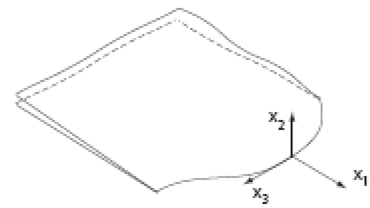
3. Stress Intensity Factors

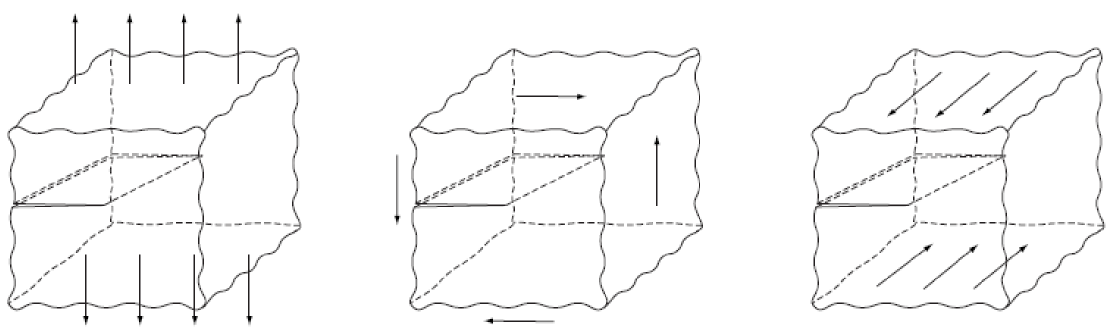
References:

Alan Zehnder, *Lecture Notes on Fracture Mechanics* (http://hdl.handle.net/1813/3075).
H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook* (1973, 1985, 2000).

Three modes of fracture. A crack in 3D may have a curved front. At any point of the crack front, a local coordinate system can be set up so that the x_1 axis is normal to the crack front and parallel to the crack surface, the x_2 axis is normal to the crack surface, and the x_3 axis is tangential to the crack front. A local cylindrical coordinate system can also be set with (r, θ) in the $x_1 - x_2$ plane.



Using the local coordinates, three fracture modes can be defined. Within the framework of linear elastic fracture mechanics (LEFM), an arbitrary crack problem (2D and 3D) can be considered as a linear combination of the three basic modes.



Mode-I (tension) Mode-II (in-plane shear) Mode-III (anti-plane shear)

The characteristic stress field near the crack tip is solved separately for each of the three fracture modes, assuming a straight crack front and semi-infinite crack length in an infinite body.

Mode I and Mode II fields. The asymptotic crack tip fields under mode I and mode II conditions were obtained by Williams (1957). Follow the general approach to solving plane elasticity problems. In the local cylindrical coordinates originated at the crack tip, the Airy stress function $\phi(r,\theta)$ satisfies the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{\partial\phi}{r\partial r}+\frac{\partial^2\phi}{r^2\partial\theta^2}\right)=0$$

By the method of separation of variables, let $\Psi(r,\theta)=\nabla^2\phi(r,\theta)=f(r)\Theta(\theta)$. The biharmonic equation becomes

$$r^2\frac{f''}{f}+r\frac{f'}{f}+\frac{\Theta'}{\Theta}=0$$

The solution to the above equation takes form: $f(r) = r^\lambda$ and $\Theta(\theta) = A\sin\lambda\theta + B\cos\lambda\theta$, where λ is an eigenvalue to be determined.

Noting that $\nabla^2\phi(r,\theta) = r^\lambda(A\sin\lambda\theta + B\cos\lambda\theta)$, let $\phi(r,\theta) = r^{\lambda+2}\Gamma(\theta)$. Then,

$$\Gamma'' + (\lambda+2)^2\Gamma = A\sin\lambda\theta + B\cos\lambda\theta$$

The general solution to the above ODE consists of a superposition of the homogeneous solution and a particular solution, namely

$$\Gamma(\theta) = C_1\sin(\lambda+2)\theta + C_2\cos(\lambda+2)\theta + C_3\sin\lambda\theta + C_4\cos\lambda\theta$$

Therefore, the original biharmonic equation is solved by a stress function in form of Williams' expansion

$$\phi(r,\theta) = r^{\lambda+2}\left[C_1\sin(\lambda+2)\theta + C_2\cos(\lambda+2)\theta + C_3\sin\lambda\theta + C_4\cos\lambda\theta\right]$$

Then, the stress components are:

$$\begin{aligned}
\sigma_r &= (\lambda+1)r^\lambda\left[-(\lambda+2)\left(C_1\sin(\lambda+2)\theta + C_2\cos(\lambda+2)\theta\right) + (2-\lambda)\left(C_3\sin\lambda\theta + C_4\cos\lambda\theta\right)\right] \\
\sigma_\theta &= (\lambda+1)(\lambda+2)r^\lambda\left[C_1\sin(\lambda+2)\theta + C_2\cos(\lambda+2)\theta + C_3\sin\lambda\theta + C_4\cos\lambda\theta\right] \\
\sigma_{r\theta} &= -(\lambda+1)r^\lambda\left[(\lambda+2)\left(C_1\cos(\lambda+2)\theta - C_2\sin(\lambda+2)\theta\right) + \lambda\left(C_3\cos\lambda\theta - C_4\sin\lambda\theta\right)\right]
\end{aligned}$$

Next, we determine the eigenvalue and the coefficients from the boundary conditions.

Mode I crack tip field. For a mode I crack, the stress field is symmetric with respect to the *x* axis, i.e.,

$$\sigma_r(r,-\theta) = \sigma_r(r,\theta),\ \sigma_\theta(r,-\theta) = \sigma_\theta(r,\theta),\ \text{and}\ \sigma_{r\theta}(r,-\theta) = -\sigma_{r\theta}(r,\theta)$$

Thus, $C_1 = C_3 = 0$. Next, apply the boundary condition at the crack surface, i.e., $\sigma_\theta = \sigma_{r\theta} = 0$ at $\theta = \pm\pi$, which leads to

$$\begin{aligned}
&(\lambda+1)(\lambda+2)\left[C_2\cos(\lambda+2)\pi + C_4\cos\lambda\pi\right] = 0 \\
&-(\lambda+1)\left[(\lambda+2)C_2\sin(\lambda+2)\pi + \lambda C_4\sin\lambda\pi\right] = 0
\end{aligned}$$

This becomes a standard eigenvalue problem. In order to have nontrivial solutions to the coefficients C_2, C_4, the determinant of the coefficient matrix of the above equations must vanish, namely

$$(\lambda+1)^2(\lambda+2)\begin{vmatrix}\cos(\lambda+2)\pi & \cos\lambda\pi \\ (\lambda+2)\sin(\lambda+2)\pi & \lambda\sin\lambda\pi\end{vmatrix} = 0$$

This reduces to a simple equation: $(\lambda+1)^2(\lambda+2)\sin 2\lambda\pi = 0$. Therefore, there exists infinite number of eigenvalues: $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \cdots$

A couple of physical conditions help us reduce the number of eigenvalues:

(1) The stress scale with r^λ, which would only make sense if $\lambda < 0$, because stress concentration (or singularity) at the crack tip is expected.
(2) The displacement scales with $r^{\lambda+1}$; to have a bounded displacement at the crack tip ($r \to 0$), it is necessary to have $\lambda + 1 > 0$.

The only eigenvalue that satisfies both the conditions is: $\lambda = -\frac{1}{2}$. Corresponding to this eigenvalue, the eigen vector gives the ratio between the two coefficients: $C_4 = 3C_2$. This leaves only one unknown parameter for the stress fields near the crack tip.

$$\sigma_r = \frac{C_2}{4\sqrt{r}}\left[-3(\cos 3\theta/2) + 5(3\cos\theta/2)\right] = \frac{3C_2}{\sqrt{r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin^2\left(\frac{\theta}{2}\right)\right]$$
$$\sigma_\theta = \frac{3C_2}{4\sqrt{r}}\left[\cos 3\theta/2 + 3\cos\theta/2\right] = \frac{3C_2}{\sqrt{r}}\cos^3\left(\frac{\theta}{2}\right)$$
$$\sigma_{r\theta} = \frac{3C_2}{4\sqrt{r}}\left[\sin 3\theta/2 + \sin\theta/2\right] = \frac{3C_2}{\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

Stress intensity factor. The remaining unknown parameter can only be determined from a remote boundary condition. On the other hand, the square root singularity as well as the circumferential distribution of the stress field near the crack tip is independent of the remote loading conditions. In other words, the mode I crack is fully described by a single parameter, which is called *stress intensity factor*. For convenience, define $K_I = 3C_2\sqrt{2\pi}$, so that the opening stress ahead of the crack tip is: $\sigma_\theta = K_I / \sqrt{2\pi r}$.

Mode I displacement field. The displacement field is determined by integrating the strain components; the latter are related to the stress components by Hooke's law.

$$u_r = K_I \frac{1+\nu}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left[(2\kappa-1)\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2}\right)\right]$$
$$u_\theta = K_I \frac{1+\nu}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left[-(2\kappa+1)\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\right]$$

where $\kappa = 3 - 4\nu$ for plane strain problems and $\kappa = (3-\nu)/(1+\nu)$ for plane stress problems.

Of particular interest is the crack opening displacement:

$$u_\theta(\theta = \pm\pi) = \pm\frac{4K_I}{E'}\left(\frac{r}{2\pi}\right)^{1/2}$$

where $E' = E/(1-\nu^2)$ for plane strain and $E' = E$ for plane stress. The opening displacement has a shape of parabola. The relative displacement of the two crack faces requires that $K_I > 0$, to avoid interpenetration of the crack faces.

Mode II crack tip field. Follow the same steps as for the mode I field. Note the different symmetry condition for Mode II. The stress field is anti-symmetric with respect to the *x* axis. Thus, in the general solution, the coefficients $C_1, C_3 \neq 0$, but $C_2 = C_4 = 0$ instead. The same eigenvalue, $\lambda = -\frac{1}{2}$, is obtained. The asymptotic stress field for a mode II crack:

$$\sigma_r = \frac{K_{II}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right)\left[1 - 3\sin^2\left(\frac{\theta}{2}\right)\right]$$
$$\sigma_\theta = -\frac{K_{II}}{\sqrt{2\pi r}}3\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$
$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 - 3\sin^2\left(\frac{\theta}{2}\right)\right]$$

K_{II} is the mode II stress intensity factor, which depends on the boundary conditions away from the crack tip. Ahead of the crack tip ($\theta = 0$), we have $\sigma_r = \sigma_\theta = 0$, but the shear stress $\sigma_{r\theta} = K_{II}/\sqrt{2\pi r}$.

The mode II displacement field is

$$u_r = K_{II}\frac{1+\nu}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left[-(2\kappa - 1)\sin\left(\frac{\theta}{2}\right) + 3\sin\left(\frac{3\theta}{2}\right)\right]$$
$$u_\theta = K_{II}\frac{1+\nu}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left[-(2\kappa + 1)\cos\left(\frac{\theta}{2}\right) + 3\cos\left(\frac{3\theta}{2}\right)\right]$$

The shearing displacements at the crack faces are

$$u_r(\theta = \pm\pi) = \mp\frac{4K_{II}}{E'}\left(\frac{r}{2\pi}\right)^{1/2}$$

The opening displacement of a mode II crack is zero, thus the possibility of contact and friction between the crack surfaces.

Mode III crack tip field. This is an anti-plane shear problem (even simpler than plane elasticity). The displacement $u_r = u_\theta = 0$, and $u_z = u_z(r,\theta)$, independent of the z coordinate. The stress components are:

$$\sigma_{zr} = \mu \frac{\partial u_z}{\partial r},\ \sigma_{z\theta} = \mu \frac{\partial u_z}{r\partial \theta}, \text{ and } \sigma_r = \sigma_\theta = \sigma_z = \sigma_{r\theta} = 0$$

Thus, the equilibrium equation becomes

$$\nabla^2 u_z = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2 \partial \theta^2} \right) u_z = 0$$

This is a harmonic equation of the anti-plane displacement. Again, using the method of separation of variables, the general solution takes the form

$$u_z = r^\lambda \left(A \sin \lambda\theta + B \cos \lambda\theta \right)$$

Apply boundary conditions: $\sigma_{z\theta} = \mu \frac{\partial u_z}{r\partial \theta} = 0$ at $\theta = \pm\pi$ (traction free crack surfaces), and we have

$$\lambda\left(A \cos \lambda\pi - B \sin \lambda\pi \right) = 0$$
$$\lambda\left(A \cos \lambda\pi + B \sin \lambda\pi \right) = 0$$

The existence of nontrivial solutions requires that $\lambda^2 \sin 2\lambda\pi = 0$. Thus, the possible eigenvaules are: $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \cdots$. With the same physical conditions for the mode I field, only one eigenvalue, $\lambda = -\frac{1}{2}$, is retained.

The asymptotic stress field for mode III is

$$\sigma_{zr} = \frac{K_{III}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right),\ \sigma_{z\theta} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)$$

K_{III} is the mode III stress intensity factor. Ahead of the crack tip ($\theta = 0$), we have $\sigma_{zr} = 0$, but $\sigma_{z\theta} = K_{III} / \sqrt{2\pi r}$.

The mode III displacement field has only one component:

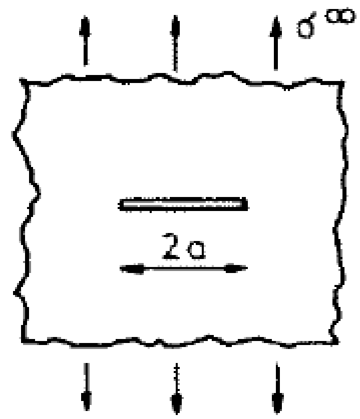
$$u_z = K_{III} \frac{4(1+\nu)}{E} \left(\frac{r}{2\pi} \right)^{1/2} \sin\left(\frac{\theta}{2}\right)$$

Summary of the linear elastic crack-tip fields. The asymptotic stress fields for the three fracture modes may be reduced to a simple, general form

$$\sigma_{ij} = \frac{K_m}{\sqrt{2\pi r}} f_{ij}^{(m)}(\theta) \text{ and } u_i = \frac{K_m}{2E}\left(\frac{r}{2\pi}\right)^{1/2} g_i^{(m)}(\theta)$$

Both the radial and circumferential distribution are fully determined. The stress intensity factors depend on the remote boundary conditions, i.e., the applied load and specimen geometry.

Calculations of stress intensity factors. Analytical methods such as the complex variables method (see Alan Zehnder, *Lecture Notes on Fracture Mechanics*, http://hdl.handle.net/1813/3075) have been used to solve full fields of elastic boundary value problems to determine the stress intensity factors. For example, the crack opening displacement of a finite crack in an infinite plate under remote tension can be obtained by the Westergaard approach of complex variable method:



$$u_y = \pm\frac{2\sigma_\infty}{E'}\sqrt{a^2 - x^2} \text{ for } |x| < a$$

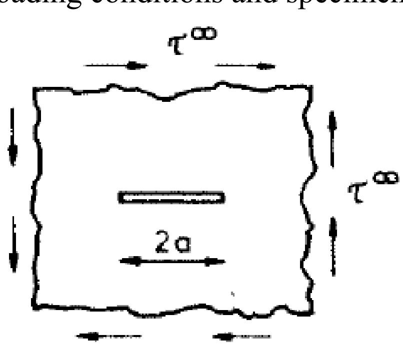
Near one crack tip (say $x = a$), the opening displacement in the local cylindrical coordinates is:

$$u_\theta = \pm\frac{2\sigma_\infty}{E'}\sqrt{(2a - r)r} \to \pm\frac{2\sigma_\infty}{E'}\sqrt{2ar}$$

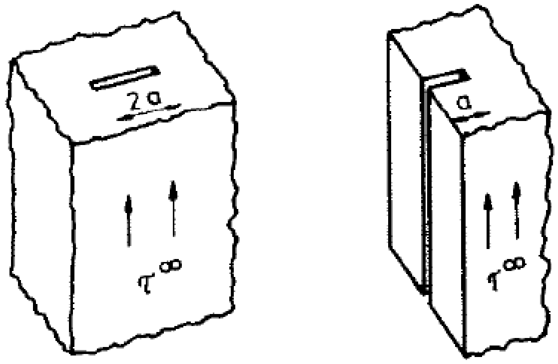
Compare the above solution to the asymptotic mode I solution, $u_\theta(\theta = \pm\pi) = \pm\frac{4K_I}{E'}\left(\frac{r}{2\pi}\right)^{1/2}$, we obtain the stress intensity factor

$$K_I = \sigma_\infty\sqrt{\pi a}$$

More generally, numerical methods (e.g., finite element method, boundary element method, etc.) are used to calculate the stress intensity factors. Some of these numerical methods will be introduced in later lectures. A large collection of stress intensity factors is available in H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook*, with a variety of loading conditions and specimen geometry. A few examples are given below.

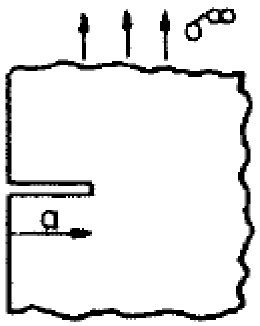


A finite crack in an infinite plate under remote shear: $K_{II} = \tau_\infty\sqrt{\pi a}$



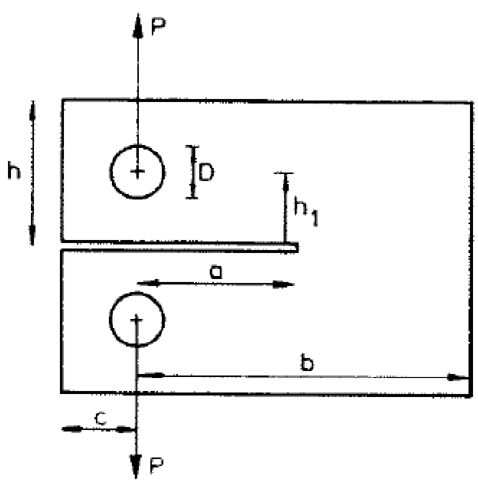
A rectangular crack in an infinite body or an edge crack in a half space under mode III shear:

$$K_{III} = \tau_\infty \sqrt{\pi a}$$



An edge crack in a half plane under remote tension:

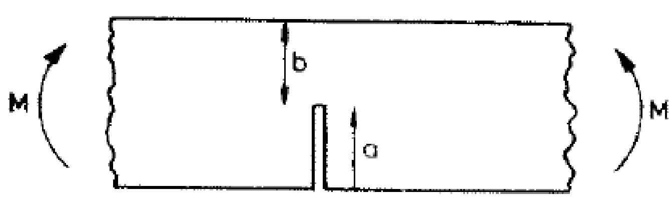
$$K_I = 1.12\sigma_\infty \sqrt{\pi a}$$



Standard ASTM compact tension specimen (h = 0.6b, h_1 = 0.275b, D = 0.25b, c = 0.25b, thickness w):

$$K_I = \frac{P\sqrt{a}}{wb} F\left(\frac{a}{b}\right)$$

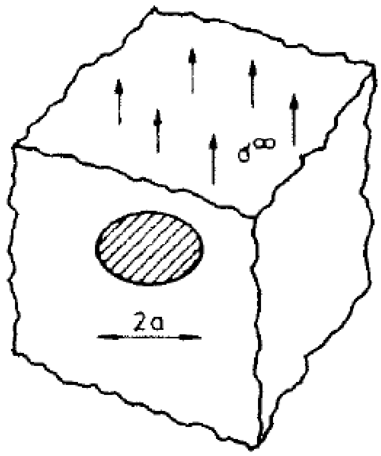
with $F(x) = 29.6 - 185.5x + 655.7x^2 - 1017x^3 + 63.9x^4$ for $0.4 < x < 0.6$.



An edge crack in a strip (thickness w) under bending:

$$K_I = \frac{M}{wb^{3/2}} F\left(\frac{a}{b}\right)$$

When a/b → 0, $F \to 11.9\sqrt{a/b}$;
When $a/b \to 1$, $F \to 3.95$.



A penny shaped crack in an infinite body under remote tension. Near crack edge field is identical to mode I plane strain with

$$K_I = \frac{2}{\pi}\sigma_\infty\sqrt{\pi a}$$

Small scale yielding and K annulus. The asymptotic crack tip field (also called K field) is only valid under two conditions:

(1) at a region close to the crack tip. Specifically, the distance to the crack tip, r, should be small compared to other length scales (e.g., crack length, specimen size). Beyond this limit, the remote boundary conditions add additional terms (non-singular) to the asymptotic field.
(2) But not too close to the crack tip. The singular stress field from the elastic solution is truncated by plastic yield or other inelastic behavior at a distance r_p.

The condition of small scale yielding (SSY) states that the plastic zone size is sufficiently small compared to the crack length and other geometrical lengths of the specimen, i.e., $r_p << L$. This ensures that the elastic K field is correct in an annular region surrounding the crack tip, i.e., $r_p < r < L$.

The plastic zone size may be estimated by comparing the asymptotic stress field to the yield strength of the material. Roughly, by $\frac{K}{\sqrt{2\pi r}} = \sigma_y$, we obtain that $r_p \sim \left(\frac{K}{\sigma_y}\right)^2$. Thus, the plastic zone size increases as the applied load increases. In most instances, SSY appears to be a reasonable assumption as long as the applied load is below about one half of the limit load for full scale plastic yielding (e.g., $\sigma_\infty < 0.5\sigma_y$).

Linear elastic fracture mechanics (LEFM) is essentially based on the assumption of SSY. Under the condition of SSY, the stress intensity factors (K_I, K_{II}, K_{III}) provide the only link between the outer boundaries and the crack tip. The fracture process at the crack tip is thus dependent on the stress intensity factors. The details of geometry and loading conditions become irrelevant as long as the stress intensity factors are known.