

4. Energy Release Rate

References:

J. W. Hutchinson, *Notes on Nonlinear Fracture Mechanics* (<http://imechanica.org/node/755>);
Alan Zehnder, *Lecture Notes on Fracture Mechanics* (<http://hdl.handle.net/1813/3075>).

As an alternative approach to studying fracture mechanics, energy approach is applicable for both linear and nonlinear elastic materials. Within linear elastic fracture mechanics (LEFM), the connection between the energy flow and the crack tip stress field provides a powerful tool (e.g., *J integral*) for the determination of stress intensity factors.

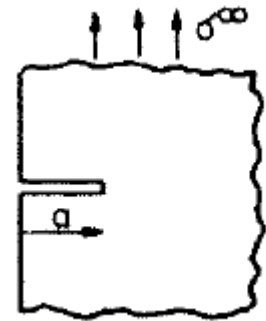
A. A. Griffith's work (1921). "The phenomena of rupture and flow in solids", *Philosophical Transactions of the Royal Society of London*, A221:163-198. (Times Cited: 2646. No. 3 in the list of the most cited, <http://imechanica.org/node/587>).



Three things known to Griffith at his time:

- (1) Linear elasticity solution of an elliptic hole in a large plate (Inglis solution);
- (2) Fracture stress measured for bulk glass is around 100 MPa, with a big scatter in data;
- (3) Theoretical strength for atomic bond breaking is much higher, ~10 GPa.

In his own experiments, Griffith measured fracture stresses of glass fibers under uniaxial tension tests and found that the measured strength increases as the fiber thickness decreases. Thus, the uniaxial tension strength depends on the specimen size, not a material property. Griffith assumed that the size-dependent strength is due to different flaw sizes in the fibers. However, the flaw size is typically unknown and not well controlled (vary from specimen to specimen). To verify his assumption, Griffith introduced a well-controlled artificial flaw-crack, much larger than other flaws in the fiber, and measured the tensile strength of specimens with different crack sizes. His experimental result showed that the measured tensile strength depends on the crack size, and the product of the strength and the square root of the crack size was nearly a constant, namely,



$$\sigma_f \sqrt{a} = C$$

The next question: what determines the constant C ? Griffith knew about *surface energy* of solids (J. W. Gibbs, 1878), and he actually did the experiments to measure the surface energy of glass at a number of high temperatures between 730°C and 900°C. Let's briefly review the concept of surface energy before we come back to Griffith's experiments.

Surface energy. The excess energy at the surface, compared to the bulk, is defined as surface energy. For a liquid, the surface energy density (per unit area of the surface) is identical to the surface tension (force per unit line length). For example, the surface tension of water is about $\gamma = 0.08 \text{ N/m}$, which is 0.08 J/m^2 for the surface energy density.

Example 1: a semi-spherical droplet. By force balance, the weight of the droplet is balanced by the surface tension around the edge, namely

$$\rho g V = 2\pi a \gamma$$

Given the mass density and surface tension of water, the radius of the droplet is predicted to be

$$a = \sqrt{\frac{3\gamma}{\rho g}}$$

Example 2: capillary effect. A thin tube (open ended) of radius a is inserted into a liquid reservoir. Let γ_1 be the surface energy density of the interface between the tube (a solid) and air, and γ_2 be the surface energy density of the interface between the tube and the liquid. The free energy of the system include the surface energy of all interfaces and the potential energy of gravity (as the liquid rise or drop in the tube), namely

$$G = G_0 - (\gamma_1 - \gamma_2)2\pi a h + \rho g \pi a^2 h \frac{h}{2}$$

where h is the height of the liquid level in the tube. The change of the liquid level outside the tube is ignored for a thin tube in a large reservoir.

By minimizing the free energy with respect to the height h , we obtain the equilibrium height,

$$h = \frac{\gamma_1 - \gamma_2}{\rho g a}$$

Thus, if $\gamma_2 < \gamma_1$ (e.g., water), the liquid rises in the tube to a positive height ($h > 0$); otherwise, if $\gamma_2 > \gamma_1$ (e.g., mercury), the liquid level drops in the tube to a negative height ($h < 0$).

Example 3: measure surface energy of a liquid membrane. A liquid membrane on a metal frame is pulled by a force on one side. At equilibrium, the force is measured to be F , and the length of the membrane is L . The width of the frame is a constant, W . The work done by the force increases the surface area of the membrane and thus the surface energy. The total free energy is a minimum at equilibrium. By a variational statement,

$$\delta G = -F \delta L + 2\gamma W \delta L = 0$$

Thus the surface energy is: $\gamma = \frac{F}{2W}$. The factor of 2 is due to the two surfaces of the membrane.

The same result can be reached by force balance with the surface tension.

The membrane method however cannot measure surface energy of a solid, because stretching of a solid membrane induces elastic energy in the bulk in addition to the change of surface energy. In general, surface energy of a solid is not necessarily identical to the surface tension. Furthermore, surface energy of a solid also depends on the deformation of the surface (strain), which leads to the concept of *surface stress*. For a reference on surface energy and surface stress of solids, see J. W. Cahn (1980) “Surface stress and the chemical-equilibrium of small crystals. I: The case of the isotropic surface.” *Acta Metall.* 28, pp. 1333–1338.

The surface energy of a solid may be measured at high temperatures, when the solid creeps. Consider a cylindrical rod of radius r and length L , under a uniaxial tension P , at a high temperature. Both the radius and the length changes due to creeping, while the volume ($V = \pi r^2 L$) remains constant. At the equilibrium, the variation of the total free energy vanishes, namely

$$\delta G = -P\delta L + \gamma\delta A = 0$$

Here, $A = 2\pi r^2 + 2\pi rL$ and $\delta A = \left(1 + \frac{2r}{L}\right)2\pi L\delta r + 2\pi r\delta L$. Meanwhile, $\delta V = 2\pi rL\delta r + \pi r^2\delta L = 0$,

which leads to $\delta r = -\frac{r}{2L}\delta L$. Inserting into the variation of the free energy, we obtain that

$$P = \gamma\pi\left(1 - \frac{2r}{L}\right)$$

Thus, measuring P , r , L at the equilibrium state determines the surface energy. The force P is also called *zero-creep* load.

Back to Griffith’s work. Griffith used a method similar to the creeping rod to measure surface energy of glass at high temperatures (a homework problem next week). To explain his experimental results on the fracture stress that depends on the crack size, Griffith took an energetic approach. Consider a reference state of the glass fiber with no crack or any other flaws. Under a uniaxial tension σ_∞ , the specimen is homogeneously loaded with an axial strain, $\varepsilon = \frac{\sigma_\infty}{E}$, where E is Young’s modulus of the material. The potential strain energy stored in this perfect specimen is then,

$$G_0 = \frac{1}{2}\sigma_\infty\varepsilon V = \frac{\sigma_\infty^2}{2E}V$$

Next, fix the remote boundary so that the applied load does not do extra work, and introduce a crack of length a into the specimen. The crack relaxes the stress thus the elastic strain energy near the crack faces, within an area scaling as $A \sim a^2$ (considering a plate of unit thickness). Meanwhile, creation of two new surfaces of the crack increases the surface energy. The change of the total free energy from the reference state due to the crack is thus

$$\Delta G = -\frac{\sigma_{\infty}^2}{2E} A + 2\gamma a = -g \frac{\sigma_{\infty}^2 a^2}{E} + 2\gamma a$$

where g is a dimensional number of order unity, not so important for the purpose of current discussion.

Sketch the change of free energy as a function of the crack length. The free energy reaches a peak at a critical crack size

$$a_c = \frac{E\gamma}{g\sigma_{\infty}^2}$$

If $a < a_c$, the free energy decreases by reducing the crack length (crack healing). If $a > a_c$, the free energy decreases by increasing the crack length (crack growing to cause fracture).

Alternatively, for a given crack length a , the critical stress to cause crack growth is

$$\sigma_c = \sqrt{\frac{E\gamma}{ga}}$$

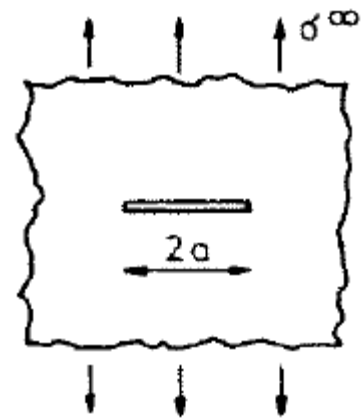
Griffith solved the elasticity problem for a finite crack in an infinite plate and obtained that

$$\sigma_c = \sqrt{\frac{2E\gamma}{\pi a}}$$

This result agrees very well with his experiments, i.e.,

$\sigma_c \sqrt{a} = C$, with the constant $C = \sqrt{\frac{2E\gamma}{\pi}}$. Taking $\gamma \sim 1 \text{ J/m}^2$

and $E \sim 9 \times 10^6 \text{ psi}$, the agreement between the predicted fracture stresses and the measured ones was excellent!

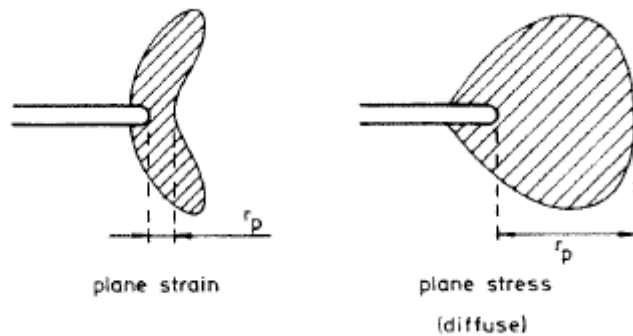


Summary of Griffith's work.

- (1) Experimental observations: the measured fracture stress of a glass fiber is much lower than the theoretical strength and depends on the specimen size.
- (2) Introduction of a crack as a controllable flaw: stress at the crack tip predicted by linear elasticity goes to infinity. Cannot directly apply the stress criterion.
- (3) An energetic approach: compare the release of elastic strain energy and the increase in the surface energy to determine the critical stress of fracture. This approach works great for the glass fibers.

George R. Irwin (1907-1998): the role of plasticity. Griffith was fortunate in working with glass, an ideal brittle material. When applying the same idea to steel, it leads to an unrealistically large surface energy for steel, $\gamma \sim 1000 \text{ J/m}^2$! On the other hand, the scaling relationship between the fracture stress and the crack size, $\sigma_c \sim a^{-1/2}$, still holds for steel and many other metals.

The role of plasticity in fracture was first realized by Irwin. Plasticity plays a negligible role in fracture of brittle materials (e.g., glass and ceramics). For ductile materials (e.g., metals), however, a finite plastic zone develops around the crack tip. The size of the plastic zone (r_p) increases as the load increases. At a critical load, the crack grows, and the material behind the crack tip unloads. The hysteresis of the loading-unloading of the materials around the crack dissipates energy (e.g., as heat), another energy term involved during fracture in addition to the surface energy and elastic strain energy. Consequently, additional energy is required for the crack growth in ductile materials. The plastic energy dissipation is small compared to the surface energy in brittle materials, but becomes significant for ductile materials, even under the small-scale yielding (SSY) condition.



Shape of plastic zone around a crack tip

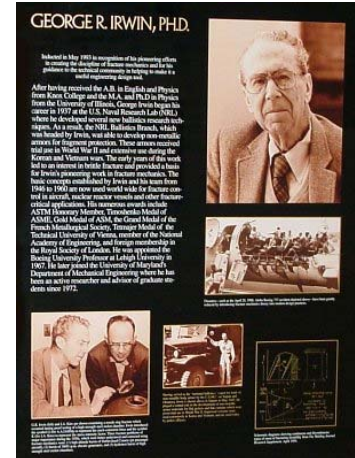
It was Irwin's idea to separate the energy into two parts:

- (1) The elastic strain energy, which releases as the crack grows. The release of the elastic energy thus defines a thermodynamic driving force for fracture;
- (2) Energy dissipation near the crack, which includes both the excess surface energy and the plastic dissipation as well as any other energy dissipation processes (e.g., phase transformation) near the crack. The lump sum of the energy dissipation defines the overall material resistance to fracture, or the fracture energy required to grow a crack.

Recognizing that the plastic deformation around the crack is controlled locally by the crack-tip stress field, independent of the sample geometry, Irwin added an additional energy term onto the surface energy density as the total fracture energy:

$$\Gamma = 2\gamma + w_p$$

where w_p is the plastic energy dissipation per unit area of crack growth. A modification to Griffith's solution for the fracture stress is then



$$\sigma_c = \sqrt{\frac{E\Gamma}{\pi a}}$$

For glass, $w_p \ll 2\gamma$, and $\Gamma \approx 2\gamma \sim 1\text{J/m}^2$; for steel, $w_p \gg 2\gamma$, and $\Gamma \sim 1000\text{J/m}^2$; epoxy has an intermediate fracture energy, $\Gamma \sim 10\text{--}1000\text{J/m}^2$.

Energy release rate. The driving force for fracture is the release of elastic strain energy. This can be obtained by solving an elasticity boundary value problem. Under the condition of small-scale yielding, the effect of plasticity is ignored in the calculations of energy release rates.

Consider a large plate with a crack of length a (or crack area $A = ah$). Apply a force P . The displacement at the loading point is Δ . Let U be the total elastic energy stored in the plate. For a fixed crack length (area), the elastic energy is a function of the applied load or displacement, i.e., $U = U(A, \Delta)$.

Now compare the elastic energy of two samples, with different crack lengths ($A_2 > A_1$), but with the same displacement Δ . The sample with the larger crack is more compliant, and the stored elastic energy is lower: $U(A_2, \Delta) < U(A_1, \Delta)$. Thus, when the crack in the first sample grows from area A_1 to area A_2 , with the displacement fixed (thus no work done by the external load), the elastic energy in the plate reduces by $-\delta U = U(A_1, \Delta) - U(A_2, \Delta)$.

The energy release rate is defined as the differential change of the elastic strain energy due to per unit area of crack growth, namely

$$G = \lim_{\delta A \rightarrow 0} \left(-\frac{\delta U}{\delta A} \right) = \lim_{A_2 \rightarrow A_1} \frac{U(A_1, \Delta) - U(A_2, \Delta)}{A_2 - A_1} = -\left(\frac{\partial U}{\partial A} \right)_\Delta$$

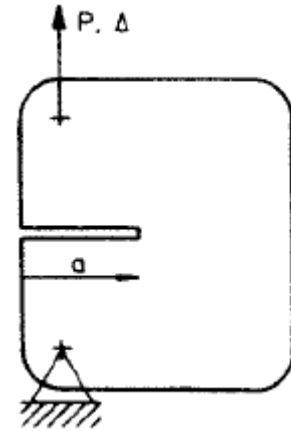
Method of compliance: measure the compliance of the sample, $C = \frac{\Delta}{P}$, as a function of the crack

size, $C = C(A)$. The elastic strain energy is then, $U = \frac{1}{2} P\Delta = \frac{\Delta^2}{2C}$. The energy release rate is thus

$$G = -\left(\frac{\partial U}{\partial A} \right)_\Delta = \frac{\Delta^2}{2C^2} \frac{dC}{dA} = \frac{P^2}{2} \frac{dC}{dA}$$

The compliance C depends on the sample geometry, crack size, and the material properties (E and ν). The energy release rate G depends on the applied force P in addition to the sample geometry and material properties.

Under a displacement control experiment, we have $dU = Pd\Delta - GdA$.



Under load control, we define a potential energy as $\Pi = U - P\Delta$. Then, $d\Pi = -\Delta dP - GdA$. Thus, an equivalent definition for the energy release rate is:

$$G = -\left(\frac{\partial \Pi}{\partial A}\right)_P$$

Noting that $\Pi = U - P\Delta = -U$ for linear elasticity problems, we have $G = \left(\frac{\partial U}{\partial A}\right)_P$. Thus, under the load control, the elastic strain energy increases as the crack grows; for a good example, consider a specimen loaded by a dead weight. Sketch the energy change under both displacement and load control.

Calculation of energy release rate. Several methods may be employed to determine energy release rate of a crack.

Energy method. Consider a double cantilever beam (DCB) specimen. Treat each arm of the specimen as a cantilever beam of length a , so that the end deflection is

$$\frac{\Delta}{2} = \frac{Pa^3}{3EI} = \frac{4Pa^3}{Eb^3}$$

The elastic strain energy stored in both arms is

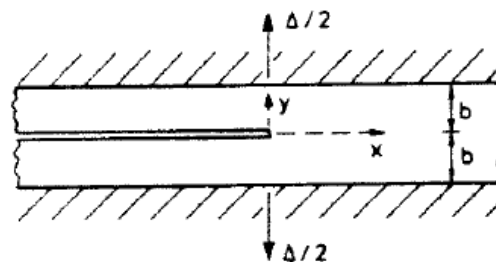
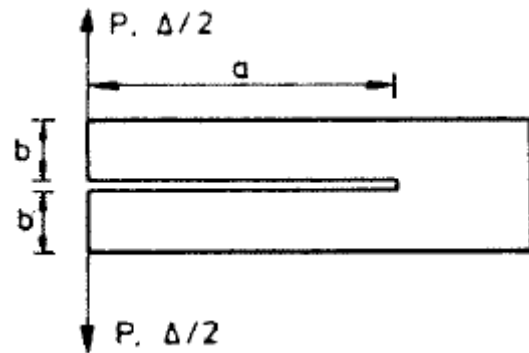
$$U = \frac{1}{2}P\Delta = \frac{4P^2a^3}{Eb^3} = \frac{Eb^3\Delta^2}{16a^3}$$

Thus, the energy release rate is

$$G = \left(\frac{\partial U}{\partial a}\right)_P = \frac{12P^2a^2}{Eb^3} \quad \text{or} \quad G = -\left(\frac{\partial U}{\partial a}\right)_\Delta = \frac{3Eb^3\Delta^2}{16a^4}$$

Confirm that the two are identical. Sketch the energy release rate as a function of the crack length under displacement and load control, respectively.

As a second example, consider a semi-infinite crack in an infinite strip held in rigid grips and subject to a prescribed separation Δ . Let the crack advance by da . The stress and strain fields are unchanged if one shifts the origin of the coordinate by da in the positive x -direction. The change of the elastic energy due to the crack growth is simply the difference between two slices of the strip, one far ahead of the crack tip ($x \rightarrow +\infty$) and the other far behind the crack tip ($x \rightarrow -\infty$). The former is uniformly strained by

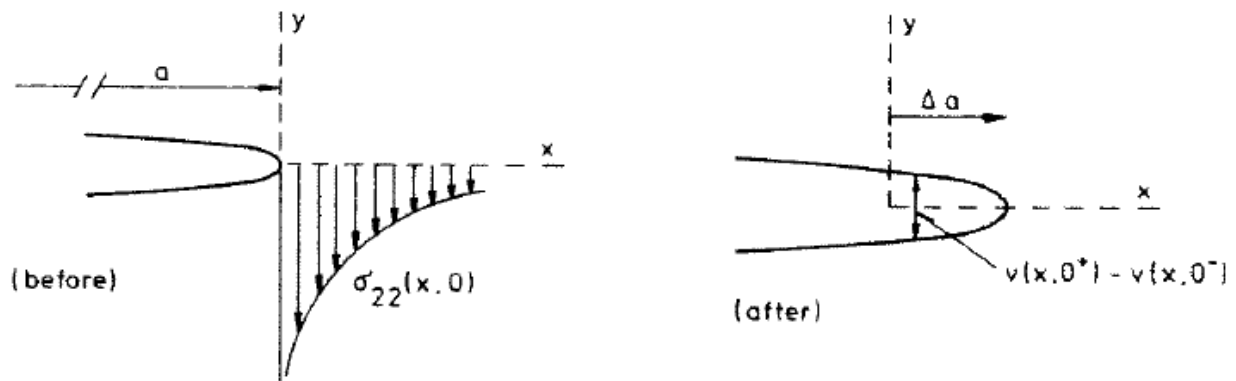


$\varepsilon_{yy} = \frac{\Delta}{2b}$ ($\varepsilon_{xx} = 0$), while the latter is unloaded by the traction-free crack surfaces. Therefore, the energy release rate for this crack is simply

$$G = 2bU(x \rightarrow +\infty)$$

Under the plane stress condition, as $x \rightarrow +\infty$, $\sigma_{yy} = \frac{E}{1-\nu^2} \frac{\Delta}{2b}$ and $\sigma_{xx} = \nu\sigma_{yy}$. Thus

$$G = 2bU(x \rightarrow +\infty) = \frac{E\Delta^2}{4b(1-\nu^2)}.$$



Crack closure integral method. Consider a mode I crack. The stress field ahead of the crack tip is

$$\sigma_{yy}(x, 0) = \frac{K_I(a)}{\sqrt{2\pi x}}.$$

Let the crack advance in the x-direction by small amount, Δa . The stress intensity factor of the new crack is, $K_I(a + \Delta a)$. The opening displacement of the crack is

$$u_y(x, 0^+) - u_y(x, 0^-) = \frac{8K_I(a + \Delta a)}{E'} \left(\frac{\Delta a - x}{2\pi} \right)^{1/2}$$

Under a prescribed displacement (no work done by external loads), the elastic strain energy release during the crack growth equals the work done by closing the crack with the traction, $-\sigma_{yy}(x, 0)$, which is

$$\begin{aligned} \Delta U &= \frac{1}{2} \int_0^{\Delta a} \sigma_{yy}(x, 0) [u_y(x, 0^+) - u_y(x, 0^-)] dx \\ &= \frac{2K_I(a)K_I(a + \Delta a)}{\pi E'} \int_0^{\Delta a} \sqrt{\frac{\Delta a - x}{x}} dx = \frac{K_I(a)K_I(a + \Delta a)}{E'} \Delta a \end{aligned}$$

By definition, the energy release rate of the initial crack is

$$G(a) = \lim_{\Delta a \rightarrow 0} \frac{\Delta U}{\Delta a} = \frac{K_I^2(a)}{E'}$$

This gives the relationship between energy release rate and stress intensity factor for a model I crack. Similar relation can be derived for mode II and mode III cracks. In general, we have

$$G = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

Recall that $E' = E/(1-\nu^2)$ for plane strain and $E' = E$ for plane stress, and $\mu = E/2(1+\nu)$ is the shear modulus.

A general form of the crack closure integral may be written as

$$G = \lim_{\Delta A \rightarrow 0} \frac{1}{\Delta A} \int_{\Delta A} \int_0^{\Delta u_i} \sigma_{2i} du_i dA$$

where $\Delta u_i = u_i^+ - u_i^-$. For linear elastic materials, the traction σ_{2i} is linearly proportional to the displacement Δu_i , thus

$$G = \lim_{\Delta A \rightarrow 0} \frac{1}{2\Delta A} \int_{\Delta A} \sigma_{2i} \Delta u_i dA$$

A revisit to Griffith's crack. As noted in a previous lecture, the full-field solution for a finite crack in an infinite plate under remote tension can be obtained by the Westergaard approach of complex variable method, which gives the crack opening displacement

$$u_y = \pm \frac{2\sigma_\infty}{E'} \sqrt{a^2 - x^2} \quad \text{for } |x| < a$$

We have found the stress intensity factor for such a crack by comparing the full-field solution to the asymptotic crack-tip solution. Let's now determine the energy release rate.

Relative to the reference state with no crack, the elastic strain energy in the cracked plate equals the negative of the work done by closing the crack so as to recover the reference state, namely,

$$U(a) = U_0 - \int_{-a}^a \frac{1}{2} \sigma_\infty (u_y^+ - u_y^-) dx = U_0 - \frac{2\sigma_\infty^2}{E'} \int_{-a}^a \sqrt{a^2 - x^2} dx = U_0 - \frac{\pi\sigma_\infty^2 a^2}{E'}$$

Then, the energy release rate for each of the two crack tips is

$$G = -\frac{1}{2} \left(\frac{\partial U}{\partial A} \right)_\Delta = \frac{\pi\sigma_\infty^2 a}{E'}$$

For a plate of finite width (2b), the energy release rate takes the form

$$G = \Omega \left(\frac{b}{a} \right) \frac{\sigma_{\infty}^2 a}{E'}$$

The dimensionless function, $\Omega \left(\frac{b}{a} \right)$, is available in the *Handbook* (Tada, Paris and Irwin), in tabulated and graphic forms.

Contour integral (J-integral) method. Consider a 2D linear elastic body. Define a contour integral

$$J = \int_C \left(w n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds$$

where $w = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$ is the elastic strain energy density, $t_i = \sigma_{ij} n_j$ is the traction along the contour C , with n_j being the unit normal of the contour path.

It can be shown that, over a closed contour over a simply-connected domain, the J-integral equals zero.

$$\begin{aligned} J &= \oint_C \left(w n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds = \oint_C \left(w \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) n_j ds \\ &= \int_A \frac{\partial}{\partial x_j} \left(w \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) dA = \int_A \left(\frac{\partial w}{\partial x_1} - \frac{\partial \sigma_{ij}}{\partial x_j} \frac{\partial u_i}{\partial x_1} - \sigma_{ij} \frac{\partial^2 u_i}{\partial x_1 \partial x_j} \right) dA \\ &= \int_A \left(\frac{\partial w}{\partial x_1} - \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} \right) dA = 0 \end{aligned}$$

In the above proof, we have used the divergence theorem converting a closed contour integral into a domain integral over the area A enclosed by the contour. In addition, we have used the equilibrium equation, $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$ (assuming no body force), and the small-strain kinematic

relation, $\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, noting the symmetry, $\sigma_{ij} \varepsilon_{ij} = \sigma_{ij} \frac{\partial u_i}{\partial x_j}$. The last step requires the

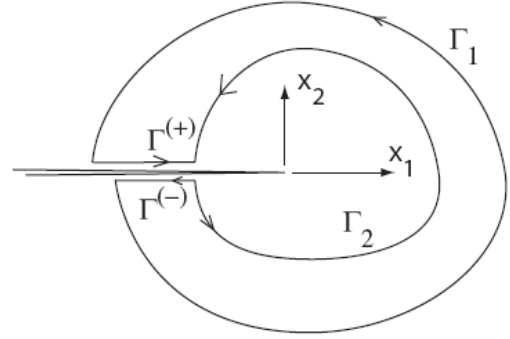
existence of a strain energy density function such that $\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}$.

Now consider J-integral around a crack tip. Over a contour path that starts on one crack face and ends on the other, the value of J-integral is non-zero and independent of the path. This can be proved by considering a closed contour formed by two different paths plus the crack

surfaces in between, i.e., $C = \Gamma_1 - \Gamma_2 + \Gamma^{(+)} + \Gamma^{(-)}$.
The J-integral over the closed contour is zero, thus

$$J_C = J_1 - J_2 + J^{(+)} + J^{(-)} = 0$$

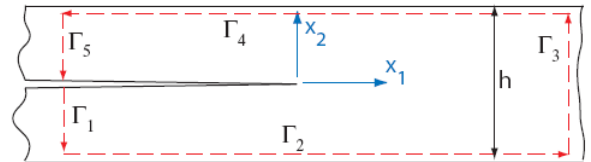
Noting that on the crack surface, $n_1 = 0$ and $t_i = n_j \sigma_{ij} = 0$, we have $J^{(+)} = J^{(-)} = 0$. Therefore, $J_1 = J_2$, i.e., the J-integral around a crack tip is path-independent.



Next we show that the J-integral over a contour around a crack tip equals the energy release rate, i.e., $J = G$. A rigorous derivation was given by Budiansky and Rice (J. Appl. Mech., pp. 201, 1973), who considered the change of the potential energy due to extension of a blunt-ended notch as a generalized crack model in nonlinear elastic solids. A similar derivation can be found in *Lecture Notes on Fracture Mechanics* by Alan Zehnder, where the energy flow during fracture is considered. Under the condition of small scale yielding, one can choose a contour path within the K-annulus, where the stress and displacement fields are given by the asymptotic crack-tip solution in terms of the stress intensity factors. Then, a direct evaluation of the J-integral verifies that $J = \frac{K^2}{E'} = G$ for mode I and mode II, and $J = \frac{K^2}{2\mu}$ for mode III.

Note: The stress intensity factors, K , and the J-integral, depend only on the current state of the crack, irrespective of how the crack may grow under the current condition. On the other hand, the energy release rate, G , is specified with respect to the direction of crack growth. In other words, for the same crack under the same loading condition, the energy release rate may be calculated as a function of the crack growth direction. *Only for the straight ahead growth, the energy release rate is related to the stress intensity factors and J-integral as developed above.*

Example: Consider a semi-infinite crack in an infinite strip held in rigid grips and subject to a prescribed separation Δ . The energy release rate for the straight ahead crack growth was obtained previously by the energy method. Now we apply the J-integral over a path as shown in the figure: $C = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$. We note the following:



- (1) Along Γ_1 and Γ_5 , far behind the crack tip, the material is completely unloaded, i.e., $\sigma_{ij} = 0$ and $w = 0$. Thus, the J-integral is zero on Γ_1 and Γ_5 : $J_1 = J_5 = 0$.
- (2) Along Γ_2 and Γ_4 , close to the interfaces with the rigid grips, we have $n_1 = 0$ and $\frac{\partial u_i}{\partial x_1} = 0$ (constant displacement). Thus, $J_2 = J_4 = 0$.

(3) Along Γ_3 , far ahead of the crack tip, the material is uniformly strained, with $\varepsilon_{22} = \frac{\Delta}{2b}$

and $\varepsilon_{11} = 0$, and also $\frac{\partial u_i}{\partial x_1} = 0$. Assuming plane stress, the elastic strain energy density is,

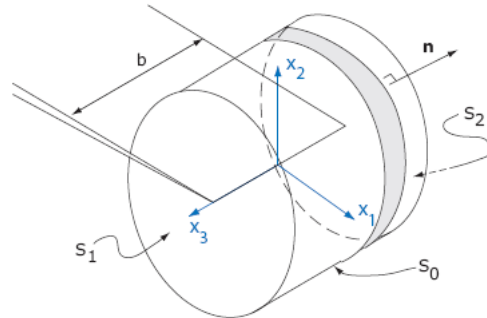
$$w = \frac{1}{2} \sigma_{22} \varepsilon_{22} = \frac{E}{2(1-\nu^2)} \left(\frac{\Delta}{2b} \right)^2. \text{ Thus,}$$

$$J_3 = \int_{\Gamma_3} \left(w n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds = \int_{-b}^b w dx_2 = 2wb = \frac{E\Delta^2}{4b(1-\nu^2)}$$

Therefore, the J-integral over the contour C is: $J = J_3 = \frac{E\Delta^2}{4b(1-\nu^2)}$, identical to the energy release rate obtained before.

J-integral in three dimensions: consider a crack in a plate of thickness b as shown. The J-integral can be evaluated over the surface surrounding the crack front line: $S = S_0 + S_1 + S_2$. However, since $n_1 = 0$ and $t_i = 0$ on S_1 and S_2 (traction free surfaces of the plate), the J-integral in this case is

$$J = \frac{1}{b} \int_{S_0} \left(w n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) dA$$



The local value of the J-integral along the crack front, $J(x_3)$, can be calculated over a slice of the cylindrical surface, $dS_0 = \Gamma dx_3$.

Summary: finding the energy release rate. The following approaches may be used to determine energy release rate for linear elastic fracture mechanics problems (small scale yielding):

- (1) Solve the full-field elasticity problem analytically, and find the energy release rate by definition.
- (2) Energy method, sometimes analytical, without solving the full-field problem.
- (3) Numerical methods (e.g., finite element method), with J-integral or crack closure integral.
- (4) Handbook solutions, available for a large number of specimen geometries.
- (5) By experiments, measuring the compliance as a function of the crack size.