\dfrac{i\omega m}{4} & \dfrac{i\omega m}{2}\\ 0 & hk & hk & 0 & 0 & 0\\ hk & 0 & 0 & 3hk & \dfrac{15i\omega m}{8} & \dfrac{i\omega m}{4}\\ 0 & -\dfrac{i\omega m}{4} & 0 & \dfrac{15i\omega m}{8} & \delta & 0\\ 0 & \dfrac{i\omega m}{2} & 0 & \dfrac{i\omega m}{4} & 0 & \delta\end{bmatrix}\begin{bmatrix}u_{1,1}\\ u_{1,2}\\ u_{2,1}\\ u_{2,2}\\ v_1\\ v_2\end{bmatrix}. \quad (5.143)$$

Problem 5.12 Show that the wave equation for pentamode materials can be expressed as

$$K\boldsymbol{S} : \boldsymbol{\nabla}[\boldsymbol{\rho}^{-1} \cdot (\boldsymbol{S} \cdot \boldsymbol{\nabla} p)] - \ddot{p} = 0\,. \tag{5.144}$$

For a plane harmonic wave with wave vector $\mathbf{k}$ incident upon a pentamode material, show that

$$[(\boldsymbol{S} \cdot \mathbf{k}) \otimes (\boldsymbol{S} \cdot \mathbf{k})] \cdot \mathbf{u} - \frac{\omega^2}{K}\boldsymbol{\rho} \cdot \mathbf{u} = \mathbf{0}\,. \tag{5.145}$$

Solution 5.12: Recall that the balance of momentum for a material with anisotropic density can be written as

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{\rho} \cdot \dot{\mathbf{v}}\,. \tag{5.146}$$

For a pentamode material

$$\boldsymbol{\sigma} = -p\,\boldsymbol{S} \tag{5.147}$$

where p is the pseudo-pressure which satisfies the relation

$$\dot{p} = -K(\boldsymbol{S} : \boldsymbol{\nabla}\mathbf{v}) \qquad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{S} = 0\,. \tag{5.148}$$

If we plug in the expression for $\boldsymbol{\sigma}$ for a pentamode material into the momentum balance equation we have

$$-\boldsymbol{\nabla} \cdot p\,\boldsymbol{S} = \boldsymbol{\rho} \cdot \dot{\mathbf{v}}\,. \tag{5.149}$$

In index notation

$$-\frac{\partial}{\partial x_j}(pS_{jk}) = \rho_{k\ell}\dot{v}_\ell\,. \tag{5.150}$$

Expanded out

$$-\left[\frac{\partial p}{\partial x_j} S_{jk} + p\,\frac{\partial S_{jk}}{\partial x_j}\right] = \rho_{k\ell}\dot{v}_\ell\,. \tag{5.151}$$

In direct notation

$$-\left[\boldsymbol{\nabla} p \cdot \boldsymbol{S} + p\,\boldsymbol{\nabla} \cdot \boldsymbol{S}\right] = \boldsymbol{\rho} \cdot \dot{\mathbf{v}}\,. \tag{5.152}$$

Since $\boldsymbol{\nabla} \cdot \boldsymbol{S} = 0$, the above simplifies to

$$-\boldsymbol{\nabla} p \cdot \boldsymbol{S} = \boldsymbol{\rho} \cdot \dot{\mathbf{v}}\,. \tag{5.153}$$

Assuming that the density tensor is invertible, we can write

$$-\boldsymbol{\rho}^{-1} \cdot (\boldsymbol{\nabla} p \cdot \boldsymbol{S}) = \dot{\mathbf{v}}\,. \tag{5.154}$$

From the symmetry of $\boldsymbol{S}$ we have

$$-\boldsymbol{\rho}^{-1} \cdot (\boldsymbol{S} \cdot \boldsymbol{\nabla} p) = \dot{\mathbf{v}}\,. \tag{5.155}$$

Taking the time derivative of the relation for the pseudo-pressure gives

$$\ddot{p} = -K(\boldsymbol{S} : \boldsymbol{\nabla}\dot{\mathbf{v}})\,. \tag{5.156}$$

Plug in the expression for $\dot{\mathbf{v}}$ to get the required relation

$$\ddot{p} = K\,\boldsymbol{S} : \boldsymbol{\nabla}\left[\boldsymbol{\rho}^{-1} \cdot (\boldsymbol{S} \cdot \boldsymbol{\nabla} p)\right] \qquad \square \tag{5.157}$$

For the plane wave equations, we once again start with the momentum balance equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{\rho} \cdot \dot{\mathbf{v}} \qquad \text{where} \quad \boldsymbol{\sigma} = K(\boldsymbol{S} : \boldsymbol{\nabla}\mathbf{u})\boldsymbol{S}\,. \tag{5.158}$$

If the material is homogeneous, K and $\boldsymbol{S}$ are constant, and we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = K\boldsymbol{\nabla}\cdot[(\boldsymbol{S}:\boldsymbol{\nabla}\mathbf{u})\boldsymbol{S}] = K[\boldsymbol{S}:\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{u})]\cdot\boldsymbol{S}\,. \tag{5.159}$$

In index notation,

$$\frac{\partial}{\partial x_j}(\sigma_{jk}) = K\,S_{mn}\,\frac{\partial^2 u_m}{\partial x_n \partial x_j}\,S_{jk}\,. \tag{5.160}$$

For plane waves with

$$\mathbf{u} = \widehat{\mathbf{u}}\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \tag{5.161}$$

we have

$$\dot{\mathbf{v}} = \ddot{\mathbf{u}} = -\omega^2\widehat{\mathbf{u}}\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]\,, \quad \boldsymbol{\nabla}\mathbf{u} = i\,\widehat{\mathbf{u}}\otimes\mathbf{k}\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]\,. \tag{5.162}$$

and

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{u}) = -\widehat{\mathbf{u}}\otimes(\mathbf{k}\otimes\mathbf{k})\,\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]\,. \tag{5.163}$$

In index notation

$$\frac{\partial^2 u_m}{\partial x_n \partial x_j} = -\widehat{u}_m k_n k_j\,\exp[i(k_\ell x_\ell - \omega t)]\,. \tag{5.164}$$

Therefore

$$\frac{\partial}{\partial x_j}(\sigma_{jk}) = -K\,S_{mn}\,(\widehat{u}_m k_n k_j)\,S_{jk}\,\exp[i(k_\ell x_\ell - \omega t)]\,. \tag{5.165}$$

Plugging into the momentum equation gives

$$-K\,S_{mn}\,(\widehat{u}_m k_n k_j)\,S_{jk}\,\exp[i(k_\ell x_\ell - \omega t)] = -\omega^2\,\rho_{kp}\widehat{u}_p\,\exp[i(k_\ell x_\ell - \omega t)] \tag{5.166}$$

or

$$K\,(S_{jk}k_j)\,(S_{mn}k_n)\widehat{u}_m = \omega^2\rho_{kp}\widehat{u}_p \tag{5.167}$$

In direct notation, using the symmetry of $\boldsymbol{S}$, we have

$$[(\boldsymbol{S}\cdot\mathbf{k})\otimes(\boldsymbol{S}\cdot\mathbf{k})]\cdot\widehat{\mathbf{u}} - \frac{\omega^2}{K}\,\boldsymbol{\rho}\cdot\widehat{\mathbf{u}} = \mathbf{0} \qquad \square \tag{5.168}$$

Chapter 6

Solutions for Exercises in Chapter 6

Problem 6.1 Use curvilinear coordinates to derive the transformation rules for electrical conductivity and the current:

$$\boldsymbol{\sigma}(\mathbf{x}) = \frac{1}{J}\,\boldsymbol{F}(\mathbf{X}) \cdot \boldsymbol{\Sigma}(\mathbf{X}) \cdot \boldsymbol{F}^T(\mathbf{X})$$

$$\mathbf{j}(\mathbf{x}) = \frac{\boldsymbol{F} \cdot \mathbf{J}(\mathbf{X})}{\det(\boldsymbol{F})}\,.$$

Solution - 6.1 : Let the natural basis for the reference state, **X**, be $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3)$ and let the coordinate curves be $(\Theta^1, \Theta^2, \Theta^3)$. For the transformed state, **x**, the natural basis is $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ and the coordinate curves are $(\theta^1, \theta^2, \theta^3)$. Assuming a sufficient degree of smoothness in the transformation, we have

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \theta^i} = \frac{\partial \mathbf{X}}{\partial \Theta^j}\,\frac{\partial \Theta^j}{\partial \theta^i} = \frac{\partial \Theta^j}{\partial \theta^i}\,\mathbf{G}_j\,. \tag{6.1}$$

The inverse process can be used to get the relationship

$$\mathbf{G}_i = \frac{\partial \theta^j}{\partial \Theta^i}\,\mathbf{g}_j\,. \tag{6.2}$$

The transformation rule for general vectors can be derived from the above relation between the basis vectors. Let,

$$\mathbf{u} = U^i \mathbf{G}_i = u^j \mathbf{g}_j\,. \tag{6.3}$$

Then,

$$U^i \mathbf{G}_i = U^i\,\frac{\partial \theta^j}{\partial \Theta^i}\,\mathbf{g}_j =: u^j \mathbf{g}_j\,. \tag{6.4}$$

Therefore,

$$u^j = \frac{\partial \theta^j}{\partial \Theta^i}\,U^i\,. \tag{6.5}$$

For a second order tensor, $\boldsymbol{\Sigma}$, we can write

$$\boldsymbol{\Sigma} = \Sigma^{k\ell}\,\mathbf{G}_k \otimes \mathbf{G}_\ell = \sigma^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j\,. \tag{6.6}$$

Using the transformation rule for $\mathbf{G}_i$, we have

$$\Sigma^{ij}\,\mathbf{G}_i \otimes \mathbf{G}_j = \Sigma^{ij}\left(\frac{\partial \theta^k}{\partial \Theta^i}\,\mathbf{g}_k\right) \otimes \left(\frac{\partial \theta^\ell}{\partial \Theta^j}\,\mathbf{g}_\ell\right) = \frac{\partial \theta^\ell}{\partial \Theta^j}\,\Sigma^{ij}\,\frac{\partial \theta^k}{\partial \Theta^i}\,\mathbf{g}_k \otimes \mathbf{g}_\ell\,. \tag{6.7}$$

Therefore,

$$\sigma^{k\ell} = \frac{\partial \theta^\ell}{\partial \Theta^j}\,\Sigma^{ij}\,\frac{\partial \theta^k}{\partial \Theta^i}\,. \tag{6.8}$$

Let us now consider the gradient of a scalar function Φ. Then, from the definition of the gradient in curvilinear coordinates

$$\boldsymbol{\nabla}_{\mathbf{X}}\,\Phi = \frac{\partial\Phi}{\partial\mathbf{X}} = \frac{\partial\Phi}{\partial\Theta^j}\,\mathbf{G}^j = \frac{\partial\phi}{\partial\theta^m}\,\frac{\partial\theta^m}{\partial\Theta^j}\,G^{jk}\,\mathbf{G}_k \tag{6.9}$$

where $G^{jk} = \mathbf{G}^j\cdot\mathbf{G}^k$. Then

$$\boldsymbol{\nabla}_{\mathbf{X}}\,\Phi = \frac{\partial\phi}{\partial\theta^m}\,\frac{\partial\theta^m}{\partial\Theta^j}\,G^{jk}\,\frac{\partial\theta^\ell}{\partial\Theta^k}\,\mathbf{g}_\ell = \frac{\partial\phi}{\partial\theta^m}\,\frac{\partial\theta^m}{\partial\Theta^j}\,G^{jk}\,\frac{\partial\theta^\ell}{\partial\Theta^k}\,g_{\ell n}\,\mathbf{g}^n\,. \tag{6.10}$$

We can now examine the expression for the power dissipated,

$$W = \int_\Omega \boldsymbol{\nabla}_{\mathbf{X}}\,\Phi\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\nabla}_{\mathbf{X}}\,\Phi\,\mathrm{d}\Omega\,. \tag{6.11}$$

The transformation rule for infinitesimal volumes is given by

$$\mathrm{d}\Omega = \sqrt{G}\,\mathrm{d}\Theta^1\,\mathrm{d}\Theta^2\,\mathrm{d}\Theta^3 = \sqrt{g}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3 \tag{6.12}$$

where $G = \det(G_{ij})$ and $g = \det(g_{ij})$. Recall that, $G_{ij} = \mathbf{G}_i\cdot\mathbf{G}_j$ and $g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j$. Therefore,

$$\begin{aligned} W &= \int_{\Omega_x} \frac{\sqrt{G}}{\sqrt{g}}\,\boldsymbol{\nabla}_{\mathbf{X}}\,\Phi\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\nabla}_{\mathbf{X}}\,\Phi\,\mathrm{d}\Omega_x \\ &= \int_{\Omega_x} \frac{\sqrt{G}}{\sqrt{g}}\left(\frac{\partial\phi}{\partial\theta^m}\,\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\,g_{q\alpha}\,\mathbf{g}^\alpha\right)\cdot\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\,\mathbf{g}_k\otimes\mathbf{g}_\ell\right) \\ &\qquad\cdot\left(\frac{\partial\phi}{\partial\theta^r}\,\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\,g_{u\beta}\,\mathbf{g}^\beta\right)\,\mathrm{d}\Omega_x \end{aligned}$$

or,

$$W = \int_{\Omega_x} \frac{\sqrt{G}}{\sqrt{g}}\left(\frac{\partial\phi}{\partial\theta^m}\,\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\,g_{qk}\right)\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\right)\left(\frac{\partial\phi}{\partial\theta^r}\,\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\,g_{u\ell}\right)\,\mathrm{d}\Omega_x \tag{6.13}$$

where we have used

$$\mathbf{g}^\alpha\cdot(\mathbf{g}_k\otimes\mathbf{g}_\ell)\cdot\mathbf{g}^\beta = (\mathbf{g}^\alpha\cdot\mathbf{g}_k)(\mathbf{g}_\ell\cdot\mathbf{g}^\beta) = \delta^\alpha_k\,\delta^\beta_\ell\,.$$

The transformation has the map

$$\begin{aligned} \boldsymbol{F} = \frac{\partial\mathbf{x}}{\partial\mathbf{X}} &= \frac{\partial\mathbf{x}}{\partial\Theta^i}\otimes\mathbf{G}^i = \frac{\partial\theta^k}{\partial\Theta^i}\,\frac{\partial\mathbf{x}}{\partial\theta^k}\otimes\mathbf{G}^i = \frac{\partial\theta^k}{\partial\Theta^i}\,\mathbf{g}_k\otimes\mathbf{G}^i \\ &= \frac{\partial\theta^k}{\partial\Theta^i}\,G^{ij}\,\mathbf{g}_k\otimes\mathbf{G}_j = \frac{\partial\theta^k}{\partial\Theta^i}\,G^{ij}\,\frac{\partial\theta^m}{\partial\Theta^j}\,\mathbf{g}_k\otimes\mathbf{g}_m\,. \end{aligned}$$

Therefore,

$$\boldsymbol{F}^T = \frac{\partial\theta^m}{\partial\Theta^i}\,G^{ij}\,\frac{\partial\theta^k}{\partial\Theta^j}\,\mathbf{g}_k\otimes\mathbf{g}_m\,.$$

Using these relations, we have

$$\begin{aligned} \boldsymbol{F}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{F}^T &= \left(\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\,\mathbf{g}_m\otimes\mathbf{g}_q\right)\cdot\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\,\mathbf{g}_k\otimes\mathbf{g}_\ell\right)\cdot\left(\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\,\mathbf{g}_u\otimes\mathbf{g}_r\right) \\ &= \left(\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\right)\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\right)\left(\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\right)g_{qk}\,g_{\ell u}\,\mathbf{g}_m\otimes\mathbf{g}_r\,. \end{aligned}$$

Furthermore,

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathbf{x}}\,\phi\cdot(\boldsymbol{F}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{F}^T)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\,\phi = &\left(\frac{\partial\phi}{\partial\theta^\alpha}\,\mathbf{g}^\alpha\right)\cdot \\
&\left[\left(\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\right)\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\right)\left(\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\right)g_{qk}\,g_{\ell u}\,\mathbf{g}_m\otimes\mathbf{g}_r\right] \\
&\cdot\left(\frac{\partial\phi}{\partial\theta^\beta}\,\mathbf{g}^\beta\right) \\
= &\left(\frac{\partial\phi}{\partial\theta^\alpha}\,\delta^\alpha_m\right)\left[\left(\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\right)\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\right)\left(\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\right)g_{qk}\,g_{\ell u}\right]\left(\frac{\partial\phi}{\partial\theta^\beta}\,\delta^\beta_r\right) \\
= &\frac{\partial\phi}{\partial\theta^m}\left[\left(\frac{\partial\theta^m}{\partial\Theta^n}\,G^{np}\,\frac{\partial\theta^q}{\partial\Theta^p}\right)\left(\frac{\partial\theta^\ell}{\partial\Theta^j}\,\Sigma^{ij}\,\frac{\partial\theta^k}{\partial\Theta^i}\right)\left(\frac{\partial\theta^r}{\partial\Theta^s}\,G^{st}\,\frac{\partial\theta^u}{\partial\Theta^t}\right)g_{qk}\,g_{\ell u}\right]\frac{\partial\phi}{\partial\theta^r}\,.
\end{aligned}$$

Comparing the above with (6.13), we see that

$$W = \int_{\Omega_x}\frac{\sqrt{G}}{\sqrt{g}}\,\boldsymbol{\nabla}_{\mathbf{x}}\,\phi\cdot(\boldsymbol{F}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{F}^T)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\,\phi\,\mathrm{d}\Omega_x\,. \tag{6.14}$$

Therefore, in the transformed coordinates, the functional $W(\phi)$ takes the form

$$W = \int_{\Omega_x}\boldsymbol{\nabla}_{\mathbf{x}}\,\phi\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\,\phi\,\mathrm{d}\Omega_x \tag{6.15}$$

with

$$\boldsymbol{\sigma} = \frac{\sqrt{G}}{\sqrt{g}}\boldsymbol{F}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{F}^T\,. \tag{6.16}$$

Noting that

$$\mathbf{g}_i = \boldsymbol{F}\cdot\boldsymbol{G}_i\,,\;\; g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j\,,\;\; G_{ij} = \boldsymbol{G}_i\cdot\boldsymbol{G}_j\,,\;\; g = \det[g_{ij}]\,,\;\; G = \det[G_{ij}] \tag{6.17}$$

we can show that

$$(\det\boldsymbol{F}^T)\,(\det\boldsymbol{F}) = J^2 = \frac{g}{G} \qquad\implies J = \sqrt{\frac{g}{G}}\,. \tag{6.18}$$

Therefore,

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{F}^T \qquad \square \tag{6.19}$$

The transformation rule for currents can be obtained using a similar process.

Problem 6.2 The Greenleaf-Lassas-Uhlman map has the form

$$\mathbf{x}(\mathbf{X}) = \begin{cases} \left(\cfrac{\|\mathbf{X}\|}{2} + 1\right) \cfrac{\mathbf{X}}{\|\mathbf{X}\|} & \text{if } \|\mathbf{X}\| < 2 \\ \mathbf{X} & \text{if } \|\mathbf{X}\| > 2 \,. \end{cases}$$

Find the expression for the electrical conductivity as a function of location for this map.

Solution - 6.2 : THE GLU map expressed in index notation w.r.t. a Cartesian basis is

$$x_i = \begin{cases} \left(\cfrac{\sqrt{X_p X_p}}{2} + 1\right) \cfrac{X_i}{\sqrt{X_p X_p}} & \text{if } \sqrt{X_p X_p} < 2 \\ X_i & \text{if } \sqrt{X_p X_p} > 2 \,. \end{cases} \tag{6.20}$$

Let us first find the quantity $\boldsymbol{F}$. In index notation

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{cases} \cfrac{\partial}{\partial X_j}\left[\cfrac{X_i}{2} + \cfrac{X_i}{\sqrt{X_p X_p}}\right] & \text{if } \sqrt{X_p X_p} < 2 \\ \cfrac{\partial X_i}{\partial X_j} & \text{if } \sqrt{X_p X_p} > 2 \,. \end{cases} \tag{6.21}$$

Now,

$$\frac{\partial X_i}{\partial X_j} = \delta_{ij} \,. \tag{6.22}$$

Also,

$$\frac{\partial}{\partial X_j}\left(\frac{X_i}{\sqrt{X_p X_p}}\right) = \frac{1}{\sqrt{X_p X_p}}\delta_{ij} + X_i \left(-\tfrac{1}{2}\,(X_p X_p)^{-3/2}\right)(2X_j) \tag{6.23}$$

Therefore,

$$F_{ij} = \begin{cases} \left(\tfrac{1}{2} + \cfrac{1}{\sqrt{X_p X_p}}\right)\delta_{ij} - \left(\cfrac{1}{\sqrt{X_p X_p}}\right)^3 X_i X_j & \text{if } \sqrt{X_p X_p} < 2 \\ \delta_{ij} & \text{if } \sqrt{X_p X_p} > 2 \,. \end{cases} \tag{6.24}$$

Reverting back to direct notation,

$$\boldsymbol{F} = \begin{cases} \left(\tfrac{1}{2} + \cfrac{1}{\|\mathbf{X}\|}\right)\boldsymbol{1} - \cfrac{1}{\|\mathbf{X}\|^3}\mathbf{X}\otimes\mathbf{X} & \text{if } \|\mathbf{X}\| < 2 \\ \boldsymbol{1} & \text{if } \|\mathbf{X}\| > 2 \,. \end{cases} \tag{6.25}$$

Therefore,

$$J = \det(\boldsymbol{F}) = \begin{cases} \left(\tfrac{1}{2} + \cfrac{1}{\|\mathbf{X}\|}\right)^3 - \cfrac{1}{\|\mathbf{X}\|^9}\det(\mathbf{X}\otimes\mathbf{X}) & \text{if } \|\mathbf{X}\| < 2 \\ 1 & \text{if } \|\mathbf{X}\| > 2 \,. \end{cases} \tag{6.26}$$

We can use these to find the expression for $\boldsymbol{\sigma}$ in terms of $\boldsymbol{\Sigma}$. The expression simplifies considerably if $\boldsymbol{\Sigma}$ is isotropic. A plot of the deformed configuration can easily be drawn in 2D. □

Problem 6.3 The space folding map of Leonhardt and Pendry can be expressed as

$$\mathbf{x} = \begin{cases} \mathbf{X} & \text{if } X_1 < 0 \\ (-X_1, X_2, X_3) & \text{if } d > X_1 > 0 \\ \mathbf{X} - (2d, 0, 0) & \text{if } X_1 > d\,. \end{cases}$$

Find the expression for the electrical conductivity as a function of location for this map.

Solution - 6.3 : Expressed in index notation, the space folding map is

$$x_i \mathbf{e}_i = \begin{cases} X_I \mathbf{E}_I & \text{if } X_1 < 0 \\ -X_1 \mathbf{E}_1 + X_2 \mathbf{E}_2 + X_3 \mathbf{E}_3 & \text{if } d > X_1 > 0 \\ (X_1 - 2d)\mathbf{E}_1 + X_2 \mathbf{E}_2 + X_3 \mathbf{E}_3 & \text{if } X_1 > d\,. \end{cases} \tag{6.27}$$

The deformation gradient $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = F_{iJ}\, \mathbf{e}_i \otimes \mathbf{E}_J = \frac{\partial x_i}{\partial X_J} \mathbf{e}_i \otimes \mathbf{E}_J \tag{6.28}$$

Therefore,

$$\boldsymbol{F} = F_{iJ}\, \mathbf{e}_i \otimes \mathbf{E}_J = \begin{cases} \dfrac{\partial X_I}{\partial X_J} \mathbf{E}_I \otimes \mathbf{E}_J & \text{if } X_1 < 0 \\ -2\dfrac{\partial X_1}{\partial X_1} \mathbf{E}_1 \otimes \mathbf{E}_1 + \dfrac{\partial X_I}{\partial X_J} \mathbf{E}_I \otimes \mathbf{E}_J & \text{if } d > X_1 > 0 \\ -2d\dfrac{\partial X_1}{\partial X_1} \mathbf{E}_1 \otimes \mathbf{E}_1 + \dfrac{\partial X_I}{\partial X_J} \mathbf{E}_I \otimes \mathbf{E}_J & \text{if } X_1 > d \end{cases} \tag{6.29}$$

or,

$$\boldsymbol{F} = \begin{cases} \delta_{IJ}\, \mathbf{E}_I \otimes \mathbf{E}_J & \text{if } X_1 < 0 \\ -2\, \mathbf{E}_1 \otimes \mathbf{E}_1 + \delta_{IJ}\, \mathbf{E}_I \otimes \mathbf{E}_J & \text{if } d > X_1 > 0 \\ -2d\, \mathbf{E}_1 \otimes \mathbf{E}_1 + \delta_{IJ}\, \mathbf{E}_I \otimes \mathbf{E}_J & \text{if } X_1 > d \end{cases} \tag{6.30}$$

or,

$$\boldsymbol{F} = \begin{cases} \mathbf{E}_I \otimes \mathbf{E}_I & \text{if } X_1 < 0 \\ -2\, \mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_I \otimes \mathbf{E}_I & \text{if } d > X_1 > 0 \\ -2d\, \mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_I \otimes \mathbf{E}_I & \text{if } X_1 > d\,. \end{cases} \tag{6.31}$$

Therefore, $J = \det \boldsymbol{F}$ is given by

$$J = \begin{cases} 1 & \text{if } X_1 < 0 \\ -1 & \text{if } d > X_1 > 0 \\ 1 - 2d & \text{if } X_1 > d\,. \end{cases} \tag{6.32}$$

The transformation equation for the conductivity is

$$\boldsymbol{\sigma} = \frac{1}{J} \boldsymbol{F} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{F}^T\,. \tag{6.33}$$

If we express $\boldsymbol{\Sigma}$ in matrix form as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{6.34}$$

we have, using the above expressions for $\boldsymbol{F}$,

$$\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma} & \text{if } X_1 < 0 \\ \begin{bmatrix} -\sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ \sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{bmatrix} & \text{if } d > X_1 > 0 \\ \begin{bmatrix} (1-2d)\sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}/(1-2d) & \sigma_{23}/(1-2d) \\ \sigma_{31} & \sigma_{32}/(1-2d) & \sigma_{33}/(1-2d) \end{bmatrix} & \text{if } X_1 > d\,. \end{cases} \tag{6.35}$$

Problem 6.4 Use curvilinear coodinates to verify the transformation rules for the magnetic permeability and the permittivity:

$$\boldsymbol{\mu}'(\mathbf{x}) = \frac{\boldsymbol{F} \cdot \boldsymbol{\mu}(\mathbf{X}) \cdot \boldsymbol{F}^T}{\det(\boldsymbol{F})} \,; \;\; \boldsymbol{\varepsilon}'(\mathbf{x}) = \frac{\boldsymbol{F} \cdot \boldsymbol{\varepsilon}(\mathbf{X}) \cdot \boldsymbol{F}^T}{\det(\boldsymbol{F})} \,.$$

Then show that Maxwell's equations are invariant under these transformations.

Solution - 6.4 : The proof in spatial curvilinear coordinates is quite tedious (even for the situation where time is not dealt with explicitly). However, proofs can be found in older textbooks on electromagnetism. The expressions for the curl given in Chapter 1 along with the approach used in Problem 1.2 can be used to verify the above relations. Showing that Maxwell's equations are invariant is a bit more involved.

Problem 6.5 Consider a cloak in the shape of an annulus of a circular cylinder with inner radius R_1 and outer radius R_2. Assume that the material properties of the cloak are derived from the transformation

$$r = R_1 + t\,R\,,\quad \theta = \Theta\,,\quad z = Z \quad \text{where} \quad t := \frac{R_2 - R_1}{R_2}\,.$$

Find the permittivity and magnetic permeability of the cloak as a function of r.

Solution - 6.5 : In cylindrical coordinates

$$\mathbf{x} = \mathbf{x}(r,\theta,z) = x_i(r,\theta,z)\,\mathbf{e}_i \tag{6.36}$$

where

$$x_1 = r\cos\theta\,,\quad x_2 = r\sin\theta\,,\quad x_3 = z\,. \tag{6.37}$$

The orthogonal basis vectors for the cylindrical coordinate system are

$$\begin{aligned} \mathbf{g}_1 &= \frac{\partial \mathbf{x}}{\partial r} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 = \mathbf{e}_r \\ \mathbf{g}_2 &= \frac{\partial \mathbf{x}}{\partial \theta} = -r\sin\theta\,\mathbf{e}_1 + r\cos\theta\,\mathbf{e}_2 = r\mathbf{e}_\theta \\ \mathbf{g}_3 &= \frac{\partial \mathbf{x}}{\partial z} = \mathbf{e}_3 = \mathbf{e}_z\,. \end{aligned} \tag{6.38}$$

The reciprocal basis vectors are given by

$$\begin{aligned} \mathbf{g}^1 &= \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 = \mathbf{e}_r \\ \mathbf{g}^2 &= -\frac{\sin\theta}{r}\,\mathbf{e}_1 + \frac{\cos\theta}{r}\,\mathbf{e}_2 = \frac{1}{r}\mathbf{e}_\theta \\ \mathbf{g}^3 &= \mathbf{e}_3 = \mathbf{e}_z\,. \end{aligned} \tag{6.39}$$

From the above we see that

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_r}{\partial z} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial z} = \frac{\partial \mathbf{e}_z}{\partial r} = \frac{\partial \mathbf{e}_z}{\partial \theta} = \frac{\partial \mathbf{e}_z}{\partial z} = 0\,. \tag{6.40}$$

Also,

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 = \mathbf{e}_\theta\,,\quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\cos\theta\,\mathbf{e}_1 - \sin\theta\,\mathbf{e}_2 = -\mathbf{e}_r\,. \tag{6.41}$$

The gradient of a scalar function in cylindrical coordinates is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \theta^i}\,\mathbf{g}^i = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z\,. \tag{6.42}$$

We can now calculate the gradient, $\boldsymbol{F}$:

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \Theta^i}\otimes\mathbf{g}^i = \frac{\partial \mathbf{x}}{\partial R}\otimes\mathbf{e}_R + \frac{1}{R}\,\frac{\partial \mathbf{x}}{\partial \Theta}\otimes\mathbf{e}_\Theta + \frac{\partial \mathbf{x}}{\partial Z}\otimes\mathbf{e}_Z \tag{6.43}$$

where $\mathbf{x} = x_r\mathbf{e}_r + x_\theta\mathbf{e}_\theta + x_z\mathbf{e}_z$. For the situation where $\mathbf{e}_r = \mathbf{e}_R$, $\mathbf{e}_\theta = \mathbf{e}_\Theta$, and $\mathbf{e}_z = \mathbf{e}_Z$, we have

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial R} &= \frac{\partial}{\partial R}(x_r\mathbf{e}_r) + \frac{\partial}{\partial R}(x_\theta\mathbf{e}_\theta) + \frac{\partial}{\partial R}(x_z\mathbf{e}_z) = \frac{\partial x_r}{\partial R}\,\mathbf{e}_r + \frac{\partial x_\theta}{\partial R}\,\mathbf{e}_\theta + \frac{\partial x_z}{\partial R}\,\mathbf{e}_z \\ \frac{\partial \mathbf{x}}{\partial \Theta} &= \frac{\partial}{\partial \Theta}(x_r\mathbf{e}_r) + \frac{\partial}{\partial \Theta}(x_\theta\mathbf{e}_\theta) + \frac{\partial}{\partial \Theta}(x_z\mathbf{e}_z) = \frac{\partial x_r}{\partial \Theta}\,\mathbf{e}_r + x_r\,\mathbf{e}_\theta + \frac{\partial x_\theta}{\partial \Theta}\,\mathbf{e}_\theta - x_\theta\,\mathbf{e}_r + \frac{\partial x_z}{\partial \Theta}\,\mathbf{e}_z \\ \frac{\partial \mathbf{x}}{\partial Z} &= \frac{\partial}{\partial Z}(x_r\mathbf{e}_r) + \frac{\partial}{\partial Z}(x_\theta\mathbf{e}_\theta) + \frac{\partial}{\partial Z}(x_z\mathbf{e}_z) = \frac{\partial x_r}{\partial Z}\,\mathbf{e}_r + \frac{\partial x_\theta}{\partial Z}\,\mathbf{e}_\theta + \frac{\partial x_z}{\partial Z}\,\mathbf{e}_z\,. \end{aligned} \tag{6.44}$$

Note that $\mathbf{X}$ has been assumed to a general vector in the above representation. The mapping considered in this problem has the form,

$$\mathbf{X} = R\mathbf{e}_r + Z\mathbf{e}_z\ ,\quad \mathbf{x} = f(R)\mathbf{e}_r + Z\mathbf{e}_z\ . \tag{6.45}$$

Therefore,

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial R} &= \frac{df}{dR}\mathbf{e}_r\\ \frac{\partial \mathbf{x}}{\partial \Theta} &= f(R)\,\mathbf{e}_\theta\\ \frac{\partial \mathbf{x}}{\partial Z} &= \mathbf{e}_z\end{aligned} \tag{6.46}$$

and we have

$$\boldsymbol{F} = \frac{df}{dR}\mathbf{e}_R\otimes\mathbf{e}_R + \frac{f(R)}{R}\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta + \mathbf{e}_Z\otimes\mathbf{e}_Z\ . \tag{6.47}$$

In this case, $f(R) = R_1 + tR$. Therefore,

$$\boldsymbol{F} = t\,\mathbf{e}_R\otimes\mathbf{e}_R + \frac{R_1+tR}{R}\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta + \mathbf{e}_Z\otimes\mathbf{e}_Z\ . \tag{6.48}$$

In matrix form

$$\boldsymbol{F} = \begin{bmatrix} t & 0 & 0\\ 0 & \dfrac{R_1+tR}{R} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{6.49}$$

and

$$J = \det\boldsymbol{F} = \frac{R_1 t}{R} + t^2\,. \tag{6.50}$$

If the reference material is isotropic and homogeneous with permittivity ε and permeability μ, the transformed material has properties

$$\boldsymbol{\varepsilon}'(r,\theta,z) = \boldsymbol{\varepsilon}\,\frac{\boldsymbol{F}\cdot\boldsymbol{F}^T}{J}\,;\ \ \boldsymbol{\mu}'(r,\theta,z) = \boldsymbol{\mu}\,\frac{\boldsymbol{F}\cdot\boldsymbol{F}^T}{J}\,. \tag{6.51}$$

with

$$\boldsymbol{T} = \frac{\boldsymbol{F}\cdot\boldsymbol{F}^T}{J} = \begin{bmatrix} \dfrac{tR}{R_1+tR} & 0 & 0\\ 0 & \dfrac{R_1+tR}{tR} & 0\\ 0 & 0 & \dfrac{R}{t(R_1+tR)}\end{bmatrix} \qquad \square \tag{6.52}$$

The inverse transformation can also be used to find the properties of the cloak. This has been used widely in the literature.

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \dfrac{R_1+tR}{tR} & 0 & 0\\ 0 & \dfrac{tR}{R_1+tR} & 0\\ 0 & 0 & \dfrac{t(R_1+TR)}{R}\end{bmatrix}. \tag{6.53}$$

Problem 6.6 The electrical conductivity equation in the presence of a current source can be expressed as

$$\boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{X}) = S(\mathbf{X}) \quad \text{with} \quad \mathbf{J}(\mathbf{X}) = \boldsymbol{\Sigma}(\mathbf{X}) \cdot \boldsymbol{\nabla}\phi(\mathbf{X})$$

where $\boldsymbol{\Sigma}(\mathbf{X})$ is the electrical conductivity tensor, $\phi(\mathbf{X})$ is a scalar potential, and $S(\mathbf{X})$ is the source term. Use the approach used to derive the transformed equations of elastodynamics to show that the conductivity equation is invariant under transformations $\mathbf{x} = \boldsymbol{F} \cdot \mathbf{X}$, $F_{ij} = \partial x_i/\partial X_j$ with

$$\boldsymbol{\sigma}(\mathbf{x}) = \frac{1}{J}\,\boldsymbol{F}(\mathbf{X}) \cdot \boldsymbol{\Sigma}(\mathbf{X}) \cdot \boldsymbol{F}^T(\mathbf{X}) \quad \text{and} \quad s(\mathbf{x}) = \frac{1}{J} S(\mathbf{X})\,.$$

Verify that $\boldsymbol{\nabla}_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{x}) = s(\mathbf{x})$ where $\mathbf{j}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\nabla}_{\mathbf{x}}\,\phi$.

Solution - 6.6 : In the elastodynamic situation we had tested the governing equations against a vector valued function. For electrical conductivity we can use a scalar value function, i.e.,

$$W = \int_\Omega [\boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{X}) - S(\mathbf{X})]\psi \,\mathrm{d}\Omega\,.$$

Using the identity $\psi\, \boldsymbol{\nabla} \cdot \mathbf{J} = \boldsymbol{\nabla} \cdot (\psi\mathbf{J}) - \mathbf{J} \cdot \boldsymbol{\nabla}\psi$ we have

$$W = \int_\Omega [\boldsymbol{\nabla} \cdot (\psi\mathbf{J}) - \mathbf{J} \cdot \boldsymbol{\nabla}\psi - S\psi]\,\mathrm{d}\Omega\,.$$

From the divergence theorem,

$$W = \int_\Gamma \psi\mathbf{J} \cdot \mathbf{n}\,\mathrm{d}\Gamma - \int_\Omega [\mathbf{J} \cdot \boldsymbol{\nabla}\psi + S\psi]\,\mathrm{d}\Omega\,.$$

Since ψ has compact support, i.e., it is zero on the boundary Γ,

$$W = -\int_\Omega [\mathbf{J} \cdot \boldsymbol{\nabla}\psi + S\psi]\,\mathrm{d}\Omega\,. \tag{6.54}$$

Plugging in the expression for $\mathbf{J}$ gives

$$W = -\int_\Omega [(\boldsymbol{\Sigma} \cdot \boldsymbol{\nabla}\phi) \cdot \boldsymbol{\nabla}\psi + S\psi]\,\mathrm{d}\Omega\,.$$

In Cartesian components,

$$W = -\int_\Omega [\Sigma_{ij}\frac{\partial\phi}{\partial X_j}\frac{\partial\psi}{\partial X_i} + S\psi]\,\mathrm{d}\Omega\,.$$

Transforming variables with $J = \det(\boldsymbol{F})$,

$$\begin{aligned} W &= -\int_{\Omega_x} \frac{1}{J}\left[\Sigma_{ij}\frac{\partial\phi}{\partial x_m}\frac{\partial x_m}{\partial X_j}\frac{\partial\psi}{\partial x_p}\frac{\partial x_p}{\partial X_i} + S\psi\right]\,\mathrm{d}\Omega_x \\ &= -\int_{\Omega_x} \frac{1}{J}\left[(F_{pi}\Sigma_{ij}F_{mj})\frac{\partial\phi}{\partial x_m}\frac{\partial\psi}{\partial x_p} + S\psi\right]\,\mathrm{d}\Omega_x\,. \end{aligned}$$

Back to direct notation,

$$W = -\int_{\Omega_x} \frac{1}{J}[\{(\boldsymbol{F} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{F}^T) \cdot \boldsymbol{\nabla}_{\mathbf{x}}\,\phi\} \cdot \boldsymbol{\nabla}_{\mathbf{x}}\,\psi + S\psi]\,\mathrm{d}\Omega_x\,. \tag{6.55}$$

Compare (6.54) and (6.55) to see that the transformed quantities are

$$\boldsymbol{\sigma}(\mathbf{x}) = \frac{1}{J}\boldsymbol{F}(\mathbf{X}) \cdot \boldsymbol{\Sigma}(\mathbf{X}) \cdot \boldsymbol{F}^T(\mathbf{X}) \quad \text{and} \quad s(\mathbf{x}) = \frac{1}{J}S(\mathbf{X}) \qquad \square$$

Verification of the divergence equation is straightforward.

Problem 6.7 Chen, H. and Chan, C. T. (2007) use a separation of variables approach to show that for a cloak based on a transformation $\boldsymbol{F}$ with

$$\boldsymbol{F}\cdot\boldsymbol{F}^T \equiv \begin{bmatrix} t^2 & 0 & 0 \\ 0 & \frac{t^2 r^2}{(r-R_1)^2} & 0 \\ 0 & 0 & \frac{t^2 r^2}{(r-R_1)^2} \end{bmatrix} \quad \text{and} \quad J = \det(\boldsymbol{F}) = \frac{t^3 r^2}{(r-R_1)^2}\,.$$

the pressure inside the cloak is

$$p(r,\theta,\phi) = \text{Re}\left[p_0 \exp\left(\frac{i(r-R_1)\cos\phi}{t}\sqrt{\frac{\omega^2\rho}{\kappa}}\right)\right]$$

where p_0 is the amplitude of the incident plane wave. Verify this result.

Solution - 6.7 : Let us use a transformation of the form given in equation (6.29), i.e.,

$$\boldsymbol{\rho}_x^{-1}(\mathbf{x}) = \frac{\rho^{-1}(\mathbf{X})}{J}\,\boldsymbol{F}\cdot\boldsymbol{F}^T \quad \text{and} \quad \kappa_x(\mathbf{x}) = J\,\kappa(\mathbf{X})\,.$$

Plugging in the expressions for $\boldsymbol{F}\cdot\boldsymbol{F}^T$ and J, we have

$$\boldsymbol{\rho}_x^{-1}(\mathbf{x}) = \frac{\rho^{-1}}{t}\begin{bmatrix} \frac{(r-R_1)^2}{r^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: \begin{bmatrix} \frac{1}{\rho_x^{rr}(r)} & 0 & 0 \\ 0 & \frac{1}{\rho_x^{\theta\theta}} & 0 \\ 0 & 0 & \frac{1}{\rho_x^{\phi\phi}} \end{bmatrix} \quad \text{and} \quad \kappa_x(\mathbf{x}) = \kappa\,\frac{r^2 t^3}{(r-R_1)^2}\,.$$

Recall that the fixed-frequency acoustic wave equation has the form

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\rho}_x^{-1}\cdot\boldsymbol{\nabla}p) + \frac{\omega^2}{\kappa_x}\,p = 0\,.$$

Since we are solving a spherically symmetric problem, the gradient and divergence operators have the form

$$\boldsymbol{\nabla}p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial\phi}\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial p}{\partial\theta}\mathbf{e}_\theta$$

and

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, v_\phi) + \frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial\theta}\,.$$

Therefore, in matrix notation,

$$\boldsymbol{\rho}_x^{-1}\cdot\boldsymbol{\nabla}p = \begin{bmatrix} \frac{1}{\rho_x^{rr}(r)} & 0 & 0 \\ 0 & \frac{1}{\rho_x^{\theta\theta}} & 0 \\ 0 & 0 & \frac{1}{\rho_x^{\phi\phi}} \end{bmatrix}\begin{bmatrix} \frac{\partial p}{\partial r} \\ \frac{1}{r\sin\phi}\frac{\partial p}{\partial\theta} \\ \frac{1}{r}\frac{\partial p}{\partial\phi} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho_x^{rr}(r)}\frac{\partial p}{\partial r} \\ \frac{1}{r\sin\phi\,\rho_x^{\theta\theta}}\frac{\partial p}{\partial\theta} \\ \frac{1}{r\rho_x^{\phi\phi}}\frac{\partial p}{\partial\phi} \end{bmatrix} =: \begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix}\,.$$

The divergence term may then be expressed as

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\rho}_x^{-1}\cdot\boldsymbol{\nabla}p) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{\partial p}{\partial r}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\phi\,\rho_x^{\theta\theta}}\frac{\partial p}{\partial\theta}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r\rho_x^{\phi\phi}}\frac{\partial p}{\partial\phi}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin^2\phi\,\rho_x^{\theta\theta}}\frac{\partial^2 p}{\partial\theta^2} + \frac{1}{r^2\sin\phi\,\rho_x^{\phi\phi}}\frac{\partial}{\partial\phi}\left(\sin\phi\,\frac{\partial p}{\partial\phi}\right)\,.\end{aligned}$$

The transformed wave equation for $p(r,\theta,\phi)$ may now be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{\partial p}{\partial r}\right)+\frac{1}{r^2\sin^2\phi\rho_x^{\theta\theta}}\frac{\partial^2 p}{\partial\theta^2}+\frac{1}{r^2\sin\phi\rho_x^{\phi\phi}}\frac{\partial}{\partial\phi}\left(\sin\phi\,\frac{\partial p}{\partial\phi}\right)+\frac{\omega^2}{\kappa_x(r)}\,p=0\,.$$

If we assume a solution of the form

$$p(r,\theta,\phi)=R(r)\,\Theta(\theta)\,\Phi(\phi)$$

we get

$$\frac{\Theta\Phi}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{dR}{dr}\right)+\frac{R\Phi}{r^2\sin^2\phi\rho_x^{\theta\theta}}\frac{d^2\Theta}{d\theta^2}+\frac{R\Theta}{r^2\sin\phi\rho_x^{\phi\phi}}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right)+\frac{\omega^2}{\kappa_x(r)}\,R\Theta\Phi=0\,.$$

Division by $R\Theta\Phi$ and multiplication by $r^2\sin^2\phi$ leads to

$$\frac{\sin^2\phi}{R}\frac{d}{dr}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{dR}{dr}\right)+\frac{1}{\rho_x^{\theta\theta}\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{\sin\phi}{\rho_x^{\phi\phi}\Phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right)+\frac{r^2\sin^2\phi\omega^2}{\kappa_x(r)}=0\,.$$

We now have a system of two ODEs, each of which evaluates to a constant m^2 (which usually also an integer squared but not in this case), i.e.,

$$\begin{aligned}&\frac{1}{\rho_x^{\theta\theta}\Theta}\frac{d^2\Theta}{d\theta^2}=-m^2\\&\frac{\sin^2\phi}{R}\frac{d}{dr}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{dR}{dr}\right)+\frac{\sin\phi}{\rho_x^{\phi\phi}\Phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right)+\frac{r^2\sin^2\phi\omega^2}{\kappa_x(r)}=m^2\,.\end{aligned}$$

Division of the second equation above by $\sin^2\phi$ leads to

$$\frac{1}{R}\frac{d}{dr}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{dR}{dr}\right)+\frac{1}{\sin\phi\rho_x^{\phi\phi}\Phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right)+\frac{r^2\omega^2}{\kappa_x(r)}=\frac{m^2}{\sin^2\phi}\,.$$

This equation can also be separated into two uncoupled ODEs,

$$\begin{aligned}&\frac{d}{dr}\left(\frac{r^2}{\rho_x^{rr}(r)}\frac{dR}{dr}\right)+\left(\frac{r^2\omega^2}{\kappa_x(r)}-C_1\right)R=0\\&\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right)-\rho_x^{\phi\phi}\left(\frac{m^2}{\sin^2\phi}-C_1\right)\Phi=0\,.\end{aligned}$$

The second equation above has the form of an associated Legendre equation and can be written in the form

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right)+\left[\nu(\nu+1)-\frac{\mu^2}{\sin^2\phi}\right]\Phi=0$$

where

$$\nu(\nu+1):=\rho_x^{\phi\phi}C_1\quad\text{and}\quad\mu^2:=\rho_x^{\phi\phi}m^2\,.$$

Note that ν and μ are not necessarily integers. The general solution of this equation involves a combination of Legendre functions $P_\nu^\mu(\cos\phi)$ which are commonly expressed in terms of hypergeometric functions. Since ϕ is an angular function, any solution of the above equation must have $\phi+2\pi n$ periodicity where n is an integer. Legendre functions satisfy these periodicity conditions whether ν and μ are integers or not.

The equation involving $R(r)$ can then be expressed as

$$\frac{d}{dr}\left[\frac{r^2}{\rho_x^{rr}(r)}\,\frac{dR}{dr}\right]+\left[\frac{r^2\omega^2}{\kappa_x(r)}-\frac{\nu(\nu+1)}{\rho_x^{\phi\phi}}\right]R=0\,.$$

Recalling that

$$\frac{1}{\rho_x^{rr}(r)}=\frac{(r-R_1)^2}{\rho r^2 t}\,,\quad \frac{1}{\rho_x^{\phi\phi}}=\frac{1}{\rho t}\qquad\text{and}\qquad \frac{1}{\kappa_x(r)}=\frac{(r-R_1)^2}{\kappa\, r^2 t^3}$$

we have

$$\frac{d}{dr}\left[\frac{(r-R_1)^2}{\rho t}\,\frac{dR}{dr}\right]+\left[\frac{\omega^2(r-R_1)^2}{\kappa t^3}-\frac{\nu(\nu+1)}{\rho t}\right]R=0$$

or

$$\frac{d}{dr}\left[(r-R_1)^2\,\frac{dR}{dr}\right]+\left[\frac{\omega^2\rho(r-R_1)^2}{\kappa t^2}-\nu(\nu+1)\right]R=0\,.$$

This has the form of Bessel's equation with a general solution that is a linear combination of spherical Bessel functions $j_\nu[\alpha(r-R_1)]$ where $\alpha^2:=\omega^2\rho/(\kappa t^2)$.

The remaining equation is

$$\frac{1}{\rho_x^{\theta\theta}\Theta}\,\frac{d^2\Theta}{d\theta^2}=-m^2\quad\rightarrow\quad\frac{d^2\Theta}{d\theta^2}+\zeta^2\Theta=0$$

where $\zeta^2:=\rho_x^{\theta\theta}m^2$. Solutions are of the form $\exp(i\zeta\theta)$ and $2\pi n$ periodicity in θ implies that ζ must be an integer. We can combine the solutions for $\Phi(\phi)$ and $\Theta(\theta)$ to get spherical harmonic solutions for the special case where $\rho_x^{\theta\theta}=\rho_x^{\phi\phi}$, i.e., when $\mu^2=\zeta^2$ and both are integers. These solutions have the form $Y_\nu^\mu(\phi,\theta)$.

If we further consider the special case where ν is an integer, the general solution for $p(r,\theta,\phi)$ can be expressed as

$$p(r,\theta,\phi)=\text{Re}\left[\sum_\nu\sum_\mu C_{\nu\mu}\left\{j_\nu[\alpha(r-R_1)]+iy_\nu[\alpha(r-R_1)]\right\}Y_\nu^\mu(\phi,\theta)\right]$$

where $C_{\nu\mu}$ are constants and $\alpha^2=\omega^2\rho/(\kappa t^2)$. Spherical Bessel function solutions of the second kind are represented by y_ν in the above equation. Note that there is a singularity at $r=R_1$ in the Hankel function solution. Also observe that j_ν is zero for $r<R_1$ and consequently the pressure in that region is zero.

Let us now consider the scattering problem for a coated sphere. Let the incident waves have the form

$$p^i(r,\theta,\phi)=\sum_\nu\sum_\mu C^i_{\nu\mu}j_\nu(\alpha t r)\,Y_\nu^\mu(\phi,\theta)\,.$$

The scattered waves in the region $r\geq R_2$ have the form

$$p^s(r,\theta,\phi)=\sum_\nu\sum_\mu C^s_{\nu\mu}h_\nu^{(0)}(\alpha t r)\,Y_\nu^\mu(\phi,\theta)$$

where $h_\nu^{(0)}$ are spherical Hankel functions of the first kind. The boundary conditions at $r=R_2$ are

$$[p]=0\qquad\Longrightarrow\qquad p^i(R_2,\theta,\phi)+p^s(R_2,\theta,\phi)=p(R_2,\theta,\phi)$$

and

$$[(\boldsymbol{\rho}^{-1}.\boldsymbol{\nabla}p)\cdot\mathbf{n}]=0\qquad\Longrightarrow\qquad\frac{1}{\rho}\left[\frac{\partial p^i}{\partial r}+\frac{\partial p^s}{\partial r}\right]_{r=R_2}=\frac{1}{\rho_x^{rr}(R_2)}\left.\frac{\partial p}{\partial r}\right|_{r=R_2}\,.$$

The first boundary condition gives

$$\sum_\nu \sum_\mu \left[C^i_{\nu\mu}\, j_\nu(\alpha t R_2) + C^s_{\nu\mu}\, h^{(0)}_\nu(\alpha t R_2)\right] Y^\mu_\nu(\phi,\theta) = \sum_\nu \sum_\mu C_{\nu\mu} j_\nu[\alpha(R_2 - R_1)] Y^\mu_\nu(\phi,\theta)$$

or,

$$\sum_\nu \sum_\mu \left[C^i_{\nu\mu}\, j_\nu(\alpha t R_2) + C^s_{\nu\mu}\, h^{(0)}_\nu(\alpha t R_2)\right] = \sum_\nu \sum_\mu C_{\nu\mu} j_\nu(\alpha t R_2)\,. \tag{6.56}$$

The second boundary condition leads to

$$\frac{\alpha t}{\rho} \sum_\nu \sum_\mu \left[C^i_{\nu\mu}\, j'_\nu(\alpha t R_2) + C^s_{\nu\mu}\, h^{(0)'}_\nu(\alpha t R_2)\right] = \frac{\alpha}{\rho^{rr}_x(R_2)} \sum_\nu \sum_\mu C_{\nu\mu} j'_\nu(\alpha t R_2)\,.$$

Using

$$\rho^{rr}_x(R_2) = \frac{\rho R_2^2 t}{(R_2 - R_1)^2} = \frac{\rho t}{t^2} = \frac{\rho}{t}$$

we have

$$\sum_\nu \sum_\mu \left[C^i_{\nu\mu}\, j'_\nu(\alpha t R_2) + C^s_{\nu\mu}\, h^{(0)'}_\nu(\alpha t R_2)\right] = \sum_\nu \sum_\mu C_{\nu\mu} j'_\nu(\alpha t R_2)\,. \tag{6.57}$$

These two conditions can be satisfied simultaneously only if $C^s_{\nu\mu} = 0$, i.e., if $C^i_{\nu\mu} = C_{\nu\mu}$, and there is no scattering from the coated sphere.

That implies that if the incident wave is plane with the wave front perpendicular to the r_ϕ plane we can write it in the form

$$p^i(r,\theta,\phi) = \text{Re}\left[p_0 \exp(i\alpha t r \cos\phi)\right]$$

using the series expansion of the exponential in terms of Bessel and Legendre functions. The the field inside the cloak can similarly be written as

$$p(r,\theta,\phi) = \text{Re}\left[p_0 \exp(i\alpha(r - R_1)\cos\phi)\right]\,.$$

Substituting the expression for α, we have

$$p(r,\theta,\phi) = \text{Re}\left[p_0 \exp\left(i\sqrt{\frac{\omega^2\rho}{\kappa}}\,\frac{(r-R_1)}{t}\,\cos\phi\right)\right] \qquad \square$$

Note the number of assumptions regarding the density that have been made in arriving at this solution.

Problem 6.8 Show that the acoustic wave equation in a medium with

$$\boldsymbol{\rho}(\mathbf{x}) = \boldsymbol{\rho}(\mathbf{r}) = \rho_r(r)\, \boldsymbol{P}_{\parallel} + \rho_\perp(r)\, \boldsymbol{P}_\perp\, ; \quad \kappa(\mathbf{x}) = \kappa(\mathbf{r})$$

can be expressed in spherical coordinates as

$$\frac{\kappa(r)}{r^2}\frac{\partial}{\partial r}\left[\frac{r^2}{\rho_r(r)}\frac{\partial p}{\partial r}\right] + \frac{\kappa(r)}{r^2\rho_\perp(r)}\nabla_\perp^2 p - \ddot{p}(r,t) = 0$$

where $\nabla_\perp^2(\bullet)$ is the angular part of the second-order Laplace-Beltrami operator in spherical coordinates. Then show that the above equation is equivalent to the uniform wave equation

$$\boldsymbol{\nabla}_{\mathbf{X}} \cdot [\boldsymbol{\nabla}_{\mathbf{X}}\, p(\mathbf{X})] - \ddot{p}(\mathbf{X}) = 0$$

only if we have a mapping $\mathbf{X} = [f(r)/r]\,\mathbf{x}$ such that

$$\rho_r = \left(\frac{r}{f}\right)^2 \frac{df}{dr}\,,\ \ \rho_\perp = \left(\frac{df}{dr}\right)^{-1}\,,\ \ \kappa = \left(\frac{r}{f}\right)^2 \left(\frac{df}{dr}\right)^{-1}\,.$$

Solution - 6.8 : The first part of this problem is similar to the previous problem. The acoustic wave equation can be expressed as

$$\kappa(\mathbf{x})\boldsymbol{\nabla}\cdot[\boldsymbol{\rho}^{-1}\cdot\boldsymbol{\nabla} p] - \ddot{p} = 0\,.$$

It is easiest if we use matrix notation (keeping in mind that κ is a scalar quantity). Then we can write

$$\boldsymbol{\rho}(\mathbf{x}) = \rho_r(r)\,\boldsymbol{P}_\parallel + \rho_\perp(r)\,\boldsymbol{P}_\perp = \rho_r(r)\,\mathbf{e}_r\otimes\mathbf{e}_r + \rho_\perp(r)\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \equiv \begin{bmatrix}\rho_r(r) & 0 & 0\\ 0 & \rho_\perp(r) & 0\\ 0 & 0 & \rho_\perp(r)\end{bmatrix}$$

and

$$\kappa(\mathbf{x}) = \kappa(r)\,.$$

Proceeding as in the previous problem we have

$$\boldsymbol{\rho}^{-1}\cdot\boldsymbol{\nabla} p = \begin{bmatrix}\frac{1}{\rho_r(r)}\dfrac{\partial p}{\partial r}\\ \frac{1}{r\sin\phi\rho_\perp(r)}\dfrac{\partial p}{\partial\theta}\\ \frac{1}{r\rho_\perp(r)}\dfrac{\partial p}{\partial\phi}\end{bmatrix} =: \begin{bmatrix}v_r\\ v_\theta\\ v_\phi\end{bmatrix}\,.$$

The divergence term is given by

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\rho}_x^{-1}\cdot\boldsymbol{\nabla} p) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho_r(r)}\frac{\partial p}{\partial r}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\phi\rho_\perp(r)}\frac{\partial p}{\partial\theta}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r\rho_\perp(r)}\frac{\partial p}{\partial\phi}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho_r(r)}\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin^2\phi\rho_\perp(r)}\frac{\partial^2 p}{\partial\theta^2} + \frac{1}{r^2\sin\phi\rho_\perp(r)}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial p}{\partial\phi}\right)\,.\end{aligned}$$

The Laplacian (second order Laplace-Beltrami operator) in spherical coordinates is

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 p}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial p}{\partial\phi}\right) =: \nabla_r^2 p + \frac{1}{r^2}\nabla_\perp^2 p\,.$$

The above split of the operator into radial and angular parts allows us to write

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\rho}_x^{-1}\cdot\boldsymbol{\nabla}p) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho_r(r)}\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\rho_\perp(r)}\,\nabla_\perp^2 p\,.$$

Therefore the acoustic wave equation can be expressed in the form

$$\frac{\kappa(r)}{r^2}\frac{\partial}{\partial r}\left[\frac{r^2}{\rho_r(r)}\frac{\partial p}{\partial r}\right] + \frac{\kappa(r)}{r^2\rho_\perp(r)}\nabla_\perp^2 p - \ddot{p}(r,t) = 0 \qquad \square$$

If we substitute the mapping

$$\rho_r = \left(\frac{r}{f}\right)^2\frac{df}{dr} = \frac{r^2 f'}{f^2}\,,\quad \rho_\perp = \left(\frac{df}{dr}\right)^{-1} = \frac{1}{f'}\,,\quad \kappa = \left(\frac{r}{f}\right)^2\left(\frac{df}{dr}\right)^{-1} = \frac{r^2}{f^2 f'}$$

into the acoustic wave equation, we have

$$\frac{1}{f^2 f'}\frac{\partial}{\partial r}\left[\frac{f^2}{f'}\frac{\partial p}{\partial r}\right] + \frac{1}{f^2}\nabla_\perp^2 p - \ddot{p}(r,t) = 0$$

or,

$$\left(\frac{2f'^2 - ff''}{ff'^3}\right)\frac{\partial p}{\partial r} + \frac{1}{f'^2}\frac{\partial^2 p}{\partial r^2} + \frac{1}{f^2}\nabla_\perp^2 p - \ddot{p}(r,t) = 0\,. \tag{6.58}$$

We now have to transform the above equation to one in $P(R)$ where $R = R(r) = f(r)$. To do that we observe that

$$\frac{\partial p}{\partial r} = \frac{\partial P}{\partial R}\frac{dR}{dr} = f'\,\frac{\partial P}{\partial R} \qquad \text{and} \qquad \frac{\partial^2 p}{\partial r^2} = \frac{\partial^2 P}{\partial R^2}\left(\frac{dR}{dr}\right)^2 + \frac{\partial P}{\partial R}\frac{d^2 R}{dr^2} = f'^2\,\frac{\partial^2 P}{\partial R^2} + f''\,\frac{\partial P}{\partial R}.$$

Plugging these into (6.58) leads to

$$\left(\frac{2f'^2 - ff''}{ff'^2}\right)\frac{\partial P}{\partial R} + \frac{\partial^2 P}{\partial R^2} + \frac{f''}{f'^2}\frac{\partial P}{\partial R} + \frac{1}{f^2}\nabla_\perp^2 P - \ddot{P}(R,t) = 0$$

or

$$\frac{\partial^2 P}{\partial R^2} + \frac{2}{f}\frac{\partial P}{\partial R} + \frac{1}{f^2}\nabla_\perp^2 P - \ddot{P}(R,t) = 0\,.$$

Alternatively,

$$\frac{\partial^2 P}{\partial R^2} + \frac{2}{R}\frac{\partial P}{\partial R} + \frac{1}{R^2}\nabla_\perp^2 P - \ddot{P}(R,t) = 0\,. \tag{6.59}$$

Recall that, in spherical coordinates,

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{X}}\cdot[\boldsymbol{\nabla}_{\mathbf{X}}\cdot P] &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial P}{\partial R}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 P}{\partial\theta^2} + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\,\frac{\partial P}{\partial\phi}\right)\\ &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial P}{\partial R}\right) + \frac{1}{R^2}\,\nabla_\perp^2 P = \frac{\partial^2 P}{\partial R^2} + \frac{2}{R}\frac{\partial P}{\partial R} + \frac{1}{R^2}\,\nabla_\perp^2 P\,.\end{aligned}$$

Comparison with (6.59) shows that

$$\boldsymbol{\nabla}_{\mathbf{X}}\cdot[\boldsymbol{\nabla}_{\mathbf{X}}\cdot P] - \ddot{P} = 0 \qquad \square$$

The necessity proof can be found in the acoustic cloaking paper by Norris (2008).

Problem 6.9 Find the transformed equations of elastodynamics for harmonic maps with the property $G_{pjj} = 0$. Show that for the special case when the equations of elastodynamics in the reference configuration have the form

$$-\boldsymbol{\nabla}_{\mathbf{X}} \cdot \boldsymbol{\Sigma} = \omega^2 \rho_X \mathbf{U} \quad \text{with} \quad \boldsymbol{\Sigma} = \kappa_X (\boldsymbol{\nabla}_{\mathbf{X}} \cdot \mathbf{U}) \boldsymbol{1}$$

the transformed stress field has the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha \boldsymbol{V} \quad \text{where} \quad \alpha := \frac{\kappa}{2J} \text{tr} \left[\boldsymbol{V} \cdot \left\{ \boldsymbol{\nabla}_{\mathbf{X}} \mathbf{u} + (\boldsymbol{\nabla}_{\mathbf{X}} \mathbf{u})^T \right\} \right]$$

where $\boldsymbol{V} = \boldsymbol{F} \cdot \boldsymbol{F}^T$. Compare the above stress-strain relation to those for pentamode materials.

Solution - 6.9 : The quantity $\mathcal{G}$ is defined as

$$\mathcal{G} \equiv G_{pk\ell} = \frac{\partial F_{pk}}{\partial X_\ell} .$$

The condition $G_{pjj} = 0$ is equivalent to

$$\frac{\partial F_{pj}}{\partial X_j} = 0 .$$

The quantity $\mathcal{G}$ appears in the transformed equations of elastodynamics in the terms

$$\begin{aligned}
\boldsymbol{\rho}_x &= \frac{\rho_X}{J} \boldsymbol{F} \cdot \boldsymbol{F}^T - \frac{1}{J\omega^2} \mathcal{G} : \mathbf{C}_X : \mathcal{G}^T \\
\mathcal{S}_x &= \frac{1}{J} (\boldsymbol{F} \boxtimes \boldsymbol{F}) : \mathbf{C}_X : \mathcal{G}^T \\
\mathcal{D}_x &= \frac{1}{J} \mathcal{G} : \mathbf{C}_X : (\boldsymbol{F}^T \boxtimes \boldsymbol{F}^T) .
\end{aligned}$$

Let us consider $\mathcal{D}_x$ in terms of Cartesian components, i.e.,

$$D_{imn} = G_{ijk} C_{jkpq} F^T_{pm} F^T_{qn} = \frac{\partial F_{ij}}{\partial X_k} F_{mp} F_{nq} C_{jkpq} .$$

For anisotropic $\mathbf{C}_X$, even for harmonic maps, $\mathcal{D}_x \neq 0$ and simple expressions the transformed equations are not obvious. But for isotropic $\mathbf{C}_X$ with *zero shear modulus*, we can write

$$C_{ijpq} = \kappa \, \delta_{ij} \, \delta_{pq}$$

where κ is the bulk modulus, and we have

$$D_{imn} = \frac{\kappa}{J} \frac{\partial F_{ij}}{\partial X_k} F_{mp} F_{nq} \delta_{jk} \delta_{pq} = \frac{\kappa}{J} \frac{\partial F_{ij}}{\partial X_j} F_{mp} F_{np} = 0 .$$

and

$$\boldsymbol{\rho}_x = \frac{\rho_X}{J} \boldsymbol{F} \cdot \boldsymbol{F}^T .$$

Similarly, since $G_{ijp} = G^T_{jpi}$ we have $G_{ijj} = G^T_{jji} = 0$ and

$$S_{ijm} = \frac{\kappa}{J} F_{ik} F_{j\ell} \delta_{k\ell} \delta_{pq} \frac{\partial F^T_{pq}}{\partial X_m} = \frac{\kappa}{J} F_{ik} F_{jk} \frac{\partial F^T_{pp}}{\partial X_m} = 0 .$$

Therefore the transformed equations of elastodynamics have the form

$$\boldsymbol{\nabla}_{\mathbf{x}} \cdot \boldsymbol{\sigma} = -\omega^2 \boldsymbol{\rho}_x \cdot \mathbf{u} \quad \text{where} \quad \boldsymbol{\sigma} = \mathbf{C}_x : \boldsymbol{\nabla}_{\mathbf{x}} \mathbf{u} \qquad \square$$

Note that the transformed equation has a tensor valued density.

The transformed stiffness tensor is given by

$$\mathsf{C}_x = \frac{1}{J}(\boldsymbol{F} \boxtimes \boldsymbol{F}) : \mathsf{C}_X : (\boldsymbol{F}^T \boxtimes \boldsymbol{F}^T)\,.$$

When $\boldsymbol{\Sigma} = \kappa_X(\boldsymbol{\nabla}_{\mathbf{X}} \cdot \mathbf{U})\boldsymbol{1}$ we have $\mathsf{C}_X = \kappa_X \boldsymbol{1} \otimes \boldsymbol{1}$ and hence

$$\mathsf{C}_x = \frac{\kappa_X}{J}(\boldsymbol{F} \boxtimes \boldsymbol{F}) : (\boldsymbol{1} \otimes \boldsymbol{1}) : (\boldsymbol{F}^T \boxtimes \boldsymbol{F}^T) = \frac{\kappa_X}{J}(\boldsymbol{F} \cdot \boldsymbol{F}^T) \otimes (\boldsymbol{F} \cdot \boldsymbol{F}^T) = \frac{\kappa_X}{J}\boldsymbol{V} \otimes \boldsymbol{V}\,.$$

Also,

$$\mathcal{S}_x = \frac{\kappa_X}{J}\,(\boldsymbol{F} \boxtimes \boldsymbol{F}) : (\boldsymbol{1} \otimes \boldsymbol{1}) : \mathcal{G}^T = \frac{\kappa_X}{J}\,(\boldsymbol{F} \cdot \boldsymbol{F}^T) \otimes (\boldsymbol{\nabla}_{\mathbf{X}} \cdot \boldsymbol{F}) = \frac{\kappa_X}{J}\,\boldsymbol{V} \otimes (\boldsymbol{\nabla}_{\mathbf{X}} \cdot \boldsymbol{F})\,.$$

Therefore

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{\kappa_X}{J}\,[(\boldsymbol{V} \otimes \boldsymbol{V}) : \boldsymbol{\nabla}_{\mathbf{x}}\,\mathbf{u} + (\boldsymbol{V} \otimes (\boldsymbol{\nabla}_{\mathbf{X}} \cdot \boldsymbol{F})) \cdot \mathbf{u}] \\
&= \frac{\kappa_X}{J}\left[\tfrac{1}{2}\boldsymbol{V} : (\boldsymbol{\nabla}_{\mathbf{x}}\,\mathbf{u} + \boldsymbol{\nabla}_{\mathbf{x}}\,\mathbf{u}^T) + \boldsymbol{F}^T : (\boldsymbol{\nabla}_{\mathbf{x}}\,\boldsymbol{F} \cdot \mathbf{u})\right]\boldsymbol{V} \\
&= \frac{\kappa_X}{J}\left[\tfrac{1}{2}\mathrm{tr}\{\boldsymbol{V} \cdot (\boldsymbol{\nabla}_{\mathbf{x}}\,\mathbf{u} + \boldsymbol{\nabla}_{\mathbf{x}}\,\mathbf{u}^T)\} + \boldsymbol{F}^T : (\boldsymbol{\nabla}_{\mathbf{x}}\,\boldsymbol{F} \cdot \mathbf{u})\right]\boldsymbol{V}\,.
\end{aligned}$$

We find that

$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha\boldsymbol{V} \quad \text{where} \quad \alpha := \frac{\kappa}{2J}\mathrm{tr}\left[\boldsymbol{V} \cdot \left\{\boldsymbol{\nabla}_{\mathbf{X}}\,\mathbf{u} + (\boldsymbol{\nabla}_{\mathbf{X}}\,\mathbf{u})^T\right\}\right]$$

only for the case where $\boldsymbol{\nabla}_{\mathbf{x}} \cdot \boldsymbol{F} = 0$.

A comparison with pentamode materials is straightforward.

Problem 6.10 Verify the relation

$$\boldsymbol{\nabla}_{\mathsf{X}} \cdot (\boldsymbol{\nabla}_{\mathsf{X}}\, p) = J\, \boldsymbol{\nabla}_{\mathsf{x}} \left(J^{-1}\boldsymbol{S}^{-1} \cdot \boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p\right) : \boldsymbol{S} \quad \text{with} \quad \boldsymbol{\nabla}_{\mathsf{x}} \cdot \boldsymbol{S} = 0\,.$$

Solution - 6.10 : Starting from equation (6.35) of the book we have

$$\begin{aligned} \boldsymbol{\nabla}_{\mathsf{X}} \cdot [\boldsymbol{\nabla}_{\mathsf{X}}\, p] &= J\boldsymbol{\nabla}_{\mathsf{x}} \cdot [J^{-1}\boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p] \\ &= J\boldsymbol{\nabla}_{\mathsf{x}} \cdot [J^{-1}\boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p] \\ &= J\boldsymbol{\nabla}_{\mathsf{x}} \cdot [J^{-1}\boldsymbol{S} \cdot \boldsymbol{S}^{-1} \cdot \boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p]\,. \end{aligned}$$

Noting that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \cdot \mathbf{b}) \equiv \frac{\partial}{\partial x_i}(b_j)\, A_{ij} + \frac{\partial}{\partial x_i}(A_{ij})\, b_j \equiv \boldsymbol{\nabla}\mathbf{b} : \boldsymbol{A}^T + (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \cdot \mathbf{b}$$

we have

$$\boldsymbol{\nabla}_{\mathsf{X}} \cdot [\boldsymbol{\nabla}_{\mathsf{X}}\, p] = J\boldsymbol{\nabla}_{\mathsf{x}}\, [J^{-1}\boldsymbol{S}^{-1} \cdot \boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p] : \boldsymbol{S}^T + (\boldsymbol{\nabla}_{\mathsf{x}} \cdot \boldsymbol{S}) \cdot [\boldsymbol{S}^{-1} \cdot \boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p]\,.$$

Since $\boldsymbol{\nabla}_{\mathsf{x}} \cdot \boldsymbol{S} = \boldsymbol{0}$, we have

$$\boldsymbol{\nabla}_{\mathsf{X}} \cdot [\boldsymbol{\nabla}_{\mathsf{X}}\, p] = J\boldsymbol{\nabla}_{\mathsf{x}}\, [J^{-1}\boldsymbol{S}^{-1} \cdot \boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p] : \boldsymbol{S}^T$$

If $\boldsymbol{S}$ is symmetric then

$$\boldsymbol{\nabla}_{\mathsf{X}} \cdot [\boldsymbol{\nabla}_{\mathsf{X}}\, p] = J\boldsymbol{\nabla}_{\mathsf{x}}\, [J^{-1}\boldsymbol{S}^{-1} \cdot \boldsymbol{V}^2 \cdot \boldsymbol{\nabla}_{\mathsf{x}}\, p] : \boldsymbol{S} \qquad \square$$

Chapter 7

Solutions for Exercises in Chapter 7

Problem 7.1 Show that for periodic composites in the quasistatic limit, the momentum equation, the stress-strain relations and the strain-displacements relations can be written as

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\sigma}}_\eta + i\widehat{\boldsymbol{\sigma}}_\eta \cdot \mathbf{k} + \omega^2 \rho_\eta \widehat{\mathbf{u}}_\eta = 0 \; ; \; \; \widehat{\boldsymbol{\sigma}}_\eta = \mathsf{C}_\eta : \widehat{\varepsilon}_\eta$$

and

$$\widehat{\varepsilon}_\eta = \tfrac{i}{2} \left(\widehat{\mathbf{u}}_\eta \otimes \mathbf{k} + \mathbf{k} \otimes \widehat{\mathbf{u}}_\eta \right) + \tfrac{1}{2} \left[\boldsymbol{\nabla} \widehat{\mathbf{u}}_\eta + (\boldsymbol{\nabla} \widehat{\mathbf{u}}_\eta)^T \right] \, .$$

Solution 7.1: The governing equations of linear elasticity in the quasistatic limit are

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_\eta = -\omega^2 \rho_\eta \mathbf{u}_\eta \; ; \quad \boldsymbol{\sigma}_\eta = \mathsf{C}_\eta : \varepsilon_\eta \; ; \quad \varepsilon_\eta = \tfrac{1}{2} [\boldsymbol{\nabla} \mathbf{u}_\eta + (\boldsymbol{\nabla} \mathbf{u}_\eta)^T] \, .$$

Bloch wave solutions of these equations for a periodic elastic composite have the form

$$\boldsymbol{\sigma}_\eta(\mathbf{x}) = \exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\boldsymbol{\sigma}}_\eta(\mathbf{x}) \, ; \quad \varepsilon_\eta(\mathbf{x}) = \exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\varepsilon}_\eta(\mathbf{x}) \, ; \quad \mathbf{u}_\eta(\mathbf{x}) = \exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\mathbf{u}}_\eta(\mathbf{x}) \, .$$

Plugging these solutions into the governing equations, we have

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot [\exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\boldsymbol{\sigma}}_\eta(\mathbf{x})] = -\omega^2 \rho_\eta(\mathbf{x}) \, \exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\mathbf{u}}_\eta(\mathbf{x}) \\
&\exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\boldsymbol{\sigma}}_\eta(\mathbf{x}) = \mathsf{C}_\eta(\mathbf{x}) : [\exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\varepsilon}_\eta(\mathbf{x})] \\
&\exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\varepsilon}_\eta(\mathbf{x}) = \tfrac{1}{2} \left\{ \boldsymbol{\nabla} \left[\exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\mathbf{u}}_\eta(\mathbf{x}) \right] + \left(\boldsymbol{\nabla} \left[\exp(i\mathbf{k} \cdot \mathbf{x}) \, \widehat{\mathbf{u}}_\eta(\mathbf{x}) \right] \right)^T \right\} \, .
\end{aligned}$$

It is easier to simplify these equations when we express them in terms of Cartesian components. For the momentum equation we have

$$\begin{aligned}
&\frac{\partial}{\partial x_j} [\exp(ik_m x_m) \, \widehat{\sigma}_{ji}] + \omega^2 \rho_\eta \, \exp(ik_m x_m) \, \widehat{u}_i = 0 \\
&\implies i \, \exp(ik_m x_m) \, k_n \delta_{nj} \, \widehat{\sigma}_{ji} + \exp(ik_m x_m) \, \frac{\partial}{\partial x_j} (\widehat{\sigma}_{ji}) + \omega^2 \rho_\eta \, \exp(ik_m x_m) \, \widehat{u}_i = 0 \\
&\implies i \, k_j \, \widehat{\sigma}_{ji} + \frac{\partial}{\partial x_j} (\widehat{\sigma}_{ji}) + \omega^2 \rho_\eta \, \widehat{u}_i = 0 \, .
\end{aligned}$$

In direct notation

$$i \, \mathbf{k} \cdot \widehat{\boldsymbol{\sigma}}_\eta + \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\sigma}}_\eta + \omega^2 \rho_\eta \, \widehat{\mathbf{u}} = \mathbf{0} \quad \square \tag{7.1}$$

The symmetry of the stress tensor allows us to write the above equation in the desired form The expression for the stress-strain relation is derived directly from the expanded stress-strain relation above to give

$$\boldsymbol{\sigma}_\eta = \mathsf{C}_\eta : \varepsilon_\eta \quad \square \tag{7.2}$$

For the strain-displacement relation we have

$$
\begin{aligned}
\exp(ik_m x_m)\,\widehat{\varepsilon}_{ij} &= \tfrac{1}{2}\left\{\frac{\partial}{\partial x_j}[\exp(ik_m x_m)\,\widehat{u}_i] + \frac{\partial}{\partial x_i}[\exp(ik_m x_m)\,\widehat{u}_j]\right\} \\
\implies \exp(ik_m x_m)\,\widehat{\varepsilon}_{ij} &= \tfrac{1}{2}\left\{i\exp(ik_m x_m)k_j\,\widehat{u}_i + \exp(ik_m x_m)\frac{\partial \widehat{u}_i}{\partial x_j}\right\} + \\
&\qquad \tfrac{1}{2}\left\{i\exp(ik_m x_m)k_i\,\widehat{u}_j + \exp(ik_m x_m)\frac{\partial \widehat{u}_j}{\partial x_i}\right\} \\
\implies \widehat{\varepsilon}_{ij} &= \tfrac{1}{2}(ik_j\widehat{u}_i + ik_i\widehat{u}_j) + \tfrac{1}{2}\left[\frac{\partial \widehat{u}_i}{\partial x_j} + \frac{\partial \widehat{u}_j}{\partial x_i}\right].
\end{aligned}
$$

In direct notation

$$
\widehat{\boldsymbol{\varepsilon}}_\eta = \frac{i}{2}(\widehat{\mathbf{u}}_\eta \otimes \mathbf{k} + \mathbf{k} \otimes \widehat{\mathbf{u}}_\eta) + \tfrac{1}{2}\left(\boldsymbol{\nabla}\widehat{\mathbf{u}}_\eta + [\boldsymbol{\nabla}\widehat{\mathbf{u}}_\eta]^T\right) \quad \square \tag{7.3}
$$

Problem 7.2 Verify that the matrix $\boldsymbol{A}$ defined by the action $\boldsymbol{A}\cdot\langle\mathbf{e}_0\rangle := (\mathbf{k}\cdot\langle\mathbf{e}_0\rangle)\mathbf{k}-(\mathbf{k}\cdot\mathbf{k})\,\langle\mathbf{e}_0\rangle$ can be expressed as $\boldsymbol{A} = \mathbf{k}\otimes\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\boldsymbol{1}$. Show that the eigenvalues of $-\boldsymbol{A}$ are

$$\lambda_1 = 0\,,\quad \lambda_2 = \lambda_3 = k_1^2 + k_2^2 + k_3^2 = \mathbf{k}\cdot\mathbf{k}\,.$$

What do these eigenvalues imply about the quantity $\mathbf{k}\cdot\langle\mathbf{e}_0\rangle$?

Solution - 7.2: If we express the equation that defined $\boldsymbol{A}$ in index notation, we have

$$\begin{aligned} A_{im}\,\langle e_m\rangle &:= (k_m\,\langle e_m\rangle)k_i - (k_pk_p)\,\langle e_i\rangle \\ &= (k_mk_i)\,\langle e_m\rangle - (k_pk_p)\,\langle e_i\rangle \\ &= [(k_mk_i) - (k_pk_p)\delta_{im}]\,\langle e_m\rangle\ . \end{aligned}$$

Therefore

$$A_{im} = k_ik_m - (k_pk_p)\delta_{im}\,.$$

In direct notation

$$\boldsymbol{A} = \mathbf{k}\otimes\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\boldsymbol{1} \quad \square \tag{7.4}$$

Writing the above out in matrix form, we have

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} - (k_1^2+k_2^2+k_3^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} k_1^2 & k_1k_2 & k_1k_3 \\ k_1k_2 & k_2^2 & k_1k_3 \\ k_1k_3 & k_2k_3 & k_3^2 \end{bmatrix} - \begin{bmatrix} k_1^2+k_2^2+k_3^2 & 0 & 0 \\ 0 & k_1^2+k_2^2+k_3^2 & 0 \\ 0 & 0 & k_1^2+k_2^2+k_3^2 \end{bmatrix} \\ &= \begin{bmatrix} -k_2^2-k_3^2 & k_1k_2 & k_1k_3 \\ k_1k_2 & -k_1^2-k_3^2 & k_1k_3 \\ k_1k_3 & k_2k_3 & -k_1^2-k_2^2 \end{bmatrix}\,. \end{aligned}$$

A straightforward eigenvalue calculation for the problem

$$-\boldsymbol{A}\cdot\langle\mathbf{e}_0\rangle = \lambda\,\langle\mathbf{e}_0\rangle$$

shows that the eigenvalues of $-\boldsymbol{A}$ are

$$\lambda_1 = 0\,,\quad \lambda_2 = \lambda_3 = (k_1^2+k_2^2+k_3^2) = \mathbf{k}\cdot\mathbf{k} \quad \square$$

Plugging in the non-zero eigenvalue into the equation gives

$$\boldsymbol{A}\cdot\langle\mathbf{e}_0\rangle = -(\mathbf{k}\cdot\mathbf{k})\,\langle\mathbf{e}_0\rangle\ .$$

If we compare this with the defining equation for $\boldsymbol{A}$ we see that

$$\mathbf{k}\cdot\langle\mathbf{e}_0\rangle = 0 \qquad \square \tag{7.5}$$

This means that the wave vector is perpendicular to the direction of the electric field vector.

Problem 7.3 Show that perturbation expansions of the Bloch wave equations for elastodynamics lead to the relations

$$\begin{aligned}\boldsymbol{\sigma}_0 &= \mathbf{C}_\eta : \varepsilon_0 \\ 0 &= \boldsymbol{\nabla}_y \mathbf{u}_0 + (\boldsymbol{\nabla}_y \mathbf{u}_0)^T \\ \varepsilon_0 &= \tfrac{i}{2}(\mathbf{u}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_0) + \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_1 + (\boldsymbol{\nabla}_y \mathbf{u}_1)^T]\,.\end{aligned}$$

Solution - 7.3: The Bloch wave equations for elastodynamics under quasistatic conditions (low frequency) are

$$\begin{aligned}&\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\sigma}}_\eta + i\widehat{\boldsymbol{\sigma}}_\eta \cdot \mathbf{k} + \omega^2 \rho_\eta \widehat{\mathbf{u}}_\eta = 0 \\ &\widehat{\boldsymbol{\varepsilon}}_\eta = \tfrac{i}{2}\left(\widehat{\mathbf{u}}_\eta \otimes \mathbf{k} + \mathbf{k} \otimes \widehat{\mathbf{u}}_\eta\right) + \tfrac{1}{2}\left[\boldsymbol{\nabla}\widehat{\mathbf{u}}_\eta + (\boldsymbol{\nabla}\widehat{\mathbf{u}}_\eta)^T\right] \\ &\widehat{\boldsymbol{\sigma}}_\eta = \mathbf{C}_\eta : \widehat{\varepsilon}_\eta\,.\end{aligned}$$

Using perturbations of the form

$$\begin{aligned}\widehat{\boldsymbol{\sigma}}_\eta(\mathbf{x}) &= \boldsymbol{\sigma}_0(\mathbf{y}) + \eta\boldsymbol{\sigma}_1(\mathbf{y}) + \eta^2\boldsymbol{\sigma}_2(\mathbf{y}) + \dots \\ \widehat{\varepsilon}_\eta(\mathbf{x}) &= \varepsilon_0(\mathbf{y}) + \eta\varepsilon_1(\mathbf{y}) + \eta^2\varepsilon_2(\mathbf{y}) + \dots \\ \widehat{\mathbf{u}}_\eta(\mathbf{x}) &= \mathbf{u}_0(\mathbf{y}) + \eta\mathbf{u}_1(\mathbf{y}) + \eta^2\mathbf{u}_2(\mathbf{y}) + \dots \\ \omega &= \omega_0^j + \eta\omega_1^j + \eta^2\omega_2^j + \dots\end{aligned}$$

where $\mathbf{y} = \mathbf{x}/\eta$, we can write the Bloch form of the momentum equation as

$$\begin{aligned}\boldsymbol{\nabla} \cdot [\boldsymbol{\sigma}_0(\mathbf{x}/\eta) + \eta\boldsymbol{\sigma}_1(\mathbf{x}/\eta) + \eta^2\boldsymbol{\sigma}_2(\mathbf{x}/\eta) + \dots] + i[\boldsymbol{\sigma}_0(\mathbf{y}) + \eta\boldsymbol{\sigma}_1(\mathbf{y}) + \eta^2\boldsymbol{\sigma}_2(\mathbf{y}) + \dots] \cdot \mathbf{k} + \\ \rho_\eta(\omega_0^j + \eta\omega_1^j + \eta^2\omega_2^j + \dots)^2[\mathbf{u}_0(\mathbf{y}) + \eta\mathbf{u}_1(\mathbf{y}) + \eta^2\mathbf{u}_2(\mathbf{y}) + \dots] = 0\,.\end{aligned}$$

Now,

$$\begin{aligned}\boldsymbol{\nabla} \cdot [\boldsymbol{\sigma}_0(\mathbf{x}/\eta)] &\equiv \frac{\partial}{\partial x_j}[\sigma_{jp}(\mathbf{x}/\eta)] = \frac{\partial}{\partial y_m}[\sigma_{jp}(\mathbf{y})]\,\frac{\partial y_m}{\partial x_j} = \frac{\partial}{\partial y_m}[\sigma_{jp}(\mathbf{y})]\,\frac{1}{\eta}\delta_{mj} = \frac{1}{\eta}\frac{\partial}{\partial y_j}[\sigma_{jp}(\mathbf{y})] \\ &\equiv \frac{1}{\eta}\boldsymbol{\nabla}_y \cdot [\boldsymbol{\sigma}_0(\mathbf{y})]\,.\end{aligned}$$

Therefore we can write the momentum equation as

$$\begin{aligned}\boldsymbol{\nabla}_y \cdot \left[\frac{1}{\eta}\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_1 + \eta\boldsymbol{\sigma}_2 + \dots\right] + i[\boldsymbol{\sigma}_0 + \eta\boldsymbol{\sigma}_1 + \eta^2\boldsymbol{\sigma}_2 + \dots] \cdot \mathbf{k} + \\ \rho_\eta(\omega_0^j + \eta\omega_1^j + \eta^2\omega_2^j + \dots)^2[\mathbf{u}_0 + \eta\mathbf{u}_1 + \eta^2\mathbf{u}_2 + \dots] = 0\,.\end{aligned}$$

If we equate terms containing the same order of η, we have

$$\begin{aligned}\boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_0 &= 0 \\ \boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_1 + i\boldsymbol{\sigma}_0 \cdot \mathbf{k} + (\omega_0^j)^2\rho_\eta\mathbf{u}_0 &= 0 \\ \boldsymbol{\nabla}_y \cdot \boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}_1 \cdot \mathbf{k} + 2\omega_0^j\omega_1^j\rho_\eta\mathbf{u}_1 &= 0 \\ &\dots\end{aligned}$$

If we follow the same procedure for the constitutive equation we get

$$\boldsymbol{\sigma}_0 + \eta\,\boldsymbol{\sigma}_1 + \dots = \mathbf{C}_\eta : (\varepsilon_0 + \eta\,\varepsilon_1 + \dots)\,.$$

The first term of the expansion lead to the required result

$$\boldsymbol{\sigma}_0 = \mathbf{C}_\eta : \varepsilon_0 \qquad \square \tag{7.6}$$

Similarly, the strain-displacement relation can be written as

$$\varepsilon_0 + \eta\,\varepsilon_1 + \dots = \frac{i}{2}[(\mathbf{u}_0 + \eta\,\mathbf{u}_1 + \dots) \otimes \mathbf{k} + \mathbf{k} \otimes (\mathbf{u}_0 + \eta\,\mathbf{u}_1 + \dots)] + \frac{1}{2\eta}\left(\boldsymbol{\nabla}_y(\mathbf{u}_0 + \eta\,\mathbf{u}_1 + \dots) + [\boldsymbol{\nabla}_y(\mathbf{u}_0 + \eta\,\mathbf{u}_1 + \dots)]^T\right)$$

Equating terms of the same order in η, we have

$$\begin{aligned} \mathbf{0} &= \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_0 + (\boldsymbol{\nabla}_y \mathbf{u}_0)^T] \\ \varepsilon_0 &= \tfrac{i}{2}(\mathbf{u}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_0) + \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_1 + (\boldsymbol{\nabla}_y \mathbf{u}_1)^T] \\ \varepsilon_1 &= \tfrac{i}{2}(\mathbf{u}_1 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_1) + \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_2 + (\boldsymbol{\nabla}_y \mathbf{u}_2)^T] \\ \dots \end{aligned}$$

The desired equations are

$$\begin{aligned} \mathbf{0} &= \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_0 + (\boldsymbol{\nabla}_y \mathbf{u}_0)^T] \\ \varepsilon_0 &= \tfrac{i}{2}(\mathbf{u}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_0) + \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_1 + (\boldsymbol{\nabla}_y \mathbf{u}_1)^T] \qquad \square \end{aligned} \tag{7.7}$$

Problem 7.4 Show that we can express the first order term in the perturbation expansion of $\widehat{\varepsilon}_\eta$ in the form

$$\boldsymbol{\varepsilon}_0 = \tfrac{1}{2}\left(\boldsymbol{\nabla}_y \widehat{\mathbf{u}}(\mathbf{y}) + [\boldsymbol{\nabla}_y \widehat{\mathbf{u}}(\mathbf{y})]^T\right)$$

where $\widehat{\mathbf{u}}(\mathbf{y}) = i(\mathbf{k}\cdot\mathbf{y})\mathbf{u}_0(\mathbf{y}) + \mathbf{u}_1(\mathbf{y})$.

Solution - 7.4: Recall that

$$\boldsymbol{\varepsilon}_0 = \tfrac{i}{2}(\mathbf{u}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_0) + \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_1 + (\boldsymbol{\nabla}_y \mathbf{u}_1)^T]\,.$$

Expressed in index notation,

$$\varepsilon^0_{ij} = \tfrac{i}{2}(u^0_i k_j + k_i u^0_j) + \tfrac{1}{2}\left[\frac{\partial u^1_i}{\partial y_j} + \frac{\partial u^1_j}{\partial y_i}\right]\,.$$

Similarly, the expression

$$\boldsymbol{\varepsilon}_0 = \tfrac{1}{2}\,\left(\boldsymbol{\nabla}_y \widehat{\mathbf{u}}(\mathbf{y}) + [\boldsymbol{\nabla}_y \widehat{\mathbf{u}}(\mathbf{y})]^T\right)$$

where $\widehat{\mathbf{u}}(\mathbf{y}) = i(\mathbf{k}\cdot\mathbf{y})\mathbf{u}_0(\mathbf{y}) + \mathbf{u}_1(\mathbf{y})$ can be written in index notation as

$$\begin{aligned}
\varepsilon^0_{ij} &= \tfrac{1}{2}\left[\frac{\partial \widehat{u}_i}{\partial y_j} + \frac{\partial \widehat{u}_j}{\partial y_i}\right] \\
&= \tfrac{1}{2}\left[\frac{\partial}{\partial y_j}(ik_p y_p u^0_i + u^1_i) + \frac{\partial}{\partial y_i}(ik_p y_p u^0_j + u^1_j)\right] \\
&= \tfrac{1}{2}\left[ik_p\delta_{pj}u^0_i + ik_p y_p\frac{\partial u^0_i}{\partial y_j} + \frac{\partial u^1_i}{\partial y_j} + ik_p\delta_{pi}u^0_j + ik_p y_p\frac{\partial u^0_j}{\partial y_i} + \frac{\partial u^1_j}{\partial y_i}\right] \\
&= \tfrac{1}{2}\left[ik_j u^0_i + ik_p y_p\frac{\partial u^0_i}{\partial y_j} + \frac{\partial u^1_i}{\partial y_j} + ik_i u^0_j + ik_p y_p\frac{\partial u^0_j}{\partial y_i} + \frac{\partial u^1_j}{\partial y_i}\right]\,.
\end{aligned}$$

Since

$$\boldsymbol{\nabla}_y \mathbf{u}_0 + (\boldsymbol{\nabla}_y \mathbf{u}_0)^T \equiv \frac{\partial u^0_i}{\partial y_j} + \frac{\partial u^0_j}{\partial y_i} = 0$$

we have

$$\varepsilon^0_{ij} = \tfrac{1}{2}\left[ik_j u^0_i + \frac{\partial u^1_i}{\partial y_j} + ik_i u^0_j + \frac{\partial u^1_j}{\partial y_i}\right] = \tfrac{i}{2}(u^0_i k_j + k_i u^0_j) + \tfrac{1}{2}\left[\frac{\partial u^1_i}{\partial y_j} + \frac{\partial u^1_j}{\partial y_i}\right]\,.$$

Therefore,

$$\boldsymbol{\varepsilon}_0 = \tfrac{i}{2}(\mathbf{u}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_0) + \tfrac{1}{2}[\boldsymbol{\nabla}_y \mathbf{u}_1 + (\boldsymbol{\nabla}_y \mathbf{u}_1)^T] = \tfrac{1}{2}\left(\boldsymbol{\nabla}_y \widehat{\mathbf{u}}(\mathbf{y}) + [\boldsymbol{\nabla}_y \widehat{\mathbf{u}}(\mathbf{y})]^T\right) \qquad \square \tag{7.8}$$

Problem 7.5 Show that the volume averaged gradient, divergence, and curl are zero for a periodic vector-valued function $\mathbf{u}(\mathbf{x})$ which satisfies the relation $\mathbf{u}(\mathbf{y}+\mathbf{a}) = \mathbf{u}(\mathbf{y})$ within a unit cell with lattice vector **a**.

Solution - 7.5: Recall that **x** is the macroscopic position and $\mathbf{y} = \mathbf{x}/\eta$ is the local (microscopic) position within the unit cell. The volume average of the vector-valued function $\mathbf{u}(\mathbf{x})$ inside in the unit cell Y is defined as

$$\langle \mathbf{u}(\mathbf{x}) \rangle = \frac{1}{V} \int_Y \mathbf{u}(\mathbf{x}, \mathbf{y}) dY \quad \text{where} \quad V = \int_Y dY\,.$$

If there are internal surfaces ∂Y_{int} in the RVE (holes, cracks etc.), then

$$\boldsymbol{\nabla}_{\mathbf{x}} \, \langle \mathbf{u}(\mathbf{x}) \rangle = \left\langle \frac{1}{\eta} \boldsymbol{\nabla}_y \mathbf{u}(\mathbf{x}, \mathbf{y}) \right\rangle - \frac{1}{V} \int_{\partial Y_{\text{int}}} \mathbf{u}(\mathbf{x}, \mathbf{y}) \otimes \mathbf{n}(\mathbf{x}, \mathbf{y}) \, dS$$

where **n** is the outward normal to the internal surfaces (away from the body). Because of periodicity, the gradient of $\langle \mathbf{u}(\mathbf{x}) \rangle = 0$ and therefore (in the absence of internal surfaces)

$$\langle \boldsymbol{\nabla}_y \mathbf{u}(\mathbf{x}, \mathbf{y}) \rangle = 0 \quad \square \tag{7.9}$$

We can use similar arguments for the divergence and the curl.

Problem 7.6 For a periodic elastic composite show that

$$\frac{1}{\langle \rho_\eta \rangle}(\mathbf{k} \cdot \mathbf{C}_{\text{eff}} \cdot \mathbf{k}) \cdot \mathbf{u}_0 = (\omega_0^j)^2 \mathbf{u}_0 \,.$$

Solution - 7.6: Let us start with the relation (7.22):

$$i \langle \boldsymbol{\sigma}_0 \rangle \cdot \mathbf{k} + \langle \rho_\eta \rangle (\omega_0^j)^2 \mathbf{u}_0 = 0$$

where

$$\langle \boldsymbol{\sigma}_0 \rangle = \frac{i}{2} \mathbf{C}_{\text{eff}} : (\mathbf{u}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{u}_0) \,.$$

Combining these we have

$$-\tfrac{1}{2} \left[\mathbf{C}_{\text{eff}} : (\widehat{\mathbf{u}}_0 \otimes \mathbf{k} + \mathbf{k} \otimes \widehat{\mathbf{u}}_0) \right] \cdot \mathbf{k} + \langle \rho_\eta \rangle \; (\omega_0^j)^2 \; \widehat{\mathbf{u}}_0 = 0 \,.$$

From the minor symmetry of the tensor $\mathbf{C}_{\text{eff}}$, i.e.,

$$C^{\text{eff}}_{ijkl} = C^{\text{eff}}_{ijlk}$$

we have

$$\mathbf{C}_{\text{eff}} : (\widehat{\mathbf{u}}_0 \otimes \mathbf{k}) = \mathbf{C}_{\text{eff}} : (\mathbf{k} \otimes \widehat{\mathbf{u}}_0) \,.$$

Hence

$$[\mathbf{C}_{\text{eff}} : (\mathbf{k} \otimes \widehat{\mathbf{u}}_0)] \cdot \mathbf{k} = \langle \rho_\eta \rangle \; (\omega_0^j)^2 \; \widehat{\mathbf{u}}_0$$

or

$$\boldsymbol{A}_{\text{eff}}(\mathbf{k}) \cdot \widehat{\mathbf{u}}_0 = (\omega_0^j)^2 \; \widehat{\mathbf{u}}_0$$

where, for any vector $\mathbf{v}$,

$$\boldsymbol{A}_{\text{eff}}(\mathbf{k}) \cdot \mathbf{v} = \frac{1}{\langle \rho_\eta \rangle} \; [\mathbf{C}_{\text{eff}} : (\mathbf{k} \otimes \mathbf{v})] \cdot \mathbf{k} \,.$$

Reorganization of the above equation shows that

$$\boldsymbol{A}_{\text{eff}}(\mathbf{k}) = \frac{1}{\langle \rho_\eta \rangle}(\mathbf{k} \cdot \mathbf{C}_{\text{eff}} \cdot \mathbf{k}) \qquad \square \tag{7.10}$$

and the required result follows. The quantity $\boldsymbol{A}_{\text{eff}}$ is the **effective acoustic tensor.**

Problem 7.7 Derive Hashin's estimate for the effective bulk modulus of a coated sphere:

$$\kappa_{\text{eff}} = \kappa_2 + \cfrac{f_1}{\cfrac{1}{\kappa_1 - \kappa_2} + \cfrac{f_2}{\kappa_2 + \cfrac{4}{3}\,\mu_2}}$$

after verifying the expressions for the radial displacements and the traction continuity conditions.

Solution - 7.7: The sample Mathematica notebook below shows the steps.

ur1 = a1r;

ur2 = a2r + b2/r^2;

ureff = aeff$r$;

sig1 = lam1(D[ur1, r] + 2ur1/r) + 2mu1D[ur1, r]

sig2 = lam2(D[ur2, r] + 2ur2/r) + 2mu2D[ur2, r]

sigeff = lameff(D[ureff, r] + 2ureff/r) + 2mueff$D$[ureff, r]

$sig1 \to 3\text{a1lam1} + 2\text{a1mu1}$

$$sig2 \to 2\text{mu2}\left(\text{a2} - \frac{2\text{b2}}{r^3}\right) + \text{lam2}\left(\text{a2} - \frac{2\text{b2}}{r^3} + \frac{2\left(\frac{\text{b2}}{r^2} + \text{a2}r\right)}{r}\right)$$

$sigeff \to 3\text{aefflameff} + 2\text{aeffmueff}$

ur1a = ur1/.$r \to$ rs

ur2a = ur2 /.$r \to$ rs

ureffb = ureff/.$r \to$ rc

ur2b = ur2/.$r \to$ rc

sig1a = sig1/.{$r \to$ rs}//FullSimplify

sig2a = sig2 /.{$r \to$ rs}//FullSimplify

sig2b = sig2/.{$r \to$ rc}//FullSimplify

a1rs

$$\frac{\text{b2}}{\text{rs}^2} + \text{a2rs}$$

aeffrc

$$\frac{\text{b2}}{\text{rc}^2} + \text{a2rc}$$

$$\text{a1}(3\text{lam1} + 2\text{mu1})$$

$$3\text{a2lam2} + 2\text{a2mu2} - \frac{4\text{b2mu2}}{\text{rs}^3}$$

$$3\text{a2lam2} + 2\text{a2mu2} - \frac{4\text{b2mu2}}{\text{rc}^3}$$

dispa = ur2a − ur1a

dispb = ureffb − ur2b

siga = sig2a − sig1a

sigb = *p* + sig2b

sigeffb = *p* + sigeff

siga0 =

siga/.{lam1 → kap1 − 2/3mu1, lam2 → kap2 − 2/3mu2,

lam3 → kap3 − 2/3mu3}//FullSimplify

sigb0 =

sigb/.{lam1 → kap1 − 2/3mu1, lam2 → kap2 − 2/3mu2,

lam3 → kap3 − 2/3mu3}//FullSimplify

sigeffb0 =

sigeffb/.{lam1 → kap1 − 2/3mu1, lam2 → kap2 − 2/3mu2,

lameff → kapeff − 2/3mueff}//FullSimplify

sol = Solve[{dispa == 0, siga0 == 0, sigb0 == 0}, {a1, a2, b2}]

a1val = a1/.sol//FullSimplify

a2val = a2/.sol//FullSimplify

b2val = b2/.sol//FullSimplify

sol1 = Solve[{sigeffb0 == 0}, {aeff}]

aeffval = aeff/.sol1

$$\frac{\text{b2}}{\text{rs}^2} - \text{a1rs} + \text{a2rs}$$

$$-\frac{\text{b2}}{\text{rc}^2}-\text{a2rc}+\text{aeffrc}$$

$$3\text{a2lam2}-\text{a1}(3\text{lam1}+2\text{mu1})+2\text{a2mu2}-\frac{4\text{b2mu2}}{\text{rs}^3}$$

$$3\text{a2lam2}+2\text{a2mu2}+p-\frac{4\text{b2mu2}}{\text{rc}^3}$$

$$3\text{aefflameff}+2\text{aeffmueff}+p$$

$$-3\text{a1kap1}+3\text{a2kap2}-\frac{4\text{b2mu2}}{\text{rs}^3}$$

$$3\text{a2kap2}+p-\frac{4\text{b2mu2}}{\text{rc}^3}$$

$$3\text{aeffkapeff}+p$$

$$\left\{a1\to-\frac{(3\text{kap2}+4\text{mu2})p\text{rc}^3}{3\text{kap2}(3\text{kap1}+4\text{mu2})\text{rc}^3+12(\text{kap1}-\text{kap2})\text{mu2rs}^3}\right\}$$

$$\left\{a2\to-\frac{(3\text{kap1}+4\text{mu2})p\text{rc}^3}{3\text{kap2}(3\text{kap1}+4\text{mu2})\text{rc}^3+12(\text{kap1}-\text{kap2})\text{mu2rs}^3}\right\}$$

$$\left\{b2\to\frac{(\text{kap1}-\text{kap2})p\text{rc}^3\text{rs}^3}{\text{kap2}(3\text{kap1}+4\text{mu2})\text{rc}^3+4(\text{kap1}-\text{kap2})\text{mu2rs}^3}\right\}$$

$$\left\{\left\{\text{aeff}\to-\frac{p}{3\text{kapeff}}\right\}\right\}$$

$$\left\{-\frac{p}{3\text{kapeff}}\right\}$$

dispb0 = dispb/.{aeff → aeffval, a2 → a2val, b2 → b2val}//

FullSimplify

soleff = Solve[dispb0 == 0, {kapeff}]

kapeffval = kapeff/.soleff//FullSimplify

kapeffval0 = kapeffval/.rs → f1^(1/3)rc//FullSimplify

Collect[Numerator[kapeffval0]/3, {f1, kap2}]

Collect[Denominator[kapeffval0]/3, {f1, kap2}]

kapeffval0/.{f1 → 0}//FullSimplify

$$\left\{\frac{1}{3}p\text{rc}\left(-\frac{1}{\text{kapeff}}+\frac{(3\text{kap1}+4\text{mu2})\text{rc}^3+3(-\text{kap1}+\text{kap2})\text{rs}^3}{\text{kap2}(3\text{kap1}+4\text{mu2})\text{rc}^3+4(\text{kap1}-\text{kap2})\text{mu2rs}^3}\right)\right\}$$

$$\left\{\left\{\text{kapeff}\to\frac{3\text{kap1kap2rc}^3+4\text{kap2mu2rc}^3+4\text{kap1mu2rs}^3-4\text{kap2mu2rs}^3}{3\text{kap1rc}^3+4\text{mu2rc}^3-3\text{kap1rs}^3+3\text{kap2rs}^3}\right\}\right\}$$

$$\left\{\frac{\text{kap2}(3\text{kap1}+4\text{mu2})\text{rc}^3+4(\text{kap1}-\text{kap2})\text{mu2rs}^3}{(3\text{kap1}+4\text{mu2})\text{rc}^3+3(-\text{kap1}+\text{kap2})\text{rs}^3}\right\}$$

$$\left\{\frac{3\text{kap1kap2}+4\text{f1kap1mu2}+4\text{kap2mu2}-4\text{f1kap2mu2}}{3\text{kap1}-3\text{f1kap1}+3\text{f1kap2}+4\text{mu2}}\right\}$$

$$\left\{\frac{1}{3}\text{kap2}(3\text{kap1}+4\text{mu2})+\text{f1}\left(\frac{4\text{kap1mu2}}{3}-\frac{4\text{kap2mu2}}{3}\right)\right\}$$

$$\left\{\text{f1}(-\text{kap1}+\text{kap2})+\frac{1}{3}(3\text{kap1}+4\text{mu2})\right\}$$

$\{\text{kap2}\}$

$$\textbf{KeffWater} = \frac{\textbf{2mu2((3 + 6f1)kap1 + 4(f2)mu2)}}{\textbf{9(f2)kap1 + 6(2 + f1)mu2}}$$

Limit[KeffWater, f1 → 0]

KeffH = kap2 + f1/(1/(kap1 − kap2) + (1 − f1)/(kap2 + 4/3mu2))

kapeffval0 − KeffH//FullSimplify

$$\frac{2\text{mu2}((3+6\text{f1})\text{kap1}+4\text{f2mu2})}{9\text{f2kap1}+6(2+\text{f1})\text{mu2}}$$

$$\frac{2\text{mu2}(3\text{kap1}+4\text{f2mu2})}{9\text{f2kap1}+12\text{mu2}}$$

$$\text{kap2} + \frac{\text{f1}}{\frac{1}{\text{kap1} - \text{kap2}} + \frac{1 - \text{f1}}{\text{kap2} + \frac{4\text{mu2}}{3}}}$$

$$\{0\}$$

invKeff = 1/kapeffval0

Limit[invKeff, kap2 → Infinity]

$$\left\{\frac{3\text{kap1} - 3\text{f1kap1} + 3\text{f1kap2} + 4\text{mu2}}{3\text{kap1kap2} + 4\text{f1kap1mu2} + 4\text{kap2mu2} - 4\text{f1kap2mu2}}\right\}$$

$$\left\{\frac{3\text{f1}}{3\text{kap1} - 4(-1 + \text{f1})\text{mu2}}\right\}$$

Problem 7.8 Show that Maxwell's equations for a Bloch periodic medium can be expressed as

$$i\mathbf{k}\cdot\mathbf{d}_\eta(\mathbf{x}) + \boldsymbol{\nabla}\cdot\mathbf{d}_\eta = 0, \qquad i\mathbf{k}\cdot\mathbf{b}_\eta(\mathbf{x}) + \boldsymbol{\nabla}\cdot\mathbf{b}_\eta = 0,$$
$$i\mathbf{k}\times\mathbf{e}_\eta(\mathbf{x}) + \boldsymbol{\nabla}\times\mathbf{e}_\eta - i\omega\mathbf{b}_\eta(\mathbf{x}) = 0, \qquad i\mathbf{k}\times\mathbf{h}_\eta(\mathbf{x}) + \boldsymbol{\nabla}\times\mathbf{h}_\eta + i\omega\mathbf{d}_\eta(\mathbf{x}) = 0\,.$$

Solution - 7.8: For a Bloch periodic medium, Maxwell's equations at fixed frequency are

$$\boldsymbol{\nabla}\cdot\mathbf{D}_\eta = 0\,; \quad \boldsymbol{\nabla}\cdot\mathbf{B}_\eta = 0\,; \quad \boldsymbol{\nabla}\times\mathbf{E}_\eta = i\omega\mathbf{B}_\eta\,; \quad \boldsymbol{\nabla}\times\mathbf{H}_\eta = -i\omega\mathbf{D}_\eta$$

and the constitutive relations are

$$\mathbf{D}_\eta = \varepsilon_\eta\,\mathbf{E}_\eta\,; \quad \mathbf{B}_\eta = \mu_\eta\,\mathbf{H}_\eta\,.$$

Periodic solutions to these equations have the form

$$\mathbf{D}_\eta = \exp(i\mathbf{k}\cdot\mathbf{x})\,\mathbf{d}_\eta\,; \quad \mathbf{B}_\eta = \exp(i\mathbf{k}\cdot\mathbf{x})\,\mathbf{b}_\eta\,; \quad \mathbf{E}_\eta = \exp(i\mathbf{k}\cdot\mathbf{x})\,\mathbf{e}_\eta\,; \quad \mathbf{H}_\eta = \exp(i\mathbf{k}\cdot\mathbf{x})\,\mathbf{h}_\eta\,.$$

Plug these solutions into Maxwell's equations to get

$$\begin{aligned}
& i\mathbf{k}\cdot\mathbf{d}_\eta + \boldsymbol{\nabla}\cdot\mathbf{d}_\eta = 0 \\
& i\mathbf{k}\cdot\mathbf{b}_\eta + \boldsymbol{\nabla}\cdot\mathbf{b}_\eta = 0 \\
& i\mathbf{k}\times\mathbf{e}_\eta(\mathbf{x}) + \boldsymbol{\nabla}\times\mathbf{e}_\eta - i\omega\mathbf{b}_\eta(\mathbf{x}) = 0 \\
& i\mathbf{k}\times\mathbf{h}_\eta(\mathbf{x}) + \boldsymbol{\nabla}\times\mathbf{h}_\eta + i\omega\mathbf{d}_\eta(\mathbf{x}) = 0\,.
\end{aligned}$$

The above calculations are straightforward in index notation.

Problem 7.9 Show, using an approach similar to that used to obtain the effective bulk modulus, that Hashin's estimate for the effective electrical permittivity of a coated sphere can be expressed as

$$\varepsilon_{\text{eff}} = \varepsilon_2 + \frac{3f_1\varepsilon_2(\varepsilon_1 - \varepsilon_2)}{3\varepsilon_2 + f_2(\varepsilon_1 - \varepsilon_2)}$$

where ε_1 is the permittivity of the sphere, ε_2 is the permittivity of the coating, and $f_1 = 1 - f_2$ is the volume fraction occupied by the sphere.

Solution - 7.9: Let us assume that the sphere is centered at the origin. We look for a solution to the time-independent Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{E} = 0\,; \;\; \boldsymbol{\nabla} \cdot \mathbf{D} = 0\,; \;\; \mathbf{D} = \epsilon\, \mathbf{E}$$

with potentials

$$\begin{aligned} \varphi_1(x) &= a_1\, r\, \cos\theta && \text{in the core} \\ \varphi_2(x) &= \left(a_2\, r + \frac{b_2}{r^2}\right) \cos\theta && \text{in the coating} \\ \varphi_{\text{eff}}(x) &= a_{\text{eff}}\, r\, \cos\theta && \text{in the effective medium.} \end{aligned}$$

Then the electric field is given by

$$\begin{aligned} \mathbf{E}_1 &= \boldsymbol{\nabla}\varphi_1(x) = a_1\, [\cos\theta\, \mathbf{e}_r - \sin\theta\, \mathbf{e}_\theta] \\ \mathbf{E}_2 &= \boldsymbol{\nabla}\varphi_2(x) = \left(a_2\, r - \frac{2b_2}{r^3}\right) \cos\theta\, \mathbf{e}_r - \left(a_2\, r + \frac{b_2}{r^3}\right) \sin\theta\, \mathbf{e}_\theta \\ \mathbf{E}_{\text{eff}} &= \boldsymbol{\nabla}\varphi_{\text{eff(x)}} = a_{\text{eff}}\, [\cos\theta\, \mathbf{e}_r - \sin\theta\, \mathbf{e}_\theta]\,. \end{aligned}$$

The potentials satisfy Laplace's equations

$$\nabla^2\varphi_1 = \nabla^2\varphi_2 = \nabla^2\varphi_{\text{eff}} = 0$$

and we only need to match the boundary conditions at the interfaces to get a solution, i.e.,

$$a_1\, r_c = a_2\, r_c + \frac{b_2}{r_c^2}\,; \;\; a_{\text{eff}}\, r_e = a_2\, r_e + \frac{b_2}{r_e^2}\,.$$

Continuity of the tangential component of the electric field at the interface implies that

$$\epsilon_1\, a_1 = \epsilon_2\, \left[a_2 - \frac{2b_2}{r_c^3}\right]\;;\;\; \epsilon_{\text{eff}}\, a_{\text{eff}} = \epsilon_2\, \left[a_2 - \frac{2b_2}{r_e^3}\right]\,.$$

Combining the above equations gives

$$\frac{b_2}{r_c^3}\left[1 + \frac{3\,\epsilon_2}{\epsilon_1 - \epsilon_2}\right] = \frac{b_2}{r_e^3}\left[1 + \frac{3\,\epsilon_2}{\epsilon_{\text{eff}} - \epsilon_2}\right]\,.$$

Defining the volume fraction f as

$$f_1 = 1 - f_2 = \frac{r_c^3}{r_e^3}$$

leads to the following expression for ϵ_{eff}:

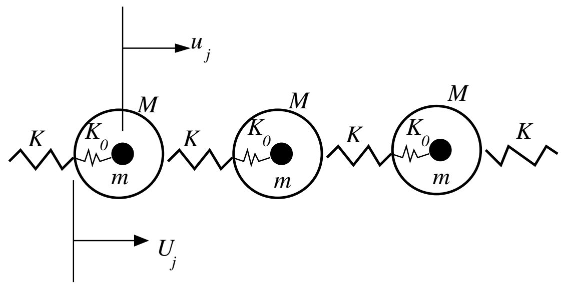
$$\epsilon_{\text{eff}} = \epsilon_2 + \frac{3\,f_1\,\epsilon_2(\epsilon_1 - \epsilon_2)}{3\,\epsilon_2 + f_2\,(\epsilon_1 - \epsilon_2)} \qquad \square \tag{7.11}$$

Problem 7.10 Consider an infinite one-dimensional lattice of circular rings connected to each other by linear springs. The mass of each ring is M and the stiffness of each spring is K. Each ring contains an additional mass m that is connected to the ring by a spring that has a stiffness K_0. Show that the dispersion relation for the lattice is

$$mM\omega^4 - [(m+M)K_0 + 2mK(1-\cos ka)] + 2KK_0(1-\cos ka) = 0$$

where a is the lattice spacing, k is the wave number, and ω is the frequency. Plot the dispersion relation for the lattice.

Solution - 7.10: Several approaches can be used to determine the equations of motion of the system. Let us follow the approach of Huang et al. (2009).



From the figure above we see that the Lagrangian of the j-th unit cell of the system is

$$\mathcal{L} = \tfrac{1}{2}K(U_j - U_{j-1})^2 + \tfrac{1}{2}K(U_{j+1} - U_j)^2 - \tfrac{1}{2}M\dot{U}_j^2 + \tfrac{1}{2}K_0(u_j - U_j)^2 - \tfrac{1}{2}m\dot{u}_j^2\,.$$

The Euler equations are

$$\frac{\partial \mathcal{L}}{\partial u_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}_i}\right) = 0\,.$$

Then, for the j-th cell, the governing equations are the Euler equations

$$\begin{aligned}&K_0(U_j - u_j) - K(U_{j-1} - 2U_j + U_{j+1}) + M\ddot{U}_j = 0\\&K_0(u_j - U_j) + m\ddot{u}_j = 0\,.\end{aligned}$$

For time-harmonic problems with frequency ω, we can write the above as

$$\begin{aligned}&K_0(U_j - u_j) - K(U_{j-1} - 2U_j + U_{j+1}) - M\omega^2 U_j = 0\\&K_0(u_j - U_j) - m\omega^2 u_j = 0\,.\end{aligned}$$

From Bloch's theorem we know that for a unit cell of length a, we have

$$U_{j-1} = \exp(-ika)\,U_j \quad \text{and} \quad U_{j+1} = \exp(ika)\,U_j\,.$$

Substituting these into the above equations, we have

$$\begin{aligned}&-K_0 u_j + \{K_0 - K[\exp(-ika) + \exp(ika) - 2] - M\omega^2\}U_j = 0\\&(K_0 - m\omega^2)u_j - K_0 U_j = 0\,.\end{aligned}$$

In matrix form,

$$\begin{bmatrix} K_0 - K[\exp(-ika) + \exp(ika) - 2] - M\omega^2 & -K_0 \\ -K_0 & K_0 - m\omega^2 \end{bmatrix} \begin{bmatrix} U_j \\ u_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\,.$$

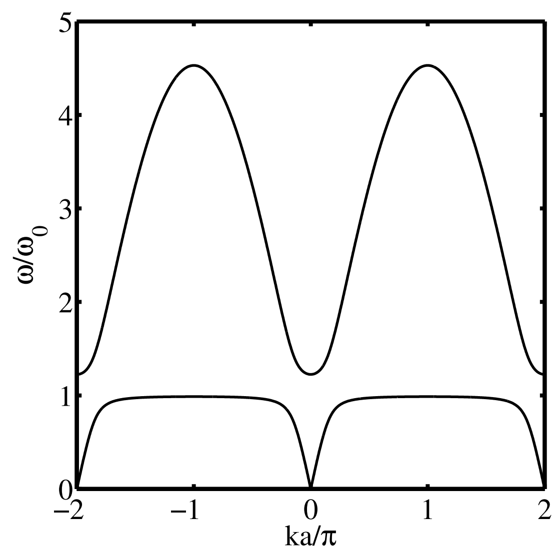
The dispersion relation is given by the determinant of the 2×2 matrix,

$$mM\omega^4 - \left[K_0(m+M) + Km\left(2 - e^{-ika} - e^{ika}\right)\right]\omega^2 + K_0K\left(2 - e^{-ika} - e^{ika}\right) = 0$$

or,

$$mM\omega^4 - \left[K_0(m+M) + 2Km\left(1 - \cos ka\right)\right]\omega^2 + 2K_0K\left(1 - \cos ka\right) = 0 \quad \square \tag{7.12}$$

The dispersion plot is shown below.



The Matlab script used to generate the above plot is given below.

```
function plotDispersion
  K0 = 0.1;
  m = 1;
  K = 1;
  M = 2;
  a = 1;
  b = pi/a;
  k = -2*b:0.01:2*b;

  for ii=1:length(k)
    [omega(ii,:)] = calcOmega(K0, K, m, M, a, k(ii));
  end
  omega0 = sqrt(K0/m);
  p1 = plot(k/b, omega(:,1)/omega0); hold on;
  p2 = plot(k/b, omega(:,2)/omega0); hold on;
  set(gca, 'LineWidth', 3, 'FontName', 'times', 'FontSize', 22);
  set([p1 p2],  'LineWidth', 2, 'Color', 'k');
  axis square;
  xlabel('ka/\pi','FontName', 'times', 'FontSize', 22);
  ylabel('\omega/\omega_0','FontName', 'times', 'FontSize', 22);
  set(gca, 'XTick', [-2 -1 0 1 2]);

function [omega] = calcOmega(K0, K, m, M, a, k)

   t1 = K0*(m+M) + 2*K*m*(1 - cos(k*a));
   t2 = -8*K0*K*m*M*(1 - cos(k*a)) + (2*K*m + K0*(m+M) - 2*K*m*cos(k*a))^2;
```

```
t3 = sqrt(2*m*M);

omega(1) = sqrt(t1 - sqrt(t2))/t3 ;
omega(2) = sqrt(t1 + sqrt(t2))/t3 ;
omega(3) = - sqrt(t1 - sqrt(t2))/t3 ;
omega(4) = - sqrt(t1 + sqrt(t2))/t3 ;
```

Problem 7.11 Plot the dispersion relation described by equation (7.44) and compare your result to the plot shown in Figure 7.11.

Solution - 7.11: A sample Matlab script is given below. Some sorting etc. are needed to clean the data.

```
function disperseNet

  calcDisperse;
  plotData;

function calcDisperse
  fid = fopen('DispersePlotDat.dat','w');
  figure;
  omega0 = 0.001:0.1:16;
  for ii=1:length(omega0)
    ii
    plotDisperse(fid, omega0(ii), ii);
  end
  fclose(fid);
  axis square;
  set(gca, 'LineWidth', 3, 'FontName', 'times', 'FontSize', 22);
  set(gca, 'XLim', [-1.41421*pi/pi 2*pi/pi], 'YLim', [0 16/pi]);
  xlabel('ka/\pi', 'FontName', 'times', 'FontSize', 22);
  ylabel('\omega/\pi', 'FontName', 'times', 'FontSize', 22);

function plotData
  load DispersePlotUniq.dat
  kk = DispersePlotUniq(:,1);
  omega = DispersePlotUniq(:,2);
  plot(kk, omega, 'k.')
  axis square;
  set(gca, 'LineWidth', 3, 'FontName', 'times', 'FontSize', 22);
  set(gca, 'XLim', [-1.41421*pi/pi 2*pi/pi], 'YLim', [0 6]);
  xlabel('ka/\pi', 'FontName', 'times', 'FontSize', 22);
  ylabel('\omega/\pi', 'FontName', 'times', 'FontSize', 22);

function plotDisperse(fid, omega0, nn)
  a = 1;
  c1 = 1;
  c2 = 5;

  % First M-Gamma
  inc = 0.1;
  int1 = 1.0:-inc:0;
  int2 = 0:inc:1.0;
  k1 = int1*pi/a;
  for ii=1:length(k1)
    omegab(ii) = bisection(k1(ii), k1(ii), a, c1, c2, omega0);
    loc(ii) = sqrt(2)*k1(ii);
    fprintf(fid, '%12.5e %12.5e\n', -loc(ii)/pi, omegab(ii)/pi);
  end
  plot(-loc/pi, real(omegab)/pi, 'k.'); hold on;

  % Then Gamma-X
  k1 = int2*pi/a;
  for ii=1:length(k1)
    omegab(ii) = bisection(k1(ii), 0, a, c1, c2, omega0);
    loc(ii) = k1(ii);
```

```
    fprintf(fid, '%12.5e %12.5e\n', loc(ii)/pi, omegab(ii)/pi);
  end
  plot(loc/pi, real(omegab)/pi, 'k.'); hold on;

  % Last X-M
  k2 = int2*pi/a;
  for ii=1:length(k2)
    omegab(ii) = bisection(pi/a, k2(ii), a, c1, c2, omega0);
    loc(ii) = pi/a+inc*(ii-1)*pi/a;
    fprintf(fid, '%12.5e %12.5e\n', loc(ii)/pi, omegab(ii)/pi);
  end
  plot(loc/pi, real(omegab)/pi, 'k.'); hold on;

function [omega] = calcOmega(k1, k2, a, c, n)

  cc = 1/2*(cos(k1*a)+cos(k2*a));
  ss = sqrt(1 -cc^2);
  tt = ss/cc;
  omega = (c/a)*acos(cc);

function [fx, dfdx] = calcFunc(omega, k1, k2, a, c1, c2)

  fx = -c2*cot((a*omega)/c1) - c1*cot((a*omega)/c2) + ...
        c2*cos(a*k1)/sin((a*omega)/c1) + c1*cos(a*k2)/sin((a*omega)/c2);
  dfdx = 0;

function [omega] = bisection(k1, k2, a, c1, c2, omega0)

  aa = omega0;
  bb = omega0 + 0.01;
  [fa, dfadx] = calcFunc(aa, k1, k2, a, c1, c2);
  [fb, dfbdx] = calcFunc(bb, k1, k2, a, c1, c2);
  while (sign(fa) == sign(fb))
    bb = bb + 0.01;
   [fb, dfbdx] = calcFunc(bb, k1, k2, a, c1, c2);
  end
  if (fa <= 0)
      lo = aa; hi = bb;
  else
      lo = bb; hi = aa;
  end
  mid = lo + (hi-lo)/2;
  while (mid ~= lo) & (mid ~=hi)
     [fmid, dfmiddx] = calcFunc(mid, k1, k2, a, c1, c2);
     if (fmid <= 0)
        lo = mid;
     else
        hi = mid;
     end
     mid = lo + (hi-lo)/2;
  end

  omega = mid;
```

Problem 7.12 Find the dispersion relations $k = k(\omega)$ for the Milton-Willis model discussed in Chapter 5. Plot the dispersion diagram for the model with $c = 1/2$.

Solution - 7.12: The solution given below may not be accurate. Please use it only a guide for the approach to be taken for this problem.

Recall that

$$\mathbf{p} \equiv \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \text{Re}\left(\left\{-i\omega m \begin{bmatrix} c^{-1}\, u_{2,2} \\ c\, u_{1,2} \end{bmatrix} + \frac{\delta}{2}\begin{bmatrix} -i\omega\, u_1 \\ -i\omega\, u_2 \end{bmatrix}\right\} e^{-i\omega t}\right)$$

or,

$$\mathbf{p} \equiv \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \text{Re}\left(\left\{\begin{bmatrix} 0 & 0 & 0 & -i\omega m c^{-1} \\ 0 & -i\omega m c & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{bmatrix} + \begin{bmatrix} \delta/2 & 0 \\ 0 & \delta/2 \end{bmatrix}\begin{bmatrix} -i\omega\, u_1 \\ -i\omega\, u_2 \end{bmatrix}\right\} e^{-i\omega t}\right)$$

Therefore, we can write the stress and momentum density equations as

$$\begin{aligned}
\sigma_{11} &= \text{Re}\left[\frac{h\kappa}{2}\left(u_{1,1} + u_{2,2}\right) e^{-i\omega t}\right] \\
\sigma_{12} &= \text{Re}\left[\frac{h\kappa}{2}\left(u_{2,1} + u_{1,2}\right) e^{-i\omega t}\right] \\
\sigma_{21} &= \text{Re}\left[\left\{\frac{h\kappa}{2}\left(u_{2,1} + u_{1,2}\right) - \omega^2 m c\, u_2\right\} e^{-i\omega t}\right] \\
\sigma_{22} &= \text{Re}\left[\left\{\frac{h\kappa}{2}\left(u_{1,1} + 3u_{2,2}\right) - \omega^2 m c^{-1}\, u_1\right\} e^{-i\omega t}\right] \\
p_1 &= \text{Re}\left[\left\{-i\omega m c^{-1}\, u_{2,2} - i\omega\frac{\delta}{2}\, u_1\right\} e^{-i\omega t}\right] \\
p_2 &= \text{Re}\left[\left\{-i\omega m c\, u_{1,2} - i\omega\frac{\delta}{2}\, u_2\right\} e^{-i\omega t}\right]
\end{aligned}$$

The equation of motion is

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{b} = \dot{\mathbf{p}}$$

Expressed in terms of components with respect to the ($\mathbf{e}_1, \mathbf{e}_2$) basis, we have

$$\sigma_{ji,j} + b_i = \dot{p}_i$$

For the two dimensional case,

$$\begin{aligned}
\sigma_{11,1} + \sigma_{21,2} + b_1 &= \dot{p}_1 \\
\sigma_{12,1} + \sigma_{22,2} + b_2 &= \dot{p}_2
\end{aligned}$$

Now,

$$\sigma_{11,1} = \text{Re}\left[\frac{h\kappa}{2}\,(u_{1,11} + u_{2,21})\,e^{-i\omega t}\right]$$
$$\sigma_{12,1} = \text{Re}\left[\frac{h\kappa}{2}\,(u_{2,11} + u_{1,21})\,e^{-i\omega t}\right]$$
$$\sigma_{21,2} = \text{Re}\left[\left\{\frac{h\kappa}{2}\,(u_{2,12} + u_{1,22}) - \omega^2 mc\,u_{2,2}\right\}\,e^{-i\omega t}\right]$$
$$\sigma_{22,2} = \text{Re}\left[\left\{\frac{h\kappa}{2}\,(u_{1,12} + 3u_{2,22}) - \omega^2 mc^{-1}\,u_{1,2}\right\}\,e^{-i\omega t}\right]$$
$$\dot{p}_1 = \text{Re}\left[\left\{-\omega^2 mc^{-1}\,u_{2,2} - \omega^2\frac{\delta}{2}\,u_1\right\}\,e^{-i\omega t}\right]$$
$$\dot{p}_2 = \text{Re}\left[\left\{-\omega^2 mc\,u_{1,2} - \omega^2\frac{\delta}{2}\,u_2\right\}\,e^{-i\omega t}\right]$$

Let us look for Bloch wave solutions (in the large wavelength limit) of the form

$$u_1(\mathbf{x}) = \widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$u_2(\mathbf{x}) = \widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}$$

Then

$$u_{1,1} = i\,k_1\,\widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{1,11} = -k_1^2\,\widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{1,12} = -k_1\,k_2\,\widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$u_{1,2} = i\,k_2\,\widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{1,21} = -k_1\,k_2\,\widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{1,22} = -k_2^2\,\widehat{u}_1\,e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$u_{2,1} = i\,k_1\,\widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{2,11} = -k_1^2\,\widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{2,12} = -k_1\,k_2\,\widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$u_{2,2} = i\,k_2\,\widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{2,21} = -k_1\,k_2\,\widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}\,;\;\; u_{2,22} = -k_2^2\,\widehat{u}_2\,e^{i\mathbf{k}\cdot\mathbf{x}}$$

Plugging into the expressions for σ_{ij} and $\dot{p}_i$, we have

$$\sigma_{11,1} = \text{Re}\left[-\frac{h\kappa}{2}\,(k_1^2\,\widehat{u}_1 + k_1\,k_2\,\widehat{u}_2)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$
$$\sigma_{12,1} = \text{Re}\left[-\frac{h\kappa}{2}\,(k_1^2\,\widehat{u}_2 + k_1\,k_2\,\widehat{u}_1)\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$
$$\sigma_{21,2} = \text{Re}\left[\left\{-\frac{h\kappa}{2}\,(k_1\,k_2\,\widehat{u}_2 + k_2^2\,\widehat{u}_1) - i\,\omega^2 mc\,k_2\,\widehat{u}_2\right\}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$
$$\sigma_{22,2} = \text{Re}\left[\left\{-\frac{h\kappa}{2}\,(k_1\,k_2\,\widehat{u}_1 + 3\,k_2^2\,\widehat{u}_2) - i\,\omega^2 mc^{-1}\,k_2\,\widehat{u}_1\right\}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$
$$\dot{p}_1 = \text{Re}\left[\left\{-i\,\omega^2 mc^{-1}\,k_2\,\widehat{u}_2 - \omega^2\frac{\delta}{2}\,\widehat{u}_1\right\}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$
$$\dot{p}_2 = \text{Re}\left[\left\{-i\,\omega^2 mc\;\,k_2\,\widehat{u}_1 - \omega^2\frac{\delta}{2}\,\widehat{u}_2\right\}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$

Plug into equations of motion (with $b_i = 0$) to get

$$\text{Re}\left[\left(-\frac{h\kappa}{2}\,\left\{(k_1^2+k_2^2)\,\widehat{u}_1 + 2\,k_1\,k_2\,\widehat{u}_2\right\} + i\,\omega^2 m(c^{-1}-c)\,k_2\,\widehat{u}_2 + \omega^2\frac{\delta}{2}\,\widehat{u}_1\right) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = 0$$
$$\text{Re}\left[\left(-\frac{h\kappa}{2}\,\left\{2\,k_1\,k_2\,\widehat{u}_1 + (k_1^2+3\,k_2^2)\,\widehat{u}_2\right\} + i\,\omega^2 m(c-c^{-1})\,k_2\,\widehat{u}_1 + \omega^2\frac{\delta}{2}\,\widehat{u}_2\right)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] = 0$$

Define

$$K := \frac{h\kappa}{2}$$

Then, in matrix form,

$$\text{Re}\left\{\begin{bmatrix} -K\,(k_1^2+k_2^2) + \omega^2\dfrac{\delta}{2} & -2\,K\,k_1\,k_2 - i\,\omega^2 m(c-c^{-1})\,k_2 \\ -2\,K\,k_1\,k_2 + i\,\omega^2 m(c-c^{-1})\,k_2 & -K\,(k_1^2+3\,k_2^2) + \omega^2\dfrac{\delta}{2}\end{bmatrix}\begin{bmatrix}\widehat{u}_1\\ \widehat{u}_2\end{bmatrix} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right\} = \begin{bmatrix}0\\0\end{bmatrix}$$

This system of equations has a nonzero solution only if $\mathbf{A}$ is singular, i.e.,

$$\det(\mathbf{A}) := \det\begin{bmatrix} K\,(k_1^2+k_2^2) - \omega^2\dfrac{\delta}{2} & 2\,K\,k_1\,k_2 + i\,\omega^2 m(c-c^{-1})\,k_2 \\ 2\,K\,k_1\,k_2 - i\,\omega^2 m(c-c^{-1})\,k_2 & K\,(k_1^2+3\,k_2^2) - \omega^2\dfrac{\delta}{2}\end{bmatrix} = 0$$

Expanded out, we have the dispersion relation

$$\det(\mathbf{A}) := \left[\frac{\delta^2}{4} - m^2\,\left(c^2+\frac{1}{c^2}-2\right)\,k_2^2\right]\omega^4 - K\,\delta\,(k_1^2+2\,k_2^2)\,\omega^2 + K^2(k_1^4+3k_2^4) = 0$$

For the case where $\delta = 0$, we have

$$\det(\mathbf{A}) = -m^2\,\left(c^2+\frac{1}{c^2}-2\right)\,k_2^2\,\omega^4 + K^2(k_1^4+3k_2^4) = 0$$

The solutions are

$$\omega^2 = \pm\frac{K\,c}{m\,k_2}\sqrt{\frac{k_1^4+3\,k_2^4}{c^4-2\,c^2+1}} \qquad \square \tag{7.13}$$

The vectors $\widehat{u}$ are given by

$$\begin{bmatrix}\widehat{u}_1\\ \widehat{u}_2\end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{-\,2k_1k_2^2 \mp i\,\sqrt{k_1^4+3k_2^4}}{(k_1^2+k_2^2)\,k_2}\end{bmatrix}$$

Remark:
Notice that the ratio $\widehat{u}_1/\widehat{u}_2$ is imaginary. This implies that the displacements in the two directions are out of phase for our model.

If $c = 0.5$ then the dispersion relations are

$$\omega^2 = \pm\frac{2\,K}{3\,m\,k_2}\,\sqrt{k_1^4+3\,k_2^4} \qquad \square \tag{7.14}$$

If $k_1 = 0$,

$$\omega^2 = \pm\frac{2\,K\,k_2}{\sqrt{3}\,m}\,.$$

and

$$\begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \mp\dfrac{i\sqrt{3}}{k_2} \end{bmatrix}$$

If $k_1 = k_2$,

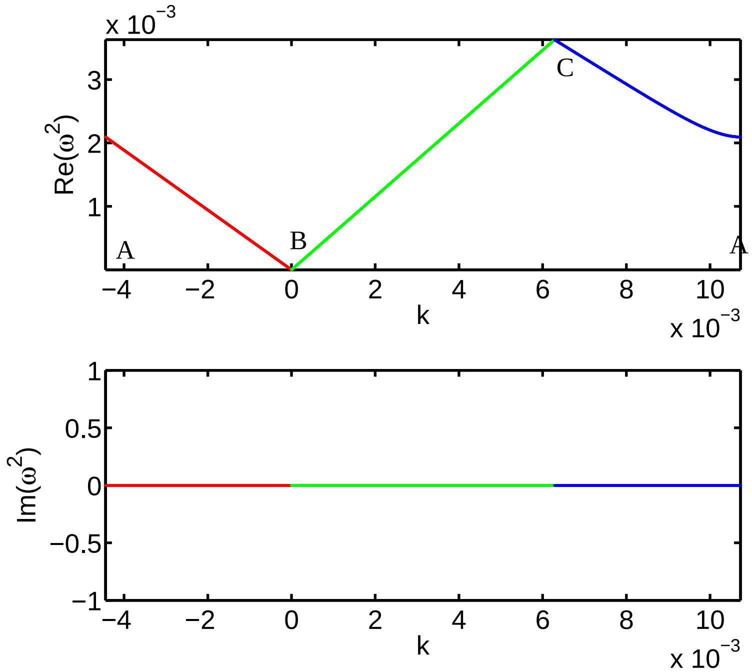
$$\omega^2 = \pm\frac{4\ K\ k_2}{3\ m}\,.$$

and

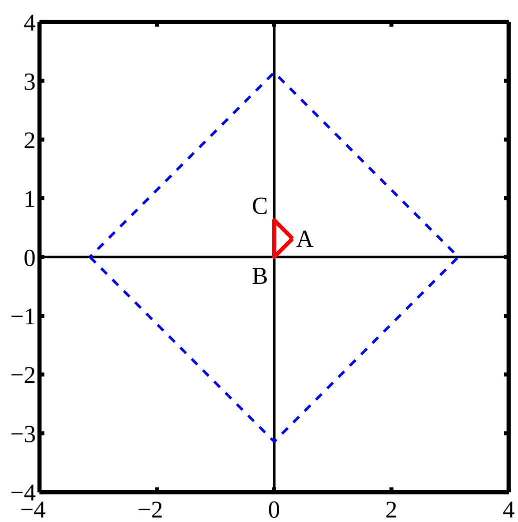
$$\begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \mp \dfrac{i}{k_2} \end{bmatrix}$$

If $k_2 = 0$ then the solution blows up.

A plot of the dispersion relation chose to $k = 0$ (i.e., for long wavelengths) is shown in Figure 7.1(a). A schematic of the reciprocal lattice and the path along which the dispersion relation has been calculated is shown in Figuer 7.1(b).



(a) Dispersion relation.



(b) Reciprocal lattice.

Figure 7.1: A material made by tiling the plane with the spring model.

Problem 7.13 The governing equation for antiplane shear (SH-waves) in an inhomogeneous, but isotropic, medium can be written as

$$\overline{\boldsymbol{\nabla}} \cdot (\mu(x_1, x_2)\overline{\boldsymbol{\nabla}} u_3(x_1, x_2)) + \omega^2 \rho(x_1, x_2) u_3(x_1, x_2) = 0\,.$$

Derive the weak form of this equation as it applies to a periodic composite by taking the product with a vector test function $\mathbf{w}(x_1, x_2)$ (with compact support) and integrating over the volume of the unit cell. Discretize the resulting equation using finite element basis functions and comment on how Bloch periodic boundary conditions may be implemented in this situation.

Solution - 7.13: Note that the weak form in this problem has a physical significance if the weighting function is scalar valued. However, for the purposes of this problem we will use a vector valued $\mathbf{w}$.

Integration over the volume after multiplication of the governing equation by a weighting function leads to

$$\int_\Omega \left[\overline{\boldsymbol{\nabla}} \cdot (\mu \overline{\boldsymbol{\nabla}} u_3)\mathbf{w} + \omega^2 \rho u_3 \mathbf{w}\right] \mathrm{d}\Omega = \mathbf{0}\,.$$

We can show that

$$\overline{\boldsymbol{\nabla}} \cdot (\mu \overline{\boldsymbol{\nabla}} u_3 \otimes \mathbf{w}) = \overline{\boldsymbol{\nabla}} \cdot (\mu \overline{\boldsymbol{\nabla}} u_3)\mathbf{w} + \mu \overline{\boldsymbol{\nabla}} \mathbf{w} \cdot \overline{\boldsymbol{\nabla}} u_3\,.$$

Using the above identity, we can write

$$\int_\Omega \left[\overline{\boldsymbol{\nabla}} \cdot (\mu \overline{\boldsymbol{\nabla}} u_3 \otimes \mathbf{w}) - \mu \overline{\boldsymbol{\nabla}} \mathbf{w} \cdot \overline{\boldsymbol{\nabla}} u_3 + \omega^2 \rho u_3 \mathbf{w}\right] \mathrm{d}\Omega = \mathbf{0}\,.$$

Using the divergence theorem, we have

$$\int_\Gamma \left[\mathbf{n} \cdot (\mu \overline{\boldsymbol{\nabla}} u_3 \otimes \mathbf{w})\right] \mathrm{d}\Gamma - \int_\Omega \left[\mu \overline{\boldsymbol{\nabla}} \mathbf{w} \cdot \overline{\boldsymbol{\nabla}} u_3 - \omega^2 \rho u_3 \mathbf{w}\right] \mathrm{d}\Omega = \mathbf{0}\,.$$

Rearranging gives the desired weak form

$$\int_\Omega \left[\mu \overline{\boldsymbol{\nabla}} \mathbf{w} \cdot \overline{\boldsymbol{\nabla}} u_3 - \omega^2 \rho u_3 \mathbf{w}\right] \mathrm{d}\Omega = \int_\Gamma \left[(\mu \overline{\boldsymbol{\nabla}} u_3 \cdot \mathbf{n})\mathbf{w}\right] \mathrm{d}\Gamma \qquad \square \tag{7.15}$$

To get the finite element discretization, we assume solutions of the form

$$u_3 = \sum_{j=1}^{n} u_j N_j(x_1, x_2) \qquad \text{and} \qquad \mathbf{w} = \sum_{k=1}^{n} \mathbf{w}_k N_k(x_1, x_2)\,.$$

Substituting these into the weak form leads to

$$\int_\Omega \left[\mu \sum_{k=1}^{n}\sum_{j=1}^{n} (\mathbf{w}_k \otimes \overline{\boldsymbol{\nabla}} N_k) \cdot (u_j \overline{\boldsymbol{\nabla}} N_j) - \omega^2 \rho (u_j N_j)(\mathbf{w}_k N_k)\right] \mathrm{d}\Omega =$$
$$\int_\Gamma \left[\sum_{k=1}^{n}\sum_{j=1}^{n} \left[\mu (u_j \overline{\boldsymbol{\nabla}} N_j) \cdot \mathbf{n}\right] (\mathbf{w}_k N_k)\right] \mathrm{d}\Gamma\,.$$

Reorganization leads to

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \int_\Omega \left[\mathbf{w}_k (\mu \overline{\boldsymbol{\nabla}} N_k \cdot \overline{\boldsymbol{\nabla}} N_j)\, u_j - \mathbf{w}_k (\omega^2 \rho N_j N_k)\, u_j\right] \mathrm{d}\Omega =$$
$$\sum_{k=1}^{n}\sum_{j=1}^{n} \int_\Gamma \left[\mathbf{w}_k (\mu\, N_k \overline{\boldsymbol{\nabla}} N_j \cdot \mathbf{n})\, u_j\right] \mathrm{d}\Gamma\,.$$

Since the weighting function is arbitrary, we have

$$\sum_{j=1}^{n} \int_{\Omega} \left[(\mu \overline{\boldsymbol{\nabla}} N_k \cdot \overline{\boldsymbol{\nabla}} N_j)\, u_j - (\omega^2 \rho N_j N_k)\, u_j\right] \mathrm{d}\Omega = \sum_{j=1}^{n} \int_{\Gamma} \left[(\mu\, N_k \overline{\boldsymbol{\nabla}} N_j \cdot \mathbf{n})\, u_j\right] \mathrm{d}\Gamma \qquad \text{for} \quad k = 1 \dots n\,.$$

Define the stiffness, mass matrix, and surface traction terms as

$$K_{kj} := \int_{\Omega} \mu \overline{\boldsymbol{\nabla}} N_k \cdot \overline{\boldsymbol{\nabla}} N_j \,\mathrm{d}\Omega\,; \quad M_{kj} := \int_{\Omega} \rho N_k N_j \,\mathrm{d}\Omega\,; \quad \text{and} \quad T_{kj} := \int_{\Gamma} \mu N_k \overline{\boldsymbol{\nabla}} N_j \cdot \mathbf{n}\,\mathrm{d}\Gamma\,.$$

Then we have

$$\sum_{j=1}^{n} \left[K_{kj} - \omega^2\, M_{kj}\right] u_j = \sum_{j=1}^{n} T_{kj}\, u_j \qquad \text{for} \quad k = 1 \dots n\,.$$

In matrix form,

$$(\underline{\underline{\mathsf{K}}} - \omega^2 \underline{\underline{\mathsf{M}}})\mathbf{u} = \underline{\underline{\mathsf{T}}}\mathbf{u} \qquad \square \tag{7.16}$$

where the quantities on the right hand side are only evaluated at the boundaries of the unit cell. In practice, it is more convenient to keep the right hand side in terms of stresses so that the dependence on the boundary displacements becomes implicit and traction free boundary conditions become easy to implement. In that case, we can write the above equation as

$$(\underline{\underline{\mathsf{K}}} - \omega^2 \underline{\underline{\mathsf{M}}})\mathbf{u} = \mathbf{f}$$

with $\mathbf{f} = 0$ for traction free boundaries. This eigenvalue problem may then be solved for $\omega(\mathbf{k})$.

One approach for implementing Bloch periodic boundary conditions for this problem can be found in Guenneau et al. (2007).

Problem 7.14 The weak form of the wave equation in a periodic elastic composite can be written as

$$\int_\Omega \left[(\boldsymbol{\nabla}\widehat{\mathbf{w}}_\eta + i\mathbf{k}\otimes\widehat{\mathbf{w}}_\eta) : \mathsf{C}_\eta : (\boldsymbol{\nabla}\widehat{\mathbf{u}}_\eta + i\mathbf{k}\otimes\widehat{\mathbf{u}}_\eta) - \omega^2\rho_\eta\widehat{\mathbf{w}}_\eta\cdot\widehat{\mathbf{u}}_\eta\right]\mathrm{d}\Omega = 0\,.$$

Show that the above equation can be discretized using finite elements in system of algebraic equations of the form

$$[\underline{\underline{\mathsf{K}}}(\mathbf{k}) - \omega^2\underline{\underline{\mathsf{M}}}]\underline{\mathsf{u}} = \underline{0}\,.$$

Solution - 7.14: We can introduce a finite element discretization of the form

$$\widehat{\mathbf{u}}_\eta = \sum_{k=1}^n \mathbf{u}_k N_k \qquad \text{and} \qquad \widehat{\mathbf{w}}_\eta = \sum_{j=1}^n \mathbf{w}_j N_j\,.$$

Plugging these into the weak form gives us

$$\sum_{j=1}^n\sum_{k=1}^n \int_\Omega \Big[(\mathbf{w}_j\otimes\boldsymbol{\nabla}N_j + i\mathbf{k}\otimes(\mathbf{w}_jN_j)) : \mathsf{C}_\eta : (\mathbf{u}_k\otimes\boldsymbol{\nabla}N_k + i\mathbf{k}\otimes(\mathbf{u}_kN_k)) - \omega^2\rho_\eta(\mathbf{w}_jN_j)\cdot(\mathbf{u}_kN_k)\Big]\mathrm{d}\Omega = 0\,.$$

Let us simplify the above equation term by term using the symmetries of C_η. For the first term, we have

$$(\mathbf{w}_j\otimes\boldsymbol{\nabla}N_j) : \mathsf{C}_\eta : (\mathbf{u}_k\otimes\boldsymbol{\nabla}N_k) = \mathbf{w}_j\cdot(\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k)\cdot\mathbf{u}_k\,.$$

For the second term,

$$(\mathbf{w}_j\otimes\boldsymbol{\nabla}N_j) : \mathsf{C}_\eta : (iN_k\mathbf{k}\otimes\mathbf{u}_k) = \mathbf{w}_j\cdot[iN_k\,\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\mathbf{k}]\cdot\mathbf{u}_k\,.$$

For the third term,

$$(iN_j\mathbf{k}\otimes\mathbf{w}_j) : \mathsf{C}_\eta : (\mathbf{u}_k\otimes\boldsymbol{\nabla}N_k) = \mathbf{w}_j\cdot[iN_j\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k]\cdot\mathbf{u}_k\,.$$

The fourth term is

$$(iN_j\mathbf{k}\otimes\mathbf{w}_j) : \mathsf{C}_\eta : (iN_k\mathbf{k}\otimes\mathbf{u}_k) = -\mathbf{w}_j\cdot[N_jN_k\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\mathbf{k}]\cdot\mathbf{u}_k\,.$$

Therefore, we can write the weak form as

$$\sum_{j=1}^n\sum_{k=1}^n \mathbf{w}_j\cdot\left\{\int_\Omega\Big(\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k + iN_k\,\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\mathbf{k} + iN_j\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k - N_jN_k\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\mathbf{k} - \omega^2\rho_\eta N_jN_k\,\boldsymbol{1}\Big)\mathrm{d}\Omega\right\}\cdot\mathbf{u}_k = 0\,.$$

The weighting function $\mathbf{w}$ can be arbitrary, therefore, for each $j = 1\dots n$, we have

$$\sum_{k=1}^n\left\{\int_\Omega\Big(\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k + iN_k\,\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\mathbf{k} + iN_j\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k - N_jN_k\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\mathbf{k} - \omega^2\rho_\eta N_jN_k\,\boldsymbol{1}\Big)\mathrm{d}\Omega\right\}\cdot\mathbf{u}_k = \mathbf{0}\,.$$

Let us define

$$\boldsymbol{K}_{jk} := \int_\Omega\Big(\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k + iN_k\,\boldsymbol{\nabla}N_j\cdot\mathsf{C}_\eta\cdot\mathbf{k} + iN_j\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\boldsymbol{\nabla}N_k - N_jN_k\,\mathbf{k}\cdot\mathsf{C}_\eta\cdot\mathbf{k}\Big)\mathrm{d}\Omega$$

$$\boldsymbol{M}_{jk} := \int_\Omega \rho_\eta N_jN_k\,\boldsymbol{1}\,\mathrm{d}\Omega\,.$$

Note that $\boldsymbol{K}_{jk}$ is a function of the wavevector $\mathbf{k}$. Then, for each j, we have

$$\sum_{k=1}^{n} (\boldsymbol{K}_{jk}(\mathbf{k}) - \omega^2 \boldsymbol{M}_{jk}) \cdot \mathbf{u}_k = \mathbf{0}\,.$$

In matrix notation,

$$[\underline{\underline{\mathsf{K}}}(\mathbf{k}) - \omega^2 \underline{\underline{\mathsf{M}}}]\underline{\underline{\mathsf{u}}} = \underline{\underline{\mathsf{0}}} \qquad \square \tag{7.17}$$

Chapter 8

Solutions for Exercises in Chapter 8

Problem 8.1 A plane wave propagating through a layered medium can be expressed in the form

$$u_2(x_1, x_3, t) = \left(A_{1j} e^{k_j x_3} + A_{2j} e^{-k_j x_3}\right) e^{i(k_1 x_1 - \omega t)} .$$

Show that if $x_3 \geq 0$ and $\text{Re}(k_j) \geq 0$ we must have $\text{Im}(k_j) \leq 0$.

Solution - 8.1 Let us start from the governing equation and let us express k_j in terms of its real and imaginary parts as

$$k_j = k_r + i k_i .$$

Then we have,

$$\frac{\partial^2 \widehat{u}_2}{\partial x_3^2} + (k_r + i k_i)^2 \, \widehat{u}_2 = 0 .$$

The solution to this equation has the form

$$\widehat{u}_2 = A_{1j} e^{i k_r x_3} e^{-k_i x_3} + A_{2j} e^{-i k_r x_3} e^{k_i x_3} .$$

Therefore, the plane wave solution can be expressed as

$$u_2(x_1, x_3, t) = \left(A_{1j} e^{i k_r x_3} e^{-k_i x_3} + A_{2j} e^{-i k_r x_3} e^{k_i x_3}\right) e^{i(k_1 x_1 - \omega t)} .$$

This solution is always bounded if $k_i = 0$, i.e., $\text{Im}(k_j) = 0$. If $x_3 > 0$ and $k_i > 0$, the second term is unbounded for $x_3 \to \infty$ unless $A_{2j} = 0$. If $x_3 > 0$ and $k_i < 0$, the first term blows up unless $A_{1j} = 0$. Therefore, one solution is valid for $k_i > 0$ and the other for $k_i < 0$, i.e.,

$$u_2(x_1, x_3, t) = \begin{cases} A_{2j} e^{-i k_r x_3} e^{k_i x_3} \, e^{i(k_1 x_1 - \omega t)} & \text{for } k_i < 0 \\ A_{1j} e^{i k_r x_3} e^{-k_i x_3} \, e^{i(k_1 x_1 - \omega t)} & \text{for } k_i > 0 . \end{cases}$$

To choose between these solutions we have to consider the case where k_j is real, $k_j > 0$, and $x_3 > 0$. For the solution to be bounded, $A_{1j} = 0$. Therefore, the solution must be

$$u_2(x_1, x_3, t) = A_{2j} e^{-i k_r x_3} e^{k_i x_3} \, e^{i(k_1 x_1 - \omega t)}$$

which is bounded only if $\text{Im}(k_j) < 0$. Hence the requirement that if $x_3 \geq 0$ and $\text{Re}(k_j) \geq 0$ we must have $\text{Im}(k_j) \leq 0$. □

Problem 8.2 Verify that the generalized reflection and transmission coefficients for TE-waves can also be applied without change to acoustic waves. What are the equivalent expressions for $R_{j,j+1}$ and $T_{j,j+1}$ for acoustic waves?

Solution - 8.2: Recall from equations (8.5) and (8.8) that the acoustic and TE wave equations for layered media can be written as

$$\left[\rho\,\frac{d}{dz}\left(\frac{1}{\rho}\,\frac{dp}{dz}\right)\right] + \left[\omega^2\frac{\rho}{\kappa} - k_1^2\right]p = 0 \quad \text{and} \quad \left[\mu\,\frac{d}{dz}\left(\frac{1}{\mu}\,\frac{dE}{dz}\right)\right] + \left[\omega^2\varepsilon\mu - k_1^2\right]E = 0\,.$$

It is more convenient to work with solutions of the form $\exp(\pm i k_z z)$ rather than $\exp(\pm k_z z)$. Note that these two forms are equivalent. Also note that for the acoustic problem

$$k_z^2 = \omega^2\frac{\rho}{\kappa} - k_1^2$$

while for the TE problem

$$k_z^2 = \omega^2\varepsilon\mu - k_1^2\,.$$

The equations are identical and the only difference can be in the boundary conditions at the interface between two layers (labeled 1 and 2. The interface conditions for acoustic waves are

$$p_1 = p_2 \quad \text{and} \quad u_{z1} = u_{z2} \qquad \Longrightarrow \qquad \frac{1}{\rho_1}\frac{\partial p_1}{\partial z} = \frac{1}{\rho_2}\frac{\partial p_2}{\partial z}\,.$$

Clearly, these have the same form as for TE waves and therefore, following the approach in Chapter 2, p. 86-88, we have

$$R_{12} = \frac{\rho_2 k_{z1} - \rho_1 k_{z2}}{\rho_2 k_{z1} + \rho_1 k_{z2}} \quad \text{and} \quad T_{12} = \frac{2\rho_2 k_{z1}}{\rho_2 k_{z1} + \rho_1 k_{z2}}\,.$$

Similarly, for a slab, we have

$$\tilde{R}_{12} = R_{12} + \frac{T_{12}T_{21}R_{23}\exp[2ik_{z2}(d_2 - d_1)]}{1 - R_{21}R_{23}\exp(2ik_{z2}(d_2 - d1)}$$
$$T_{13} = \frac{T_{12}T_{23}R_{23}\exp[2ik_{z2}(d_2 - d_1)]}{1 - R_{21}R_{23}\exp(2ik_{z2}(d_2 - d1)}\,.$$

The expressions for $R_{j,j+1}, T_{j,j+1}$ are clearly

$$R_{j,j+1} = \frac{\rho_{j+1}k_{z,j} - \rho_j k_{z,j+1}}{\rho_{j+1}k_{z,j} + \rho_j k_{z,j+1}} \quad \text{and} \quad T_{j,j+1} = \frac{2\rho_{j+1}k_{z,j}}{\rho_{j+1}k_{z,j} + \rho_j k_{z,j+1}} \quad \square$$

Problem 8.3 Find the matrix $\underline{\underline{\mathsf{H}}}$ and the associated state vector $\underline{\mathsf{v}}$ for the propagation of P-SV waves in a layered medium.

Solution - 8.3 This problem involves linear elastic waves in a layered medium. P-SV waves are a superposition of P-waves and SV-waves and can occur at interfaces between homogeneous isotropic media. When these waves occur at a surface they are called Rayleigh waves.

The displacement field in the layered medium has the form

$$\mathbf{u}(x_1, x_3, t) = \widehat{\mathbf{u}}(x_3) \exp[i(k_1 x_1 - \omega t)] \,.$$

For P-SV waves, the non-zero components of displacement are

$$u_1(x_1, x_3, t) = \widehat{u}_1(x_3) \exp[i(k_1 x_1 - \omega t)] \quad \text{and} \quad u_3(x_1, x_3, t) = \widehat{u}_3(x_3) \exp[i(k_1 x_1 - \omega t)] \,.$$

Let us assume that each layer in the medium is composed of an isotropic material. Then the stress-displacement relation in a layer is

$$\sigma_{ij} = \lambda \, \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \,.$$

Then the stress-displacement relations are

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu) \, \frac{\partial u_1}{\partial u_1} + \lambda \, \frac{\partial u_3}{\partial u_3} = \left[\lambda \, \frac{d\widehat{u}_3}{dx_3} + ik_1(\lambda + 2\mu)\widehat{u}_1(x_3) \right] \exp[i(k_1 x_1 - \omega t)] \\
\sigma_{22} &= \lambda \left(\frac{\partial u_1}{\partial u_1} + \frac{\partial u_3}{\partial u_3} \right) = \lambda \left[\frac{d\widehat{u}_3}{dx_3} + ik_1 \, \widehat{u}_1(x_3) \right] \exp[i(k_1 x_1 - \omega t)] \\
\sigma_{33} &= (\lambda + 2\mu) \, \frac{\partial u_3}{\partial u_3} + \lambda \, \frac{\partial u_1}{\partial u_1} = \left[(\lambda + 2\mu) \, \frac{d\widehat{u}_3}{dx_3} + ik_1 \lambda \widehat{u}_1(x_3) \right] \exp[i(k_1 x_1 - \omega t)] \\
\sigma_{31} &= \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \mu \left[\frac{d\widehat{u}_1}{dx_3} + ik_1 \widehat{u}_3(x_3) \right] \exp[i(k_1 x_1 - \omega t)] \\
\sigma_{23} &= \mu \, \frac{\partial u_3}{\partial x_2} = 0 \quad \text{and} \quad \sigma_{12} = \mu \, \frac{\partial u_1}{\partial x_2} = 0 \,.
\end{aligned}$$

Since the stresses must also be harmonic, we can write

$$\sigma_{31} = \widehat{\sigma}_{31}(x_3) \, \exp[i(k_1 x_1 - \omega t)] \quad \text{and} \quad \sigma_{33} = \widehat{\sigma}_{33}(x_3) \, \exp[i(k_1 x_1 - \omega t)]$$

where

$$\widehat{\sigma}_{31} = \mu \, \frac{d\widehat{u}_1}{dx_3} + ik_1 \mu \widehat{u}_3(x_3) \quad \text{and} \quad \widehat{\sigma}_{33} = (\lambda + 2\mu) \, \frac{d\widehat{u}_3}{dx_3} + ik_1 \lambda \widehat{u}_1(x_3) \,.$$

Also, the momentum equation is

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \rho \, \frac{\partial^2 u_j}{\partial t^2} \,.$$

In terms of the non-zero components of stress,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{31}}{\partial x_3} = \rho \, \frac{\partial^2 u_1}{\partial t^2} \quad \text{and} \quad \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \, \frac{\partial^2 u_3}{\partial t^2} \,.$$

If we substitute the expressions for σ_{ij} into the momentum equation, we have

$$\begin{aligned}
& ik_1 \left[\lambda(x_3) \, \frac{d\widehat{u}_3}{dx_3} + ik_1 [\lambda(x_3) + 2\mu(x_3)] \widehat{u}_1(x_3) \right] + \frac{d\widehat{\sigma}_{31}}{dx_3} = -\omega^2 \rho(x_3) \widehat{u}_1(x_3) \\
& ik_1 \mu(x_3) \left[\frac{d\widehat{u}_1}{dx_3} + ik_1 \widehat{u}_3(x_3) \right] + \frac{d\widehat{\sigma}_{33}}{dx_3} = -\omega^2 \rho(x_3) \widehat{u}_3(x_3) \,.
\end{aligned}$$

Rearrangement leads to

$$
\begin{aligned}
ik_1\lambda(x_3)\frac{d\widehat{u}_3}{dx_3} + \frac{d\widehat{\sigma}_{31}}{dx_3} &= \left[k_1^2[\lambda(x_3)+2\mu(x_3)] - \omega^2\rho(x_3)\right]\widehat{u}_1(x_3) \\
ik_1\mu(x_3)\frac{d\widehat{u}_1}{dx_3} + \frac{d\widehat{\sigma}_{33}}{dx_3} &= \left[k_1^2\mu(x_3) - \omega^2\rho(x_3)\right]\widehat{u}_3(x_3)\,.
\end{aligned}
$$

Rearrangement of the expressions for $\widehat{\sigma}_{ij}$ gives us

$$
\begin{aligned}
\mu(x_3)\,\frac{d\widehat{u}_1}{dx_3} &= \widehat{\sigma}_{31}(x_3) - ik_1\mu(x_3)\widehat{u}_3(x_3) \\
[\lambda(x_3)+2\mu(x_3)]\,\frac{d\widehat{u}_3}{dx_3} &= \widehat{\sigma}_{33}(x_3) - ik_1\lambda(x_3)\widehat{u}_1(x_3)\,.
\end{aligned}
\tag{8.1}
$$

Using these to remove the derivatives of $\widehat{u}_i$ in the momentum equations, we get

$$
\begin{aligned}
\frac{d\widehat{\sigma}_{31}}{dx_3} &= -\frac{ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)}\left[\widehat{\sigma}_{33}(x_3) - ik_1\lambda(x_3)\widehat{u}_1(x_3)\right] + \\
&\qquad \left[k_1^2[\lambda(x_3)+2\mu(x_3)] - \omega^2\rho(x_3)\right]\widehat{u}_1(x_3) \\
\frac{d\widehat{\sigma}_{33}}{dx_3} &= -ik_1\left[\widehat{\sigma}_{31}(x_3) - ik_1\mu(x_3)\widehat{u}_3(x_3)\right] + \left[k_1^2\mu(x_3) - \omega^2\rho(x_3)\right]\widehat{u}_3(x_3)
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{d\widehat{\sigma}_{31}}{dx_3} &= -\frac{ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)}\,\widehat{\sigma}_{33}(x_3) + \left[\frac{4k_1^2\mu(x_3)[\lambda(x_3)+\mu(x_3)]}{\lambda(x_3)+2\mu(x_3)} - \omega^2\rho(x_3)\right]\widehat{u}_1(x_3) \\
\frac{d\widehat{\sigma}_{33}}{dx_3} &= -ik_1\widehat{\sigma}_{31}(x_3) - \omega^2\rho(x_3)\widehat{u}_3(x_3)\,.
\end{aligned}
\tag{8.2}
$$

Combining the four equations and expressing them in matrix form, we have

$$
\frac{d}{dx_3}\begin{bmatrix}\widehat{u}_1\\ \widehat{u}_3\\ \widehat{\sigma}_{31}\\ \widehat{\sigma}_{33}\end{bmatrix} = \begin{bmatrix} 0 & -ik_1 & \frac{1}{\mu(x_3)} & 0 \\ \frac{-ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)} & 0 & 0 & \frac{1}{\lambda(x_3)+2\mu(x_3)} \\ \left[\frac{4k_1^2\mu(x_3)[\lambda(x_3)+\mu(x_3)]}{\lambda(x_3)+2\mu(x_3)} - \omega^2\rho(x_3)\right] & 0 & 0 & -\frac{ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)} \\ 0 & -\omega^2\rho(x_3) & -ik_1 & 0 \end{bmatrix}\begin{bmatrix}\widehat{u}_1\\ \widehat{u}_3\\ \widehat{\sigma}_{31}\\ \widehat{\sigma}_{33}\end{bmatrix}.
$$

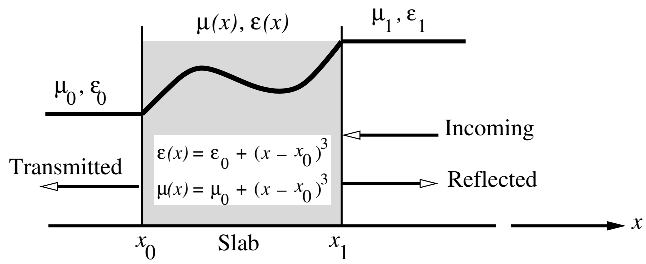
Therefore, the $\underline{\underline{\mathsf{H}}}$ matrix and the $\underline{\underline{\mathsf{v}}}$ vector are

$$
\underline{\underline{\mathsf{H}}} = \begin{bmatrix} 0 & -ik_1 & \frac{1}{\mu(x_3)} & 0 \\ \frac{-ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)} & 0 & 0 & \frac{1}{\lambda(x_3)+2\mu(x_3)} \\ \left[\frac{4k_1^2\mu(x_3)[\lambda(x_3)+\mu(x_3)]}{\lambda(x_3)+2\mu(x_3)} - \omega^2\rho(x_3)\right] & 0 & 0 & -\frac{ik_1\lambda(x_3)}{\lambda(x_3)+2\mu(x_3)} \\ 0 & -\omega^2\rho(x_3) & -ik_1 & 0 \end{bmatrix}
\tag{8.3}
$$

and

$$
\underline{\underline{\mathsf{v}}} = \begin{bmatrix}\widehat{u}_1\\ \widehat{u}_3\\ \widehat{\sigma}_{31}\\ \widehat{\sigma}_{33}\end{bmatrix} \qquad \square
\tag{8.4}
$$

Problem 8.4 Use the WKBJ method to find the transmission coefficient for a plane wave propagating through the graded slab shown in the figure below.



Solution - 8.4 Let us consider the TE-wave propagation problem for this slab. Recall that the TE-Wave equation in this medium can be expressed as

$$\mu(x)\,\frac{d}{dx}\left[\frac{1}{\mu(x)}\,\frac{dE_y}{dx}\right] + [\omega^2\varepsilon(x)\mu(x) - k_z^2]\,E_y = 0\,.$$

Therefore, $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \varepsilon$. Since μ is not constant, we define $\psi(x) = E_y(x)/\sqrt{\mu(x)}$ to get the equation

$$\frac{d^2\psi}{dx^2} + \left[k_x^2 - \sqrt{\mu}\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{\mu}}\right)\right]\psi = 0$$

where $k_x^2(x) = \omega^2\varepsilon(x)\mu(x) - k_z^2$. Note that, with $\mu = \mu_0[1 + (x - x0)^3]$,

$$k_\mu^2(x) := k_x^2 - \sqrt{\mu}\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{\mu}}\right) = k_x^2 - \frac{3(x - x_0)[5(x - x_0)^3 - 4]}{4[(x - x0)^3 + 1]^2}$$

we have the equation

$$\frac{d^2\psi}{dx^2} + k_\mu^2(x)\psi(x) = 0\,.$$

Using $\omega^2 s^2(x) = k_\mu^2(x)$, and the WKBJ solution of the above equation can be found using the approach given in the text. Note that closed form solutions of the problem are not possible and numerical solutions are needed. Use some reasonable values of the parameters to arrive at your solution.

The unknown $E_y(x)$ can be found once $\psi(x)$ is known and the boundary conditions at the incident boundary of the slab have been applied. The ratio of the magnitudes of the field at the two boundaries gives the transmission coefficient. □

Problem 8.5 Let z_0 and z be the locations of the top and bottom of an isotropic elastic layer. Show that the propagator matrix for antiplane shear waves in the layer can be expressed as

$$\underline{\underline{\mathsf{P}}}(z, z_0) = \begin{bmatrix} \cos k_z(z - z_0) & (\mu k_z)^{-1} \sin k_z(z - z_0) \\ -\mu k_z \sin k_z(z - z_0) & \cos k_z(z - z_0) \end{bmatrix}.$$

Calculate the propagator matrix and then verify your result using the matrix exponential solution and Sylvester's formula. Note that the eigenvalues of a square matrix of the form $\alpha\underline{\underline{\mathsf{A}}}$ where α is a scalar are given by $\alpha\lambda_j$ where λ_j are the eigenvalues of $\underline{\underline{\mathsf{A}}}$.

Solution - 8.5 Recall that the propagator matrix for a homogeneous layer is given by

$$\underline{\underline{\mathsf{P}}}(z, z_0) = \exp[(z - z_0)\underline{\underline{\mathsf{H}}}] .$$

In the case of antiplane shear waves in an isotropic linear elastic medium

$$\underline{\underline{\mathsf{H}}} = \begin{bmatrix} 0 & \dfrac{1}{\mu(z)} \\ -k_z^2\mu(z) & 0 \end{bmatrix} \quad \text{with} \quad k_z^2 = \omega^2 \frac{\rho(z)}{\mu(z)} - k_1^2 .$$

The eigenvalues of $\underline{\underline{\mathsf{H}}}$ are $\lambda = \pm i k_z$. Using the Taylor expansion of the exponential around z_0, we have

$$\underline{\underline{\mathsf{P}}}(z, z_0) = \underline{\underline{\mathsf{I}}} + (z - z_0)\underline{\underline{\mathsf{H}}} + \frac{1}{2!}(z - z_0)^2\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}} + \frac{1}{3!}(z - z_0)^3\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}} + \frac{1}{4!}(z - z_0)^4\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}} + \frac{1}{5!}(z - z_0)^5\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}}\underline{\underline{\mathsf{H}}} + \dots$$

Explicitly,

$$\underline{\underline{\mathsf{P}}}(z, z_0) = \begin{bmatrix} 1 - \frac{k_z^2(z-z_0)^2}{2!} + \frac{k_z^4(z-z_0)^4}{4!} + \dots & \frac{1}{k_z\mu}(\frac{k_z(z-z_0)}{1!} - \frac{k_z^3(z-z_0)^3}{3!} + \frac{k_z^5(z-z_0)^5}{5!} + \dots) \\ -k_z\mu(\frac{k_z(z-z_0)}{1!} - \frac{k_z^3(z-z_0)^3}{3!} + \frac{k_z^5(z-z_0)^5}{5!} + \dots) & 1 - \frac{k_z^2(z-z_0)^2}{2!} + \frac{k_z^4(z-z_0)^4}{4!} + \dots \end{bmatrix}$$

Noting that the expressions above are just series expansions of sines and cosines, we have

$$\underline{\underline{\mathsf{P}}}(z, z_0) = \begin{bmatrix} \cos[k_z(z - z_0)] & \frac{1}{\mu k_z} \sin[k_z(z - z_0)] \\ -\mu k_z \sin[k_z(z - z_0)] & \cos[k_z(z - z_0)] \end{bmatrix} \qquad \square \tag{8.5}$$

Verification using Sylvester's formula (p. 305 of the text) is straightforward.

Problem 8.6 Derive the relations between the $\mathbf{E}_\mathrm{z}, \mathbf{H}_\mathrm{z}$ and $\mathbf{E}_\mathrm{s}, \mathbf{H}_\mathrm{s}$. Also, find the explicit form of the matrix $\underline{\underline{\mathsf{H}}}$.

Solution - 8.6 Recall that Maxwell's equations have the form

$$\boldsymbol{\nabla} \times \mathbf{E} = i\omega\boldsymbol{\mu} \cdot \mathbf{H} \quad \text{and} \quad \boldsymbol{\nabla} \times \mathbf{H} = -i\omega\boldsymbol{\varepsilon} \cdot \mathbf{E}\,.$$

We partition the field vectors and the material tensors into tangential and normal components:

$$\mathbf{E} = \begin{bmatrix} \underline{\underline{\mathsf{E}_s}} \\ E_z \end{bmatrix} \;;\; \mathbf{H} = \begin{bmatrix} \underline{\underline{\mathsf{H}_s}} \\ H_z \end{bmatrix} \boldsymbol{\mu} = \begin{bmatrix} \underline{\underline{\boldsymbol{\mu}_{ss}}} & \underline{\underline{\boldsymbol{\mu}_{sz}}} \\ \underline{\underline{\boldsymbol{\mu}_{zs}}} & \mu_{zz} \end{bmatrix} \;,\; \boldsymbol{\varepsilon} = \begin{bmatrix} \underline{\underline{\boldsymbol{\varepsilon}_{ss}}} & \underline{\underline{\boldsymbol{\varepsilon}_{sz}}} \\ \underline{\underline{\boldsymbol{\varepsilon}_{zs}}} & \varepsilon_{zz} \end{bmatrix}\,.$$

Now,

$$\boldsymbol{\nabla} \times \mathbf{E} = \begin{bmatrix} \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \\ \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\mu} \cdot \mathbf{H} = \begin{bmatrix} \mu_{xx}H_x + \mu_{xy}H_y + \mu_{xz}H_z \\ \mu_{yx}H_x + \mu_{yy}H_y + \mu_{yz}H_z \\ \mu_{zx}H_x + \mu_{zy}H_y + \mu_{zz}H_z \end{bmatrix} = \begin{bmatrix} \underline{\underline{\boldsymbol{\mu}_{ss}}}\,\underline{\underline{\mathsf{H}_s}} + H_z\underline{\underline{\boldsymbol{\mu}_{sz}}} \\ \underline{\underline{\boldsymbol{\mu}_{zs}}}\,\underline{\underline{\mathsf{H}_s}} + H_z\mu_{zz} \end{bmatrix}\,.$$

Similarly,

$$\boldsymbol{\nabla} \times \mathbf{H} = \begin{bmatrix} \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} \\ \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} \cdot \mathbf{E} = \begin{bmatrix} \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z \\ \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z \\ \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z \end{bmatrix} = \begin{bmatrix} \underline{\underline{\boldsymbol{\varepsilon}_{ss}}}\,\underline{\underline{\mathsf{E}_s}} + E_z\underline{\underline{\boldsymbol{\varepsilon}_{sz}}} \\ \underline{\underline{\boldsymbol{\varepsilon}_{zs}}}\,\underline{\underline{\mathsf{E}_s}} + E_z\varepsilon_{zz} \end{bmatrix}\,.$$

From the z-components of Maxwell's equations, we then have

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\left(\underline{\underline{\boldsymbol{\mu}_{zs}}}\,\underline{\underline{\mathsf{H}_s}} + H_z\mu_{zz}\right) \quad \text{and} \quad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\left(\underline{\underline{\boldsymbol{\varepsilon}_{zs}}}\,\underline{\underline{\mathsf{E}_s}} + E_z\varepsilon_{zz}\right).$$

From the definition of the tangential gradient operator, we see that

$$\boldsymbol{\nabla}_s \times \mathbf{E}_s = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{e}_z \quad \text{and} \quad \boldsymbol{\nabla}_s \times \mathbf{H}_s = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{e}_z\,.$$

Therefore we can write, reverting back to vector notation,

$$\boldsymbol{\nabla}_s \times \mathbf{E}_s = i\omega\left(\boldsymbol{\mu}_{zs} \cdot \mathbf{H}_s + H_z\mu_{zz}\mathbf{e}_z\right) \quad \text{and} \quad \boldsymbol{\nabla}_s \times \mathbf{H}_s = -i\omega\left(\boldsymbol{\varepsilon}_{zs} \cdot \mathbf{E}_s + E_z\varepsilon_{zz}\mathbf{e}_z\right).$$

Rearranging, and noting that $\mathbf{H}_z = H_z\mathbf{e}_z, \mathbf{E}_z = E_z\mathbf{e}_z$,

$$\mathbf{H}_z = \frac{1}{i\omega\mu_{zz}}\boldsymbol{\nabla}_s \times \mathbf{E}_s - \frac{1}{\mu_{zz}}\,\boldsymbol{\mu}_{zs} \cdot \mathbf{H}_s \quad \text{and} \quad \mathbf{E}_z = -\frac{1}{i\omega\varepsilon_{zz}}\boldsymbol{\nabla}_s \times \mathbf{H}_s - \frac{1}{\varepsilon_{zz}}\,\boldsymbol{\varepsilon}_{zs} \cdot \mathbf{E}_s \quad \square \tag{8.6}$$

Let us now consider the x- and y-components of Maxwell's equations. We have,

$$\begin{bmatrix} \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \\ \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} \end{bmatrix} = i\omega(\underline{\underline{\boldsymbol{\mu}_{ss}}}\,\underline{\underline{\mathsf{H}_s}} + H_z\underline{\underline{\boldsymbol{\mu}_{sz}}}) \quad \text{and} \quad \begin{bmatrix} \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} \\ \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} \end{bmatrix} = -i\omega(\underline{\underline{\boldsymbol{\varepsilon}_{ss}}}\,\underline{\underline{\mathsf{E}_s}} + E_z\underline{\underline{\boldsymbol{\varepsilon}_{sz}}})$$

or,

$$\begin{bmatrix} \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \\ \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} \end{bmatrix} = i\omega \left[\underline{\underline{\boldsymbol{\mu}_{ss}}}\, \underline{\underline{\mathsf{H}_s}} + \left(\frac{1}{i\omega\mu_{zz}} [\boldsymbol{\nabla}_s \times \mathbf{E}_s]_z - \frac{1}{\mu_{zz}} \underline{\underline{\boldsymbol{\mu}_{zs}}}\, \underline{\underline{\mathsf{H}_s}} \right) \underline{\underline{\boldsymbol{\mu}_{sz}}} \right]$$

and

$$\begin{bmatrix} \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} \\ \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} \end{bmatrix} = -i\omega \left[\underline{\underline{\boldsymbol{\varepsilon}_{ss}}}\, \underline{\underline{\mathsf{E}_s}} - \left(\frac{1}{i\omega\varepsilon_{zz}} [\boldsymbol{\nabla}_s \times \mathbf{H}_s]_z + \frac{1}{\varepsilon_{zz}} \underline{\underline{\boldsymbol{\varepsilon}_{zs}}}\, \underline{\underline{\mathsf{E}_s}} \right) \underline{\underline{\boldsymbol{\varepsilon}_{sz}}} \right] .$$

For a infinite layered medium with only z variation in the fields, the above equations simplify to

$$\begin{bmatrix} -\dfrac{\partial E_y}{\partial z} \\ \dfrac{\partial E_x}{\partial z} \end{bmatrix} = i\omega \left[\underline{\underline{\boldsymbol{\mu}_{ss}}}\, \underline{\underline{\mathsf{H}_s}} - \frac{1}{\mu_{zz}} \underline{\underline{\boldsymbol{\mu}_{zs}}}\, \underline{\underline{\mathsf{H}_s}}\, \underline{\underline{\boldsymbol{\mu}_{sz}}} \right] = i\omega \left[\underline{\underline{\boldsymbol{\mu}_{ss}}} - \frac{1}{\mu_{zz}} \underline{\underline{\boldsymbol{\mu}_{sz}}}\, \underline{\underline{\boldsymbol{\mu}_{zs}}} \right] \underline{\underline{\mathsf{H}_s}}$$

and

$$\begin{bmatrix} -\dfrac{\partial H_y}{\partial z} \\ \dfrac{\partial H_x}{\partial z} \end{bmatrix} = -i\omega \left[\underline{\underline{\boldsymbol{\varepsilon}_{ss}}}\, \underline{\underline{\mathsf{E}_s}} - \frac{1}{\varepsilon_{zz}} \underline{\underline{\boldsymbol{\varepsilon}_{zs}}}\, \underline{\underline{\mathsf{E}_s}}\, \underline{\underline{\boldsymbol{\varepsilon}_{sz}}} \right] = -i\omega \left[\underline{\underline{\boldsymbol{\varepsilon}_{ss}}} - \frac{1}{\varepsilon_{zz}} \underline{\underline{\boldsymbol{\varepsilon}_{sz}}}\, \underline{\underline{\boldsymbol{\varepsilon}_{zs}}} \right] \underline{\underline{\mathsf{E}_s}} .$$

Expanded out,

$$\frac{\partial}{\partial z} \begin{bmatrix} E_y \\ E_x \end{bmatrix} = i\omega \begin{bmatrix} \dfrac{\mu_{xz}\mu_{zx}}{\mu_{zz}} - \mu_{xx} & \dfrac{\mu_{xz}\mu_{zy}}{\mu_{zz}} - \mu_{xy} \\ \mu_{yx} - \dfrac{\mu_{yz}\mu_{zx}}{\mu_{zz}} & \mu_{yy} - \dfrac{\mu_{yz}\mu_{zy}}{\mu_{zz}} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix}$$

and

$$\frac{\partial}{\partial z} \begin{bmatrix} H_y \\ H_x \end{bmatrix} = i\omega \begin{bmatrix} \varepsilon_{xx} - \dfrac{\varepsilon_{xz}\varepsilon_{zx}}{\varepsilon_{zz}} & \varepsilon_{xy} - \dfrac{\varepsilon_{xz}\varepsilon_{zy}}{\varepsilon_{zz}} \\ \dfrac{\varepsilon_{yz}\varepsilon_{zx}}{\varepsilon_{zz}} - \varepsilon_{yx} & \dfrac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}} - \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} .$$

Combining the equations together,

$$\frac{\partial}{\partial z} \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = i\omega \begin{bmatrix} 0 & 0 & \mu_{yx} - \dfrac{\mu_{yz}\mu_{zx}}{\mu_{zz}} & \mu_{yy} - \dfrac{\mu_{yz}\mu_{zy}}{\mu_{zz}} \\ 0 & 0 & \dfrac{\mu_{xz}\mu_{zx}}{\mu_{zz}} - \mu_{xx} & \dfrac{\mu_{xz}\mu_{zy}}{\mu_{zz}} - \mu_{xy} \\ \dfrac{\varepsilon_{yz}\varepsilon_{zx}}{\varepsilon_{zz}} - \varepsilon_{yx} & \dfrac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}} - \varepsilon_{yy} & 0 & 0 \\ \varepsilon_{xx} - \dfrac{\varepsilon_{xz}\varepsilon_{zx}}{\varepsilon_{zz}} & \varepsilon_{xy} - \dfrac{\varepsilon_{xz}\varepsilon_{zy}}{\varepsilon_{zz}} & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix}$$

The $\underline{\underline{\mathsf{H}}}$ matrix is $i\omega$ times the 4×4 matrix above. □

Problem 8.7 In the Schoenberg-Sen model of a periodic layered medium, the slowness component s_z is related to the angle of incidence θ_i by $s_z = \sin\theta_i/c_0$ where c_0 is the phase speed in the medium of incidence. Show that, in the low frequency limit, the effective angle of transmission (θ_t) into the layered medium is given by

$$\theta_t = \tan^{-1}\left\{\sin\theta_i\left[\langle\rho\rangle^{\frac{1}{2}}\langle 1/\rho\rangle^{\frac{1}{2}}\left(c_0^2\frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle} - \sin^2\theta_i\right)^{\frac{1}{2}}\right]^{-1}\right\}.$$

Show that the same expression is obtained for the angle of transmission in a homogeneous medium with

$$\kappa = \frac{1}{\langle 1/\kappa\rangle} \quad \text{and} \quad \boldsymbol{\rho} \equiv \begin{bmatrix}\langle\rho\rangle & 0 & 0\\ 0 & 1/\langle 1/\rho\rangle & 0\\ 0 & 0 & 1/\langle 1/\rho\rangle\end{bmatrix}.$$

Solution - 8.7 We can express the slowness components in polar coordinates (r,θ) with θ measured counter-clockwise from the x-axis. Then,

$$s_x^{\text{eff}} = s(\theta)\cos\theta \quad \text{and} \quad s_z = s(\theta)\sin\theta$$

where $s(\theta)$ is the slowness surface (in 2D). Recall that

$$\frac{1}{\langle 1/\kappa\rangle\langle\rho\rangle}(s_x^{\text{eff}})^2 + \frac{\langle 1/\rho\rangle}{\langle 1/\kappa\rangle}s_z^2 = 1\,.$$

In polar coordinates,

$$\frac{1}{\langle 1/\kappa\rangle\langle\rho\rangle}s^2(\theta)\cos^2\theta + \frac{\langle 1/\rho\rangle}{\langle 1/\kappa\rangle}s^2(\theta)\sin^2\theta = 1$$

or

$$s(\theta) = \left[\frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle\left(\frac{\cos^2\theta}{\langle\rho\rangle\langle 1/\rho\rangle} + \sin^2\theta\right)}\right]^{1/2}.$$

If θ_i is the angle of incidence and θ_t^{eff} is the angle of transmission, Snell's law requires that

$$\frac{\sin\theta_i}{c_0} = \frac{\sin\theta_t}{c_t^{\text{eff}}} = s(\theta_t)\sin\theta_t = s_z(\theta_t)$$

where c_t^{eff} is the effective speed of sound in the layered medium. Therefore,

$$\frac{\sin\theta_i}{c_0} = \sin\theta_t\left[\frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle\left(\frac{\cos^2\theta_t}{\langle\rho\rangle\langle 1/\rho\rangle} + \sin^2\theta_t\right)}\right]^{1/2}.$$

Rearranging,

$$\frac{\sin^2\theta_i}{c_0^2}\left(\frac{\cos^2\theta_t}{\langle\rho\rangle\langle 1/\rho\rangle} + \sin^2\theta_t\right) = \sin^2\theta_t\frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle}$$

or

$$\frac{\sin^2\theta_i}{c_0^2}\left(\frac{1}{\langle\rho\rangle\langle 1/\rho\rangle} + \tan^2\theta_t\right) = \tan^2\theta_t\frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle}$$

or

$$\tan^2\theta_t\left[c_0^2\frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle} - \sin^2\theta_i\right]\langle\rho\rangle\langle 1/\rho\rangle = \sin^2\theta_i\,.$$

Therefore,

$$\theta_t = \tan^{-1}\left\{\sin\theta_i \left[\langle\rho\rangle^{\frac{1}{2}} \langle 1/\rho\rangle^{\frac{1}{2}} \left(c_0^2 \frac{\langle 1/\kappa\rangle}{\langle 1/\rho\rangle} - \sin^2\theta_i\right)^{\frac{1}{2}}\right]^{-1}\right\} \qquad \square \tag{8.7}$$

The second part of the problem can be solved using the techniques discussed in earlier chapters. For a isotropic incidence medium over a medium with a transversely anisotropic density and bulk modulus κ, we can show that the transmission angle is

$$\theta_t = \tan^{-1}\left\{\sin\theta_i \left[(\rho_x/\rho_z)^{\frac{1}{2}} \left(\frac{\rho_z\, c_0^2}{\kappa} - \sin^2\theta_i\right)^{\frac{1}{2}}\right]^{-1}\right\}$$

which is the relation that we seek. $\square$

Problem 8.8 Verify that the effective stiffness of an elastic laminate can be expressed as

$$\begin{aligned}
\underline{\underline{\mathbf{C}}}^{nn}_{\text{eff}} &= \left\langle (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \right\rangle^{-1} \\
\underline{\underline{\mathbf{C}}}^{nt}_{\text{eff}} &= \left\langle (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{nt} \right\rangle \\
\underline{\underline{\mathbf{C}}}^{tt}_{\text{eff}} &= \left\langle \underline{\underline{\mathbf{C}}}^{tt} - \underline{\underline{\mathbf{C}}}^{tn} \cdot (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{nt} \right\rangle + \\
&\qquad \left\langle \underline{\underline{\mathbf{C}}}^{tn} \cdot (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \right\rangle \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{nn})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{nt} \right\rangle .
\end{aligned}$$

Consider the three-layered laminate shown in the figure below and assume that the x-direction is the direction of lamination and the y, z-directions are in the plane of the laminate. The stiffness of layer i is labeled $\mathbf{C}_i$ in the figure.

Assume that layers 1 and 3 are made of a transversely isotropic AS/3501 (carbon fiber/epoxy) composite with Young's moduli $E_x = 9$ GPa, $E_y = E_z = 140$ GPa, Poisson's ratios $\nu_{xy} = 0.1$, $\nu_{yz} = \nu_{xz} = 0.3$, and shear moduli $G_{xz} = G_{yz} = 7$ GPa. Assume that layer 2 is isotropic with Young's modulus $E = 70$ GPa and Poisson's ratio $\nu = 0.2$. What are the quasistatic effective elastic stiffnesses of the three layer composite if the thicknesses of layers 1 and 3 are 2 mm and that of layer 2 is 5 mm?

Solution - 8.8 Recall that

$$\begin{bmatrix} -\left\langle \underline{\boldsymbol{\varepsilon}}^{\text{n}} \right\rangle \\ \left\langle \underline{\boldsymbol{\sigma}}^{\text{t}} \right\rangle \end{bmatrix} = \begin{bmatrix} -\left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle & \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle \\ \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nt}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle & \left\langle \underline{\underline{\mathbf{C}}}^{\text{tt}} - (\underline{\underline{\mathbf{C}}}^{\text{nt}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle \end{bmatrix} \begin{bmatrix} -\underline{\boldsymbol{\sigma}}^{\text{n}} \\ \underline{\boldsymbol{\varepsilon}}^{\text{t}} \end{bmatrix}$$

and

$$\begin{bmatrix} -\left\langle \underline{\boldsymbol{\varepsilon}}^{\text{n}} \right\rangle \\ \left\langle \underline{\boldsymbol{\sigma}}^{\text{t}} \right\rangle \end{bmatrix} = \begin{bmatrix} -(\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} & (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} \\ (\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} & \underline{\underline{\mathbf{C}}}^{\text{tt}}_{\text{eff}} - (\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} \end{bmatrix} \begin{bmatrix} -\underline{\boldsymbol{\sigma}}^{\text{n}} \\ \underline{\boldsymbol{\varepsilon}}^{\text{t}} \end{bmatrix} .$$

Therefore,

$$(\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} = \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \qquad \square \tag{8.8}$$

and

$$\begin{aligned}
(\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} &= \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle \\
(\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} &= \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nt}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \\
\underline{\underline{\mathbf{C}}}^{\text{tt}}_{\text{eff}} - (\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} &= \left\langle \underline{\underline{\mathbf{C}}}^{\text{tt}} - (\underline{\underline{\mathbf{C}}}^{\text{nt}})^{T} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle .
\end{aligned}$$

Then from the first equation above,

$$\left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} = \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle$$

or

$$\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} = \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle \qquad \square \tag{8.9}$$

From the second equation and the symmetry of $\mathbf{C}$,

$$(\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}})^T \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} = \left\langle \underline{\underline{\mathbf{C}}}^{\text{tn}} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle$$

Therefore,

$$(\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}})^T \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} = \left\langle \underline{\underline{\mathbf{C}}}^{\text{tn}} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle$$

and from the third equation we have

$$\underline{\underline{\mathbf{C}}}^{\text{tt}}_{\text{eff}} = \left\langle \underline{\underline{\mathbf{C}}}^{\text{tt}} - \underline{\underline{\mathbf{C}}}^{\text{tn}} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle + \left\langle \underline{\underline{\mathbf{C}}}^{\text{tn}} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle \qquad \square \quad (8.10)$$

In standard engineering notation, the strain-stress relation for an orthotropic material is

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}.$$

For a transversely isotropic material that is isotropic in the $2-3$ plane, the above relation simplifies to

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{xy}}{E_x} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{yz}}{E_y} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}.$$

where $G_{yz} = E_y/(2(1+\nu_{yz}))$. Inversion of the above relation gives us

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E_x^2(1-\nu_{yz})}{\Delta} & \frac{E_x E_y \nu_{xy}}{\Delta} & \frac{E_x E_y \nu_{xy}}{\Delta} & 0 & 0 & 0 \\ \frac{E_x E_y \nu_{xy}}{\Delta} & \frac{E_x E_y - E_y^2 \nu_{xy}^2}{\Delta(1+\nu_{yz})} & \frac{E_y^2 \nu_{xy}^2 + E_x E_y \nu_{yz}}{\Delta(1+\nu_{yz})} & 0 & 0 & 0 \\ \frac{E_x E_y \nu_{xy}}{\Delta} & \frac{E_y^2 \nu_{xy}^2 + E_x E_y \nu_{yz}}{\Delta(1+\nu_{yz})} & \frac{E_x E_y - E_y^2 \nu_{xy}^2}{\Delta(1+\nu_{yz})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E_y}{2(1+\nu_{yz})} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{xy} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{xy} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

where $\Delta := E_x(1-\nu_{yz}) - 2E_y\nu_{xy}^2$. Plugging in the values of the elastic moduli of the two materials we have

$$\underline{\underline{\mathbf{C}}}_1 = \underline{\underline{\mathbf{C}}}_3 = \begin{bmatrix} 16.2 & 36 & 36 & 0 & 0 & 0 \\ 36 & 233.8 & 126.2 & 0 & 0 & 0 \\ 36 & 126.2 & 233.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 53.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix} ; \underline{\underline{\mathbf{C}}}_2 = \begin{bmatrix} 77.8 & 19.4 & 19.4 & 0 & 0 & 0 \\ 19.4 & 77.8 & 19.4 & 0 & 0 & 0 \\ 19.4 & 19.4 & 77.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 29.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 29.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 29.2 \end{bmatrix}.$$

Now, for layers 1 and 3,

$$\underline{\underline{\mathbf{C}}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1122} & C_{2233} & C_{2222} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{2222} - C_{2233}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix}$$

and for layer 2

$$\underline{\underline{\mathbf{C}}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1122} & C_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{1111} - C_{1122}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{1111} - C_{1122}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{1111} - C_{1122}) \end{bmatrix}$$

Therefore, we have

$$\begin{aligned} & C^{(1,3)}_{1111} = 16.2\,, C^{(1,3)}_{2222} = C^{(1,3)}_{3333} = 233.8 \\ & C^{(1,3)}_{1122} = C^{(1,3)}_{1133} = C^{(1,3)}_{2211} = C^{(1,3)}_{3311} = 36\,, C^{(1,3)}_{2233} = C^{(1,3)}_{3322} = 126.2 \\ & C^{(1,3)}_{2323} = 53.8\,, C^{(1,3)}_{3131} = C^{(1,3)}_{1212} = 7 \end{aligned}$$

and

$$\begin{aligned} & C^{(2)}_{1111} = C^{(2)}_{2222} = C^{(2)}_{3333} = 77.8 \\ & C^{(2)}_{1122} = C^{(2)}_{1133} = C^{(2)}_{2211} = C^{(2)}_{3311} = C^{(2)}_{2233} = C^{(2)}_{3322} = 19.4 \\ & C^{(2)}_{2323} = C^{(2)}_{3131} = C^{(2)}_{1212} = 29.2\,. \end{aligned}$$

All other components of the stiffness tensors are zero. Assuming that the plane of isotropy is the y-z plane and that the normal to the layers is in the x-direction, we can partition the stiffness matrix using the definitions

$$\underline{\underline{\mathbf{C}}}^{\text{nn}} = \begin{bmatrix} C_{1111} & \sqrt{2}\,C_{1112} & \sqrt{2}\,C_{1113} \\ \sqrt{2}\,C_{1211} & 2\,C_{1212} & 2\,C_{1213} \\ \sqrt{2}\,C_{1311} & 2\,C_{1312} & 2\,C_{1313} \end{bmatrix} \;;\; \underline{\underline{\mathbf{C}}}^{\text{nt}} = \begin{bmatrix} C_{1122} & C_{1133} & \sqrt{2}\,C_{1123} \\ \sqrt{2}\,C_{1222} & \sqrt{2}\,C_{1233} & 2\,C_{1223} \\ \sqrt{2}\,C_{1322} & \sqrt{2}\,C_{1333} & 2\,C_{1323} \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{tn}} = \begin{bmatrix} C_{2211} & \sqrt{2}\,C_{2212} & \sqrt{2}\,C_{2213} \\ C_{3311} & \sqrt{2}\,C_{3312} & \sqrt{2}\,C_{3313} \\ \sqrt{2}\,C_{2311} & 2\,C_{2312} & 2\,C_{2313} \end{bmatrix} \;;\; \underline{\underline{\mathbf{C}}}^{\text{tt}} = \begin{bmatrix} C_{2222} & C_{2233} & \sqrt{2}\,C_{2223} \\ C_{3322} & C_{3333} & \sqrt{2}\,C_{3323} \\ \sqrt{2}\,C_{2322} & \sqrt{2}\,C_{2333} & 2\,C_{2323} \end{bmatrix}.$$

Plugging in the values of the stiffness matrix components, we have

$$\underline{\underline{\mathbf{C}}}^{\text{nn}}_1 = \underline{\underline{\mathbf{C}}}^{\text{nn}}_3 = \begin{bmatrix} 16.2 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} \,,\quad \underline{\underline{\mathbf{C}}}^{\text{nt}}_1 = \underline{\underline{\mathbf{C}}}^{\text{nt}}_3 = \begin{bmatrix} 36 & 36 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{tn}}_1 = \underline{\underline{\mathbf{C}}}^{\text{tn}}_3 = \begin{bmatrix} 36 & 0 & 0 \\ 36 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \,,\quad \underline{\underline{\mathbf{C}}}^{\text{tt}}_1 = \underline{\underline{\mathbf{C}}}^{\text{tt}}_3 = \begin{bmatrix} 233.8 & 126.2 & 0 \\ 126.2 & 233.8 & 0 \\ 0 & 0 & 107.7 \end{bmatrix}$$

and

$$\underline{\underline{\mathbf{C}}}^{\text{nn}}_2 = \begin{bmatrix} 77.8 & 0 & 0 \\ 0 & 58.3 & 0 \\ 0 & 0 & 58.3 \end{bmatrix} \,,\quad \underline{\underline{\mathbf{C}}}^{\text{nt}}_2 = \begin{bmatrix} 19.4 & 19.4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{tn}}_2 = \begin{bmatrix} 19.4 & 0 & 0 \\ 19.4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \,,\quad \underline{\underline{\mathbf{C}}}^{\text{tt}}_2 = \begin{bmatrix} 77.8 & 19.4 & 0 \\ 19.4 & 77.8 & 0 \\ 0 & 0 & 58.3 \end{bmatrix}.$$

Now we have to computing volume averages. For the first partition, we have, with $L = L_1 + L_2 + L_3$,

$$\begin{aligned}(\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} &= \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \\ &= \frac{1}{V}\left[\int_{V_1} (\underline{\underline{\mathbf{C}}}^{\text{nn}}_1)^{-1} dV + \int_{V_2} (\underline{\underline{\mathbf{C}}}^{\text{nn}}_2)^{-1} dV + \int_{V_3} (\underline{\underline{\mathbf{C}}}^{\text{nn}}_3)^{-1} dV\right] \\ &= \frac{1}{L}\left[L_1(\underline{\underline{\mathbf{C}}}^{\text{nn}}_1)^{-1} + L_2(\underline{\underline{\mathbf{C}}}^{\text{nn}}_2)^{-1} + L_3(\underline{\underline{\mathbf{C}}}^{\text{nn}}_3)^{-1}\right] .\end{aligned}$$

Plugging in the values of the layer thicknesses and the stiffness matrices, we have

$$(\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}})^{-1} = \begin{bmatrix} 0.034 & 0 & 0 \\ 0 & 0.041 & 0 \\ 0 & 0 & 0.041 \end{bmatrix} .$$

Therefore,

$$\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}} = \begin{bmatrix} 28.9 & 0 & 0 \\ 0 & 24.2 & 0 \\ 0 & 0 & 24.2 \end{bmatrix} \text{GPa} .$$

Similarly, for the second partition, we have

$$\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} = \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle = \begin{bmatrix} 32.6 & 32.6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{GPa}$$

and

$$\underline{\underline{\mathbf{C}}}^{\text{tn}}_{\text{eff}} = \begin{bmatrix} 32.6 & 0 & 0 \\ 32.6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{GPa} .$$

Finally, plugging in values for the transverse components of the matrix, we have

$$\underline{\underline{\mathbf{C}}}^{\text{tt}}_{\text{eff}} = \begin{bmatrix} 145.6 & 65.3 & 0 \\ 65.3 & 145.6 & 0 \\ 0 & 0 & 80.3 \end{bmatrix} \text{GPa} .$$

Writing in engineering notation again, we have

$$\underline{\underline{\mathbf{C}}}_{\text{eff}} = \begin{bmatrix} 28.9 & 32.6 & 32.6 & 0 & 0 & 0 \\ 32.6 & 145.6 & 65.3 & 0 & 0 & 0 \\ 32.6 & 65.3 & 145.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 40.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.1 \end{bmatrix} \text{GPa} \quad \square$$

Problem 8.9 Show that for a laminate made of isotropic linear elastic layers, the effective stiffness tensor has components

$$\begin{aligned}
&C^{\text{eff}}_{1111} = \left\langle \tfrac{1}{\lambda+2\mu} \right\rangle^{-1} ~;~ C^{\text{eff}}_{1122} = C^{\text{eff}}_{1133} = \left\langle \tfrac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \tfrac{1}{\lambda+2\mu} \right\rangle^{-1} \\
&C^{\text{eff}}_{1212} = C^{\text{eff}}_{1313} = \left\langle \tfrac{1}{\mu} \right\rangle^{-1} ~;~ C^{\text{eff}}_{2323} = \langle \mu \rangle ~;~ C^{\text{eff}}_{1112} = C^{\text{eff}}_{1113} = C^{\text{eff}}_{1123} = 0 \\
&C^{\text{eff}}_{2222} = C^{\text{eff}}_{3333} = \left\langle \tfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \right\rangle + \left\langle \tfrac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \tfrac{1}{\lambda+2\mu} \right\rangle^{-1} \\
&C^{\text{eff}}_{2233} = \left\langle \tfrac{2\mu\lambda}{\lambda+2\mu} \right\rangle + \left\langle \tfrac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \tfrac{1}{\lambda+2\mu} \right\rangle^{-1} ~.
\end{aligned}$$

Solution - 8.9: Recall that

$$\begin{aligned}
\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}} &= \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \\
\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} &= \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle \\
\underline{\underline{\mathbf{C}}}^{\text{tt}}_{\text{eff}} &= \left\langle \underline{\underline{\mathbf{C}}}^{\text{tt}} - \underline{\underline{\mathbf{C}}}^{\text{tn}} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle + \\
&\qquad \left\langle \underline{\underline{\mathbf{C}}}^{\text{tn}} \cdot (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \right\rangle^{-1} \cdot \left\langle (\underline{\underline{\mathbf{C}}}^{\text{nn}})^{-1} \cdot \underline{\underline{\mathbf{C}}}^{\text{nt}} \right\rangle
\end{aligned}$$

where

$$\underline{\underline{\mathbf{C}}}^{\text{nn}} = \begin{bmatrix} C_{1111} & \sqrt{2}\, C_{1112} & \sqrt{2}\, C_{1113} \\ \sqrt{2}\, C_{1211} & 2\, C_{1212} & 2\, C_{1213} \\ \sqrt{2}\, C_{1311} & 2\, C_{1312} & 2\, C_{1313} \end{bmatrix} ~;~ \underline{\underline{\mathbf{C}}}^{\text{nt}} = \begin{bmatrix} C_{1122} & C_{1133} & \sqrt{2}\, C_{1123} \\ \sqrt{2}\, C_{1222} & \sqrt{2}\, C_{1233} & 2\, C_{1223} \\ \sqrt{2}\, C_{1322} & \sqrt{2}\, C_{1333} & 2\, C_{1323} \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{tn}} = \begin{bmatrix} C_{2211} & \sqrt{2}\, C_{2212} & \sqrt{2}\, C_{2213} \\ C_{3311} & \sqrt{2}\, C_{3312} & \sqrt{2}\, C_{3313} \\ \sqrt{2}\, C_{2311} & 2\, C_{2312} & 2\, C_{2313} \end{bmatrix} ~;~ \underline{\underline{\mathbf{C}}}^{\text{tt}} = \begin{bmatrix} C_{2222} & C_{2233} & \sqrt{2}\, C_{2223} \\ C_{3322} & C_{3333} & \sqrt{2}\, C_{3323} \\ \sqrt{2}\, C_{2322} & \sqrt{2}\, C_{2333} & 2\, C_{2323} \end{bmatrix} ~.$$

If the material in each layer is isotropic, then the constitutive relation is

$$C_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\,.$$

Then the components of the stiffness tensor are

$$\begin{aligned}
&C_{1111} = \lambda + 2\mu\,; \quad C_{1112} = 0\,; \quad C_{1113} = 0\,; \quad C_{1211} = 0 \\
&C_{1212} = \mu\,; \quad C_{1213} = 0\,; \quad C_{1311} = 0\,; \quad C_{1312} = 0\,; \quad C_{1313} = \mu \\
&C_{1122} = \lambda\,; \quad C_{1133} = \lambda\,; \quad C_{1123} = 0\,; \quad C_{1222} = 0 \\
&C_{1233} = 0\,; \quad C_{1223} = 0\,; \quad C_{1322} = 0\,; \quad C_{1333} = 0 \quad C_{1323} = 0 \\
&C_{2211} = \lambda\,; \quad C_{2212} = 0\,; \quad C_{2213} = 0\,; \quad C_{3311} = \lambda \\
&C_{3312} = 0\,; \quad C_{3313} = 0\,; \quad C_{2311} = 0\,; \quad C_{2312} = \lambda \quad C_{2313} = 0 \\
&C_{2222} = \lambda + 2\mu\,; \quad C_{2233} = \lambda\,; \quad C_{2223} = 0\,; \quad C_{3322} = \lambda \\
&C_{3333} = \lambda + 2\mu\,; \quad C_{3323} = 0\,; \quad C_{2322} = 0\,; \quad C_{2333} = 0\,; \quad C_{2323} = \mu\,.
\end{aligned}$$

Therefore,

$$\underline{\underline{\mathbf{C}}}^{\text{nn}} = \begin{bmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} ~;~ \underline{\underline{\mathbf{C}}}^{\text{nt}} = \begin{bmatrix} \lambda & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{tn}} = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; ; \; \underline{\underline{\mathbf{C}}}^{\text{tt}} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} .$$

Substituting into the expressions for the effective stiffness,

$$\underline{\underline{\mathbf{C}}}^{\text{nn}}_{\text{eff}} = \begin{bmatrix} \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & 0 & 0 \\ 0 & \left\langle \frac{1}{2\mu} \right\rangle^{-1} & 0 \\ 0 & 0 & \left\langle \frac{1}{2\mu} \right\rangle^{-1} \end{bmatrix} = \begin{bmatrix} C^{\text{eff}}_{1111} & 0 & 0 \\ 0 & 2\,C^{\text{eff}}_{1212} & 0 \\ 0 & 0 & 2\,C^{\text{eff}}_{1313} \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{nt}}_{\text{eff}} = \begin{bmatrix} \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & 0 & 0 \\ 0 & \left\langle \frac{1}{2\mu} \right\rangle^{-1} & 0 \\ 0 & 0 & \left\langle \frac{1}{2\mu} \right\rangle^{-1} \end{bmatrix} \begin{bmatrix} \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} C^{\text{eff}}_{1122} & C^{\text{eff}}_{1133} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\mathbf{C}}}^{\text{tt}}_{\text{eff}} = \begin{bmatrix} \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle & \left\langle \frac{2\lambda\mu}{\lambda+2\mu} \right\rangle & 0 \\ \left\langle \frac{2\lambda\mu}{\lambda+2\mu} \right\rangle & \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle & 0 \\ 0 & 0 & \langle 2\mu \rangle \end{bmatrix} +$$

$$\begin{bmatrix} \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & 0 & 0 \\ \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & 0 & 0 \\ 0 & \left\langle \frac{1}{2\mu} \right\rangle^{-1} & 0 \\ 0 & 0 & \left\langle \frac{1}{2\mu} \right\rangle^{-1} \end{bmatrix} \begin{bmatrix} \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & \left\langle \frac{2\lambda\mu}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & 0 \\ \left\langle \frac{2\lambda\mu}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} & 0 \\ 0 & 0 & \langle 2\mu \rangle \end{bmatrix}$$

$$= \begin{bmatrix} C^{\text{eff}}_{2222} & C^{\text{eff}}_{2233} & 0 \\ C^{\text{eff}}_{3322} & C^{\text{eff}}_{3333} & 0 \\ 0 & 0 & 2\,C^{\text{eff}}_{2323} \end{bmatrix} .$$

Therefore,

$$\begin{aligned} & C^{\text{eff}}_{1111} = \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} \; ; \; C^{\text{eff}}_{1122} = C^{\text{eff}}_{1133} = \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} \\ & C^{\text{eff}}_{1212} = C^{\text{eff}}_{1313} = \left\langle \frac{1}{\mu} \right\rangle^{-1} \; ; \; C^{\text{eff}}_{2323} = \langle \mu \rangle \; ; \; C^{\text{eff}}_{1112} = C^{\text{eff}}_{1113} = C^{\text{eff}}_{1123} = 0 \\ & C^{\text{eff}}_{2222} = C^{\text{eff}}_{3333} = \left\langle \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} \\ & C^{\text{eff}}_{2233} = \left\langle \frac{2\mu\lambda}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} . \end{aligned} \qquad \square \qquad (8.11)$$

Problem 8.10 Verify that the quasistatic effective permittivity of a laminate is given by the relations

$$\begin{aligned}
\varepsilon_{11}^{\text{eff}} &= \left\langle \tfrac{1}{\varepsilon_{11}} \right\rangle^{-1} \\
\varepsilon_{1j}^{\text{eff}} &= \left\langle \tfrac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \tfrac{\varepsilon_{1j}}{\varepsilon_{11}} \right\rangle \\
\varepsilon_{ij}^{\text{eff}} &= \left\langle \varepsilon_{ij} - \tfrac{\varepsilon_{i1}\,\varepsilon_{1j}}{\varepsilon_{11}} \right\rangle + \left\langle \tfrac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \tfrac{\varepsilon_{i1}}{\varepsilon_{11}} \right\rangle \left\langle \tfrac{\varepsilon_{1j}}{\varepsilon_{11}} \right\rangle \quad ,i,j \neq 1\,.
\end{aligned}$$

Solution - 8.10 Recall that

$$\underline{\underline{\mathbf{L}}}(\mathbf{x}) = \frac{1}{\varepsilon_{11}} \begin{bmatrix} -1 & \underline{\underline{\boldsymbol{\varepsilon}}}^{1t} \\ \underline{\underline{\boldsymbol{\varepsilon}}}^{t1} & \varepsilon_{11}\underline{\underline{\boldsymbol{\varepsilon}}}^{tt} - \underline{\underline{\boldsymbol{\varepsilon}}}^{t1} \cdot \underline{\underline{\boldsymbol{\varepsilon}}}^{1t} \end{bmatrix}.$$

where

$$\underline{\underline{\boldsymbol{\varepsilon}}}^{1t} = \begin{bmatrix} \varepsilon_{12} & \varepsilon_{13} \end{bmatrix} \;;\; \underline{\underline{\boldsymbol{\varepsilon}}}^{t1} = \begin{bmatrix} \varepsilon_{21} \\ \varepsilon_{31} \end{bmatrix} \;;\; \underline{\underline{\boldsymbol{\varepsilon}}}^{tt} = \begin{bmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}.$$

Expanding the terms

$$\underline{\underline{\mathbf{L}}}(\mathbf{x}) = \begin{bmatrix} -\frac{1}{\varepsilon_{11}} & \frac{\varepsilon_{12}}{\varepsilon_{11}} & \frac{\varepsilon_{13}}{\varepsilon_{11}} \\ \frac{\varepsilon_{21}}{\varepsilon_{11}} & \frac{-\varepsilon_{12}\varepsilon_{21}+\varepsilon_{11}\varepsilon_{22}}{\varepsilon_{11}} & \frac{-\varepsilon_{13}\varepsilon_{21}+\varepsilon_{11}\varepsilon_{23}}{\varepsilon_{11}} \\ \frac{\varepsilon_{31}}{\varepsilon_{11}} & \frac{-\varepsilon_{12}\varepsilon_{31}+\varepsilon_{11}\varepsilon_{32}}{\varepsilon_{11}} & \frac{-\varepsilon_{13}\varepsilon_{31}+\varepsilon_{11}\varepsilon_{33}}{\varepsilon_{11}} \end{bmatrix}.$$

Also recall that

$$\underline{\underline{\mathbf{L}}}_{\text{eff}} = \left\langle \underline{\underline{\mathbf{L}}}(\mathbf{x}) \right\rangle$$

where

$$\underline{\underline{\mathbf{L}}}_{\text{eff}} = \frac{1}{\varepsilon_{11}^{\text{eff}}} \begin{bmatrix} -1 & \underline{\underline{\boldsymbol{\varepsilon}}}^{1t}_{\text{eff}} \\ \underline{\underline{\boldsymbol{\varepsilon}}}^{t1}_{\text{eff}} & \varepsilon_{11}^{\text{eff}}\underline{\underline{\boldsymbol{\varepsilon}}}^{tt}_{\text{eff}} - \underline{\underline{\boldsymbol{\varepsilon}}}^{t1}_{\text{eff}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}}^{1t}_{\text{eff}} \end{bmatrix}.$$

where

$$\underline{\underline{\boldsymbol{\varepsilon}}}^{1t}_{\text{eff}} = \begin{bmatrix} \varepsilon_{12}^{\text{eff}} & \varepsilon_{13}^{\text{eff}} \end{bmatrix} \;;\; \underline{\underline{\boldsymbol{\varepsilon}}}^{t1}_{\text{eff}} = \begin{bmatrix} \varepsilon_{21}^{\text{eff}} \\ \varepsilon_{31}^{\text{eff}} \end{bmatrix} \;;\; \underline{\underline{\boldsymbol{\varepsilon}}}^{tt}_{\text{eff}} = \begin{bmatrix} \varepsilon_{22}^{\text{eff}} & \varepsilon_{23}^{\text{eff}} \\ \varepsilon_{32}^{\text{eff}} & \varepsilon_{33}^{\text{eff}} \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} -\frac{1}{\varepsilon_{11}^{\text{eff}}} & \frac{\varepsilon_{12}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} & \frac{\varepsilon_{13}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} \\ \frac{\varepsilon_{21}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} & \frac{-\varepsilon_{12}^{\text{eff}}\varepsilon_{21}^{\text{eff}}+\varepsilon_{11}^{\text{eff}}\varepsilon_{22}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} & \frac{-\varepsilon_{13}^{\text{eff}}\varepsilon_{21}^{\text{eff}}+\varepsilon_{11}^{\text{eff}}\varepsilon_{23}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} \\ \frac{\varepsilon_{31}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} & \frac{-\varepsilon_{12}^{\text{eff}}\varepsilon_{31}^{\text{eff}}+\varepsilon_{11}^{\text{eff}}\varepsilon_{32}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} & \frac{-\varepsilon_{13}^{\text{eff}}\varepsilon_{31}^{\text{eff}}+\varepsilon_{11}^{\text{eff}}\varepsilon_{33}^{\text{eff}}}{\varepsilon_{11}^{\text{eff}}} \end{bmatrix} = \begin{bmatrix} -\left\langle\frac{1}{\varepsilon_{11}}\right\rangle & \left\langle\frac{\varepsilon_{12}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{\varepsilon_{13}}{\varepsilon_{11}}\right\rangle \\ \left\langle\frac{\varepsilon_{21}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{12}\varepsilon_{21}+\varepsilon_{11}\varepsilon_{22}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{13}\varepsilon_{21}+\varepsilon_{11}\varepsilon_{23}}{\varepsilon_{11}}\right\rangle \\ \left\langle\frac{\varepsilon_{31}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{12}\varepsilon_{31}+\varepsilon_{11}\varepsilon_{32}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{13}\varepsilon_{31}+\varepsilon_{11}\varepsilon_{33}}{\varepsilon_{11}}\right\rangle \end{bmatrix}.$$

From the (1,1) terms of the equation we see that

$$\varepsilon_{11}^{\text{eff}} = \left\langle \tfrac{1}{\varepsilon_{11}} \right\rangle^{-1}.$$

Therefore, we can write

$$\begin{aligned}
&\begin{bmatrix} 1 & \varepsilon_{12}^{\text{eff}} & \varepsilon_{13}^{\text{eff}} \\ \varepsilon_{21}^{\text{eff}} & -\varepsilon_{12}^{\text{eff}}\varepsilon_{21}^{\text{eff}} + \varepsilon_{11}^{\text{eff}}\varepsilon_{22}^{\text{eff}} & -\varepsilon_{13}^{\text{eff}}\varepsilon_{21}^{\text{eff}} + \varepsilon_{11}^{\text{eff}}\varepsilon_{23}^{\text{eff}} \\ \varepsilon_{31}^{\text{eff}} & -\varepsilon_{12}^{\text{eff}}\varepsilon_{31}^{\text{eff}} + \varepsilon_{11}^{\text{eff}}\varepsilon_{32}^{\text{eff}} & -\varepsilon_{13}^{\text{eff}}\varepsilon_{31}^{\text{eff}} + \varepsilon_{11}^{\text{eff}}\varepsilon_{33}^{\text{eff}} \end{bmatrix} \\
&\qquad = \left\langle \tfrac{1}{\varepsilon_{11}} \right\rangle^{-1} \begin{bmatrix} -\left\langle\frac{1}{\varepsilon_{11}}\right\rangle & \left\langle\frac{\varepsilon_{12}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{\varepsilon_{13}}{\varepsilon_{11}}\right\rangle \\ \left\langle\frac{\varepsilon_{21}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{12}\varepsilon_{21}+\varepsilon_{11}\varepsilon_{22}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{13}\varepsilon_{21}+\varepsilon_{11}\varepsilon_{23}}{\varepsilon_{11}}\right\rangle \\ \left\langle\frac{\varepsilon_{31}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{12}\varepsilon_{31}+\varepsilon_{11}\varepsilon_{32}}{\varepsilon_{11}}\right\rangle & \left\langle\frac{-\varepsilon_{13}\varepsilon_{31}+\varepsilon_{11}\varepsilon_{33}}{\varepsilon_{11}}\right\rangle \end{bmatrix}.
\end{aligned}$$

The required equations are obtained by solving the above system of equations. □

Problem 8.11 Use the Backus approach to find expressions for the effective magnetic permeability of a laminate with lamination in the x_1-direction.

Solution - 8.11 To find the effective material properties of the laminate we take advantage of the fact that since the tangential components (parallel to the layers) of the magnetic field (**H**) are piecewise constant and continuous across the interfaces between the layers, these tangential components must be constant, i.e., H_2 and H_3 are constant in the laminate. Similarly, the continuity of the normal magnetic induction field (**B**) across the interfaces and the fact that this field is constant in each layer implies that the component B_1 is constant in the laminate.

Recall that the constitutive relation between **B** and **H** is

$$\mathbf{B} = \boldsymbol{\mu} \cdot \mathbf{H} \,.$$

Let us rewrite the constitutive relation in matrix form (with respect to the rectangular Cartesian basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$) so that constant fields appear on the right hand side. We start by breaking up the matrix representation of the constitutive relation into the form

$$\begin{bmatrix} B_1 \\ \underline{\underline{\mathsf{B}}}^{\text{t}} \end{bmatrix} = \begin{bmatrix} \mu_{11} & \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \\ \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} & \underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} \end{bmatrix} \begin{bmatrix} H_1 \\ \underline{\underline{\mathsf{H}}}^{\text{t}} \end{bmatrix}$$

where

$$\underline{\underline{\mathsf{B}}}^{\text{t}} = \begin{bmatrix} D_2 \\ D_3 \end{bmatrix} \;;\; \underline{\underline{\mathsf{H}}}^{\text{t}} = \begin{bmatrix} H_2 \\ H_3 \end{bmatrix}$$

and

$$\underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} = \begin{bmatrix} \mu_{12} & \mu_{13} \end{bmatrix} \;;\; \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} = \begin{bmatrix} \mu_{21} \\ \mu_{31} \end{bmatrix} \;;\; \underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} = \begin{bmatrix} \mu_{22} & \mu_{23} \\ \mu_{32} & \mu_{33} \end{bmatrix} \,.$$

Note that the constant fields are B_1 and $\underline{\underline{\mathsf{H}}}^{\text{t}}$. We want to rewrite the equation so that these constant fields appear on the right hand side. From the first row we get

$$B_1 = \mu_{11}\, H_1 + \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \cdot \underline{\underline{\mathsf{H}}}^{\text{t}}$$

or,

$$H_1 = (\mu_{11})^{-1}\, B_1 - (\mu_{11})^{-1} \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \cdot \underline{\underline{\mathsf{H}}}^{\text{t}} \,.$$

From the second row we get

$$\underline{\underline{\mathsf{B}}}^{\text{t}} = \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1}\, H_1 + \underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} \cdot \underline{\underline{\mathsf{H}}}^{\text{t}} \,.$$

Substitution gives

$$\underline{\underline{\mathsf{B}}}^{\text{t}} = (\mu_{11})^{-1} \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1}\, B_1 - (\mu_{11})^{-1} (\underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} \cdot \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}}) \cdot \underline{\underline{\mathsf{H}}}^{\text{t}} + \underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} \cdot \underline{\underline{\mathsf{H}}}^{\text{t}}$$

or,

$$\underline{\underline{\mathsf{B}}}^{\text{t}} = (\mu_{11})^{-1} \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1}\, B_1 + \left[\underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} - (\mu_{11})^{-1} (\underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} \cdot \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}}) \right] \cdot \underline{\underline{\mathsf{H}}}^{\text{t}} \,.$$

Collecting and rearranging leads to

$$\begin{bmatrix} -H_1 \\ \underline{\underline{\mathsf{B}}}^{\text{t}} \end{bmatrix} = \frac{1}{\mu_{11}} \begin{bmatrix} -1 & \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \\ \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} & \mu_{11} \underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} - \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} \cdot \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \end{bmatrix} \begin{bmatrix} -B_1 \\ \underline{\underline{\mathsf{H}}}^{\text{t}} \end{bmatrix}$$

where the negative signs on H_1 and B_1 are used to make sure that the signs of the off diagonal terms are identical. Define

$$\underline{\underline{\mathsf{L}}}(\mathbf{x}) := \frac{1}{\mu_{11}} \begin{bmatrix} -1 & \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \\ \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} & \mu_{11} \underline{\underline{\boldsymbol{\mu}}}^{\text{tt}} - \underline{\underline{\boldsymbol{\mu}}}^{\text{t}1} \cdot \underline{\underline{\boldsymbol{\mu}}}^{1\text{t}} \end{bmatrix} \,.$$

Then we have,

$$\begin{bmatrix} -H_1 \\ \underline{\underline{\mathsf{B}}}^{\mathrm{t}} \end{bmatrix} = \underline{\underline{\mathsf{L}}}(\mathbf{x}) \cdot \begin{bmatrix} -B_1 \\ \underline{\underline{\mathsf{H}}}^{\mathrm{t}} \end{bmatrix} .$$

Since the vector on the right hand side is constant, a volume average gives

$$\begin{bmatrix} -\langle H_1 \rangle \\ \left\langle \underline{\underline{\mathsf{B}}}^{\mathrm{t}} \right\rangle \end{bmatrix} = \left\langle \underline{\underline{\mathsf{L}}}(\mathbf{x}) \right\rangle \cdot \begin{bmatrix} -B_1 \\ \underline{\underline{\mathsf{H}}}^{\mathrm{t}} \end{bmatrix} .$$

Let us define the effective magnetic permeability of the laminate, $\underline{\underline{\boldsymbol{\mu}}}^{\mathrm{eff}}$, via the relation

$$\langle \mathbf{B} \rangle = \boldsymbol{\mu}_{\mathrm{eff}} \cdot \langle \mathbf{H} \rangle \ .$$

Since the tangential components of $\mathbf{H}$ are constant in the laminate, the average values $\langle H_2 \rangle$ and $\langle H_3 \rangle$ must also be constant. Similarly, the average value $\langle B_1 \rangle$ must be constant. Therefore we can use the same arguments as we used before to write the effective constitutive relation in the form

$$\begin{bmatrix} -\langle H_1 \rangle \\ \left\langle \underline{\underline{\mathsf{B}}}^{\mathrm{t}} \right\rangle \end{bmatrix} = \underline{\underline{\mathbf{L}}}_{\mathrm{eff}} \cdot \begin{bmatrix} -\langle B_1 \rangle \\ \left\langle \underline{\underline{\mathsf{H}}}^{\mathrm{t}} \right\rangle \end{bmatrix} = \underline{\underline{\mathbf{L}}}_{\mathrm{eff}} \cdot \begin{bmatrix} -B_1 \\ \underline{\underline{\mathsf{H}}}^{\mathrm{t}} \end{bmatrix}$$

where

$$\underline{\underline{\mathbf{L}}}_{\mathrm{eff}} := \frac{1}{\mu_{11}^{\mathrm{eff}}} \begin{bmatrix} -1 & \underline{\underline{\boldsymbol{\mu}}}_{\mathrm{eff}}^{1\mathrm{t}} \\ \underline{\underline{\boldsymbol{\mu}}}_{\mathrm{eff}}^{\mathrm{t}1} & \mu_{11}^{\mathrm{eff}} \underline{\underline{\boldsymbol{\mu}}}_{\mathrm{eff}}^{\mathrm{tt}} - \underline{\underline{\boldsymbol{\mu}}}_{\mathrm{eff}}^{\mathrm{t}1} \cdot \underline{\underline{\boldsymbol{\mu}}}_{\mathrm{eff}}^{1\mathrm{t}} \end{bmatrix}$$

and $\boldsymbol{\mu}_{\mathrm{eff}}$ has been decomposed in exactly the same manner as $\boldsymbol{\mu}$. If we compare the two forms of the effective relations we get a formula for determining the effective permittivity of the laminate.

$$\underline{\underline{\mathbf{L}}}_{\mathrm{eff}} = \left\langle \underline{\underline{\mathsf{L}}}(\mathbf{x}) \right\rangle \ .$$

Expanding out the terms, we have

$$\begin{aligned} \mu_{11}^{\mathrm{eff}} &= \left\langle \tfrac{1}{\mu_{11}} \right\rangle^{-1} \\ \mu_{1j}^{\mathrm{eff}} &= \left\langle \tfrac{1}{\mu_{11}} \right\rangle^{-1} \left\langle \tfrac{\mu_{1j}}{\mu_{11}} \right\rangle \qquad \square \\ \mu_{ij}^{\mathrm{eff}} &= \left\langle \mu_{ij} - \tfrac{\mu_{i1}\ \mu_{1j}}{\mu_{11}} \right\rangle + \left\langle \tfrac{1}{\mu_{11}} \right\rangle^{-1} \left\langle \tfrac{\mu_{i1}}{\mu_{11}} \right\rangle \left\langle \tfrac{\mu_{1j}}{\mu_{11}} \right\rangle \quad , i, j \neq 1 \end{aligned} \tag{8.12}$$

Problem 8.12 Use the relation

$$\left[\boldsymbol{S}_{\text{eff}} - \boldsymbol{P}_{\|}(\mathbf{n})\right]^{-1} = \left\langle \left[\boldsymbol{S}(\mathbf{x}) - \boldsymbol{P}_{\|}(\mathbf{n})\right]^{-1} \right\rangle$$

to find the effective permittivity of a laminate that is oriented in the x_2-direction.

Solution - 8.12 From the above relation we see that

$$\boldsymbol{S}_{\text{eff}} = \boldsymbol{P}_{\|}(\mathbf{n}) + \left\langle \left[\boldsymbol{S}(\mathbf{x}) - \boldsymbol{P}_{\|}(\mathbf{n})\right]^{-1} \right\rangle^{-1} .$$

Recall that

$$\boldsymbol{S}(\mathbf{x}) = \varepsilon_0[\varepsilon_0 \boldsymbol{1} - \boldsymbol{\varepsilon}(\mathbf{x})]^{-1} \quad \text{and} \quad \boldsymbol{S}_{\text{eff}} = \varepsilon_0[\varepsilon_0 \boldsymbol{1} - \boldsymbol{\varepsilon}_{\text{eff}}]^{-1} .$$

Therefore,

$$\varepsilon_0 \boldsymbol{S}_{\text{eff}}^{-1} = \varepsilon_0 \boldsymbol{1} - \boldsymbol{\varepsilon}_{\text{eff}} \qquad \Longrightarrow \qquad \boldsymbol{\varepsilon}_{\text{eff}} = \varepsilon_0(\boldsymbol{1} - \boldsymbol{S}_{\text{eff}}^{-1}) .$$

Plugging in the expression for $\boldsymbol{S}_{\text{eff}}$, we have

$$\boldsymbol{\varepsilon}_{\text{eff}} = \varepsilon_0 \left[\boldsymbol{1} - \left[\boldsymbol{P}_{\|}(\mathbf{n}) + \left\langle \left[\boldsymbol{S}(\mathbf{x}) - \boldsymbol{P}_{\|}(\mathbf{n})\right]^{-1} \right\rangle^{-1}\right]^{-1}\right] .$$

For simplicity, let us consider a rank-1 laminate with

$$\boldsymbol{\varepsilon}_1 = \begin{bmatrix} \varepsilon_1^{11} & 0 & 0 \\ 0 & \varepsilon_1^{22} & 0 \\ 0 & 0 & \varepsilon_1^{33} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon}_2 = \begin{bmatrix} \varepsilon_2^{11} & 0 & 0 \\ 0 & \varepsilon_2^{22} & 0 \\ 0 & 0 & \varepsilon_2^{33} \end{bmatrix} .$$

Assuming $\varepsilon_0 = 1$, we have

$$\boldsymbol{S}(x) = [\boldsymbol{1} - \boldsymbol{\varepsilon}(\mathbf{x})]^{-1} = \begin{bmatrix} \frac{1}{1-\varepsilon_{11}} & 0 & 0 \\ 0 & \frac{1}{1-\varepsilon_{22}} & 0 \\ 0 & 0 & \frac{1}{1-\varepsilon_{33}} \end{bmatrix} .$$

With $\mathbf{n} = (0, 1, 0)$, we have

$$[\boldsymbol{S}(x) - \boldsymbol{P}_{\|}(\mathbf{n})]^{-1} = \begin{bmatrix} 1-\varepsilon_{11} & 0 & 0 \\ 0 & \frac{1-\varepsilon_{22}}{\varepsilon_{22}} & 0 \\ 0 & 0 & 1-\varepsilon_{33} \end{bmatrix} .$$

Therefore,

$$\left\langle [\boldsymbol{S}(x) - \boldsymbol{P}_{\|}(\mathbf{n})]^{-1} \right\rangle = \begin{bmatrix} 1 - f_1\varepsilon_1^{11} - f_2\varepsilon_2^{22} & 0 & 0 \\ 0 & \frac{f_1}{\varepsilon_1^{22}} + \frac{f_2}{\varepsilon_2^{22}} - 1 & 0 \\ 0 & 0 & 1 - f_1\varepsilon_1^{33} - f_2\varepsilon_2^{33} \end{bmatrix}$$

where $f_1 + f_2 = 1$ are the volume fractions of the two materials. Inverting,

$$\left\langle [\boldsymbol{S}(x) - \boldsymbol{P}_{\|}(\mathbf{n})]^{-1} \right\rangle^{-1} = \begin{bmatrix} \frac{1}{1-f_1\varepsilon_1^{11}-f_2\varepsilon_2^{22}} & 0 & 0 \\ 0 & \frac{1}{\frac{f_1}{\varepsilon_1^{22}}+\frac{f_2}{\varepsilon_2^{22}}-1} & 0 \\ 0 & 0 & \frac{1}{1-f_1\varepsilon_1^{33}-f_2\varepsilon_2^{33}} \end{bmatrix} .$$

Therefore,

$$\boldsymbol{P}_{\|}(\mathbf{n}) + \left\langle [\boldsymbol{S}(x) - \boldsymbol{P}_{\|}(\mathbf{n})]^{-1} \right\rangle^{-1} = \begin{bmatrix} \frac{1}{1-f_1\varepsilon_1^{11}-f_2\varepsilon_2^{22}} & 0 & 0 \\ 0 & \frac{f_1\varepsilon_2^{22}+f_2\varepsilon_1^{22}}{f_1\varepsilon_2^{22}+f_2\varepsilon_1^{22}-\varepsilon_1^{22}\varepsilon_2^{22}} & 0 \\ 0 & 0 & \frac{1}{1-f_1\varepsilon_1^{33}-f_2\varepsilon_2^{33}} \end{bmatrix} .$$

Inverting again,

$$\left[\boldsymbol{P}_{\|}(\mathbf{n}) + \left\langle [\boldsymbol{S}(x) - \boldsymbol{P}_{\|}(\mathbf{n})]^{-1} \right\rangle^{-1}\right]^{-1} = \begin{bmatrix} 1 - f_1\varepsilon_1^{11} - f_2\varepsilon_2^{22} & 0 & 0 \\ 0 & \frac{f_1\varepsilon_2^{22} + f_2\varepsilon_1^{22} - \varepsilon_1^{22}\varepsilon_2^{22}}{f_1\varepsilon_2^{22} + f_2\varepsilon_1^{22}} & 0 \\ 0 & 0 & 1 - f_1\varepsilon_1^{33} - f_2\varepsilon_2^{33} \end{bmatrix}.$$

Therefore,

$$\boldsymbol{\varepsilon}\text{eff} = \begin{bmatrix} f_1\varepsilon_1^{11} + f_2\varepsilon_2^{22} & 0 & 0 \\ 0 & 1 - \frac{f_1\varepsilon_2^{22} + f_2\varepsilon_1^{22} - \varepsilon_1^{22}\varepsilon_2^{22}}{f_1\varepsilon_2^{22} + f_2\varepsilon_1^{22}} & 0 \\ 0 & 0 & f_1\varepsilon_1^{33} + f_2\varepsilon_2^{33} \end{bmatrix}$$

or,

$$\boldsymbol{\varepsilon}_{\text{eff}} = \begin{bmatrix} f_1\varepsilon_1^{11} + f_2\varepsilon_2^{22} & 0 & 0 \\ 0 & \frac{\varepsilon_1^{22}\varepsilon_2^{22}}{f_1\varepsilon_2^{22} + f_2\varepsilon_1^{22}} & 0 \\ 0 & 0 & f_1\varepsilon_1^{33} + f_2\varepsilon_2^{33} \end{bmatrix} \qquad \square \qquad (8.13)$$

Problem 8.13 Show that the Tartar-Murat-Lurie-Cherkaev formula for the effective permittivity of a rank-1 laminate

$$f_1\varepsilon_2[\varepsilon_2\boldsymbol{1} - \boldsymbol{\varepsilon}_{\text{eff}}]^{-1} = \varepsilon_2[\varepsilon_2\boldsymbol{1} - \boldsymbol{\varepsilon}_1]^{-1} - f_2\boldsymbol{P}_{\|}(\mathbf{n})$$

is equivalent to the formula derived using the Backus method.

Solution - 8.13: Reorganization of the TMLC formula gives us

$$\boldsymbol{\varepsilon}_{\text{eff}} = \varepsilon_2\boldsymbol{1} - \left[\frac{1}{f_1}\left[\varepsilon_2\boldsymbol{1} - \boldsymbol{\varepsilon}_1\right]^{-1} - \frac{f_2}{f_1\varepsilon_2}\,\boldsymbol{P}_{\|}(\mathbf{n})\right]^{-1} .$$

Assuming that $\mathbf{n} = \mathbf{e}_1 = (1, 0, 0)$, we have

$$\boldsymbol{P}_{\|}(\mathbf{n}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Let us assume that the permittivity tensor in each layer is symmetric. Expressing everything in matrix form and running the calculation through Mathematica shows that (with $f_1 = 1 - f_2$)

$$\begin{aligned}
\varepsilon_{11}^{\text{eff}} &= \varepsilon_2 + \frac{(\varepsilon_1^{11} - \varepsilon_2)\varepsilon_2 f_1}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1} \\
\varepsilon_{22}^{\text{eff}} &= \varepsilon_1^{22} f_1 - \varepsilon_2 f_2 - \frac{\varepsilon_1^{12}\varepsilon_1^{12} f_1 f_2}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1} \\
\varepsilon_{33}^{\text{eff}} &= \varepsilon_1^{33} f_1 - \varepsilon_2 f_2 - \frac{\varepsilon_1^{13}\varepsilon_1^{13} f_1 f_2}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1} \\
\varepsilon_{23}^{\text{eff}} &= \varepsilon_1^{23} f_1 - \frac{\varepsilon_1^{12}\varepsilon_1^{13} f_1 f_2}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1} \\
\varepsilon_{13}^{\text{eff}} &= \frac{\varepsilon_1^{13}\varepsilon_2 f_1}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1} \\
\varepsilon_{12}^{\text{eff}} &= \frac{\varepsilon_1^{12}\varepsilon_2 f_1}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1} .
\end{aligned}$$

The corresponding Backus solution is

$$\begin{aligned}
\varepsilon_{11}^{\text{eff}} &= \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} \\
\varepsilon_{22}^{\text{eff}} &= \left\langle \varepsilon_{22} - \frac{\varepsilon_{12}\varepsilon_{12}}{\varepsilon_{11}} \right\rangle + \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \frac{\varepsilon_{12}}{\varepsilon_{11}} \right\rangle \left\langle \frac{\varepsilon_{12}}{\varepsilon_{11}} \right\rangle \\
\varepsilon_{33}^{\text{eff}} &= \left\langle \varepsilon_{33} - \frac{\varepsilon_{13}\varepsilon_{13}}{\varepsilon_{11}} \right\rangle + \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \frac{\varepsilon_{13}}{\varepsilon_{11}} \right\rangle \left\langle \frac{\varepsilon_{13}}{\varepsilon_{11}} \right\rangle \\
\varepsilon_{23}^{\text{eff}} &= \left\langle \varepsilon_{23} - \frac{\varepsilon_{12}\varepsilon_{13}}{\varepsilon_{11}} \right\rangle + \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \frac{\varepsilon_{12}}{\varepsilon_{11}} \right\rangle \left\langle \frac{\varepsilon_{13}}{\varepsilon_{11}} \right\rangle \\
\varepsilon_{13}^{\text{eff}} &= \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \frac{\varepsilon_{13}}{\varepsilon_{11}} \right\rangle \\
\varepsilon_{12}^{\text{eff}} &= \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} \left\langle \frac{\varepsilon_{12}}{\varepsilon_{11}} \right\rangle .
\end{aligned}$$

For a rank-2 laminate with layer properties $\boldsymbol{\varepsilon}_1$ and $\varepsilon_2\boldsymbol{1}$,

$$\left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} = \left[\frac{1}{V}\int_V \frac{1}{\varepsilon_{11}}\,dV\right]^{-1} = \left[\frac{1}{V}\int_{V_1} \frac{1}{\varepsilon_1^{11}}\,dV + \frac{1}{V}\int_{V_2} \frac{1}{\varepsilon_2}\,dV\right]^{-1} = \left[\frac{f_1}{\varepsilon_1^{11}} + \frac{f_2}{\varepsilon_2}\right]^{-1}$$

or,

$$\varepsilon_{11}^{\text{eff}} = \left\langle \frac{1}{\varepsilon_{11}} \right\rangle^{-1} = \frac{\varepsilon_1^{11}\varepsilon_2}{\varepsilon_2 f_1 + \varepsilon_1^{11} f_2}\,.$$

From the TMLC equations, using $f_1 + f_2 = 1$, we have

$$\varepsilon_{11}^{\text{eff}} = \frac{\varepsilon_1^{11}\varepsilon_2}{\varepsilon_1^{11} f_2 + \varepsilon_2 f_1}\,.$$

Hence the Backus and TMLC results agree for $\varepsilon_{11}^{\text{eff}}$. The same procedure can be used to verify the equivalence of the other relations. □

Problem 8.14 Show that, for a rank-1 laminate with layer permittivities ε_1 and ε_2 and volume fractions f_1 and f_2, Milton's relation reduces to

$$f_1(\varepsilon_{\text{eff}} - \varepsilon_2)^{-1} = (\varepsilon_1 - \varepsilon_2)^{-1} + f_2 \boldsymbol{P}_{\mathbf{n}} \quad \text{where} \quad \boldsymbol{P}_{\mathbf{n}} = \frac{\mathbf{n} \otimes \mathbf{n}}{\mathbf{n} \cdot \varepsilon_2 \cdot \mathbf{n}} .$$

Solution - 8.14: Recall Milton's relation:

$$\left[(\varepsilon_{\text{eff}} - \varepsilon_0)^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} = \left\langle \left[\{\varepsilon(\mathbf{x}) - \varepsilon_0\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} \right\rangle$$

where

$$\boldsymbol{P}_{\mathbf{n}} = \frac{\mathbf{n} \otimes \mathbf{n}}{\mathbf{n} \cdot \varepsilon_0 \cdot \mathbf{n}} .$$

Clearly, if $\varepsilon_0 = \varepsilon_2$, the average is undefined for regions with $\varepsilon(\mathbf{x}) = \varepsilon_2$. Instead, we use a small δ, and take the limit as $\delta \to 0$. With $\varepsilon_0 = \varepsilon_2 - \delta \boldsymbol{1}$, we have

$$\left[(\varepsilon_{\text{eff}} - \varepsilon_2 + \delta \boldsymbol{1})^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} = \left\langle \left[\{\varepsilon(\mathbf{x}) - \varepsilon_2 + \delta \boldsymbol{1}\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} \right\rangle$$

where

$$\boldsymbol{P}_{\mathbf{n}} = \frac{\mathbf{n} \otimes \mathbf{n}}{\mathbf{n} \cdot (\varepsilon_2 - \delta \boldsymbol{1}) \cdot \mathbf{n}} .$$

The right hand side can be written as

$$\begin{aligned}
\left\langle \left[\{\varepsilon(\mathbf{x}) - \varepsilon_2 + \delta \boldsymbol{1}\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} \right\rangle &= \frac{1}{V} \int_V \left[\{\varepsilon(\mathbf{x}) - \varepsilon_2 + \delta \boldsymbol{1}\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} dV \\
&= \frac{1}{V} \left[\int_{V_1} \left[\{\varepsilon_1 - \varepsilon_2 + \delta \boldsymbol{1}\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} dV + \int_{V_2} \left[\{\delta \boldsymbol{1}\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} dV\right] \\
&= f_1 \left[\{\varepsilon_1 - \varepsilon_2 + \delta \boldsymbol{1}\}^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} + f_2 \left[\delta^{-1} \boldsymbol{1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} .
\end{aligned}$$

The apparently problematic term in the above expression is $f_2 \left[\delta^{-1} \boldsymbol{1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1}$. However, if we express everything in matrix form, take the inverse and then take a limit as $\delta \to 0$, we find that

$$\lim_{\delta \to 0} \left[\delta^{-1} \boldsymbol{1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} = \boldsymbol{0} .$$

Therefore, in the limit $\delta \to 0$, we have

$$\left[(\varepsilon_{\text{eff}} - \varepsilon_2)^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} = f_1 \left[(\varepsilon_1 - \varepsilon_2)^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]^{-1} \quad \text{where} \quad \boldsymbol{P}_{\mathbf{n}} = \frac{\mathbf{n} \otimes \mathbf{n}}{\mathbf{n} \cdot \varepsilon_2 \cdot \mathbf{n}} .$$

Taking the inverse of both sides,

$$(\varepsilon_{\text{eff}} - \varepsilon_2)^{-1} + \boldsymbol{P}_{\mathbf{n}} = \frac{1}{f_1} \left[(\varepsilon_1 - \varepsilon_2)^{-1} + \boldsymbol{P}_{\mathbf{n}}\right]$$

or,

$$f_1 (\varepsilon_{\text{eff}} - \varepsilon_2)^{-1} = (\varepsilon_1 - \varepsilon_2)^{-1} + (1 - f_1) \boldsymbol{P}_{\mathbf{n}} = (\varepsilon_1 - \varepsilon_2)^{-1} + f_2 \boldsymbol{P}_{\mathbf{n}} \quad \square \tag{8.14}$$

Problem 8.15 Show that for an isotropic reference material with

$$\mathsf{C}_0 = \lambda_0 \boldsymbol{1} \otimes \boldsymbol{1} + 2\mu_0 \mathbf{1}$$

the Cartesian components of the operator $\mathsf{P}_\mathbf{n}$ have the form

$$(\mathsf{P}_\mathbf{n})_{ijk\ell} = \left(\tfrac{1}{\lambda_0+2\mu_0} - \tfrac{1}{\mu_0}\right) n_i n_j n_k n_\ell \\ + \tfrac{1}{4\mu_0}(n_i n_\ell \delta_{jk} + n_i n_k \delta_{j\ell} + n_j n_\ell \delta_{ik} + n_j n_k \delta_{i\ell})\,.$$

Solution - 8.15 Recall that

$$(\mathsf{P}_\mathbf{n})_{ijk\ell} = \frac{1}{4}\left(C^{-1}_{j\ell} n_i n_k + C^{-1}_{jk} n_i n_\ell + C^{-1}_{i\ell} n_j n_k + C^{-1}_{ik} n_j n_\ell\right]$$

where

$$C_{ij} = n_p C^0_{pijq} n_q\,.$$

For an isotropic reference material

$$C^0_{pijq} = \lambda_0 \delta_{pi}\delta_{jq} + \mu_0(\delta_{pj}\delta_{iq} + \delta_{pq}\delta_{ij})\,.$$

Therefore,

$$C_{ij} = \lambda_0 n_i n_j + \mu_0(n_i n_j + n_p n_p \delta_{ij})\,.$$

Since **n** is a unit vector, $n_p n_p = 1$, and we have

$$C_{ij} = \lambda_0 n_i n_j + \mu_0(n_i n_j + \delta_{ij})\,.$$

To find the inverse of $\boldsymbol{C}$ it is convenient to use matrix notation. Thus

$$\underline{\underline{\mathsf{C}}} = \begin{bmatrix} \mu_0 + (\lambda_0+\mu_0)n_1^2 & (\lambda_0+\mu_0)n_1 n_2 & (\lambda_0+\mu_0)n_1 n_3 \\ (\lambda_0+\mu_0)n_1 n_2 & \mu_0 + (\lambda_0+\mu_0)n_2^2 & (\lambda_0+\mu_0)n_2 n_3 \\ (\lambda_0+\mu_0)n_1 n_3 & (\lambda_0+\mu_0)n_2 n_3 & \mu_0 + (\lambda_0+\mu_0)n_3^2 \end{bmatrix}\,.$$

Inverting the matrix and expressing the components in index notation, we have

$$C^{-1}_{ij} = \frac{[(\lambda_0+\mu_0)n_p n_p + \mu_0]\delta_{ij} - (\lambda_0+\mu_0)n_i n_j}{\mu_0[(\lambda_0+\mu_0)n_p n_p + \mu_0]}\,.$$

Again using $n_p n_p = 1$, we have

$$C^{-1}_{ij} = \frac{(\lambda_0+2\mu_0)\delta_{ij} - (\lambda_0+\mu_0)n_i n_j}{\mu_0(\lambda_0+2\mu_0)} = \frac{1}{\mu_0}\,\delta_{ij} - \frac{(\lambda_0+\mu_0)}{\mu_0(\lambda_0+2\mu_0)}\,n_i n_j\,.$$

Define

$$A := \frac{1}{\mu_0} \quad \text{and} \quad B := -\frac{(\lambda_0+\mu_0)}{\mu_0(\lambda_0+2\mu_0)} = \frac{1}{\lambda_0+2\mu_0} - \frac{1}{\mu_0}$$

to write

$$C^{-1}_{ij} = A\,\delta_{ij} + B\,n_i n_j\,.$$

Substitute into the expression for $(\mathsf{P}_\mathbf{n})_{ijk\ell}$ to get

$$(\mathsf{P}_\mathbf{n})_{ijk\ell} = \frac{1}{4}\Big(A\,\delta_{j\ell} n_i n_k + B\,n_j n_\ell n_i n_k + A\,\delta_{jk} n_i n_\ell + B\,n_j n_k n_i n_\ell + \\ A\,\delta_{i\ell} n_j n_k + B\,n_i n_\ell n_j n_k + A\,\delta_{ik} n_j n_\ell + B\,n_i n_k n_j n_\ell\Big)$$

or

$$(\mathsf{P}_\mathbf{n})_{ijk\ell} = \frac{1}{4}\left[A\,(\delta_{j\ell} n_i n_k + \delta_{jk} n_i n_\ell + \delta_{i\ell} n_j n_k + \delta_{ik} n_j n_\ell) + 4B\,n_i n_j n_k n_\ell\right]$$

Plugging back the expressions for A and B gives the desired result. □