The mathematical foundations of anelasticity: Existence of smooth global intermediate configurations*

Christian Goodbrake† Alain Goriely‡ Arash Yavari§

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Abstract

A central tool of nonlinear anelasticity is the multiplicative decomposition of the deformation tensor that assumes that the deformation gradient can be decomposed as a product of an elastic and an anelastic tensor. It is usually justified by the existence of an intermediate configuration. Yet, this configuration cannot exist in Euclidean space, in general, and the mathematical basis for this assumption is on unsatisfactory ground. Here, we derive a sufficient condition for the existence of global intermediate configurations, starting from a multiplicative decomposition of the deformation gradient. We show that these global configurations are unique up to isometry. We examine the result of isometrically embedding these configurations in higher dimensional Euclidean space, and construct multiplicative decompositions of the deformation gradient reflecting these embeddings. As an example, for a family of radially-symmetric deformations, we construct isometric embeddings of the resulting intermediate configurations, and compute the residual stress fields explicitly.

Keywords: Riemannian Geometry, Anelasticity, Intermediate Configuration, Multiplicative Decomposition.

1 Introduction

The theory of nonlinear elasticity is a field theory that describes elastic deformations in continua. A motion is modeled as a smooth isotopy, parameterized by time, such that for each time t, the induced diffeomorphism $\varphi_t : \mathcal{B} \to \mathcal{S}$ gives an embedding of the body, modeled as a smooth manifold $\mathcal{B}$ into the fixed ambient space $\mathcal{S}$. When the kinematics is considered, a value of t is implicitly fixed, and the image of φ_t is denoted $\mathcal{C}$. Hence at each moment of time, a deformation is typically modeled as a diffeomorphism between an initial stress-free configuration $\mathcal{B}$ and the current configuration $\mathcal{C}$, both assumed to be smooth manifolds. In principle, $\mathcal{C} \subset \mathcal{S}$ depends on time, but for the remainder of this paper we shall suppress the time dependence, since we are concerned with the kinematics at each moment in time, and not the dynamics of how a body evolves through time. With a slight abuse of notation, we write this as

$$\varphi : \mathcal{B} \to \mathcal{C}. \tag{1}$$

The central object of nonlinear elasticity is the so-called deformation gradient $\mathbf{F}$, which is determined by the tangent map of φ. Technically, $\mathbf{F}(X) = T\varphi|_{\pi^{-1}(X)}$, where π is the natural projection in the tangent bundle onto the base space; $\mathbf{F}(X)$ is the restriction of $T\varphi$ to the fiber over X. Note that $T\varphi$ is a vector bundle morphism mapping the tangent bundle $T\mathcal{B}$ to the tangent bundle $T\mathcal{C}$, hence it also includes φ as

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†Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK.

‡Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK.

§School of Civil and Environmental Engineering & The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA. E-mail: arash.yavari@ce.gatech.edu.

the map on the base space. Interested readers will find Husemöller [1994] to be an invaluable resource for a complete treatment of vector bundles and vector bundle morphisms; this treatment is done in categorical language, hence Riehl [2016] or Mac Lane et al. [1998] may be a useful resource to readers. Because the manifolds $\mathcal{B}$ and $\mathcal{C}$ are parallelizable, their tangent bundles are trivial, hence we can write the vector bundle morphism $T\varphi$ as $T\varphi = (\varphi, \mathbf{F})$, where here we consider $\mathbf{F}$ as a tensor field mapping tangent vector fields on $\mathcal{B}$ to tangent vector fields on $\mathcal{C}$. We consider $\mathbf{F}(X)$ as the restriction of this tensor field to the fiber over X. Geometrically, $\mathbf{F}(X)$ is a linear map that sends the vector $\mathbf{v} \in T_X\mathcal{B}$ to $\mathbf{F}(X)\mathbf{v} \in T_{\varphi(X)}\mathcal{C}$, where in coordinates $\{X^A\} : \mathcal{B} \to \mathbb{R}^n$ and $\{x^a\} : \mathcal{C} \to \mathbb{R}^n$, one has

$$\mathbf{F}(X) = \left(\frac{\partial \varphi^a}{\partial X^A}\bigg|_X \right) \frac{\partial}{\partial x^a} \otimes dX^A. \tag{2}$$

The deformation φ maps points in $\mathcal{B}$ to points in $\mathcal{C}$, and the deformation gradient $\mathbf{F}(X)$ maps tangent vectors of $\mathcal{B}$ at the point X to tangent vectors of $\mathcal{C}$ at the point $\varphi(X)$. To guarantee that matter does not penetrate itself, it is required that $\det \mathbf{F} > 0$ everywhere, i.e., $\det \mathbf{F}(X) > 0$ for all $X \in \mathcal{B}$, which ensures that φ is locally invertible, and is orientation preserving.

An important tenet of the theory of elasticity is that there exists a reference configuration that is stress-free. However, there are plenty of physical situations where either this configuration is not explicitly known or stresses are created by other physical processes than elastic deformations. For instance, in a growth process different elements of a body change in size or relative position which induces strains and associated stresses even in the absence of applied loads or body force [Goriely, 2017]. Similarly, additional strains appear in thermoelasticity [Sadik and Yavari, 2017b], accretion [Sozio and Yavari, 2017, 2019], and defect mechanics [Yavari and Goriely, 2013, 2012a,b]. We call *anelastic strains*, strains that are not created through elastic deformations. They are also known in various communities as eigenstrains [Mura, 1982], initial strains [Kondo, 1949], inherent strains [Ueda et al., 1975], transformation strains [Eshelby, 1957] or residual strains [Ambrosi et al., 2019].

A central assumption of *nonlinear anelasticity* is that the anelastic strains are solely created by a local anelastic deformation tensor field and the deformation gradient can be decomposed multiplicatively as

$$\mathbf{F} = \mathbf{A}\mathbf{G}, \tag{3}$$

where the field $\mathbf{G}$ generates purely anelastic strain and the field $\mathbf{A}$ generates purely elastic strain. The corresponding conceptual hypothesis, that anelastic contribution can be taken into account through a *multiplicative decomposition of the deformation gradient*, follows from early work in different communities [Sadik and Yavari, 2017a]: The earliest work is due to Eckart [1948] who introduced a framework for anelasticity based on "relaxability-in-the-small". In polymer swelling it was first discussed by Flory [1956]; in the theory of defects by Bilby et al. [1957]; in elastoplasticity by Kröner [1958], [Kröner and Seeger, 1959, p.100], [Kröner, 1960, p.286]; and later was popularized by [Lee, 1969]. In the theory of thermoelasticity, it was properly formalized by Stojanović et al. [Stojanovic, 1969, Stojanovic et al., 1964] and in the context of biological tissues, the multiplicative decomposition was independently proposed in Russia by Kondaurov and Nikitin [1987] and in Japan by Takamizawa et al. [Takamizawa, 1991, Takamizawa and Hayashi, 1987, Takamizawa and Matsuda, 1990] who used it to characterize residual stresses in arteries. The same conceptual ideas can also be found in the work of Tranquillo and Murray on wound healing [Tranquillo and Murray, 1993, 1992]. It became a central concept of biomechanics following the seminal work of Rodriguez et al. [1994] who showed how to translate growth processes in terms of the tensor field $\mathbf{G}$. There has been recent interest in understanding different aspects of this decomposition [Neff, 2008, Neff et al., 2009, Reina and Conti, 2014, Casey, 2017, Del Piero, 2018, Du et al., 2018].

A typical conceptual sketch of the multiplicative decomposition is given in Fig. 1. From a practical point of view, this decomposition is perfectly suitable to define all kinematic and mechanical quantities as well as to obtain the governing equations for anelasticity, from which theoretical and computational progress can be achieved. Over the years, it has become a popular tool in the mechanics of large deformations, especially in the biological context where anelastic strains are generated by growth, remodeling, or active processes [Goriely, 2017]. However, from a mathematical point of view there are a number of less-than-satisfactory

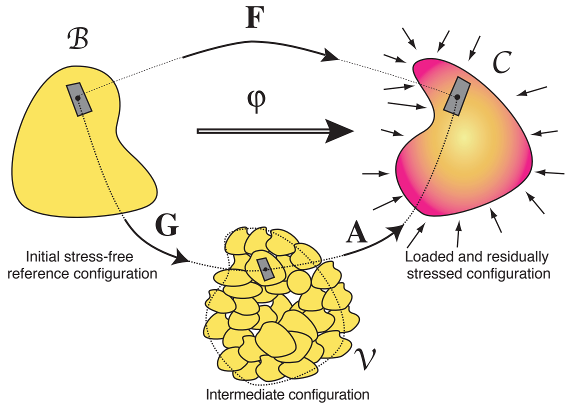

Figure 1: The multiplicative decomposition: starting from a reference stress-free configuration in Euclidean space, a local deformation $\mathbf{G}$ is applied to each material point, creating an intermediate configuration that is further deformed by $\mathbf{A}$, to recover the integrity of the body, into a residually stressed configuration in Euclidean space.

aspects to this hypothesis. For instance, it was realized early [Casey and Naghdi, 1980] that a decomposition generating the required strain is not unique, as for any isometry $\mathbf{Q}(X)$, the alternative decomposition $\mathbf{F}(X) = (\mathbf{A}(X)\mathbf{Q}(X))(\mathbf{Q}^{-1}(X)\mathbf{G}(X))$, is equally valid. In practice, it is not a problem as the final solution for the stress is independent of $\mathbf{Q}(X)$. A more serious criticism is related to the mathematical status of this so-called intermediate configuration. Indeed, while the field $\mathbf{F}$ is the derivative of φ (or "gradient" of φ), in general, by construction neither $\mathbf{A}$ nor $\mathbf{G}$ can be written as derivative maps (or "gradients"). This property is called incompatibility or non-integrability and is at the heart of anelasticity: there are no maps between the reference and intermediate configurations in Euclidean spaces and the usual picture of a dislocated configuration viewed as a collection of sub-bodies or a disjoint union of vector spaces shown in Fig. 1 is unsatisfactory as there could be infinitely many of these pieces and their connections must be somehow specified.

The purpose of this paper is to show that, using the proper geometric setting, the intermediate configuration can be properly defined. Since we want this intermediate configuration to be in some sense equivalent to the multiplicative decomposition we begin with, we exploit the non-uniqueness of the multiplicative decomposition to attempt to construct a different decomposition $\mathbf{F} = \tilde{\mathbf{A}}\tilde{\mathbf{G}}$ satisfying the following two properties:

i) The new decomposition generates the same anelastic strain as the original, i.e.,

$$\mathbf{G}^{\mathsf{T}}\mathbf{G} = \tilde{\mathbf{G}}^{\mathsf{T}}\tilde{\mathbf{G}}. \tag{4}$$

ii) The decomposition is induced by the composition of maps between Riemannian manifolds, i.e.,

$$\varphi = \alpha \circ \gamma : (\mathcal{B}, \mathbf{M}) \xrightarrow{\gamma} (\mathcal{M}, \mathbf{K}) \xrightarrow{\alpha} (\mathcal{C}, \mathbf{m}), \quad \text{with} \quad T\varphi = (\varphi, \mathbf{F}), \quad T\alpha = (\alpha, \tilde{\mathbf{A}}), \quad T\gamma = (\gamma, \tilde{\mathbf{G}}). \tag{5}$$

We seek to identify when this is possible, and when it is, to what extent the maps α and γ are uniquely defined. The Riemannian manifold $(\mathcal{M}, \mathbf{K})$ we will ultimately construct is the intermediate configuration appearing in geometric theories, hence the problem we seek to solve can be stated as follows: Given a multiplicative decomposition $\mathbf{F} = \mathbf{A}\mathbf{G}$ of the deformation gradient field, does there exist an intermediate configuration $(\mathcal{M}, \mathbf{K})$ such that the deformation φ can be factored through $(\mathcal{M}, \mathbf{K})$, with the tangent maps induced by this factoring generating a multiplicative decomposition of the deformation gradient that generates the same strain as the original decomposition? Whenever this is possible, theories based on the multiplicative decomposition can be reformulated as equivalent theories based on Riemannian intermediate configurations.

2 Preliminaries

We suppose that we are given a factorization of the tensor field $\mathbf{F}$, and desire to construct a factorization of $\varphi : (\mathcal{B}, \mathbf{M}) \to (\mathcal{C}, \mathbf{m})$ through some Riemannian manifold $(\mathcal{M}, \mathbf{K})$ that reflects this decomposition. We first have to establish the nature of the objects $\mathbf{G}$ and $\mathbf{A}$. When one writes, $\mathbf{F} = \mathbf{A}\mathbf{G}$, very little is said about $\mathbf{G}$ and $\mathbf{A}$. We make a number of implicit assumptions about the structure of the configurations and these tensors. As a first step, it is important to discuss these assumptions explicitly. Since $\mathbf{F}(X)$ is invertible at each point X, we know that $\mathbf{G}(X)$ has a left inverse and $\mathbf{A}(X)$ has a right inverse: $\mathbf{G}(X)$ is injective and $\mathbf{A}(X)$ is surjective. Explicitly, the right inverse of $\mathbf{A}(X)$ and the left inverse of $\mathbf{G}(X)$ are

$$(\mathbf{A})_R^{-1}(X) = \mathbf{G}(X)\mathbf{F}^{-1}(X), \quad (\mathbf{G})_L^{-1}(X) = \mathbf{F}^{-1}(X)\mathbf{A}(X). \tag{6}$$

Secondly, we have considered $\mathbf{F}(X)$ as the restriction of the tangent map $T\varphi$ to the fiber over the point X, so we ought to frame $\mathbf{G}(X)$ and $\mathbf{A}(X)$ as the restriction of vector bundle morphisms to fibers over the point X as well. Doing this, we define the two vector bundle morphisms $(\mathrm{id}_{\mathcal{B}}, \mathbf{G})$ and $(\varphi, \mathbf{A})$, the composition of which yields the vector bundle morphism $(\varphi, \mathbf{F})$, where again, the triviality of the tangent bundle of $\mathcal{B}$ lets us decompose these morphisms into the component on the base space and the component on the fibers. In principle, any two diffeomorphisms γ and α satisfying $\varphi = \alpha \circ \gamma$ could be used to extend $\mathbf{G}$ and $\mathbf{A}$ into vector bundle morphisms. The non-integrability of the fields $\mathbf{G}$ and $\mathbf{A}$ is equivalent to stating that these vector bundle morphisms do not lie in the image of the tangent bundle functor.

Reframing the maps $\mathbf{A}$ and $\mathbf{G}$ as vector bundle morphisms has already solved one key problem with the multiplicative decomposition, namely that we now preserve the underlying topology of the body throughout the decomposition. In this language the requirement (5) can be interpreted as requiring the existence of vector bundle morphisms $(\varphi, \mathbf{F}) = (\alpha, \tilde{\mathbf{A}}) \circ (\gamma, \tilde{\mathbf{G}})$ such that $(\alpha, \tilde{\mathbf{A}})$ and $(\gamma, \tilde{\mathbf{G}})$ are not arbitrary vector bundle morphisms, but morphisms that lie in the image of the tangent functor. Further, Since we are interested in the notion of strain, we require that the manifolds $\mathcal{B}$ and $\mathcal{C}$ are equipped with metrics say, $\mathbf{M}$ and $\mathbf{m}$, respectively. Therefore, we replace the smooth manifold $\mathcal{B}$ with the Riemannian manifold $(\mathcal{B}, \mathbf{M})$, and the smooth manifold $\mathcal{C}$ with the Riemannian manifold $(\mathcal{C}, \mathbf{m})$. With this, we consider φ as the diffeomorphism

$$\varphi : (\mathcal{B}, \mathbf{M}) \to (\mathcal{C}, \mathbf{m}), \tag{7}$$

which lets us compute the Lagrangian strain field induced by φ as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^\mathsf{T}\mathbf{F} - \mathbf{I}\right), \tag{8}$$

where $\mathbf{I}$ is the identity tensor. We emphasize that $\varphi : \mathcal{B} \to \mathcal{C}$ is viewed as a morphism in a different category than $\varphi : (\mathcal{B}, \mathbf{M}) \to (\mathcal{C}, \mathbf{m})$, namely we consider the category whose objects are not merely smooth manifolds, but Riemannian manifolds, and whose morphisms are smooth maps between Riemannian manifolds. In this category, two Riemannian manifolds are isomorphic if they are diffeomorphic, even if they are not isometric, much like how two vector spaces may be isomorphic as vector spaces, even if they have different metric structures. Likewise, as general vector bundles can be given Riemannian metrics, two metrized vector bundles may be isomorphic as vector bundles even if their respective metric structures do not agree. Since $\mathbf{G}(X)$ can also generate strain, we require that its codomain has an inner product. Again, this inner product is implicit in the definition of $\mathbf{G}^\mathsf{T}(X)$. Therefore, it must be initially prescribed along with the decomposition $\mathbf{F}(X) = \mathbf{A}(X)\mathbf{G}(X)$, though in practice it is often implicitly taken to be the standard inner product on $\mathbb{R}^m$. Labeling the codomain of $\mathbf{G}(X)$ as U_X, we have $\mathbf{F}(X) = \mathbf{A}(X)\mathbf{G}(X)$, with

$$\mathbf{G}(X) : T_X(\mathcal{B}, \mathbf{M}) \to U_X, \tag{9}$$

from the tangent space at X to some inner product space U_X and

$$\mathbf{A}(X) : U_X \to T_{\varphi(X)}(\mathcal{C}, \mathbf{m}). \tag{10}$$

Because $\mathbf{G}(X)$ is injective, its image is a subspace of U_X with the same dimension as $T_X(\mathcal{B}, \mathbf{M})$. If we consider

$$\hat{\mathbf{G}}(X) : T_X(\mathcal{B}, \mathbf{M}) \to \mathrm{im}(\mathbf{G}), \tag{11}$$

such that $\mathbf{G}(X) = \iota_X \circ \hat{\mathbf{G}}(X)$ (ι_X being the inclusion map $\mathrm{im}(\mathbf{G}(X)) \hookrightarrow U_X$), we have an invertible linear transformation. If we then restrict the domain of $\mathbf{A}(X)$ to $\mathrm{im}(\mathbf{G}(X))$, we obtain another invertible linear transformation

$$\hat{\mathbf{A}}(X) : \mathrm{im}(\mathbf{G}(X)) \to T_{\varphi(X)}(\mathcal{C}, \mathbf{m}). \tag{12}$$

The composition of the field $\hat{\mathbf{A}}\hat{\mathbf{G}}$ is another multiplicative decomposition of the field $\mathbf{F}$, but consisting of invertible factors. Each factor is still not generally integrable, so this is not the decomposition we ultimately seek. Next, we turn our attention to the strain induced by the field $\mathbf{G}$. The inner product structure on U_X, being a positive-definite symmetric, bilinear map, admits a representation as a positive-definite symmetric tensor

$$\mathbf{h} : U_X \otimes U_X \to \mathbb{R}. \tag{13}$$

Since we can parameterize the disjoint union of inner products $\mathbf{h}$ by points in $\mathcal{B}$, we can consider $\mathbf{h}$ as a field, though it may be highly irregular. With this, we can now choose a basis $\{\mathbf{e}_\alpha\}$, and its corresponding dual basis $\{\boldsymbol{\vartheta}^\alpha\}$, for U_X and express the requirement (4) in components as

$$M^{AD} G^\alpha{}_D h_{\alpha\beta} G^\beta{}_B = M^{AD} \tilde{G}^\alpha{}_D h_{\alpha\beta} \tilde{G}^\beta{}_B, \tag{14}$$

explicitly showing how the inner product structure on U_X is implicitly required to examine the strain induced by $\mathbf{G}$. This structure generally varies from point to point. For now, we make no assumptions about the smoothness of $\mathbf{h}$, $\mathbf{G}$, or $\mathbf{A}$, other than noticing that the product $\mathbf{AG}$ is $\mathbf{F}$, which is smooth, being the induced tangent map of the diffeomorphism φ. Using $\mathbf{h}$, we can write U_X as the direct sum of orthogonal subspaces, i.e.,

$$U_X = \mathrm{im}(\mathbf{G}(X)) \oplus \mathrm{im}(\mathbf{G}(X))^\perp, \tag{15}$$

and we can construct the orthogonal projection

$$\boldsymbol{\pi}_X : U_X \to \mathrm{im}(\mathbf{G}(X)). \tag{16}$$

If we denote the inclusion $\iota_X^\perp : \mathrm{im}(\mathbf{G}(X))^\perp \to U_X$ and the inclusion $\iota_X : \mathrm{im}(\mathbf{G}(X)) \to U_X$ as before, this projection satisfies $\boldsymbol{\pi}_X \circ \iota_X = \mathrm{id}_{\mathrm{im}(\mathbf{G}(X))}$, $\iota_X \circ \boldsymbol{\pi}_X \circ \mathbf{G}(X) = \mathbf{G}(X)$, and $\boldsymbol{\pi}_X \circ \iota_X^\perp = \mathbf{0}$. With these definitions, we have

$$\hat{\mathbf{G}}(X) = \boldsymbol{\pi}_X \circ \mathbf{G}(X), \qquad \hat{\mathbf{A}}(X) = \mathbf{A}(X) \circ \iota_X. \tag{17}$$

The strain field induced by $\mathbf{G}$ is then

$$\mathbf{E}_\mathbf{G} = \frac{1}{2}\left(\mathbf{G}^\mathsf{T}\mathbf{G} - \mathbf{I}\right), \tag{18}$$

where again $\mathbf{I}$ is the identity tensor field. Since $\mathbf{G}(X)$ is injective, the field $\mathbf{E}_\mathbf{G}$ is equal to the strain induced by $\hat{\mathbf{G}}$, which is given by the expression (18) with $\mathbf{G}$ replaced by $\hat{\mathbf{G}}$. The main assumption for the rest of this paper is that the field $\mathbf{G}^\mathsf{T}\mathbf{G}$ is at least C^1.

3 Construction of an Intermediate Configuration

We can now take advantage of the geometric structure described in the previous section to construct the intermediate configuration. Given $\mathbf{G}$ and $\mathbf{h}$, we can construct a positive-definite, symmetric tensor field $\mathbf{H}$ on $\mathcal{B}$ by requiring

$$\mathbf{H}(\mathbf{u}, \mathbf{v}) = \mathbf{h}(\mathbf{G}\mathbf{u}, \mathbf{G}\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \Gamma\left(T(\mathcal{B}, \mathbf{M})\right), \tag{19}$$

where $\Gamma\left(T\left(\mathcal{B}, \mathbf{M}\right)\right)$ is the space of sections of the tangent bundle of $(\mathcal{B}, \mathbf{M})$, i.e., the space of tangent vector fields. The tensor field $\mathbf{H} = \mathbf{G}^*\mathbf{h}$ is the pull-back of $\mathbf{h}$ under $\mathbf{G}$, hence can be used to compute the inner product fields of the vector fields $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{G}$ is injective, and $\mathbf{h}$ is positive-definite and symmetric, $\mathbf{H}$ is a positive-definite and symmetric tensor field. If $\mathbf{H}$ is smooth with respect to X, it satisfies the definition of a metric tensor field, and we can build the Riemannian manifold $(\mathcal{B}, \mathbf{H})$. We will show next that this manifold is the intermediate configuration.

The minimal degree of smoothness required for the fundamental theorem of Riemannian geometry to hold is C^1 and the structures that can be defined on the intermediate configuration will depend on the smoothness of $\mathbf{H}$. Indeed, if $\mathbf{H}(X)$ is C^1, we can construct the Levi-Civita connection on $(\mathcal{B}, \mathbf{H})$, if it is C^2, we can define the Riemann curvature, and if it is C^k with $k \geq 3$, we can construct an isometric embedding in a sufficiently high dimensional Euclidean space that is also C^k [Nash, 1956]. Here, we are interested in the case where $\mathbf{H}$ is at least C^1 which is guaranteed by our assumption that $\mathbf{G}^\mathsf{T}\mathbf{G}$ is C^1. Note that if the anelastic strain $\mathbf{E}_\mathbf{G}$ is directly prescribed instead of a multiplicative decomposition, we instead examine the smoothness of $\mathbf{H} = (2\mathbf{E}_\mathbf{G} + \mathbf{I})^\flat$, and proceed mutatis mutandis. In general, $(\mathcal{B}, \mathbf{M})$ and $(\mathcal{B}, \mathbf{H})$ are different Riemannian manifolds, since their specific geometric structures are encapsulated in the distinct metrics $\mathbf{M}$ and $\mathbf{H}$. However, their topologies are the same, since the metric topology of any Riemannian manifold agrees with its underlying manifold topology [Lee, 2001], which is the same for both $(\mathcal{B}, \mathbf{M})$ and $(\mathcal{B}, \mathbf{H})$.

Having defined the two distinct but diffeomorphic Riemannian manifolds, $(\mathcal{B}, \mathbf{M})$ and $(\mathcal{B}, \mathbf{H})$, we need to construct a map between them. Because they both have the smooth manifold $\mathcal{B}$ at their core, we can begin with the identity morphism on $\mathcal{B}$. We can then transform this morphism, which is a morphism in the category of smooth manifolds, into a morphism in our category of Riemannian manifolds by equipping the domain and the codomain with Riemannian metrics. It is important to note that even though our original morphism is the identity, we need not use the same metric for the domain and the codomain. Doing this with the two metric tensor fields $\mathbf{M}$ and $\mathbf{H}$, we consider the map $\mathrm{id}_\mathcal{B} : (\mathcal{B}, \mathbf{M}) \to (\mathcal{B}, \mathbf{H})$.

From a categorical perspective, there is a forgetful functor from the category of Riemannian manifolds described earlier to the category of smooth manifolds that forgets the metric structures. The map $\mathrm{id}_\mathcal{B}$: $(\mathcal{B}, \mathbf{M}) \to (\mathcal{B}, \mathbf{H})$ is a morphism in the category of Riemannian manifolds which under this forgetful functor becomes the identity map on $\mathcal{B}$ in the category of smooth manifolds. This mapping maps all subsets of points to themselves but with a different geometry by replacing the metric tensor $\mathbf{M}$ by $\mathbf{H}$. This map is defined at the level of points and is smooth. The total deformation then factors as

$$(\mathcal{B}, \mathbf{M}) \xrightarrow{\mathrm{id}_\mathcal{B}} (\mathcal{B}, \mathbf{H}) \xrightarrow{\tilde{\varphi}} (\mathcal{C}, \mathbf{m}), \tag{20}$$

where $\tilde{\varphi}$ is the same map as $\varphi : (\mathcal{B}, \mathbf{M}) \to (\mathcal{C}, \mathbf{m})$ at the level of points, but with the manifold $(\mathcal{B}, \mathbf{H})$ as its domain, i.e., φ maps from the Riemannian manifold $(\mathcal{B}, \mathbf{M})$, and $\tilde{\varphi}$ maps from $(\mathcal{B}, \mathbf{H})$, but both of these morphisms become $\varphi : \mathcal{B} \to \mathcal{C}$ under the forgetful functor that forgets metric tensor fields. Additionally, by construction, the map $\mathrm{id}_\mathcal{B}$ defined above induces the same strain as $\mathbf{G}$, since $\mathbf{H}$ is the pullback of $\mathbf{h}$ under $\mathbf{G}$, as can be explicitly checked in components

$$\delta^A{}_B H_{AC} \delta^C{}_D = G^\alpha{}_B h_{\alpha\beta} G^\beta{}_D. \tag{21}$$

This last expression is written in terms of the disjoint union of the frames $\{\mathbf{e}_\alpha\}$ [Sozio and Yavari, 2020]; the interpretation of these as a moving frame depends on the smoothness of $\mathbf{G}$. Specifically, $\mathbf{G}$ can be (highly) discontinuous in which case we just have a collection of frames, one frame for each tangent space, with no general relationship between the frames at different points. As a pathological example, $\mathbf{G}$ could be square and invertible on points with rational coordinates, and non-square (but still full rank) on the other points, in which case $\{\mathbf{e}_\alpha\}$ at each point would not even have a consistent number of elements, let alone be interpretable as a moving frame. In this case, $\mathbf{h}$ would not be consistently the same size, but $\mathbf{H}$, its pull back, would be, which is why the smoothness of $\mathbf{H}$ is the important criterion, not the smoothness of $\mathbf{h}$ or $\mathbf{G}$ separately.

Note that the induced factorization $T\varphi = T\tilde{\varphi} \circ T\,\mathrm{id}_\mathcal{B}$ generates the same strain as the original multiplicative decomposition $\mathbf{F} = \mathbf{A}\mathbf{G}$. As desired, it is induced by the composition of maps between Riemannian manifolds. Therefore, taking

$$\gamma = \mathrm{id}_\mathcal{B}, \quad \alpha = \tilde{\varphi}, \tag{22}$$

satisfies both of our desired conditions (4) and (5), since

$$(\varphi, \mathbf{F}) = T\varphi = T(\alpha \circ \gamma) = T\alpha \circ T\gamma = (\alpha, \tilde{\mathbf{A}}) \circ (\gamma, \tilde{\mathbf{G}}) = (\alpha \circ \gamma, \tilde{\mathbf{A}}\tilde{\mathbf{G}}). \tag{23}$$

Here, $\tilde{\mathbf{G}}(X)$ maps a vector in $T_X(\mathcal{B}, \mathbf{M})$ to the corresponding vector in $T_X(\mathcal{B}, \mathbf{H})$, which despite appearing like the identity, generates strain because the inner product $\mathbf{H}(X)$ is typically different than the inner product $\mathbf{M}(X)$. If these inner product structures are dropped, then $\tilde{\mathbf{G}}(X)$ does become the identity on $T_X\mathcal{B}$. The other factor is therefore $\tilde{\mathbf{A}}(X) = \mathbf{F}(X)\tilde{\mathbf{G}}^{-1}(X)$, which clearly exists since $\tilde{\mathbf{G}}(X)$ is trivially invertible.

We now wish to determine the precise relationship between $\mathbf{G}(X)$ and $\tilde{\mathbf{G}}(X)$. We have already seen that they generate the same strain, but $\tilde{\mathbf{G}}(X)$ is induced by a map between Riemannian manifolds, while $\mathbf{G}(X)$ is simply a linear map applied to the tangent space of $(\mathcal{B}, \mathbf{M})$ at X. Given a basis $\{\mathbf{E}_i\}$, $i = 1, ..., n$ for $T_X(\mathcal{B}, \mathbf{M})$, there is a natural induced basis for $\mathrm{im}(\mathbf{G}(X))$ given as

$$\bar{\mathbf{e}}_i = \mathbf{G}(X)\mathbf{E}_i, \tag{24}$$

i.e., we use the images of the basis vectors of $T_X(\mathcal{B}, \mathbf{M})$ as the basis for $\mathrm{im}(\mathbf{G}(X))$. On these bases, $\mathbf{G}(X)$ has the particularly nice form

$$\mathbf{G}(X) = \delta^{\alpha}{}_{A}\bar{\mathbf{e}}_{\alpha} \otimes \mathbf{E}^{A}, \tag{25}$$

where $\{\mathbf{E}^A\}$ is the dual basis of $\{\mathbf{E}_A\}$. The linear map

$$\begin{aligned} \boldsymbol{\varepsilon}_X : \mathrm{im}(\mathbf{G}(X)) &\to T_X(\mathcal{B}, \mathbf{H}), \\ \bar{\mathbf{e}}_i &\mapsto \mathbf{E}_i, \end{aligned} \tag{26}$$

is then an isometry for every X. Note that $\boldsymbol{\varepsilon}_X$ depends on the specific decomposition, but the composition $\boldsymbol{\varepsilon}_X \circ \boldsymbol{\pi}_X \circ \mathbf{G}(X) = \tilde{\mathbf{G}}(X)$, only depends on the strain generated. Therefore, provided that $\mathbf{H}$ is smooth enough for $(\mathcal{B}, \mathbf{H})$ to be a Riemannian manifold, we can take the disjoint images of the tangent spaces of the manifold $(\mathcal{B}, \mathbf{M})$ under the maps $\mathbf{G}(X)$, and embed them *isometrically* into the tangent bundle $T(\mathcal{B}, \mathbf{H})$ via the maps $\boldsymbol{\varepsilon}_X$. These images then inherit the unique Levi-Civita connection based on the metric tensor $\mathbf{H}$. If $\boldsymbol{\varepsilon}_X$ are interpreted passively as a change of basis, the decomposition $\mathbf{F}(X) = \hat{\mathbf{A}}(X)\hat{\mathbf{G}}(X)$ can be interpreted as the same decomposition as $\mathbf{F}(X) = \tilde{\mathbf{A}}(X)\tilde{\mathbf{G}}(X) = (\hat{\mathbf{A}}(X) \circ \boldsymbol{\varepsilon}_X^{-1})(\boldsymbol{\varepsilon}_X \circ \hat{\mathbf{G}}(X))$, expressed in terms of an anholonomic basis on $(\mathcal{B}, \mathbf{H})$. Notice that postcomposition by $\boldsymbol{\varepsilon}_X \circ \boldsymbol{\pi}_X$ effectively removes any nonsmoothness or discontinuities present in the field $\mathbf{G}$. If $\hat{\mathbf{G}}$ is C^1, we can determine the object of anholonomicity for the anholonomic basis on $(\mathcal{B}, \mathbf{H})$ described above, though this is superfluous to the main result. The construction of the intermediate configuration and its associated maps is schematically shown in Figure 2.

Next, we wish to see to what extent the factorization $\varphi = \alpha \circ \gamma : (\mathcal{B}, \mathbf{M}) \xrightarrow{\gamma} (\mathcal{B}, \mathbf{H}) \xrightarrow{\alpha} (\mathcal{C}, \mathbf{m})$ is unique. Suppose we have a different intermediate configuration, $(\mathcal{M}, \mathbf{K})$ that satisfies the two requirements (4) and (5). We then write $\varphi = \alpha' \circ \gamma'$ with

$$\begin{aligned} \gamma' &: (\mathcal{B}, \mathbf{M}) \to (\mathcal{M}, \mathbf{K}), \\ \alpha' &: (\mathcal{M}, \mathbf{K}) \to (\mathcal{C}, \mathbf{m}). \end{aligned} \tag{27}$$

Denoting the tangent maps as

$$T\gamma' = (\gamma', \mathbf{G}'), \quad T\alpha' = (\alpha', \mathbf{A}'), \tag{28}$$

the anelastic strain requirement (4) demands

$$\tilde{\mathbf{G}}^{\mathsf{T}}\tilde{\mathbf{G}} = \mathbf{G}'^{\mathsf{T}}\mathbf{G}', \tag{29}$$

which in components reads

$$\delta^{A}{}_{B} H_{AC} \delta^{C}{}_{D} = G'^{A}{}_{B} K_{AC} G'^{C}{}_{D}. \tag{30}$$

This, however, is exactly the condition for $(\mathcal{B}, \mathbf{H})$ and $(\mathcal{M}, \mathbf{K})$ to be isometric under the map $\gamma' \circ \gamma^{-1}$. Hence, the intermediate configuration we constructed is unique up to isometry and it is sensible to speak of *the* intermediate configuration rather than *an* intermediate configuration that may be one of many. Taken together, we have established the main result:

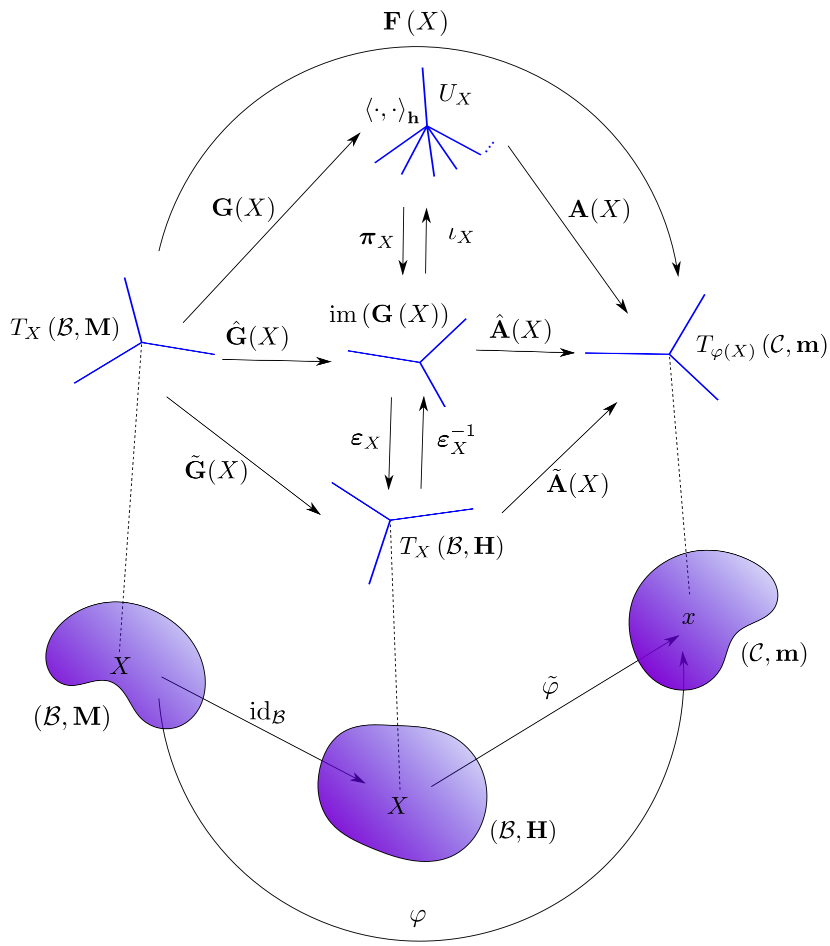

Figure 2: Construction of a global intermediate configuration. Starting from $\mathbf{F} = \mathbf{AG}$, and knowledge of the inner products on U_X, provided the pullback of $\mathbf{h}$ is C^1, this diagram can be constructed. The maps $\boldsymbol{\varepsilon}_X$ and ι_X are isometries, and all paths commute, apart from those starting at U_X.

Theorem (Isometric Integrability Theorem)**.** *Given a deformation $\varphi : (\mathcal{B},\mathbf{M}) \to (\mathcal{C},\mathbf{m})$ with the tangent map $T\varphi = (\varphi,\mathbf{F})$ satisfying $\det\mathbf{F} > 0$, and a multiplicative decomposition $\mathbf{F} = \mathbf{AG}$, if the codomain of $\mathbf{G}$ has an inner product, and the pullback of this inner product under $\mathbf{G}$, or equivalently $\mathbf{G}^\mathsf{T}\mathbf{G}$, is at least C^1, there exists a Riemannian manifold $(\mathcal{B},\mathbf{H})$, unique up to isometry, such that the composition of maps $\varphi = \alpha\circ\gamma : (\mathcal{B},\mathbf{M}) \xrightarrow{\gamma} (\mathcal{B},\mathbf{H}) \xrightarrow{\alpha} (\mathcal{C},\mathbf{m})$ induces a factorization $\mathbf{F} = \tilde{\mathbf{A}}\tilde{\mathbf{G}}$ satisfying $\mathbf{G}^\mathsf{T}\mathbf{G} = \tilde{\mathbf{G}}^\mathsf{T}\tilde{\mathbf{G}}$.*

4 Isometric Embeddings of $(\mathcal{B},\mathbf{H})$

Considered intrinsically, we are finished, since we have constructed the intermediate configuration as an abstract Riemannian manifold. However, the intrinsic approach to Riemannian geometry is notoriously difficult to visualize, hence one may want to produce an isometric embedding of this intermediate configuration in some higher dimensional Euclidean space, much like how often $(\mathcal{B},\mathbf{M})$ and $(\mathcal{C},\mathbf{m})$ are thought of as isometrically embedded submanifolds of the Euclidean space $\mathbb{E}^n$. It is natural to ask under what conditions the constructed manifold $(\mathcal{B},\mathbf{H})$ permits an isometric embedding in $\mathbb{E}^m$ as an n-dimensional submanifold for some higher dimensional Euclidean space. In this situation, a natural multiplicative decomposition of $\mathbf{F}$

would provide pointwise a sequence of maps $\mathbb{E}^n \to \mathbb{E}^m \to \mathbb{E}^n$ that not only gives the local change in geometry caused by $\mathbf{G}$, but also provides the local orientation under the embedding of $(\mathcal{B}, \mathbf{H})$ into $\mathbb{E}^m$. This would be an example of a decomposition where $\mathbf{A}$ and $\mathbf{G}$ are not square, hence the projection onto the image of $\mathbf{G}$ is nontrivial.

In general, the problem of isometrically embedding an arbitrary Riemannian manifold $(\mathcal{M}, \mathbf{K})$ in Euclidean space $\mathbb{E}^m$ is multifaceted, in that there may be topological obstructions in addition to the geometric obstructions that must be overcome. The topological problem was solved by Whitney [1944], showing that a manifold of dimension n can be embedded in Euclidean space of dimension $2n$. Additionally, this bound is sharp, as there are n-dimensional manifolds that cannot be embedded in Euclidean space of dimension $2n-1$; $\mathbb{RP}^{2^p}$ cannot be embedded in $\mathbb{E}^{2^{p+1}-1}$. Nash [1954] solved the geometric problem for C^1 isometric embeddings, and later for C^k isometric embeddings with $k \geq 3$ [Nash, 1956]. Specifically to construct these isometric embeddings, Nash assumes the existence of a smooth embedding of $(\mathcal{M}, \mathbf{K})$ in $\mathbb{E}^m$, and homogeneously scales it to obtain a short embedding, one where all distances are shorter than they need to be. He then constructs an isometric embedding by successively applying corrections, with the minimal dimension m needed depending on both the desired smoothness of the limit embedding, and the dimension of $\mathcal{M}$.

For C^k isometric embeddings, Nash's construction requires $m \geq \frac{n}{2}(3n+11)$ for compact manifolds, and $m \geq \frac{n}{2}\left(3n^2+14n+11\right)$ for noncompact manifolds. The large number of dimensions required for these constructions limits the usefulness of these embeddings, as they are difficult to visualize. However, for C^1 embeddings, Nash only requires $m \geq n+2$, which, together with Whitney's results, implies that all 2-dimensional manifolds permit C^1 isometric embeddings in $\mathbb{E}^4$. Kuiper [1955] sharpened Nash's construction, reducing the minimal number of extra dimensions to $n+1$ by altering the iteration device. This does not guarantee C^1 embeddings of all 2-dimensional manifolds in $\mathbb{E}^3$, since as in the case of the Klein bottle, topological obstructions may still be present. However, provided that a smooth embedding in $\mathbb{E}^3$ exists, an isometric embedding can be constructed.

We seek to take advantage of this construction explicitly. When $(\mathcal{B}, \mathbf{M}), (\mathcal{C}, \mathbf{m}) \subset \mathbb{E}^2$, the intermediate configuration constructed is a two-dimensional manifold, and hence, topology allowing, the Nash-Kuiper construction permits an isometric embedding in $\mathbb{E}^3$ that we can in principle visualize. We want to realize extrinsically the geometry of the intermediate configurations obtained from multiplicative decompositions of two-dimensional deformation gradients by obtaining such an embedding in $\mathbb{E}^3$, which would allow us to interpret the map γ as a map into $\mathbb{E}^3$ yielding this embedding. To do this, we construct the intrinsic geometry of the intermediate configuration as above to obtain $(\mathcal{B}, \mathbf{H})$, and if a smooth embedding of $(\mathcal{B}, \mathbf{H})$ in $\mathbb{E}^3$ exists, so does a C^1 isometric embedding. Specifically, in our case, assuming $(\mathcal{B}, \mathbf{M}) \subset \mathbb{E}^2$ we can trivially construct a short embedding of $(\mathcal{B}, \mathbf{H})$ in $\mathbb{E}^3$ by considering $(\mathcal{B}, \mathbf{M}) \hookrightarrow \mathbb{E}^2$ as a smooth embedding of $(\mathcal{B}, \mathbf{H})$. This embedding can then be appropriately scaled to become short, and then embedded into $\mathbb{E}^3$ via the inclusion $\mathbb{E}^2 \hookrightarrow \mathbb{E}^3$. Hence, for two-dimensional deformations, we can always in principle isometrically embed our constructed intermediate configuration $(\mathcal{B}, \mathbf{H})$ in $\mathbb{E}^3$, by the Nash-Kuiper algorithm.

These Nash-Kuiper embeddings are often difficult to construct in practice, and the resulting embeddings, being the limit of increasingly complicated embeddings, are typically impossible to present in a closed form. To circumvent this and for visualization purposes, we consider embeddings that can be solved by semi-inverse methods, and simply acknowledge that, in general, C^1 isometric embeddings exist, though they may be prohibitively difficult to build explicitly. Additionally, to illustrate that the initial decomposition may be particularly pathological, and yet generate a "nice" intermediate configuration, we will start on purpose with a multiplicative decomposition with highly discontinuous factors, and a correspondingly pathological set of inner products $\mathbf{h}$, that generates a C^1 metric tensor field $\mathbf{H}$. The pathology we introduce here is admittedly contrived; it serves to highlight the fact that the smoothness of $\mathbf{H}$ determines when an intermediate configuration can be constructed, rather than the regularity of other quantities appearing in our analysis thus far.

4.1 A Radially-Symmetric Example

Consider a radially-symmetric deformation given as

$$r = f(R), \qquad \theta = \Theta, \tag{31}$$

where r and R are the radial coordinates in the current and reference configurations, respectively, and θ and Θ are the respective angular coordinates. We will prescribe a multiplicative decomposition of the associated deformation gradient $\mathbf{F}$ that places the unknown function $f(R)$ and its derivatives entirely in the elastic factor $\mathbf{A}$. This ensures that $\mathbf{G}$ is specified, and upon the imposition of a collection of inner products $\mathbf{h}$, fully specifies the data necessary to define the intermediate configuration. We will then impose zero boundary traction, specify a strain-energy density function, and compute the residual stress. The factorization of this map through an isometric embedding of the intermediate configuration yields the interpretation of this factorization as the reference configuration anelastically evolving into the curved surface obtained from this embedding, and this surface being subsequently smashed flat elastically.

Geometry. The deformation gradient generated by this map has components

$$[F^a{}_A] = \begin{bmatrix} \frac{df}{dR} & 0 \\ 0 & 1 \end{bmatrix}. \tag{32}$$

Consider the factorization

$$\begin{bmatrix} \frac{df}{dR} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{df}{dR}\left(1 + I_Q\left(R\right)\right)^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} 1 + I_Q\left(R\right) & 0 \\ 0 & R \end{bmatrix}, \tag{33}$$

where I_Q is the indicator function of the rationals:

$$I_Q\left(R\right) = \begin{cases} 1 & \text{if } R \in \mathbb{Q} \\ 0 & \text{if } R \notin \mathbb{Q} \end{cases}. \tag{34}$$

Additionally, we impose the inner products

$$\left[h_{\alpha\beta}\right] = \begin{bmatrix} \left(1 + I_Q\left(R\right)\right)^{-2} & 0 \\ 0 & 1 + a\left(R^2 - R\right) \end{bmatrix}, \tag{35}$$

with $0 < a < 4$, a positive constant. Notice in particular, that each factor of this decomposition is nowhere continuous, and that the collection of inner products is also nowhere continuous. This pathology is academic in nature, since despite $\mathbf{G}$ and $\mathbf{A}$ being nowhere continuous, $\mathbf{F}$ is smooth, and by construction the discontinuities in $\mathbf{G}$ are canceled by discontinuities in $\mathbf{h}$. We do this to obtain the smooth metric tensor field $\mathbf{H}$ to show that our construction is well defined so long as $\mathbf{H}$ is smooth, even when $\mathbf{G}$, $\mathbf{A}$, and $\mathbf{h}$ are not, though in practice, all of these quantities will likely be smooth everywhere. Demonstrating this, we compute $\mathbf{H}$, which has components

$$\left[H_{AB} = G^\alpha{}_A G^\beta{}_B h_{\alpha\beta}\right] = \begin{bmatrix} 1 & 0 \\ 0 & R^2\left(1 + a\left(R^2 - R\right)\right) \end{bmatrix}, \tag{36}$$

and is continuous, and positive definite. Geometrically, the map $\mathrm{id}_{\mathcal{B}} : (\mathcal{B}, \mathbf{M}) \to (\mathcal{B}, \mathbf{H})$ generates angular contraction within the unit disk and angular stretching outside the unit disk, while keeping radial distances preserved. In terms of principal stretches, we have

$$\lambda_R = 1, \qquad \lambda_\Theta = 1 + a\left(R^2 - R\right), \tag{37}$$

the latter of which takes its minimum value of $1 - \frac{a}{4}$ at $R = \frac{1}{2}$.

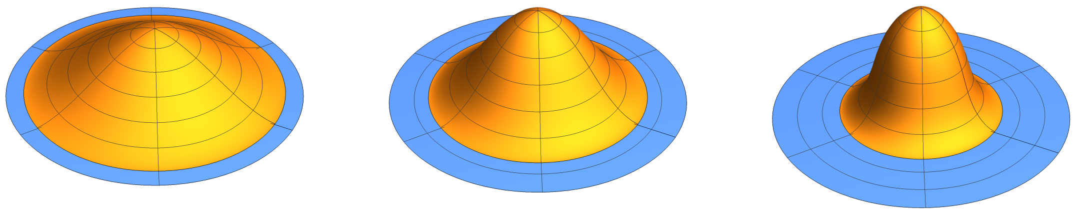

Figure 3: Selecting values for a, the blue disk $R \le R^*(a)$ gets mapped to the orange intermediate configuration. In the region $R > R^*(a)$, the symmetric ansatz breaks down, and some kind of wrinkling is necessary to accommodate the extra length in the angular direction.

Embedding. The symmetry present in $\mathbf{H}$ suggests an isometric embedding in $\mathbb{E}^3$ may take the form

$$\mathbf{x} = r(R)\mathbf{e}_r(\Theta) + z(R)\mathbf{e}_z. \tag{38}$$

As we will soon see, the solvability of this ansatz depends on the value of a, and the domain chosen; we will consider the disk shaped domains $R \le R^*$ for some R^* yet to be determined (see Figure 3). Computing the metric induced by this embedding, and demanding it be equal to $\mathbf{H}$, one obtains

$$r = R\sqrt{1 + a\left(R^2 - R\right)}, \qquad z'(R)^2 = 1 - \frac{\left[2 + aR\left(4R - 3\right)\right]^2}{4 + 4aR\left(R - 1\right)}. \tag{39}$$

Notice that the denominator appearing in the equation for $z'(R)$ is identically positive for $0 < a < 4$. Hence there is no singularity within that range. Additionally, $z(R)$ must be a real function, so it is clear that when the right-hand side of (39) becomes negative, our ansatz ceases to be valid. Collecting the right-hand side of (39) yields the rational expression

$$z'(R)^2 = \frac{aR(-16aR^3 + 24aR^2 - 9aR - 12R + 8)}{4\left[1 + a(R^2 - R)\right]}. \tag{40}$$

Clearly, the right-hand side vanishes at $R = 0$, and the remaining cubic factor is positive at $R = 0$. Because the denominator is identically positive, this means that $z'(R)$ is real valued in some finite sized disk. Additionally, the discriminant in R of the cubic factor is $-6912(a-4)^2 a$, which being negative, indicates that the cubic factor only has one real zero, i.e., our ansatz is only solvable in some finite disk rather than in a finite disk plus some larger annular region. We compute this real root R^* and obtain

$$R^*\left(a\right) = \frac{1}{4}\left[2 + \frac{4-a}{\left[(a-4)(a^2 + 2a^{3/2})\right]^{1/3}} - \frac{\left[(a-4)(a^2+2a^{3/2})\right]^{1/3}}{a}\right]. \tag{41}$$

Hence, we shall, for any particular value of a, restrict our attention to embeddings of the disk $R \le R^*\left(a\right)$. Finally, we choose a sign for $z'(R)$ and integrate it by quadrature to obtain the embeddings

$$\mathbf{x}(R,\Theta) = R\sqrt{1 + a\left(R^2 - R\right)}\mathbf{e}_r(\Theta) \pm \int_R^{R^*} \sqrt{\frac{a\rho(-16a\rho^3 + 24a\rho^2 - 9a\rho - 12\rho + 8)}{4\left[1 + a(\rho^2 - \rho)\right]}}\, d\rho\, \mathbf{e}_z, \tag{42}$$

with the choice of sign indicating a reflection across the plane $z = 0$, and the bounds on the integral chosen so that the boundary of the embedded disk lies in the plane $z = 0$. Taking the gradient of this embedding yields the nonsquare tensor $\bar{\mathbf{G}}$

$$\begin{aligned}\bar{\mathbf{G}} = &\left(\frac{2 + aR(4R-3)}{2\sqrt{1 + aR(R-1)}}\mathbf{e}_r \pm \sqrt{\frac{aR(-16aR^3 + 24aR^2 - 9aR - 12R + 8)}{4\left[1 + a(R^2 - R)\right]}}\mathbf{e}_z\right) \otimes \mathbf{E}_R \\ &+ \sqrt{1 + a\left(R^2 - R\right)}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta,\end{aligned} \tag{43}$$

which, in components, reads

$$\left[\bar{G}^{\alpha}{}_{A}\right]=\begin{bmatrix}r'(R) & 0\\ 0 & 1\\ z'(R) & 0\end{bmatrix}=\begin{bmatrix}\frac{2+aR(4R-3)}{2\sqrt{1+aR(R-1)}} & 0\\ 0 & 1\\ \pm\sqrt{\frac{aR(-16aR^3+24aR^2-9aR-12R+8)}{4[1+a(R^2-R)]}} & 0\end{bmatrix}. \tag{44}$$

We demand the corresponding factor $\bar{A}^{a}{}_{\alpha}$ be valid in that

$$F^{a}{}_{A}=\bar{A}^{a}{}_{\alpha}\bar{G}^{\alpha}{}_{A}, \tag{45}$$

and that it also projects vector fields onto the tangent plane of the surface, i.e.,

$$\bar{\mathbf{A}}\mathbf{n}=\mathbf{0}, \tag{46}$$

where $\mathbf{n}=r'(R)\mathbf{e}_z-z'(R)\mathbf{e}_r$ is the normal vector to the embedding. We note that because $\bar{\mathbf{G}}$ only maps into the tangent spaces of our surface, this requirement is not strictly necessary, because only the action of $\bar{\mathbf{A}}$ on the range spaces of $\bar{\mathbf{G}}$ matters. We add this requirement to uniquely determine $\bar{\mathbf{A}}$, with the acknowledgement that another choice could be made for $\bar{\mathbf{A}}$ that still yields $\mathbf{F}$ upon multiplication by $\bar{\mathbf{G}}$. This yields

$$\begin{aligned}\left[\bar{A}^{a}{}_{\alpha}\right]&=\begin{bmatrix}f'(R)r'(R) & 0 & -f'(R)z'(R)\\ 0 & 1 & 0\end{bmatrix}\\ &=\begin{bmatrix}f'(R)\frac{2+aR(4R-3)}{2\sqrt{1+aR(R-1)}} & 0 & \mp f'(R)\sqrt{\frac{aR(-16aR^3+24aR^2-9aR-12R+8)}{4[1+a(R^2-R)]}}\\ 0 & 1 & 0\end{bmatrix},\end{aligned} \tag{47}$$

where the new inner product $\bar{h}_{\alpha\beta}$ is the standard Euclidean inner product on $\mathbb{E}^3$, which in cylindrical coordinates reads

$$\left[\bar{h}_{\alpha\beta}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & r^2=R^2\left(1+a\left(R^2-R\right)\right) & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{48}$$

Residual Stress. The total deformation in general depends on the choice of strain energy density, and there will be residual stress when boundary tractions are removed, since the nonzero Gaussian curvature present in the intermediate configuration indicates that additional strain is necessary to embed it in the plane. The Cauchy stress $\boldsymbol{\sigma}$ for a two-dimensional isotropic hyperelastic solid with strain-energy density function $W\left(I_1,I_2\right)$ takes the form

$$\boldsymbol{\sigma}=\frac{2}{\sqrt{I_2}}\left(\frac{\partial W}{\partial I_1}\mathbf{B}+I_2\frac{\partial W}{\partial I_2}\mathbf{I}\right), \tag{49}$$

where $I_1=\mathrm{tr}(\mathbf{B})$, $I_2=\det(\mathbf{B})$, and $\mathbf{B}$ is the left elastic Cauchy-Green tensor, which has components

$$B^{a}{}_{b}=F^{a}{}_{A}H^{AB}F^{d}{}_{B}m_{db}=A^{a}{}_{\alpha}h^{\alpha\beta}A^{d}{}_{\beta}m_{db}=\bar{A}^{a}{}_{\alpha}\bar{h}^{\alpha\beta}\bar{A}^{d}{}_{\beta}m_{db}. \tag{50}$$

In this case, we have

$$\left[B^{a}{}_{b}\right]=\begin{bmatrix}f'(R)^2 & 0\\ 0 & \frac{f(R)^2}{R^2(1+a(R^2-R))}\end{bmatrix}, \tag{51}$$

hence, $I_1=f'(R)^2+f(R)^2/(R^2\left(1+a\left(R^2-R\right)\right))$ and $I_2=f'(R)^2f(R)^2/(R^2\left(1+a\left(R^2-R\right)\right))$.

For incompressible materials, the Cauchy stress has the following representation:

$$\boldsymbol{\sigma}=2\frac{\partial W}{\partial I_1}\mathbf{B}-p\mathbf{I}, \tag{52}$$

where p is the Lagrange multiplier associated with incompressibility.

For an incompressible two-dimensional neo-Hookean solid $W = \frac{\mu}{2}(I_1 - 2)$ and we find $f'(R)f(R) = R\sqrt{1 + a(R^2 - R)}$. This equation can be integrated to obtain

$$f(R) = \sqrt{2\int_0^R \rho\sqrt{1 + a(\rho^2 - \rho)}d\rho}. \tag{53}$$

Having obtained the deformation mapping, we only have to solve for the residual stress. We insert the deformation map into the equilibrium equations $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$, and obtain the following expression for the radial derivative of p, the Lagrange multiplier field corresponding to the incompressibility constraint:

$$\frac{dp}{dR} = \mu f'(R)\left[\frac{f(R)}{R^2(1 + a(R^2 - R))} - \frac{f'(R)^2}{f(R)} - 2f''(R)\right], \tag{54}$$

which can be integrated by quadrature and coupled with the zero boundary traction condition to obtain the full residual stress field, depicted in Figure 4.

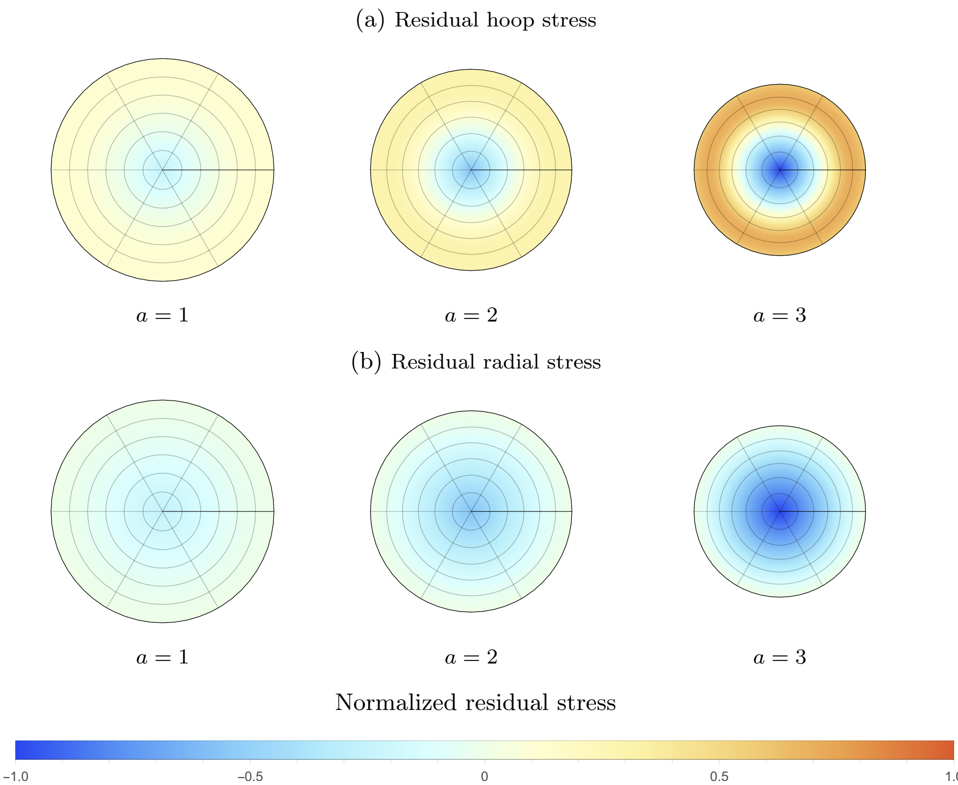

Figure 4: Notice that the hoop stress and radial stress become equal at $R = 0$, indicating that the residual stress at the center of the disk is a pure pressure, as would be expected by the symmetry of the problem.

In general to obtain the residual stress field, one needs to solve the traction-free boundary-value problem $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$, noting that the divergence operator depends on the local geometry of the ambient space of the current configuration, and the Cauchy stress involves the elastic stretch generated by embedding the intermediate configuration into this ambient space.

In summary, we started with a pathological multiplicative decomposition of a two-dimensional deformation, with a fully specified anelastic factor, and obtained an intrinsic description of the geometry of the intermediate configuration. We then visualized this configuration by isometrically embedding it in $\mathbb{E}^3$. From this embedding, we computed a different multiplicative decomposition with nonsquare factors that gives not only the local strain, but also the local orientation of the embedded tangent spaces. More specifically, since we have an embedding, we can identify each tangent plane as an affine subspace of $\mathbb{E}^3$, and in particular, we know how this plane is positioned and oriented as a subspace. This is equivalent to knowing the normal vector to the embedded surface at each point. We then assumed an incompressible neo-Hookean material and computed the residual stress in the current configuration by solving the equilibrium equation $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ with vanishing boundary tractions. This construction can be interpreted as "smashing" the isometric embedding of the intermediate configuration into a flat plane, hence creating stresses in the process.

5 Concluding Remarks

In this paper, we presented a sufficient condition for global intermediate configurations to be constructed together with an explicit construction to build it. We demonstrated that when this sufficient condition is satisfied for decompositions of two-dimensional deformations, the resulting intermediate configurations can in principle be isometrically embedded in $\mathbb{E}^3$, and provided an example. Additionally, we have shown that these intermediate configurations are unique up to isometry, which may be interpreted passively as a change of coordinates.

Here, we have made no assumptions on the nature of any particular source of anelasticity, as our discussion was purely geometric in nature, and therefore is generally applicable to any particular anelastic process. Care should be taken however, since we began with some notion of anelastic strain as the primitive quantity, that if a different measure is taken to be primitive, such as residual stress, then the intermediate configuration is only uniquely determined to the same extent that the anelastic strain is uniquely determined.

Alternatively, our construction could be repeated using elastic strain as the primitive strain measure rather than the anelastic strain. The construction then proceeds as before, *mutatis mutandis*, and one obtains an intermediate configuration as we have done. While this similar construction may appear to be of no additional use, we note that it may be the case that $\mathbf{H}$ is not C^1, while the analogous quantity obtained by using the elastic strain is C^1, hence gaining on regularity. For example, in the case of cavitation, there is a subdomain on which $\mathbf{F}$ is singular. Suppose that under the factorization $\mathbf{F} = \mathbf{AG}$, where $\mathbf{A}$ and $\mathbf{G}$ are square, these singularities exist only in $\mathbf{G}$ and not in $\mathbf{A}$. Provided that the collection of inner products $\mathbf{h}$ is well behaved, their pull back under $\mathbf{G}$ will not be C^1, but their pull back under $\mathbf{A}^{-1}$ will be. This approach allows one to construct an intermediate configuration based on the topology of $\mathcal{C}$ rather than $\mathcal{B}$, letting one distinguish between elastic and anelastic cavitation. This example, in addition to the fact that in some experiments elastic strain may be easier to measure, demonstrates that this alternative construction can be useful, despite being nearly identical in form, since in certain cases it confers advantages over the construction utilizing $\mathbf{G}$. This is done at the cost of having to (generalized-)invert $\mathbf{A}$, which must be done with care when $\mathbf{A}$ is not square since the generalized inverse $\mathbf{A}_g^{-1}$ must map entirely into $\operatorname{im}(\mathbf{G})$ to accurately capture the elastic strain. When $\det \mathbf{F} > 0$, using the right inverse $\mathbf{A}_g^{-1} = (\mathbf{A})_R^{-1} = \mathbf{GF}^{-1}$ ensures this.

Since the intermediate configurations constructed here are typically not Euclidean, they are not generally preserved under a nontrivial action of the special Euclidean group. Hence, in the development of physical theories involving this configuration, the requirement of equivariance under nontrivial actions of the special Euclidean group is inappropriate, and should be replaced instead with equivariance under actions of the symmetry group of the intermediate configuration, which may only be the trivial group. We further comment on this notion of equivariance. Suppose we have a map $f : A \to B$, where A and B are G-spaces for some topological group G (G-spaces are those possessing a group action, i.e., a map $G \times A \to A$ that agrees with the group structure). The map f is a G-equivariant map if it "commutes" with the group action, i.e., choosing an element $g \in G$, and denoting the group action on A by $\star_A$ and the group action on B by $\star_B$, we have

$$g \star_B f(x) = f(g \star_A x), \quad \forall x \in A.$$

As an example, in Euclidean space the Cauchy Stress is an $\mathrm{SE}(n)$-equivariant function of the deformation φ

via the relation

$$\boldsymbol{\sigma}((\mathbf{Q}|c)\star_A\varphi)=\boldsymbol{\sigma}(\mathbf{Q}\varphi+c)=\mathbf{Q}\boldsymbol{\sigma}(\varphi)\mathbf{Q}^{\mathsf{T}}=(\mathbf{Q}|c)\star_B(\boldsymbol{\sigma}(\varphi)),\quad\forall(\mathbf{Q}|c)\in\mathrm{SE}(n).$$

Notice how the action on $\varphi\in\mathrm{Hom}\left((\mathcal{B},\mathbf{M}),\mathbb{E}^n\right)$ is different than the action on $\boldsymbol{\sigma}\in T\mathbb{E}^n\otimes T^*\mathbb{E}^n$ (alternatively $\boldsymbol{\sigma}\in T\mathbb{E}^n\otimes T\mathbb{E}^n$, $\boldsymbol{\sigma}\in T^*\mathbb{E}^n\otimes T^*\mathbb{E}^n$, or $\boldsymbol{\sigma}\in\Omega^1(\mathbb{E}^n)\otimes\Omega^{n-1}(\mathbb{E}^n)$ via application of musical and Hodge star isomorphisms; see Kanso et al. [2007] for a discussion of these representations.) Both of these actions are ultimately induced by the defining action of $\mathrm{SE}(n)$ on Euclidean space. The lack of such a global Euclidean structure in the intermediate configuration is why the imposition of "invariance" under superposed rigid body motions on the intermediate configuration in Casey and Naghdi [1980] is inappropriate. The proper "invariance" requirements can be obtained by taking the action of the intermediate configuration's isometry group (which may be trivial) and prolonging it to actions on the configuration's tangent bundle, cotangent bundle, and the various vector bundles obtained by taking the tensor product bundles of these bundles. The proper "invariance" requirements then amount to requiring that constitutive laws be group equivariant maps with respect to these induced actions. Other examples of equivariance requirements are material symmetry, where the group in question is the material symmetry group at each point, and all invariance requirements are special cases of equivariance requirements where $\star_B$ is the trivial action $g\star_B x=x$, $\forall g\in G$, $\forall x\in B$. See Husemöller [1994] for a detailed treatment of G-spaces and group actions.

Finally, here we have only considered the regular case of C^1 embeddings that prohibits explicitly singularities in the anelastic deformation tensor. In particular, if the metric tensor $\mathbf{H}$ either becomes singular or loses positive definiteness on some subset, the topologies of $(\mathcal{B},\mathbf{M})$ and $(\mathcal{B},\mathbf{H})$ no longer necessarily agree. The singular case is particularly interesting as it can be used to treat a number of highly relevant problems in mechanics such as cavitation, accretive growth, point defects, and fracture. All these effects can, in principle, be modeled by using a singular multiplicative decomposition, though the construction of the appropriate intermediate configuration is more involved and falls outside the scope of the analysis presented here.

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