Finite Deformation: Special Cases

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Outline

Topics

- Definintions
- Considère condition
- Helmholtz free energy
- Stability of equilibrium
- Time-dependent, inhomogeneous deformation
- Virtual work formulation

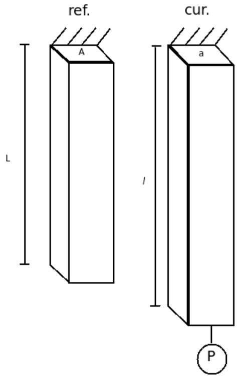
Examples

- Truss example
- Necking example
- Coexistent phases example
- Critical force and Gibbs free energy example
- Gibbs free energy of spherical balloon example
- Wave in pre-stressed bar example

Overview

- For small deformations, we applied F=ma in the reference frame
- For large deformations, this is no longer a valid approximation
- If we upgrade our definitions to handle more situations...
- ...formulate new material models using the free energy density...
- ...and link it to the finite element method...
- ...we can explore all kinds of new phenomena!

Strains



- Stretch: $\lambda = \frac{l}{L}$
- Engineering: $e = \frac{l-L}{L} = \lambda - 1$
- Natural: $\varepsilon = log(\frac{l}{L}) = log(\lambda)$
- Lagrange: $\eta = \frac{1}{2}\left[\left(\frac{l}{L}\right)^2 - 1\right] = \frac{1}{2}\left(\lambda^2 - 1\right)$

NOTE: These are all functions of the stretch (λ)

Stresses

- True/Cauchy: $\sigma = \frac{P}{a}$
- Nominal/1st Piola Kirchoff: $s = \frac{P}{A}$

NOTE: Others can be defined as well

Increments and Work-Conjugates

Increments

Taking the derivates of our strain definitions w.r.t. λ gives us

- $\delta\lambda = \frac{\delta l}{L}$
- $\delta e = \delta\lambda$
- $\delta\varepsilon = \frac{\delta\lambda}{\lambda}$
- $\delta\eta = \lambda\delta\lambda$

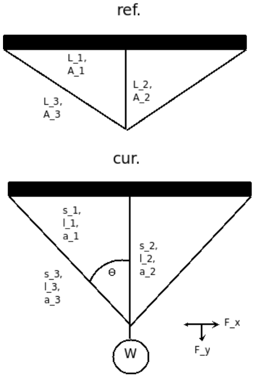
Work

- An increment of work is given by $P\delta l$

Work-conjugates

- Find pairs that give $\frac{\text{"incr. of work in cur."}}{\text{"volume in ref."}} = \frac{P\delta l}{AL}$
- With nominal stress, $\frac{P\delta l}{AL} = s\delta\lambda = s\delta e$
- With true stress, $\frac{P\delta l}{aL} = \sigma\delta\varepsilon$
- What about Lagrange? $\frac{P\delta l}{AL} = \delta\eta(?) = \lambda\delta\lambda(?)$
- $(?) = S = \frac{s}{\lambda}$, the 2nd Piola-Kirchoff Stress!

Truss Example



Deformation geometry

$\lambda_1 = \frac{l_1}{L_1} = 1,\ \lambda_2 = \frac{l_2}{L_2},\ \lambda_3 = \frac{l_3}{L_3} = \frac{\sqrt{l_1^2 + l_2^2}}{\sqrt{L_1^2 + L_2^2}}$

Note: only unknown here is l_2

Material Model: Neo-Hookean

$s_i = \mu(\lambda_i - \lambda_i^{-2}),\ i = 1,2,3,\ s_1 = 0$

Note: nominal stress form

Force Balance

F_x : By symmetry, this is 0

F_y : $W - 2(cos(\theta) s_3 a_3) - s_2 a_2 = 0$

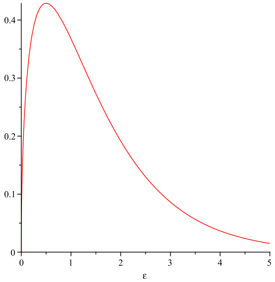
Incompressibility

$AL = al \Rightarrow a = \frac{AL}{l}$

Mixing results in...

$W - 2\left(\frac{l_2}{l_3} s_3 \frac{A_3 L_3}{l_3}\right) - s_2 \frac{A_2 L_2}{l_2} = 0$, which is 1 eqn. for 1 unk.

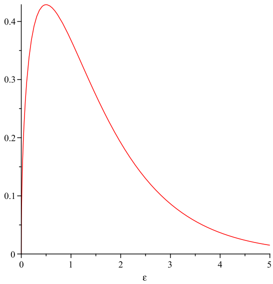
Necking 1



x-axis: ε, y-axis: P/AK

- What is the force/strain relation for a power law material, such as aluminum?
- One possible formulation using natural strain / true stress is:
  - $\varepsilon = log(\lambda)$, $P = \sigma a$
- A power law material model is used, which fits the experimental $\varepsilon \rightarrow \sigma$ curves well.
  - $\sigma = K\varepsilon^N$, $N = \frac{1}{2}$ for our example
- Incompressibility: $AL = al \Rightarrow a = Ae^{-\varepsilon}$
- Mixing these gives $\sigma = \frac{P}{a} = \frac{P}{Ae^{-\varepsilon}} = K\varepsilon^N$

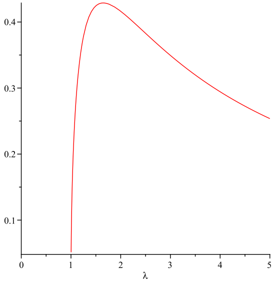
Necking 2



x-axis: ε, y-axis: P/AK

- $\frac{dP}{d\varepsilon} = AK(e^{-\varepsilon}N\varepsilon^{N-1} - \varepsilon^{N} - e^{-\varepsilon} = a(\sigma' - \sigma) = 0$ at the maximum
- a cannot be zero, and thus $\sigma' = \sigma$ (the Considère condition)
- Applying our material model, this gives $\varepsilon = N$
- Note that $\sigma'$is the tangent modulus
- Material hardening / geometric softening
- Model is invalid after this condition, as this is only good for homogenous deformation

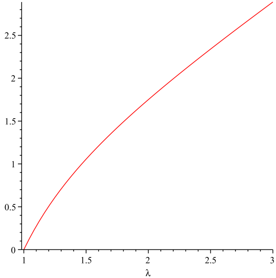
Necking 3



x-axis: λ, y-axis:P/AK

- We can also formulate this using s, λ
- $\lambda = \frac{l}{L}$, $P = sA$
- Using $\varepsilon = log(\lambda)$ and $AL = al$,the power law material model becomes
  - $s = K\frac{log(\lambda)^N}{\lambda}$
- Mixing these gives $P = KA\frac{log(\lambda)^N}{\lambda} = As(\lambda)$
- $\frac{dP}{d\lambda} = As' \Rightarrow s' = 0$ (Considère condition)
- Taking this derivative from the material model gives $N = log(\lambda) \Leftrightarrow \lambda = e^N$

Necking 4



x-axis: λ, y-axis: $P/A\mu$

- What about for Neo-Hookean materials?
- Formulate this using s, λ
- $\lambda = \frac{l}{L}$, $P = sA$
- Material model becomes $s = \mu(\lambda - \lambda^{-2})$
- $P = \mu A(\lambda - \lambda^{-2})$
- $\frac{dP}{d\lambda} = \mu A\left(1 + \frac{2}{\lambda^3}\right)$, $\left(1 + \frac{2}{\lambda^3}\right) = 0$ (Considère condition)
- This is satisfied when $\lambda = \sqrt[3]{-2}$, which is not valid.
- Conclusion: no necking in Neo-Hookean materials.

Helmholtz free energy 1

- $\theta$is temp., l is length
- Given $F(l, \theta)$,$\delta F = \frac{\partial F}{\partial l}\delta l + \frac{\partial F}{\partial \theta}\delta\theta = P\delta l + \eta\delta\theta$
- Assuming that temperate change is negligible, i.e. adiobatic, and thus consider $\delta F = P\delta l$
- Recall $P = sA$ and $l = \lambda L \Rightarrow \frac{\delta F}{AL} = \frac{P\delta l}{AL}$
- $\delta W = s\delta\lambda \Rightarrow s(\lambda) = \frac{dW}{d\lambda}$, where W is the energy density

Helmholtz free energy 2

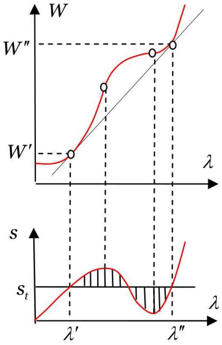
- Using our material models, we can write them in terms of energy density
- For neo-Hookean, $s(\lambda) = \mu(\lambda - \lambda^{-2}) = \frac{dW}{d\lambda}$
- Integrating gives $W(\lambda) = \frac{\mu}{2}\left(\lambda^2 + 2\lambda^{-1} - c\right), W(1) = 0 \Rightarrow c = 3$
- For power law, $s = K\frac{log(\lambda)^N}{\lambda} = \frac{dW}{d\lambda}$
- Integrating gives $W(\lambda) = \frac{K}{N+1} log(\lambda)^{N+1}$

Stability of equilibrium

- If force is static, then object may equilibrate. If so, what is λ in this state? Is it stable?
- Gibbs free energy = potential energy=Helmholtz free energy - work done by external force
- $G = ALW(\lambda) - PL \cdot (\lambda - 1) = F(\lambda) - P \cdot (l - L)$
- Using thermodynamics, we can show that the equilibrium state is reached when G is minimized.
- Evaluating G at $\lambda + \delta\lambda$ by expanding a Taylor series about λ $(x_0 = \lambda, x - x_0 = \delta\lambda)$
- $G(\lambda + \delta\lambda) = G(\lambda) + G'(\lambda)(\delta\lambda) + \frac{G''(\lambda)}{2}(\delta\lambda)^2 + \ldots$
- Setting $G'(\lambda) = ALW' - PL = 0 \Rightarrow W' = \frac{P}{A} = s$ (stress is recovered)
- This is a stable equillibrium if $G''(\lambda) > 0$ i.e. if $W'' > 0$

NOTE: Recall from necking that $W'' = s' = 0$ where maximal force is achieved.

Coexistent phases example

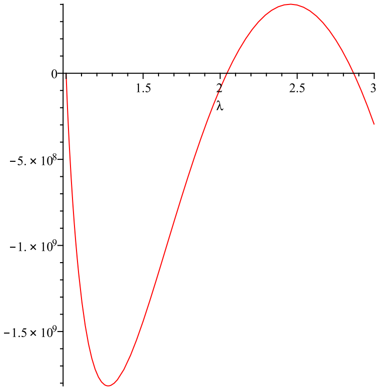


- Mixed phases → non-convex energy density → material model not one to one.
- $L' + L'' = L$, $l' + l'' = l$, $\lambda' L' + \lambda'' L'' = l$
- Problem: For some nonconvex $W(\lambda)$,find the unstable stress regime.
- $s_t = \frac{W''-W'}{\lambda''-\lambda'} = \frac{dW}{d\lambda}$ is the tangent line that passes through the points λ' and λ''
- Maxwell's rule:
  - $\frac{dW}{d\lambda} = s_t$at λ' and λ''
  - $\int_{\lambda'}^{\lambda''} s d\lambda = 0$
  - The area under the curve is equal
- $F = W(\lambda')AL' + W(\lambda'')AL''$
- Summary: In equilibrium the phases separate, but if metastable, they coexist.

Gibbs example 1

- Problem: For a power law material, determine $P_c$and plot $G(\lambda)$ for value around it.
- Power law in terms of nominal stress: $s = \frac{K}{\lambda} log(\lambda)^N$
- Considère condition: $s' = 0 \Rightarrow \frac{ds}{d\lambda} = \frac{K(log(\lambda)^{N-1}N - log(\lambda)^N}{\lambda^2} = 0$
- $\lambda = 1$ or $\lambda = e^N$
- $P = kKA\frac{log(\lambda)^N}{\lambda} = sA \Rightarrow P_c = \frac{KAN^N}{e^N}$

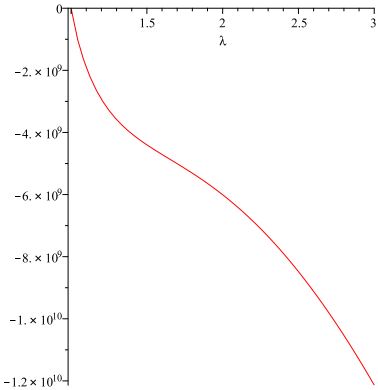
Gibbs example 2



x-axis: λ, y-axis: $G(\lambda)$, $P_c < P$

- Let's examine the behaviour around $P_C$of $G(\lambda)$
- Recall $W(\lambda) = \frac{K}{N+1} log(\lambda)^{N+1}$
- $G(\lambda) = ALW(\lambda) - PL \cdot (\lambda - 1)$
- For the above G(lambda), the force is less than the critical force, and for small lambda near 1, the helmholtz free energy is less than the work being done.

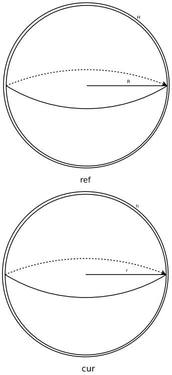
Gibbs example 3



x-axis: λ, y-axis: $G(\lambda)$, $P_c > P$

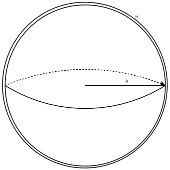
- $G(\lambda) = ALW(\lambda) - PL \cdot (\lambda - 1)$
- For Pmore, the work being done is always greater than the Helmholz free energy.

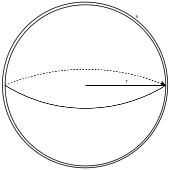
Spherical balloon example 1



- Problem: Discuss Gibbs free energy of balloon
- $G = 4\pi r^2 HW - p\frac{4}{3}\pi r^3$
- Using three ingredients:
  - Def Geom: $\lambda_1 = \lambda_2 = \frac{2\pi r}{2\pi R} = \frac{r}{R}$, $\lambda_3 = \frac{h}{H}$
  - Incompressibility: $4\pi R^2 H = 4\pi r^2 h \Rightarrow \lambda_3 = \lambda_1^{-2}$
  - Force balance: $\sigma_3 \approx 0$, $\sigma_1 = \sigma_2 = \frac{Pr}{2h}$, (biaxial state)
    - Consider a half sphere: $2\pi rh\sigma_1 = \pi r^2 p$
    - $\sigma = (\sigma_1, \sigma_2, 0)$ and add hydrostatic pressure to get $(0, 0, -\sigma_1)$
  - Material model (true stress): $-\sigma_1 = \mu \cdot (\lambda^2 - \lambda^{-1})$

Spherical balloon example 2



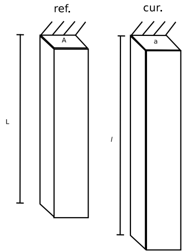


- Result: $p = \frac{-2H}{R}(\lambda_1^{-7} - \lambda_1^{-1})$
- We can find critical pressure from here, and plot $G(\lambda)$as before.

Time-dependent, inhomogeneous deformation

- Now we move to a more complicated model and consider time and space variation
- 4 ingredients:
  - Def Geom: $\lambda(X,t) = \frac{x(X+dX,t)-x(X,t)}{dX} = \frac{\partial x(X,t)}{dX}$
  - Cons. of Mass: $\rho(X)$ only.
  - Cons. of Momentum (Force Balance):
    $\frac{\partial}{\partial X}(s(X,t)A(X))dX + B(X,t)A(X)dX = \rho(X)\frac{\partial^2 x(X,t)}{\partial t}A(X)dX$
  - Mat. Model: $s = g(\lambda)$

Wave in pre-stressed bar example



Using the ingredients:

- Def. Geom: $x(X,t) = \lambda_0 X + u(X,t)$, $\lambda = \frac{\partial x}{\partial X} = \lambda_0 + \frac{\partial u}{\partial X}$
- Mat. Model: Expand $s = g(\lambda)$ using TS to find:
  - $s \approx g(\lambda_0) + g'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2} g''(\lambda_0)(\lambda - \lambda_0)^2 + \ldots$
- Cons. of Momentum: $\frac{\partial}{\partial X} s(X,t) = \rho \frac{\partial^2 x(X,t)}{\partial t^2}$

By keeping up to linear terms from TS, we can mix ingredients to find:

- $\frac{\partial^2 u}{\partial t^2} = \frac{g'(\lambda_0)}{\rho} \frac{\partial^2 u}{\partial t^2}$
- $c = \sqrt{\frac{g'(\lambda_0)}{\rho}}$

Virtual work formulation

- Define virtual displacement: $\delta x = \delta x(X)$
- Virtual stretch: $\delta\lambda = \frac{\partial(\delta\lambda)}{\partial X}$
- Using conservation of momentum, we can write an expression for virtual work:
  - $As\delta x|_{X_1}^{X_2} + \int_{X_1}^{X_2} \frac{\partial}{\partial X}(sA)\delta x dX$
  - Integrating by parts gives us: $\int_{X_1}^{X_2} As\frac{\partial}{\partial X}(\delta x)dX$
- For an arbitrary segment, $s\delta\lambda = s\frac{\partial}{\partial X}(\delta x)$ is the virtual work done by all forces on the segment.
- This is the basis for the finite element method.

Research

- Biomaterials are complicated, inhomogenous, anisotropic materials
- Tissues, in particular, require consideration of special material models
- Tissues also grow and this addition of mass and volume change must be considered
- My ongoing work will consider various models for growth
- Applications to limb, root, and cell, and embryo growth.

The end

- Thank you all for a great semester.