On the Direct and Reverse Multiplicative Decompositions of Deformation Gradient in Nonlinear Anisotropic Anelasticity*

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Abstract

In this paper we discuss nonlinear anisotropic anelasticity formulated based on the two multiplicative decompositions $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ and $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$. Using the Bilby-Kröner-Lee decomposition $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ one can define a Riemannian material manifold (the natural configuration of an anelastic body) whose metric explicitly depends on the anelastic deformation $\overset{a}{\mathbf{F}}$. We call this the *global material intermediate configuration*. Deformation is a map from this Riemannian manifold to the flat ambient space. Using the reverse decomposition $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$, the reference configuration is a (flat) submanifold of the Euclidean ambient space, while the global intermediate configuration is a Riemannian manifold whose metric explicitly depends on the elastic deformation $\overset{e}{\mathbb{F}}$. We call this the *global spatial intermediate configuration*. We show that the direct $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ and reverse $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$ decompositions correspond to the same anelastic motion if and only if $\overset{e}{\mathbf{F}}$ and $\overset{e}{\mathbb{F}}$ are equal up to local isometries of the reference configuration. We discuss the constitutive equations of anisotropic anelastic solids in terms of both intermediate configurations. It is shown that the two descriptions of anelasticity are equivalent in the sense that the Cauchy stresses calculated using them are identical. We note that, unlike isotropic solids, for an anisotropic solid the material metric is not sufficient for describing the constitutive behavior of the solid; the energy function explicitly depends on $\overset{a}{\mathbf{F}}$ (or $\overset{a}{\mathbb{F}}$) through the structural tensors.

Keywords: Anelasticity; Intermediate Configuration; Residual Stress; Finite Plasticity; Structural Tensors.

Contents

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1 Introduction

Anelasticity is the study of finite deformations of bodies that, in addition to elastic deformations, undergo non-elastic deformations or microstructural changes due to other physical, chemical, or biological processes, e.g., bulk growth and remodelling, accretion, swelling in gels, plasticity, thermal expansion/contraction, diffusion, etc. We refer to strains due to non-elastic deformations as *anelastic strains* or *eigenstrains*.[1] As an example, in bulk growth different material points may change in size or shape even in the absence of external loads [Epstein and Maugin, 2000, Ben Amar and Goriely, 2005, Yavari, 2010, Goriely, 2017]. Other examples of anelastic strains appear in accretion [Tomassetti et al., 2016, Sozio and Yavari, 2017, 2019, Zurlo and Truskinovsky, 2017, 2018, Truskinovsky and Zurlo, 2019], thermoelasticity [Ozakin and Yavari, 2010, Sadik and Yavari, 2017b], and solids with distributed defects [Yavari and Goriely, 2013b, 2012a,b]. One should note that stress in anelasticity explicitly depends on the elastic strain, and not the total strain.

A fundamental assumption of nonlinear anelasticity of simple materials[2] is that locally the elastic and anelastic deformations can be decoupled through a multiplicative decomposition of the deformation gradient into elastic and anelastic parts: $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$, where the anelastic strains are induced from the material tensor field $\overset{a}{\mathbf{F}}$ while the elastic strains explicitly depend on the two-point tensor field $\overset{e}{\mathbf{F}}$. It has been known that the decomposition $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ is not unique as $\mathbf{F} = (\overset{e}{\mathbf{F}}\mathbf{Q})(\mathbf{Q}^{-1}\overset{a}{\mathbf{F}})$ is another equivalent decomposition for any isometry $\mathbf{Q}$ [Casey and Naghdi, 1980].[3] Although the total strain is compatible, neither the elastic nor the anelastic part needs to be compatible. The incompatibility of elastic strain (and consequently anelastic strain) is the source of residual stresses in anelastic bodies. Unlike elastic bodies that have a stress-free reference configuration that can be isometrically embedded into the Euclidean ambient space, anelastic bodies do not have such Euclidean reference configurations, in general. Anelastic bodies are non-Euclidean in this sense. Non-Euclidean solids—a term that was coined by Henri Poincaré [Poincaré, 1905]—has been used interchangeably for anelastic bodies in the recent literature [Zurlo and Truskinovsky, 2017, 2018, Truskinovsky and Zurlo, 2019].

The ideas leading to the decomposition $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ originated from different scientific communities [Sadik and Yavari, 2017a]. The first systematic study of nonlinear anelasticity is due to Eckart [1948]. Eckart suggested that a theory of anelasticity can be formulated by modifying two fundamental assumptions of the classical theory of elasticity that he called "principle of a constant relaxed state", and the "principle of relaxability-in-the-large". The first "principle" refers to assuming a fixed stress-free reference configuration independent of applied loads and the history of deformation. The second "principle" is equivalent to assuming a Euclidean stress-free reference configuration. Motivated by earlier works of geometers [Eisenhart, 1926], Eckart suggested replacing "relaxability-in-the-large" by "relaxability-in-the-small". He clearly saw the connection between anelasticity and Riemannian geometry, and explicitly modeled anelastic strains by a Riemannian metric. Independently, Kondo [1949] suggested that the natural framework for formulating the mechanics of residually-stressed bodies is Riemannian geometry. Kondo [1949] used the terms "free manifold" and "free space" for the natural configuration of a residually-stressed body. Later on he coined the term "material manifold" [Kondo, 1950a]. Kondo in his attempts of modeling plasticity using Riemannian and

[1] *Eigenstrain* is a hybrid German-English term whose origin is in the pioneering paper of Hans Reissner [Reissner, 1931] (Eigenspannung means proper or self stress). A few decades after the work of Reissner, Mura [Kinoshita and Mura, 1971, Mura, 1982] popularized this term. In the mechanics literature, for the same concept, a few other terms have been used: *initial strain* [Kondo, 1949], *nuclei of strain* [Mindlin and Cheng, 1950], *transformation strain* [Eshelby, 1957], *inherent strain* [Ueda et al., 1975], and *residual strain* [Ambrosi et al., 2019]. For infinite bodies, and in the setting of linear elasticity, the first systematic study of eigenstrains and the stresses they induce is due to Eshelby [1957].

[2] A material whose elastic response at any point depends only on the first deformation gradient (and its evolution) at that point is called *simple* [Noll, 1958].

[3] Casey and Naghdi [1980] claimed that there is an $SO(3)$-ambiguity in the decomposition $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$. However, this is not true for elastically anisotropic anelastic solids. Assuming that $\overset{e}{\mathbf{F}}\mathbf{Q}$ is an elastic deformation gradient implies that $W(\overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(\overset{e}{\mathbf{F}}\mathbf{Q}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$, where W is the energy function, implies that $\mathbf{Q}$ is a material symmetry. Denoting the material symmetry group by $\mathring{\mathcal{G}}$, for an anisotropic solid there is a $\mathring{\mathcal{G}}$-ambiguity (and not an $SO(3)$-ambiguity) in the multiplicative decomposition. In particular, this implies that for triclinic solids the multiplicative decomposition is unique.

non-Riemannian geometries [Kondo, 1950a,b, 1952] was motivated by the works of Élie Cartan [Cartan, 1926, 1928].[4]

These early works focused on modeling anelastic strains as Riemannian metrics. A one-dimensional analogue of $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{s}{\mathbf{F}}$, where $\overset{s}{\mathbf{F}}$ is the swelling part of the deformation gradient was first introduced by Flory and Rehner [1944], see also [Duda et al., 2010]. In finite plasticity the multiplicative decomposition first appeared in [Bilby et al., 1957, Page 41, Eq. (12)], and in [Kröner, 1959, Page 286, Eq. (4)]. This decomposition was popularized in the plasticity literature by Lee and Liu [1967] and Lee [1969]. In nonlinear thermoelasticity it is due to Stojanović et al. [1964] and Stojanović [1969]. In the mechanics of bulk growth it is due to Kondaurov and Nikitin [1987], Takamizawa and Hayashi [1987], Takamizawa and Matsuda [1990], and Takamizawa [1991]. Similar ideas can also be found in [Tranquillo and Murray, 1992, 1993]. The multiplicative decomposition was popularized in biomechanics by Rodriguez et al. [1994]. In the past two decades the multiplicative decomposition of deformation gradient has become a popular modeling tool in nonlinear solid mechanics, and especially in biomechanics [Goriely, 2017]. We should mention that there have been several recent works on different aspects of the multiplicative decomposition of the deformation gradient [Neff, 2008, Neff et al., 2009, Reina and Conti, 2014, Casey, 2017, Del Piero, 2018, Du et al., 2018, Goodbrake et al., 2021].

In linear anelasticity, the total linearized strain is additively decomposed into elastic and anelastic strains: $\boldsymbol{\epsilon} = \overset{e}{\boldsymbol{\epsilon}} + \overset{a}{\boldsymbol{\epsilon}}$. This decomposition is unambiguous for linearized strain. However, this is not the case for nonlinear anelasticity as there are different measures of strain and even for a given measure of strain there is more than one possible decomposition [Nemat-Nasser, 1979]. In the case of deformation gradient another possibility is $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$,[5] which following Lubarda [1999] we call the *reverse decomposition*. Clifton [1972] considered the polar decompositions of the different elastic and anelastic deformation gradients and under certain assumptions concluded that for isotropic solids the two decompositions are equivalent. Lubarda [1999] restricted his analysis of the reverse decomposition to isotropic solids and assumed that $\overset{e}{\mathbb{F}} = \overset{e}{\mathbf{F}}$.[6] He showed that there is a duality between the constitutive formulations of finite plasticity using the two decompositions for isotropic solids. He also concluded that the Bilby-Kröner-Lee decomposition is preferable in the case of anisotropic solids. In [Davoli and Francfort, 2015] it was concluded that the reverse decomposition corresponds to a more natural dissipation functional.

The main contributions of this paper can be summarized as follows:

- A global spatial intermediate configuration is constructed for anisotropic anelasticity.
- The relation between the spatial intermediate configuration and the material intermediate configuration (material manifold) is established.
- In the decompositions $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$, a priori there is no relation between $\overset{e}{\mathbf{F}}$ and $\overset{e}{\mathbb{F}}$ (or between $\overset{a}{\mathbf{F}}$ and $\overset{a}{\mathbb{F}}$). When the direct and reverse decompositions represent the same anelstic deformation, we find such a relationship between $\overset{e}{\mathbf{F}}$ and $\overset{e}{\mathbb{F}}$, see (3.9). This result is summarized in Theorem 3.1.
- Constitutive equations of anisotropic solids are formulated with respect to the global spatial intermediate configuration.
- We show that for anisotropic solids the two decompositions are equivalent, i.e., Cauchy stresses calculated with respect to the two decompositions are identical (Theorem 3.5). This is a generalization of the works of Clifton [1972] and Lubarda [1999].

This paper is organized as follows. Nonlinear elasticity is tersely reviewed in §2. In §3 the geometry and the constitutive equations of nonlinear anelasticity are discussed. Material metric and some strain tensors are defined in §3.1 and §3.2. In §3.3 the global material intermediate configuration is discussed. In §3.4 we construct a global spatial intermediate configuration that reflects the reverse multiplicative decomposition

[4] Interestingly, in his development of non-Riemannian geometries, and more specifically torsion of a connection, Cartan [1922] was motivated by the work of Cosserat brothers on generalized continua [Cosserat and Cosserat, 1909]. See [Scholz, 2019] for a detailed history and discussion.

[5] Note that $\mathbf{F} = (\overset{a}{\mathbb{F}}\mathbf{Q}^{-1})(\mathbf{Q}\overset{e}{\mathbb{F}})$ is an equivalent decomposition for any invertible $\mathbf{Q}$. Material-frame-indifference implies that the two multiplicative decompositions $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$ and $\mathbf{F} = (\overset{a}{\mathbb{F}}\mathbf{Q}^{-1})(\mathbf{Q}\overset{e}{\mathbb{F}})$ are equivalent for any $\mathbf{Q} \in SO(T_x\mathcal{C}) = SO(3)$. We observe that for anisotropic solids there is much more freedom in choosing the elastic and anelastic deformations in the reverse decomposition compared to the Bilby-Kröner-Lee decomposition.

[6] Note that $\overset{e}{\mathbb{F}}$ and $\overset{e}{\mathbf{F}}$ have the same tensorial character and $\overset{e}{\mathbb{F}} = \overset{e}{\mathbf{F}}$ makes sense intrinsically.

$\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$. We make a connection between the geometries of the two intermediate configurations. The constitutive equations of elastically anisotropic anelastic bodies are discussed in §3.5, and it is shown that the two decompositions are equivalent for anisotropic solids. The concluding remarks are given in §4.

2 Nonlinear anisotropic elasticity

2.1 Kinematics and strain tensors

Deformation of an elastic body is a time-dependent map $\varphi_t : (\mathcal{B}, \mathring{\mathbf{G}}) \to (\mathcal{S}, \mathring{\mathbf{g}})$, where $\mathcal{S}$ is the Euclidean ambient space, and $\mathring{\mathbf{g}}$ is the Euclidean background metric. $\mathcal{B}$ is a submanifold of $\mathcal{S}$, and $\mathring{\mathbf{G}} = \mathring{\mathbf{g}}|_{\mathcal{B}}$.[7] In elasticity (and anelasticity) the local change of length is a quantity of interest and that is why the reference configuration and the ambient space are equipped with Riemannian metrics. For a fixed value of t we denote $\varphi = \varphi_t$, and $\mathcal{C} = \varphi(\mathcal{B}) \subset \mathcal{S}$. Therefore, at any instant of time, deformation is a map from the stress-free reference configuration $\mathcal{B}$ to the current configuration $\mathcal{C}$. Therefore, we write deformation as the map $\varphi : (\mathcal{B}, \mathring{\mathbf{G}}) \to (\mathcal{C}, \mathring{\mathbf{g}})$.

In nonlinear elasticity deformation gradient $\mathbf{F}$ is the tangent map of the deformation map φ. More precisely, $\mathbf{F}(X) = T\varphi|_{\pi^{-1}(X)}$, where $\pi : T\mathcal{B} \to \mathcal{B}$ is the natural projection in the tangent bundle onto the base space. In other words, $\mathbf{F}(X)$ is the restriction of $T\varphi$ to the fiber over X. The tangent map $T\varphi$ is a vector bundle morphism that maps the tangent bundle $T\mathcal{B}$ to the tangent bundle $T\mathcal{C}$, and hence it also includes φ as the map on the base space. As $\mathcal{S}$, and consequently $\mathcal{B}$ and $\mathcal{C}$, are parallelizable, $T\mathcal{B}$ and $T\mathcal{C}$ are trivial, and hence one can write $T\varphi = (\varphi, \mathbf{F})$, where $\mathbf{F}$ is understood as a tensor field that maps tangent vector fields on $\mathcal{B}$ to tangent vector fields on $\mathcal{C}$. Note that $\mathbf{F}(X)$ is a linear mapping that maps the vector $\mathbf{U} \in T_X\mathcal{B}$ to $\mathbf{F}(X)\mathbf{U} \in T_{\varphi(X)}\mathcal{C}$. Let us consider coordinate charts $\{X^A\} : \mathcal{B} \to \mathbb{R}^n$ and $\{x^a\} : \mathcal{C} \to \mathbb{R}^n$, for $\mathcal{B}$ and $\mathcal{C}$, respectively ($n = 2$ or 3). With respect to these coordinate charts deformation gradient has the following representation

$$\mathbf{F}(X) = \frac{\partial \varphi^a(X)}{\partial X^A} \frac{\partial}{\partial x^a} \otimes dX^A = F^a{}_A(X) \frac{\partial}{\partial x^a} \otimes dX^A . \tag{2.1}$$

To avoid self-penetration of matter, a necessary condition is that $\det \mathbf{F}(X) > 0$, $\forall X \in \mathcal{B}$, i.e., φ is locally invertible and orientation preserving. The dual of $\mathbf{F}$ is defined as

$$\mathbf{F}^\star : T^*_{\varphi(X)}\mathcal{C} \to T^*_X\mathcal{B} , \quad \langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{V} \rangle = \langle \mathbf{F}^\star \boldsymbol{\alpha}, \mathbf{V} \rangle , \quad \forall \mathbf{V} \in T_X\mathcal{B},\ \boldsymbol{\alpha} \in T^*_x\mathcal{C} , \tag{2.2}$$

where $T^*_{\varphi(X)}\mathcal{C}$ and $T^*_X\mathcal{B}$ denote the cotangent spaces of $T_{\varphi(X)}\mathcal{C}$ and $T_X\mathcal{B}$, respectively, and $\langle .,. \rangle$ is the natural pairing of a 1-form and a vector: $\langle \boldsymbol{\alpha}, \mathbf{v} \rangle = \alpha_a\, v^a$. $\mathbf{F}^\star$ has the following coordinate representation

$$\mathbf{F}^\star(X) = F^a{}_A(X)\, dX^A \otimes \frac{\partial}{\partial x^a} . \tag{2.3}$$

The transpose of deformation gradient is defined as[8] $\mathbf{F}^{\mathring{\mathsf{T}}} : T_x\mathcal{C} \to T_X\mathcal{B}$, $\langle\!\langle \mathbf{F}\mathbf{V},\ \mathbf{v} \rangle\!\rangle_{\mathring{\mathbf{g}}} = \langle\!\langle \mathbf{V}, \mathbf{F}^{\mathring{\mathsf{T}}}\mathbf{v} \rangle\!\rangle_{\mathring{\mathbf{G}}}$, $\forall \mathbf{V} \in T_X\mathcal{B}$, $\mathbf{v} \in T_x\mathcal{C}$, where $\langle\!\langle , \rangle\!\rangle_{\mathring{\mathbf{G}}}$ and $\langle\!\langle , \rangle\!\rangle_{\mathring{\mathbf{g}}}$ are the inner products induced by the metrics $\mathring{\mathbf{G}}$ and $\mathring{\mathbf{g}}$, respectively. $\mathbf{F}^{\mathring{\mathsf{T}}}$ has the following representation

$$\mathbf{F}^{\mathring{\mathsf{T}}}(X) = (F^{\mathring{\mathsf{T}}}(X))^A{}_a \frac{\partial}{\partial X^A} \otimes dx^a = \mathring{g}_{ab}(\mathbf{x})\, F^b{}_B(X)\, \mathring{G}^{AB}(X) \frac{\partial}{\partial X^A} \otimes dx^a . \tag{2.4}$$

Note that $\mathbf{F}^{\mathring{\mathsf{T}}} = \mathring{\mathbf{G}}^\sharp \mathbf{F}^\star \mathring{\mathbf{g}}$, where $\mathring{\mathbf{G}}^\sharp$ is the inverse of $\mathring{\mathbf{G}}$, i.e., $\mathring{G}^{AC}\, \mathring{G}_{CB} = \delta^A{}_B$. For $\mathbf{V}$ a vector field on $\mathcal{B}$, $\varphi_*\mathbf{V} = T\varphi \cdot \mathbf{V} \circ \varphi^{-1} = \mathbf{F} \cdot \mathbf{V} \circ \varphi^{-1}$ is a vector field on $\mathcal{C} \subset \mathcal{S}$—the push-forward of $\mathbf{V}$ by φ. Similarly, if $\mathbf{v}$ is

[7] Our notation is slightly different from that of Marsden and Hughes [1994]. We use $\mathring{\mathbf{g}}$ and $\mathring{\mathbf{G}}$ for metrics of the Euclidean ambient space and the induced Euclidean metric in the reference configuration. We reserve $\mathbf{g}$ and $\mathbf{G}$ for the (non-flat) Riemannian metrics of the spatial and material intermediate configurations, respectively.

[8] We use $(.)^{\mathring{\mathsf{T}}}$ when the metrics $\mathring{\mathbf{G}}$ and $\mathring{\mathbf{g}}$ are used in calculating the transpose. In §3, we will use the notation $(.)^{\mathsf{T}}$ for transpose calculated using the metrics $\mathbf{G}$ and $\mathring{\mathbf{g}}$, where $\mathbf{G}$ is the material metric that as we will see in §3 explicitly depends on the local anelastic deformation.

a vector field on $\mathcal{C}=\varphi(\mathcal{B})$, the pull-back of $\mathbf{v}$ by φ is defined as $\varphi^*\mathbf{v}=T(\varphi^{-1})\cdot\mathbf{v}\circ\varphi=\mathbf{F}^{-1}\cdot\mathbf{v}\circ\varphi$, which is a vector field on $\mathcal{B}$. The pull-back and push-forward of tensor fields are defined similarly.

The right Cauchy-Green strain is defined as $\mathbf{C}^\flat=\varphi^*\mathring{\mathbf{g}}=\mathbf{F}^\star\,\mathring{\mathbf{g}}\,\mathbf{F}$. Note that the two Riemannian manifolds $(\mathcal{C},\mathring{\mathbf{g}})$ and $(\mathcal{B},\mathbf{C}^\flat)$ are isometric. Therefore, the deformation can be equivalently described by the map $\mathrm{id}_{\mathcal{B}}:(\mathcal{B},\mathring{\mathbf{G}})\to(\mathcal{B},\mathbf{C}^\flat)$. The left Cauchy-Green deformation tensor is defined as $\mathbf{B}^\sharp=\varphi^*\mathring{\mathbf{g}}^\sharp$, and in components $B^{AB}=(F^{-1})^A{}_a(F^{-1})^B{}_b\,\mathring{g}^{ab}$, where $\mathring{\mathbf{g}}^\sharp$ is the inverse of $\mathring{\mathbf{g}}$, i.e., $\mathring{g}^{ac}\,\mathring{g}_{cb}=\delta^a{}_b$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are denoted by $\mathbf{c}^\flat$ and $\mathbf{b}^\sharp$ (the Finger deformation tensor), respectively, and are defined as $\mathbf{c}^\flat=\varphi_*\mathring{\mathbf{G}}$ and $\mathbf{b}^\sharp=\varphi_*\mathring{\mathbf{G}}^\sharp$, with their corresponding components $c_{ab}=(F^{-1})^A{}_a(F^{-1})^B{}_b\,\mathring{G}_{AB}$ and $b^{ab}=F^a{}_A\,F^b{}_B\,\mathring{G}^{AB}$, respectively. Note that $\mathbf{c}^\flat=\varphi_*\mathring{\mathbf{G}}$ is the spatial analogue of $\mathbf{C}^\flat$. This means that the two Riemannian manifolds $(\mathcal{B},\mathring{\mathbf{G}})$ and $(\mathcal{C},\mathbf{c}^\flat)$ are isometric. Therefore, the deformation can be equivalently described by the map $\mathrm{id}_{\mathcal{C}}:(\mathcal{C},\mathbf{c}^\flat)\to(\mathcal{C},\mathring{\mathbf{g}})$. In summary, one has the following measures of strain:

$$
\begin{aligned}
&\mathbf{C}^\flat=\varphi^*\mathring{\mathbf{g}}\,, && \mathbf{C}=\mathring{\mathbf{G}}^\sharp\mathbf{C}^\flat=\mathbf{F}^{\mathring{\mathsf{T}}}\mathbf{F}\,,\\
&\mathbf{b}^\sharp=\varphi_*\mathring{\mathbf{G}}^\sharp\,, && \mathbf{b}=\mathbf{b}^\sharp\mathring{\mathbf{g}}=\mathbf{F}\mathbf{F}^{\mathring{\mathsf{T}}}\,,\\
&\mathbf{B}^\sharp=\varphi^*\mathring{\mathbf{g}}^\sharp\,, && \mathbf{B}=\mathbf{B}^\sharp\mathring{\mathbf{g}}=\mathbf{F}^{-1}\mathbf{F}^{-\mathring{\mathsf{T}}}\,,\\
&\mathbf{c}^\flat=\varphi_*\mathring{\mathbf{G}}\,, && \mathbf{c}=\mathring{\mathbf{g}}^\sharp\mathbf{c}^\flat=\mathbf{F}^{-\mathring{\mathsf{T}}}\mathbf{F}^{-1}\,.
\end{aligned}
\tag{2.5}
$$

Note $\mathbf{B}=\mathbf{C}^{-1}$ and $\mathbf{b}=\mathbf{c}^{-1}$. The following commutative diagram summarizes the three equivalent descriptions of motion in nonlinear elasticity.

$$
\begin{tikzcd}
(\mathcal{B},\mathring{\mathbf{G}}) \arrow[r, "\mathrm{id}_{\mathcal{B}}"] \arrow[d, dotted, "\mathrm{id}_{\mathcal{B}}"] & (\mathcal{B},\mathbf{C}^\flat) \arrow[d, dotted, "\varphi"] \\
(\mathcal{B},\mathring{\mathbf{G}}) \arrow[r, "\varphi"] \arrow[d, dotted, "\varphi"] & (\mathcal{C},\mathring{\mathbf{g}}) \arrow[d, dotted, "\mathrm{id}_{\mathcal{C}}"] \\
(\mathcal{C},\mathbf{c}^\flat) \arrow[r, "\mathrm{id}_{\mathcal{C}}"] & (\mathcal{C},\mathring{\mathbf{g}})
\end{tikzcd}
\tag{2.6}
$$

The three horizontal maps describe the same elastic deformation while the vertical maps are isometries.[9] In this paper we use dotted arrows to emphasize that a map is an isometry.

The principal invariants of $\mathbf{b}$ (and $\mathbf{C}$) are defined as [Ogden, 1997]: $I_1=\operatorname{tr}\mathbf{b}=b^a{}_a=b^{ab}\,\mathring{g}_{ab}$, $I_2=\frac{1}{2}\left(I_1^2-\operatorname{tr}\mathbf{b}^2\right)=\frac{1}{2}\left(I_1^2-b^a{}_b\,b^b{}_a\right)=\frac{1}{2}\left(I_1^2-b^{ab}b^{cd}\,\mathring{g}_{ac}\,\mathring{g}_{bd}\right)$, and $I_3=\det\mathbf{b}$. Note that

$$
\begin{aligned}
\operatorname{tr}\mathbf{b}&=\operatorname{tr}_{\mathring{\mathbf{g}}}(\varphi_*\mathring{\mathbf{G}}^\sharp)=\operatorname{tr}_{\varphi^*\mathring{\mathbf{g}}}\mathring{\mathbf{G}}^\sharp=\operatorname{tr}_{\mathring{\mathbf{G}}^\sharp}\varphi^*\mathring{\mathbf{g}}=\operatorname{tr}_{\mathring{\mathbf{G}}^\sharp}\mathbf{C}^\flat=\operatorname{tr}\mathbf{C}\,,\\
\det\mathbf{b}&=\det(\mathbf{b}^\sharp\mathring{\mathbf{g}})=\det\mathbf{b}^\sharp\det\mathring{\mathbf{g}}=\det(\varphi_*\mathring{\mathbf{G}}^\sharp)\det\mathring{\mathbf{g}}=\det(\mathbf{F}\mathring{\mathbf{G}}^\sharp\mathbf{F}^\star)\det\mathring{\mathbf{g}}\\
&=\det\mathring{\mathbf{G}}^\sharp\det(\mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F})=\det(\mathbf{C}^\flat\mathring{\mathbf{G}}^\sharp)=\det\mathbf{C}\,.
\end{aligned}
\tag{2.7}
$$

We assume a hyperelastic solid, i.e., there exists an energy function $\mathring{W}=\mathring{W}(X,\mathbf{F},\mathring{\mathbf{G}},\mathring{\mathbf{g}})$. As the focus of this paper is on kinematics and constitutive equations of anelasticity we will not discuss the balance laws (see Sozio and Yavari [2020] for discussions on balance laws in anelasticity).

2.2 Constitutive equations in nonlinear elasticity

We restrict ourselves to hyper-elastic solids, i.e., assume the existence of an energy function that for a simple material depends on the deformation gradient. However, as deformation gradient is a two-point tensor, the energy function (which is a scalar) must explicitly depend on the metrics of the reference and current configurations as well, i.e., $\mathring{W}=\mathring{W}(X,\mathbf{F},\mathring{\mathbf{G}},\mathring{\mathbf{g}})$.

[9] The Lagrangian strain calculated using the first two rows is $\frac{1}{2}\big(\mathbf{C}^\flat-\mathring{\mathbf{G}}\big)$, while using the third row one obtains its pushforward, i.e., $\frac{1}{2}\big(\mathrm{id}_{\mathcal{C}}^*\,\mathring{\mathbf{g}}-\mathbf{c}^\flat\big)=\frac{1}{2}\big(\mathring{\mathbf{g}}-\mathbf{c}^\flat\big)=\frac{1}{2}\varphi_*\big(\mathbf{C}^\flat-\mathring{\mathbf{G}}\big)$.

2.2.1 Material symmetry in elasticity

The material symmetry group $\mathring{\mathcal{G}}_X$ of an elastic body made of a solid with the energy function $\mathring{W}$ at a point X with respect to the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ is defined as [Šilhavý, 2013]

$$\mathring{W}(X, \mathbf{F}\mathring{\mathbf{K}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \,, \qquad \forall\, \mathring{\mathbf{K}} \in \mathring{\mathcal{G}}_X \leqslant \mathrm{Orth}(\mathring{\mathbf{G}}) \,, \tag{2.8}$$

for all deformation gradients $\mathbf{F}$, where $\mathring{\mathbf{K}} : T_X\mathcal{B} \to T_X\mathcal{B}$ is an invertible linear transformation, and $\mathrm{Orth}(\mathring{\mathbf{G}}) = \{\mathring{\mathbf{Q}} : T_X\mathcal{B} \to T_X\mathcal{B} \mid \mathring{\mathbf{Q}}^{\mathsf{T}} = \mathring{\mathbf{Q}}^{-1}\}$. The condition $\mathring{\mathbf{Q}}^{\mathsf{T}} = \mathring{\mathbf{Q}}^{-1}$ is equivalent to $\mathring{\mathbf{Q}}_* \mathring{\mathbf{G}} = \mathring{\mathbf{G}}$, or $\mathring{\mathbf{Q}}^{-\star} \mathring{\mathbf{G}} \mathring{\mathbf{Q}}^{-1} = \mathring{\mathbf{G}}$. When $\mathscr{G}$ is a subgroup of $\mathscr{H}$, this is denoted as $\mathscr{G} \leqslant \mathscr{H}$. For hyperelastic solids, objectivity (material-frame-indifference) requires that the energy function depend on the deformation through the right Cauchy-Green deformation tensor $\mathbf{C}^\flat$, i.e., $W = W(X, \mathbf{C}^\flat, \mathring{\mathbf{G}})$. This implies that the material symmetry group $\mathring{\mathcal{G}}_X$ of a hyperelastic solid is the subgroup of $\mathring{\mathbf{G}}$-orthogonal transformations $\mathrm{Orth}(\mathring{\mathbf{G}})$ such that

$$W(X, \mathring{\mathbf{K}}^* \mathbf{C}^\flat, \mathring{\mathbf{G}}) = W(X, \mathbf{C}^\flat, \mathring{\mathbf{G}}) \,, \qquad \forall \mathring{\mathbf{K}} \in \mathring{\mathcal{G}}_X \leqslant \mathrm{Orth}(\mathring{\mathbf{G}}) \,, \tag{2.9}$$

where $\mathring{\mathbf{K}}^* \mathbf{C}^\flat = \mathring{\mathbf{K}}^\star \mathbf{C}^\flat \mathring{\mathbf{K}}$. The material symmetry group can be characterized using a finite collection of structural tensors $\mathring{\boldsymbol{\zeta}}_i$ of order μ_i, $i = 1, \dots, N$ [Liu, 1982, Boehler, 1987, Zheng and Spencer, 1993, Zheng, 1994, Lu and Papadopoulos, 2000, Mazzucato and Rachele, 2006]

$$\mathring{\mathbf{Q}} \in \mathring{\mathcal{G}} \leqslant \mathrm{Orth}(\mathring{\mathbf{G}}) \iff \langle \mathring{\mathbf{Q}} \rangle_{\mu_1} \mathring{\boldsymbol{\zeta}}_1 = \mathring{\boldsymbol{\zeta}}_1 \,, \dots \,, \langle \mathring{\mathbf{Q}} \rangle_{\mu_N} \mathring{\boldsymbol{\zeta}}_N = \mathring{\boldsymbol{\zeta}}_N \,. \tag{2.10}$$

The set of structural tensors is a basis for the space of tensors that are invariant under the action of the group $\mathring{\mathcal{G}}$. The μ-th power Kronecker product $\langle \mathring{\mathbf{Q}} \rangle_\mu$ of a $\mathring{\mathbf{G}}$-orthogonal transformation $\mathring{\mathbf{Q}}$ for a μ-th order tensor $\mathring{\boldsymbol{\zeta}}$ is defined as $(\langle \mathring{\mathbf{Q}} \rangle_\mu \mathring{\zeta})^{\bar{A}_1 \dots \bar{A}_\mu} = \mathring{Q}^{\bar{A}_1}{}_{A_1} \dots \mathring{Q}^{\bar{A}_\mu}{}_{A_\mu} \mathring{\zeta}^{A_1 \dots A_\mu}$. Notice that $\langle \mathring{\mathbf{Q}} \rangle_m (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_m) = \mathring{\mathbf{Q}}\mathbf{v}_1 \otimes \cdots \otimes \mathring{\mathbf{Q}}\mathbf{v}_m$, where $\mathbf{v}_i \in T_X\mathcal{B}$, $i = 1, \dots, m$, are arbitrary vectors. Eq. (2.10) tells us that the material symmetry group $\mathring{\mathcal{G}}$ is the invariance group of the set of the structural tensors $\mathring{\boldsymbol{\zeta}}_i$, $i = 1, \dots, N$. The energy function has the following functional form

$$W = \widehat{W}(X, \mathbf{C}^\flat, \mathring{\mathbf{G}}, \mathring{\boldsymbol{\zeta}}_1, \dots, \mathring{\boldsymbol{\zeta}}_N) \,. \tag{2.11}$$

When structural tensors are considered as arguments of the energy function, the energy function becomes an isotropic function of its arguments—the so-called *principle of isotropy of space* [Boehler, 1979].

Instead of using the set of tensors $\{\mathbf{C}^\flat, \mathring{\mathbf{G}}, \mathring{\boldsymbol{\zeta}}_1, \dots, \mathring{\boldsymbol{\zeta}}_N\}$, one can use a corresponding set of isotropic invariants. According to a theorem by Hilbert for any finite collection of tensors there exists a finite set of isotropic invariants—the *integrity basis* for the set of isotropic invariants of the collection [Spencer, 1971]. Let us denote the integrity basis by I_j, $j = 1, \dots, m$. Thus, one can write $W = W(X, I_1, ..., I_m)$. In terms of the integrity basis, the second Piola-Kirchhoff stress tensor has the following representation [Doyle and Ericksen, 1956, Marsden and Hughes, 1994]

$$\mathbf{S} = 2\frac{\partial \widehat{W}}{\partial \mathbf{C}^\flat} = \sum_{j=1}^{m} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat} \,, \qquad W_j = W_j(X, I_1, ..., I_m) := \frac{\partial W}{\partial I_j} \,, \quad j = 1, \dots, m \,, \tag{2.12}$$

where the second Piola-Kirchhoff stress $\mathbf{S}$ has the following relationship with the first Piola-Kirchhoff and Cauchy stresses: $S^{AB} = (F^{-1})^A{}_a P^{aB} = J (F^{-1})^A{}_a (F^{-1})^B{}_b \, \sigma^{ab}$.

2.2.2 Covariant constitutive equations

In nonlinear elasticity, conservation of mass and the balance of linear and angular momenta can be derived by postulating the balance of energy and its invariance under rigid body translations and rotations of the Euclidean ambient space [Green and Rivlin, 1964]. This approach was generalized to nonlinear elasticity in a Riemannian ambient space by Hughes and Marsden [1977] and led to a covariant formulation of nonlinear elasticity. Covariant elasticity was further developed in [Marsden and Hughes, 1994, Simo and Marsden, 1984, Yavari et al., 2006, Yavari and Golgoon, 2019].

Let us consider a body with an energy function $W = \mathring{W}(X, \mathbf{F}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$, and a diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$, which can be thought of a change of coordinates in the ambient space or a mapping of the ambient space to itself. Spatial covariance of the energy function is the invariance of the energy function under any such diffeomprphism, i.e., $\mathring{W}(X, \xi_{t*}\mathbf{F}, \mathring{\mathbf{G}}, \xi_{t*}\mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$ (note that $\xi_{t*}\mathring{\mathbf{G}} = \mathring{\mathbf{G}}$). Marsden and Hughes [1994] proved that spatial covariance of energy function implies that $\mathring{W}(X, \mathbf{F}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(X, \mathbf{C}^\flat, \mathring{\mathbf{G}})$.

Let us next consider a material (referential) diffeomorphism $\Xi : \mathcal{B} \to \mathcal{B}$. A homogeneous energy function $W(\mathbf{C}^\flat, \mathring{\mathbf{G}})$ is materially covariant if it is invariant under any material diffeomorphism Ξ, i.e., $W(\Xi_*\mathbf{C}^\flat, \Xi_*\mathring{\mathbf{G}}) = W(\mathbf{C}^\flat, \mathring{\mathbf{G}})$. For inhomogeneous bodies material covariance is defined locally and using local diffeomorphisms such that $\Xi(X) = X$. Marsden and Hughes [1994] showed that material covariance of energy function implies isotropy. They suggested that a covariant description of non-isotropic materials requires some additional variables in the energy function. Material covariance was further studied by Lu and Papadopoulos [2000] and Lu [2012]. Lu [2012] showed that when structural tensors are included as arguments of the energy function, in addition to the material metric and the right Cauchy-Green tensor, the energy function becomes a materially covariant function, i.e.,

$$\widehat{W}(X, \Xi_*\mathbf{C}^\flat, \Xi_*\mathring{\mathbf{G}}, \Xi_*\mathring{\boldsymbol{\zeta}}_1, \ldots, \Xi_*\mathring{\boldsymbol{\zeta}}_N) = \widehat{W}(X, \mathbf{C}^\flat, \mathring{\mathbf{G}}, \mathring{\boldsymbol{\zeta}}_1, \ldots, \mathring{\boldsymbol{\zeta}}_N)\,. \tag{2.13}$$

Lu [2012] also showed that spatial and material covariance of an energy function imply the *principle of isotropy of space* [Boehler, 1979], i.e., the energy function of an anisotropic solid is an isotropic function of its arguments when structural tensors are included.

3 Nonlinear anisotropic anelasticity

In this section we first review the global intermediate configuration of nonlinear anelasticity corresponding to the Bilby-Kröner-Lee decomposition $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ following an approach similar to that of [Goodbrake et al., 2021]. We next construct a global intermediate configuration that corresponds to the reverse multiplicative decomposition $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$. We then make a connection between the two intermediate manifolds. Finally, we will discuss constitutive equations for elastically anisotropic anelastic solids and will show the equivalence of the two decompositions.

For factorizations of the tensor field $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$ we are interested in constructing the corresponding factorizations of $\varphi : (\mathcal{B}, \mathring{\mathbf{G}}) \to (\mathcal{S}, \mathring{\mathbf{g}})$ through some Riemannian manifolds that we call global material and spatial intermediate configurations. We assume that the maps

$$\begin{aligned} &\overset{a}{\mathbf{F}}(X) : T_X\mathcal{B} \to T_X\mathcal{B}\,, && \overset{e}{\mathbf{F}}(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{C}\,,\\ &\overset{e}{\mathbb{F}}(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{C}\,, && \overset{a}{\mathbb{F}}(X) : T_{\varphi(X)}\mathcal{C} \to T_{\varphi(X)}\mathcal{C}\,, \end{aligned} \tag{3.1}$$

are invertible. Here, we have assumed that in both decompositions the local elastic deformations are two-point tensors. In the direct decomposition instead of $(3.1)_1$ one can assume that $\overset{a}{\mathbf{F}}(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{C}$, and $\overset{e}{\mathbf{F}}(X) : T_{\varphi(X)}\mathcal{C} \to T_{\varphi(X)}\mathcal{C}$. Similarly, in the reverse decomposition one can assume that $\overset{e}{\mathbb{F}}(X) : T_X\mathcal{B} \to T_X\mathcal{B}$, and $\overset{a}{\mathbb{F}}(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{C}$. It is important to clearly define the tensor character of the different fields.[10]

3.1 Material metric in anelasticity

The rest (natural) configuration of an anelastic body cannot be isometrically embedded into the Euclidean ambient space, in general. In this sense an anelastic body is non-Euclidean and the natural distances in its natural configuration are measured using a metric—the material metric—that explicitly depends on the local anelastic deformations. Knowing that $\overset{a}{\mathbf{F}}$ is a linear map from $T_X\mathcal{B}$ to itself and given the (Euclidean) metric $\mathring{\mathbf{G}}$ one can define another metric $\mathbf{G} = \overset{a}{\mathbf{F}}{}^*\mathring{\mathbf{G}}$. Anelastic strain (or eigenstrain) can be visualized as

[10] An example of ignoring the tensor character of tensor fields in elasticity is how deformation gradient has been related to the displacement field in the literature. This has led to the incorrect view that linear elasticity is not frame indifferent, see [Steigmann, 2007, Yavari and Ozakin, 2008].

follows [Yavari, 2021]. Given a vector $\mathbf{W} \in T_X\mathcal{B}$, it has the length square $\langle\!\langle \mathbf{W}, \mathbf{W} \rangle\!\rangle_{\mathring{\mathbf{G}}}$, where $\langle\!\langle , \rangle\!\rangle_{\mathring{\mathbf{G}}}$ is the inner product induced by $\mathring{\mathbf{G}}$. The anelastic deformation maps this vector to the vector $\overset{a}{\mathbf{F}}\mathbf{W}$ in the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$ and its length square is $\langle\!\langle \overset{a}{\mathbf{F}}\mathbf{W}, \overset{a}{\mathbf{F}}\mathbf{W} \rangle\!\rangle_{\mathring{\mathbf{G}}}$. This is the natural square length of the vector. From the definition of pull-back of a metric, one has $\langle\!\langle \overset{a}{\mathbf{F}}\mathbf{W}, \overset{a}{\mathbf{F}}\mathbf{W} \rangle\!\rangle_{\mathring{\mathbf{G}}} = \langle\!\langle \mathbf{W}, \mathbf{W} \rangle\!\rangle_{\overset{a}{\mathbf{F}}^*\mathring{\mathbf{G}}}$, where $\mathbf{G} = \overset{a}{\mathbf{F}}^*\mathring{\mathbf{G}} = \overset{a}{\mathbf{F}}^\star\,\mathring{\mathbf{G}}\,\overset{a}{\mathbf{F}}$. In components, $(\overset{a}{\mathbf{F}}^*\mathring{\mathbf{G}})_{AB} = \overset{a}{F}{}^M{}_A\,\overset{a}{F}{}^N{}_B\,\mathring{G}_{MN}$, where $\{X^A\}$ is a coordinate chart for $\mathcal{B}$.[11] In summary, the material metric is related to the flat Euclidean metric $\mathring{\mathbf{G}}$ and the local anelastic deformation $\overset{a}{\mathbf{F}}$ as $\mathbf{G} = \overset{a}{\mathbf{F}}^*\mathring{\mathbf{G}}$.

3.2 Strain tensors in anelasticity

In anelasticity, the elastic strains are defined by replacing $\mathbf{F}$ with $\overset{e}{\mathbf{F}}$ in (2.5). In the geometric approach, strain tensors are defined by replacing $\mathring{\mathbf{G}}$ with $\mathbf{G}$ in (2.5), while the transpose operator $(.)^\mathsf{T}$ is defined using $\mathring{\mathbf{g}}$ and $\mathbf{G}$. Hence, one has the following strains

$$
\begin{aligned}
&\overset{e}{\mathbf{C}}{}^\flat = \overset{e}{\mathbf{F}}{}^*\mathring{\mathbf{g}}\,, && \overset{e}{\mathbf{C}} = \mathring{\mathbf{G}}^\sharp\overset{e}{\mathbf{C}}{}^\flat = \overset{e}{\mathbf{F}}{}^{\mathring{\mathsf{T}}}\overset{e}{\mathbf{F}}\,, && \mathbf{C}^\flat = \varphi^*\mathring{\mathbf{g}}\,, && \mathbf{C} = \mathbf{G}^\sharp\mathbf{C}^\flat = \mathbf{F}^\mathsf{T}\mathbf{F}\,,\\
&\overset{e}{\mathbf{b}}{}^\sharp = \overset{e}{\mathbf{F}}{}_*\mathring{\mathbf{G}}^\sharp\,, && \overset{e}{\mathbf{b}} = \overset{e}{\mathbf{b}}{}^\sharp\mathring{\mathbf{g}} = \overset{e}{\mathbf{F}}\overset{e}{\mathbf{F}}{}^{\mathring{\mathsf{T}}}\,, && \mathbf{b}^\sharp = \varphi_*\mathbf{G}^\sharp\,, && \mathbf{b} = \mathbf{b}^\sharp\mathring{\mathbf{g}} = \mathbf{F}\mathbf{F}^\mathsf{T}\,,\\
&\overset{e}{\mathbf{B}}{}^\sharp = \overset{e}{\mathbf{F}}{}^*\mathring{\mathbf{g}}^\sharp\,, && \overset{e}{\mathbf{B}} = \overset{e}{\mathbf{B}}{}^\sharp\mathring{\mathbf{G}} = \overset{e}{\mathbf{F}}{}^{-1}\overset{e}{\mathbf{F}}{}^{-\mathring{\mathsf{T}}}\,, && \mathbf{B}^\sharp = \varphi^*\mathring{\mathbf{g}}^\sharp\,, && \mathbf{B} = \mathbf{B}^\sharp\mathbf{G} = \mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\,,\\
&\overset{e}{\mathbf{c}}{}^\flat = \overset{e}{\mathbf{F}}{}_*\mathring{\mathbf{G}}\,, && \overset{e}{\mathbf{c}} = \mathring{\mathbf{g}}^\sharp\overset{e}{\mathbf{c}}{}^\flat = \overset{e}{\mathbf{F}}{}^{-\mathring{\mathsf{T}}}\overset{e}{\mathbf{F}}{}^{-1}\,, && \mathbf{c}^\flat = \varphi_*\mathbf{G}\,, && \mathbf{c} = \mathring{\mathbf{g}}^\sharp\mathbf{c}^\flat = \mathbf{F}^{-\mathsf{T}}\mathbf{F}^{-1}\,.
\end{aligned}
\tag{3.2}
$$

It should be noticed that the strain tensors obtained starting from the material metric tensors $\mathring{\mathbf{G}}$ and $\mathbf{G}$ are the same, viz.

$$
\overset{e}{\mathbf{b}} = \mathbf{b}\,,\quad \overset{e}{\mathbf{b}}{}^\sharp = \mathbf{b}^\sharp\,,\quad \overset{e}{\mathbf{c}} = \mathbf{c}\,,\quad \overset{e}{\mathbf{c}}{}^\flat = \mathbf{c}^\flat\,. \tag{3.3}
$$

This is not the case for the strain tensors obtained by pulling back the Euclidean ambient metric $\mathring{\mathbf{g}}$, as

$$
\overset{e}{\mathbf{B}} = \overset{a}{\mathbf{F}}{}_*\mathbf{B}\,,\quad \overset{e}{\mathbf{B}}{}^\sharp = \overset{a}{\mathbf{F}}{}_*\mathbf{B}^\sharp\,,\quad \overset{e}{\mathbf{C}} = \overset{a}{\mathbf{F}}{}_*\mathbf{C}\,,\quad \overset{e}{\mathbf{C}}{}^\flat = \overset{a}{\mathbf{F}}{}_*\mathbf{C}^\flat\,. \tag{3.4}
$$

3.3 The global material intermediate configuration

One can write the following factorization of the total deformation:

$$
(\mathcal{B}, \mathring{\mathbf{G}}) \xrightarrow{\mathrm{id}_\mathcal{B}} (\mathcal{B}, \mathbf{G}) \xrightarrow{\varphi} (\mathcal{C}, \mathring{\mathbf{g}})\,. \tag{3.5}
$$

We call $(\mathcal{B}, \mathbf{G})$ the *global material intermediate configuration*. The intermediate configuration is unique up to isometry by construction (equality of elastic strain). This configuration is what has also been called the *material manifold* in the literature [Kondo, 1950a, Ozakin and Yavari, 2010, Yavari, 2010, Lu, 2012, Yavari and Goriely, 2012a,b, 2013b]. Note that the Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is the natural configuration of the anelastic body. The local anelastic deformations are encoded in the metric $\mathbf{G} = \overset{a}{\mathbf{F}}{}^*\mathring{\mathbf{G}}$ [Yavari, 2021]. Note that the local anelastic deformations are fully encoded in the material metric only in the isotropic case [Sozio and Yavari, 2020]; for elastically anisotropic anelastic solids one would need to include structural tensors that explicitly depend on $\overset{a}{\mathbf{F}}$ as we will discuss in §3.5.

It should be noted that, in general, $\overset{a}{\mathbf{F}}$ is incompatible, i.e., it is not the tangent of any map from $\mathcal{B}$ to itself. Incompatibility of $\overset{a}{\mathbf{F}}$ is a necessary condition for non-flatness of the material metric $\mathbf{G}$, which is

[11] In nonlinear elasticity strain is usually defined using line elements in the reference and current configurations. Instead of using vectors, one can define the material metric in terms of line elements as follows. The line element at $X \in \mathcal{B}$ associated with $\mathring{\mathbf{G}}$ is written as $d\mathring{s}^2 = \mathring{G}_{AB}(X)\,dX^A dX^B$. This is the natural line element in the absence of eigenstrains (the natural distance of two points with coordinates X^A and $X^A + dX^A$ is $d\mathring{s}$). Let us imagine that an infinitesimal segment dX^A is detached from the rest of the body and is allowed to relax in the Euclidean space $(\mathcal{B}, \mathring{\mathbf{G}})$. One obtains a segment $\overset{a}{F}{}^A{}_M\,dX^M$, with length square $ds^2 = \mathring{G}_{AB}\,(\overset{a}{F}{}^A{}_M\,dX^M)\,(\overset{a}{F}{}^B{}_N\,dX^N) = G_{AB}\,dX^A dX^B = (\mathring{G}_{AB}\,\overset{a}{F}{}^A{}_M\,\overset{a}{F}{}^B{}_N)\,dX^M dX^N$. The new metric $G_{AB} = \mathring{G}_{AB}\,\overset{a}{F}{}^A{}_M\,\overset{a}{F}{}^B{}_N$ is the material metric and $ds^2 = G_{AB}\,dX^A dX^B$ is the natural line element in the presence of eigenstrains.

the source of residual stresses. However, it is not sufficient; there are incompatible distributions of $\overset{a}{\mathbf{F}}$ that do not induce residual stresses—contorted aeolotropy [Noll, 1967] or zero-stress (impotent) eigenstrains (or impotent dislocation distributions in the case of plasticity) [Mura, 1989, Sozio and Yavari, 2021].

3.4 The global spatial intermediate configuration

Let us next consider the reverse decomposition $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$. Similarly to (3.2), we define the following strain tensors:

$$
\begin{aligned}
&\overset{e}{\mathbb{C}}^\flat = \overset{e}{\mathbb{F}}^*\mathring{\mathbf{g}}\,, && \overset{e}{\mathbb{C}} = \mathring{\mathbf{G}}^\sharp\overset{e}{\mathbb{C}}^\flat = \overset{e}{\mathbb{F}}^{\mathring{\mathsf{T}}}\overset{e}{\mathbb{F}}\,,\\
&\overset{e}{\mathbb{b}}^\sharp = \overset{e}{\mathbb{F}}_*\mathring{\mathbf{G}}^\sharp\,, && \overset{e}{\mathbb{b}} = \overset{e}{\mathbb{b}}^\sharp\mathring{\mathbf{g}} = \overset{e}{\mathbb{F}}\overset{e}{\mathbb{F}}^{\mathring{\mathsf{T}}}\,,\\
&\overset{e}{\mathbb{B}}^\sharp = \overset{e}{\mathbb{F}}^*\mathring{\mathbf{g}}^\sharp\,, && \overset{e}{\mathbb{B}} = \overset{e}{\mathbb{B}}^\sharp\mathring{\mathbf{G}} = \overset{e}{\mathbb{F}}^{-1}\overset{e}{\mathbb{F}}^{-\mathring{\mathsf{T}}}\,,\\
&\overset{e}{\mathbb{c}}^\flat = \overset{e}{\mathbb{F}}_*\mathring{\mathbf{G}}\,, && \overset{e}{\mathbb{c}} = \mathring{\mathbf{g}}^\sharp\overset{e}{\mathbb{c}}^\flat = \overset{e}{\mathbb{F}}^{-\mathring{\mathsf{T}}}\overset{e}{\mathbb{F}}^{-1}\,.
\end{aligned}
\tag{3.6}
$$

First note that while $\overset{a}{\mathbf{F}}$ is a material tensor, $\overset{a}{\mathbb{F}}$ is a spatial tensor. Also notice that both $\overset{e}{\mathbf{F}}$ and $\overset{e}{\mathbb{F}}$ are two-point tensors. Let us consider a vector $\mathbf{W}$ in the reference configuration. It has the natural length square $\langle\!\langle \mathbf{W}, \mathbf{W}\rangle\!\rangle_{\mathring{\mathbf{G}}}$. The local elastic deformation $\overset{e}{\mathbb{F}}$ maps the vector $\mathbf{W}$ in the reference configuration to $\mathbf{w} = \overset{e}{\mathbb{F}}\mathbf{W}$ in the ambient space. Thus, $\langle\!\langle \mathbf{W}, \mathbf{W}\rangle\!\rangle_{\mathring{\mathbf{G}}} = \langle\!\langle \overset{e}{\mathbb{F}}^{-1}\mathbf{w}, \overset{e}{\mathbb{F}}^{-1}\mathbf{w}\rangle\!\rangle_{\mathring{\mathbf{G}}} = \langle\!\langle \mathbf{w}, \mathbf{w}\rangle\!\rangle_{\overset{e}{\mathbb{F}}_*\mathring{\mathbf{G}}} = \langle\!\langle \mathbf{w}, \mathbf{w}\rangle\!\rangle_{\mathbf{g}}$, where $\mathbf{g} = \overset{e}{\mathbb{F}}_*\mathring{\mathbf{G}} = \overset{e}{\mathbb{c}}^\flat$ is the push-forward of the Euclidean metric of the reference configuration by the local elastic deformation. We call $\mathbf{g}$ the spatial material metric. Therefore, we have the following factorization of the total deformation

$$
(\mathcal{B}, \mathring{\mathbf{G}}) \xrightarrow{\varphi} (\mathcal{C}, \mathbf{g}) \xrightarrow{\mathrm{id}_{\mathcal{C}}} (\mathcal{C}, \mathring{\mathbf{g}})\,. \tag{3.7}
$$

We call $(\mathcal{C}, \mathbf{g})$ the *global spatial intermediate configuration*, which is unique up to isometry by construction.

In summary, we have the following material and spatial factorizations of the total deformation

$$
\begin{tikzcd}
(\mathcal{B}, \mathring{\mathbf{G}}) \arrow[r, "\mathrm{id}_{\mathcal{B}}"] \arrow[d, dotted, "\mathrm{id}_{\mathcal{B}}"] & (\mathcal{B}, \mathbf{G}) \arrow[r, "\varphi"] \arrow[d, "\varphi"] & (\mathcal{C}, \mathring{\mathbf{g}}) \arrow[d, dotted, "\mathrm{id}_{\mathcal{C}}"] \\
(\mathcal{B}, \mathring{\mathbf{G}}) \arrow[r, "\varphi"] & (\mathcal{C}, \mathbf{g}) \arrow[r, "\mathrm{id}_{\mathcal{C}}"] & (\mathcal{C}, \mathring{\mathbf{g}})
\end{tikzcd}
\tag{3.8}
$$

In each of the two rows, the elastic deformation is represented by a map from the material intermediate configuration $(\mathcal{B}, \mathbf{G})$ and from spatial intermediate configuration $(\mathcal{C}, \mathring{\mathbf{g}})$, respectively, to the Euclidean ambient space. Hence, in order for the two intermediate configurations to represent the same anelastic process, they must be isometric. In other words, the multiplicative decompositions $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$ have the same "elastic strain" if and only if $\mathbf{g} = \varphi_*\mathbf{G}$.[12] This means that the metric $\mathbf{g} = \overset{e}{\mathbb{F}}_*\mathring{\mathbf{G}}$ can be written as $\mathbf{g} = \mathbf{F}_*\mathbf{G} = \overset{e}{\mathbf{F}}_*\mathring{\mathbf{G}}$. Thus, $\overset{e}{\mathbb{F}}^*\overset{e}{\mathbf{F}}_*\mathring{\mathbf{G}} = (\overset{e}{\mathbb{F}}^{-1}\overset{e}{\mathbf{F}})_*\mathring{\mathbf{G}} = \mathring{\mathbf{G}}$. This implies that $\overset{e}{\mathbb{F}}^{-1}\overset{e}{\mathbf{F}} = \mathbf{Q}$ is an isometry for $\mathring{\mathbf{G}}$. Therefore,

$$
\overset{e}{\mathbb{F}} = \overset{e}{\mathbf{F}}\boldsymbol{\mathcal{Q}}\,, \qquad \boldsymbol{\mathcal{Q}} \in \mathcal{I}(\mathcal{B}, \mathring{\mathbf{G}})\,, \tag{3.9}
$$

where $\mathcal{I}(\mathcal{B}, \mathring{\mathbf{G}})$ is the isometry group of $(\mathcal{B}, \mathring{\mathbf{G}})$. In components, $F^a{}_A = \overset{e}{F}{}^a{}_B\,\overset{a}{F}{}^B{}_A = \overset{a}{\mathbb{F}}{}^a{}_b\,\overset{e}{\mathbb{F}}{}^b{}_A$, and $\overset{e}{\mathbb{F}}{}^a{}_A = \overset{e}{F}{}^a{}_B\,Q^B{}_A$. The previous discussion can be summarized in the following result.

Theorem 3.1. *The direct* $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ *and reverse* $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$ *decompositions are equivalent if and only if* $\overset{e}{\mathbf{F}}_*\mathring{\mathbf{G}} = \overset{e}{\mathbb{F}}_*\mathring{\mathbf{G}}$*, i.e.,* $\overset{e}{\mathbf{F}}$ *and* $\overset{e}{\mathbb{F}}$ *are equal up to local isometries of the reference configuration* $(\mathcal{B}, \mathring{\mathbf{G}})$.

Finally, it should be noticed that a deformation in terms of the global material intermediate configuration can be described fully referentially. Similarly, it can be described fully spatially using the global spatial

[12] Lagrange strain is calculated using the first row as $\mathbf{E} = \frac{1}{2}(\varphi^*\mathring{\mathbf{g}} - \mathbf{G})$. Similarly, using the second row spatial strain is calculated as $\mathbf{e} = \frac{1}{2}(\mathring{\mathbf{g}} - \mathbf{g}) = \varphi_*\mathbf{E}$.

intermediate configuration. This is shown in the following commutative diagram, which is a generalization of (2.6) to anelasticity.

$$
\begin{tikzcd}
(\mathcal{B}, \mathring{\mathbf{G}}) \arrow[r, "\mathrm{id}_{\mathcal{B}}"] \arrow[d, dotted, "\mathrm{id}_{\mathcal{B}}"] & (\mathcal{B}, \mathbf{G}) \arrow[r, "\mathrm{id}_{\mathcal{B}}"] \arrow[d, dotted, "\mathrm{id}_{\mathcal{B}}"] & (\mathcal{B}, \mathbf{C}^\flat) \arrow[d, dotted, "\varphi"] \\
(\mathcal{B}, \mathring{\mathbf{G}}) \arrow[r, "\mathrm{id}_{\mathcal{B}}"] \arrow[d, dotted, "\mathrm{id}_{\mathcal{B}}"] & (\mathcal{B}, \mathbf{G}) \arrow[r, "\varphi"] \arrow[d, dotted, "\varphi"] & (\mathcal{C}, \mathring{\mathbf{g}}) \arrow[d, dotted, "\mathrm{id}_{\mathcal{C}}"] \\
(\mathcal{B}, \mathring{\mathbf{G}}) \arrow[r, "\varphi"] \arrow[d, dotted, "\varphi"] & (\mathcal{C}, \mathbf{g}) \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, dotted, "\mathrm{id}_{\mathcal{C}}"] & (\mathcal{C}, \mathring{\mathbf{g}}) \arrow[d, dotted, "\mathrm{id}_{\mathcal{C}}"] \\
(\mathcal{C}, \mathbf{c}^\flat) \arrow[r, "\mathrm{id}_{\mathcal{C}}"] & (\mathcal{C}, \mathbf{g}) \arrow[r, "\mathrm{id}_{\mathcal{C}}"] & (\mathcal{C}, \mathring{\mathbf{g}})
\end{tikzcd}
\tag{3.10}
$$

Notice that the first and second rows of the above commutative diagram describe an anelastic deformation using the global material intermediate configuration. The third and fourth rows describe the same deformation using the global spatial intermediate configuration. Note that all the vertical maps are isometries.

Example 3.2 (Radially-symmetric finite eigenstrains in a spherical ball)**.** Consider a homogeneous spherical ball of radius R_o made of a nonlinear elastic solid. The metric of the eigenstrain-free reference configuration $\mathring{\mathbf{G}}$ in the spherical coordinates (R, Θ, Φ) has the representation $\mathring{\mathbf{G}} = \mathrm{diag}\left(1, R^2, R^2 \sin^2\Theta\right)$. For the Euclidean ambient space we choose the spherical coordinates (r, θ, ϕ). The Euclidean metric of the ambient space $\mathring{\mathbf{g}}$ has the representation $\mathring{\mathbf{g}} = \mathrm{diag}\left(1, r^2, r^2 \sin^2\theta\right)$. We assume that the ball has a radially-symmetric distribution of radial $\omega_R = \omega_R(R)$ and circumferential $\omega_\Theta = \omega_\Theta(R)$ finite eigenstrains. This means that with respect to the spherical coordinates (R, Θ, Φ), $\overset{a}{\mathbf{F}}$ has the following representation [Yavari and Goriely, 2013a, Golgoon and Yavari, 2018a, Yavari, 2021][13]

$$
\overset{a}{\mathbf{F}} = \overset{a}{\mathbf{F}}(R) = \begin{bmatrix} e^{\omega_R(R)} & 0 & 0 \\ 0 & e^{\omega_\Theta(R)} & 0 \\ 0 & 0 & e^{\omega_\Theta(R)} \end{bmatrix}. \tag{3.11}
$$

Therefore, metric of the global material intermediate configuration (material metric) reads

$$
\mathbf{G} = \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}} = \begin{bmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R)}\, R^2 & 0 \\ 0 & 0 & e^{2\omega_\Theta(R)}\, R^2 \sin^2\Theta \end{bmatrix}. \tag{3.12}
$$

We assume radial deformations, i.e., $(r, \theta, \phi) = (r(R), \Theta, \Phi)$. Thus, with respect to the spherical coordinates (R, Θ, Φ) and (r, θ, ϕ), the total deformation gradient has the representation $\mathbf{F} = \mathrm{diag}\left(r'(R), 1, 1\right)$. Metric of the global spatial intermediate configuration is written as

$$
\mathbf{g} = \mathbf{F}_*\mathbf{G} = \begin{bmatrix} \frac{e^{2\omega_R(R)}}{r'^2(R)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R)}\, R^2 & 0 \\ 0 & 0 & e^{2\omega_\Theta(R)}\, R^2 \sin^2\Theta \end{bmatrix}. \tag{3.13}
$$

Example 3.3 (Radially-symmetric eigentwists in a circular cylindrical bar)**.** Let us consider a circular cylindrical bar with a radial distribution of eigentwists. This problem was analyzed for isotropic solids in [Yavari and Goriely, 2015] and for orthotropic solids in [Yavari, 2021]. We assume an eigentwist distribution $\psi(R)$. In cylindrical coordinates (R, Θ, Z), $\mathring{\mathbf{G}} = \mathrm{diag}\left(1, R^2, 1\right)$, and $\overset{a}{\mathbf{F}}$ has the following representation

$$
\overset{a}{\mathbf{F}} = \overset{a}{\mathbf{F}}(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi(R) \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.14}
$$

[13] Goodbrake et al. [2020] showed that the eigenstrain distributions (3.11) are the only universal eigenstrains that are consistent with Family 4 universal deformations of incompressible isotropic spherical shells [Ericksen, 1954].

Therefore, metric of the global material intermediate configuration (material metric) reads

$$\mathbf{G} = \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & \psi(R)R^2 \\ 0 & \psi(R)R^2 & 1+\psi^2(R)R^2 \end{bmatrix} . \tag{3.15}$$

For the Euclidean ambient space we choose the cylindrical coordinates (r, θ, z). The Euclidean metric of the ambient space $\mathring{\mathbf{g}}$ has the representation $\mathring{\mathbf{g}} = \mathrm{diag}(1, r^2, 1)$. We assume deformations of the form: $(r, \theta, z) = (r(R), \Theta + \tau Z, \lambda Z)$, where τ and λ are some unknown constants to be determined. The deformation gradient is written as

$$\mathbf{F} = \begin{bmatrix} r'(R) & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & \lambda \end{bmatrix} . \tag{3.16}$$

Metric of the global spatial intermediate configuration reads

$$\mathbf{g} = \mathbf{F}_{*}\mathbf{G} = \begin{bmatrix} \frac{1}{r'^2(R)} & 0 & 0 \\ 0 & R^2 & \lambda^{-1}R^2[\psi(R)-\tau] \\ 0 & \lambda^{-1}R^2[\psi(R)-\tau] & \lambda^{-2}\left[R^2[\tau-\psi(R)]^2+1\right] \end{bmatrix} . \tag{3.17}$$

Remark 3.4. Goodbrake et al. [2021] provided an interpretation of Kondo [1950a]'s material manifold of nonlinear anelasticity, and found that any Riemannian manifold $(\mathcal{M}, \mathbf{H})$ is a global intermediate configuration if there exist two maps $\overset{a}{\varphi} : \mathcal{B} \to \mathcal{M}$ and $\overset{e}{\varphi} : \mathcal{M} \to \mathcal{C}$ such that i) $\varphi = \overset{e}{\varphi} \circ \overset{a}{\varphi}$, and ii) $\mathbf{H} = \overset{a}{\varphi}_{*}\mathbf{G}$. Hence, the global material intermediate configuration corresponds to the choices $\overset{a}{\varphi} = \mathrm{id}_{\mathcal{B}}$ and $\overset{e}{\varphi} = \varphi$, while the global spatial intermediate configuration corresponds to $\overset{a}{\varphi} = \varphi$ and $\overset{e}{\varphi} = \mathrm{id}_{\mathcal{C}}$. Note that the fact that $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{C}, \mathbf{g})$ are isometric reflects the principle that Goodbrake et al. [2021] called "equality of anelastic strain". Note also that although $T\overset{a}{\varphi}$ and $T\overset{e}{\varphi}$ are compatible by construction, the incompatibility of the local anelastic deformations is reflected in the non-flatness of the intermediate manifold $(\mathcal{M}, \mathbf{H})$.

3.5 Constitutive equations in nonlinear anelasticity

In §2.2 for an elastic solid an energy function of the form $\mathring{W} = \mathring{W}(X, \mathbf{F}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$ was assumed. In the presence of local anelastic deformations, energy explicitly depends on the local elastic deformation, i.e., $W = \mathring{W}(X, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$. Objectivity implies that this can be put in the form $\widehat{W}(X, \overset{e}{\mathbf{C}}{}^{\flat}, \mathring{\mathbf{G}})$, where $\overset{e}{\mathbf{C}}{}^{\flat} = \overset{e}{\mathbf{F}}{}^{*}\mathring{\mathbf{g}}$. Changing variables, one has

$$\widehat{W}(X, \overset{e}{\mathbf{C}}{}^{\flat}, \mathring{\mathbf{G}}) = \widehat{W}(X, \overset{a}{\mathbf{F}}_{*}\mathbf{C}^{\flat}, \overset{a}{\mathbf{F}}_{*}\mathbf{G}) . \tag{3.18}$$

This cannot be put in any of the forms $W = \check{W}(X, \mathbf{C}^{\flat}, \mathbf{G}) = \widehat{W}(X, \mathbf{C}^{\flat}, \mathring{\mathbf{G}}) = \overline{W}(X, \overset{e}{\mathbf{C}}{}^{\flat}, \mathbf{G})$. In other words, the material metric $\mathbf{G}$ is not enough to describe the constitutive behavior of an anisotropic material, and an explicit $\overset{a}{\mathbf{F}}$-dependence is unavoidable:

$$W = \widetilde{W}(X, \mathbf{C}^{\flat}, \mathbf{G}, \overset{a}{\mathbf{F}}) = \widehat{W}(X, \mathbf{C}^{\flat}, \mathring{\mathbf{G}}, \overset{a}{\mathbf{F}}) = \overline{W}(X, \overset{e}{\mathbf{C}}{}^{\flat}, \mathbf{G}, \overset{a}{\mathbf{F}}) . \tag{3.19}$$

Let us define $W(X, \mathbf{F}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(X, \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$.

3.5.1 Material symmetry in anelasticity

Note that

$$\begin{aligned} W(X, \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) &= \mathring{W}(X, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \overset{e}{\mathbf{F}}\mathring{\mathbf{K}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(X, \overset{e}{\mathbf{F}}\mathring{\mathbf{K}}\overset{a}{\mathbf{F}}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \\ &= W(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathbf{K}}\overset{a}{\mathbf{F}}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(X, \mathbf{F}\mathbf{K}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) , \qquad \forall\, \mathring{\mathbf{K}} \in \mathring{\mathcal{G}}_X , \end{aligned} \tag{3.20}$$

where $\mathbf{K} = \overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathbf{K}}\overset{a}{\mathbf{F}}$. This means that $\mathcal{G}_X = \overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathcal{G}}_X\overset{a}{\mathbf{F}}$, which is Noll's rule [Noll, 1958, Coleman and Noll, 1959, 1963, 1964]. Material symmetry can be expressed in the following three equivalent forms:[14]

$$W(X, \mathbf{F}, \mathring{\mathbf{K}}\overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}(\mathring{\mathbf{K}}\overset{a}{\mathbf{F}})^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathbf{K}}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$$

[14] One should note that for any invertible tensor $\mathbf{A}$ on has $W(\mathbf{F}\mathbf{A}, \overset{a}{\mathbf{F}}\mathbf{A}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(\mathbf{F}\mathbf{A}(\overset{a}{\mathbf{F}}\mathbf{A})^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(\mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(\mathbf{F}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}})$.

$$= W(X, \mathbf{F}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \,, \qquad \forall\, \mathring{\mathbf{K}} \in \mathring{\mathcal{G}}_X \,, \tag{3.21}$$

$$\begin{aligned} W(X, \mathbf{F}\mathbf{K}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) &= \mathring{W}(X, \mathbf{F}\mathbf{K}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathbf{K}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \\ &= W(X, \mathbf{F}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \,, \qquad \forall\, \mathbf{K} \in \mathcal{G}_X \,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} W(X, \mathbf{F}, \overset{a}{\mathbf{F}}\mathbf{K}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) &= \mathring{W}(X, \mathbf{F}(\overset{a}{\mathbf{F}}\mathbf{K})^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\mathbf{K}^{-1}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathbf{K}}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \\ &= \mathring{W}(X, \mathbf{F}\overset{a}{\mathbf{F}}{}^{-1}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) = W(X, \mathbf{F}, \overset{a}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathring{\mathbf{g}}) \,, \qquad \forall\, \mathbf{K} \in \mathcal{G}_X \,. \end{aligned} \tag{3.23}$$

We assume that the energy function is materially covariant. Using (2.13) this implies that

$$\begin{aligned} W &= \mathsf{W}(X, \overset{e}{\mathbf{C}}{}^\flat, \mathring{\mathbf{G}}, \mathring{\boldsymbol{\zeta}}_1, \dots, \mathring{\boldsymbol{\zeta}}_N) \\ &= \mathsf{W}(X, \overset{a}{\mathbf{F}}_*\mathbf{C}^\flat, \overset{a}{\mathbf{F}}_*\mathbf{G}, \mathring{\boldsymbol{\zeta}}_1, \dots, \mathring{\boldsymbol{\zeta}}_N) \\ &= \mathsf{W}(X, \mathbf{C}^\flat, \mathbf{G}, \overset{a}{\mathbf{F}}{}^*\mathring{\boldsymbol{\zeta}}_1, \dots, \overset{a}{\mathbf{F}}{}^*\mathring{\boldsymbol{\zeta}}_N) \\ &= \mathsf{W}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_N) \,, \end{aligned} \tag{3.24}$$

where $\boldsymbol{\zeta}_i = \overset{a}{\mathbf{F}}{}^*\mathring{\boldsymbol{\zeta}}_i$, $i = 1, \dots N$. Hence, the material symmetry group of an anelastic body can be characterized using the structural tensors $\boldsymbol{\zeta}_i$ of order μ_i, $i = 1, \dots, N$:

$$\mathbf{Q} \in \mathcal{G} \leqslant \mathrm{Orth}(\mathbf{G}) \iff \langle \mathbf{Q} \rangle_{\mu_1} \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_1 \,, \dots \,, \langle \mathbf{Q} \rangle_{\mu_N} \boldsymbol{\zeta}_N = \boldsymbol{\zeta}_N \,, \tag{3.25}$$

where $\mathrm{Orth}(\mathbf{G}) = \{\mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \mid \mathbf{Q}^\mathsf{T} = \mathbf{Q}^{-1}\}$, and $(.)^\mathsf{T}$ is defined with respect to $\mathbf{G}$, and not $\mathring{\mathbf{G}}$. In other words, the material symmetry group of an anelastic body is written with respect to the material manifold $(\mathcal{B}, \mathbf{G})$. Note that Noll's rule implies that $\mathbf{Q} = \overset{a}{\mathbf{F}}{}^{-1}\mathring{\mathbf{Q}}\overset{a}{\mathbf{F}}$ and induces an isomorphism between $\mathrm{Orth}(\mathbf{G})$ and $\mathrm{Orth}(\mathring{\mathbf{G}})$. In summary, (3.24) implies that in the classical anisotropic constitutive equations of elasticity if one replaces $\mathring{\mathbf{G}}$ by $\mathbf{G}$ one finds the corresponding anelastic constitutive equation.[15]

3.5.2 Constitutive equations written with respect to $(\mathcal{C}, \mathbf{g})$

Next, we would like to write the constitutive equations with respect to the global intermediate configuration $(\mathcal{C}, \mathbf{g})$. Let us denote the symmetry group of the material relative to $(\mathcal{C}, \mathbf{g})$ by $\widetilde{\mathcal{G}}$. Note that $\mathcal{C} = \varphi(\mathcal{B})$ and $\mathbf{g} = \varphi_*\mathbf{G}$. Thus, Noll's rule [Noll, 1958, Coleman and Noll, 1959, 1963, 1964] tells us that

$$\widetilde{\mathcal{G}} = \varphi_*\mathcal{G} = \mathbf{F}\,\mathcal{G}\,\mathbf{F}^{-1} \,. \tag{3.26}$$

This means that $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are conjugate subgroups of the general linear group, and hence, are isomorphic. The relation (3.26) holds if and only if it holds for all the generators of the group $\mathcal{G}$. Let us assume that $\mathcal{G}$ is finitely generated and denote the generating sets of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ by $\{\mathbf{Q}_1, \dots, \mathbf{Q}_m\}$ and $\{\widetilde{\mathbf{Q}}_1, \dots, \widetilde{\mathbf{Q}}_m\}$, respectively. In this case, (3.26) holds if and only if $\widetilde{\mathbf{Q}}_j = \mathbf{F}\,\mathbf{Q}_j\,\mathbf{F}^{-1}$, $j = 1, \dots, m$. The material symmetry group with respect to $(\mathcal{C}, \mathbf{g})$ is characterized using the structural tensors $\widetilde{\boldsymbol{\zeta}}_i$ of order μ_i, $i = 1, \dots, N$:

$$\widetilde{\mathbf{Q}} \in \widetilde{\mathcal{G}} \leqslant \mathrm{Orth}(\mathbf{g}) \iff \langle \widetilde{\mathbf{Q}} \rangle_{\mu_1} \widetilde{\boldsymbol{\zeta}}_1 = \widetilde{\boldsymbol{\zeta}}_1 \,, \dots \,, \langle \widetilde{\mathbf{Q}} \rangle_{\mu_N} \widetilde{\boldsymbol{\zeta}}_N = \widetilde{\boldsymbol{\zeta}}_N \,, \tag{3.27}$$

where using (3.26) and (2.10), one has $\widetilde{\mathbf{Q}} = \varphi_*\mathbf{Q} = \mathbf{F}\,\mathbf{Q}\,\mathbf{F}^{-1}$, and $\widetilde{\boldsymbol{\zeta}}_i = \varphi_*\boldsymbol{\zeta}_i$, $i = 1, \dots, N$. This, in particular, implies that the type of the symmetry group of the material with respect to $(\mathcal{C}, \mathbf{g})$ and $(\mathcal{B}, \mathbf{G})$ is the same. The energy function written with respect to the global intermediate configuration $(\mathcal{C}, \mathbf{g})$ is related to that written with respect to $(\mathcal{B}, \mathbf{G})$, e.g., (3.24), by push forward:

$$\mathsf{w} = \varphi_* W = \mathsf{W}(\varphi^{-1}(x), \varphi_*\mathbf{C}^\flat, \varphi_*\mathbf{G}, \varphi_*\boldsymbol{\zeta}_1, \dots, \varphi_*\boldsymbol{\zeta}_N) =: \widehat{\mathsf{w}}(x, \mathring{\mathbf{g}}, \mathbf{g}, \widetilde{\boldsymbol{\zeta}}_1, \dots, \widetilde{\boldsymbol{\zeta}}_N) \,. \tag{3.28}$$

In particular, for an elastically isotropic anelastic solid from (3.28) one has $\mathsf{w} = \widehat{\mathsf{w}}(x, \mathring{\mathbf{g}}, \mathbf{g})$. For an isotropic elastic solid this is reduced to $\mathsf{w} = \widehat{\mathsf{w}}(x, \mathring{\mathbf{g}}, \mathbf{c}^\flat)$.

[15] This is what was done in [Golgoon and Yavari, 2018a,b].

Theorem 3.5. *The direct* $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ *and reverse* $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$ *decompositions corresponding to the same local elastic deformation have identical corresponding Cauchy stresses. In this sense, for anisotropic solids the two decompositions of deformation gradient are equivalent.*

Proof. For an anisotropic solid, the second Piola-Kirchhoff with respect to $(\mathcal{C}, \mathbf{g})$ is written as

$$\widetilde{\mathbf{s}} = 2\varphi_* \frac{\partial \mathsf{W}}{\partial \mathbf{C}^\flat} = 2\frac{\partial\, \varphi_* \mathsf{W}}{\partial\, \varphi_* \mathbf{C}^\flat} = 2\frac{\partial \widehat{\mathsf{w}}}{\partial \mathring{\mathbf{g}}}\,. \tag{3.29}$$

The Cauchy stress is calculated as

$$\widetilde{\boldsymbol{\sigma}} = 2\widetilde{J}\,(T\,\mathrm{id}_{\mathcal{C}})\,\widetilde{\mathbf{s}}\,(T\,\mathrm{id}_{\mathcal{C}})^\star = 2\widetilde{J}\,\frac{\partial \widehat{\mathsf{w}}}{\partial \mathring{\mathbf{g}}}\,. \tag{3.30}$$

Note that $\det(T\,\mathrm{id}_{\mathcal{C}}) = 1$, and hence,

$$\widetilde{J} = \sqrt{\frac{\det \mathring{\mathbf{g}}}{\det \mathbf{g}}} = \sqrt{\frac{\det \mathring{\mathbf{g}}}{\det \mathbf{G}}}\, \det \mathbf{F} = J\,. \tag{3.31}$$

Therefore,

$$\widetilde{\boldsymbol{\sigma}} = 2J\,\frac{\partial \widehat{\mathsf{w}}}{\partial \mathring{\mathbf{g}}} = \boldsymbol{\sigma}\,, \tag{3.32}$$

where the Doyle-Ericksen formula [Doyle and Ericksen, 1956, Marsden and Hughes, 1994, Yavari et al., 2006] was used in the second equality. □

Example 3.6 (Elastically isotropic anelastic solids)**.** One can show that the principal invariants using the two intermediate configurations are identical. The first invariant with respect to the two intermediate configurations is calculated as

$$\begin{aligned} I_1 &= \mathrm{tr}\,\mathbf{C} = \mathrm{tr}_{\mathbf{G}^\sharp}(\mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F}) = \mathbf{G}^\sharp : \mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F}\,,\\ \widetilde{I}_1 &= \mathrm{tr}_{\mathbf{g}^\sharp}\,\mathring{\mathbf{g}} = \mathrm{tr}_{\mathring{\mathbf{g}}}\,\mathbf{g}^\sharp = \mathrm{tr}_{\mathring{\mathbf{g}}}(\mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star) = \mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star : \mathring{\mathbf{g}} = \mathbf{G}^\sharp : \mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F}\,. \end{aligned} \tag{3.33}$$

Thus, $\widetilde{I}_1 = I_1$. We know that $I_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}\,\mathbf{C}^2\right)$, and $\widetilde{I}_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}\,\mathring{\mathbf{g}}^2\right)$. But notice that

$$\begin{aligned} \mathrm{tr}\,\mathbf{C}^2 &= \mathrm{tr}_{\mathbf{G}^\sharp}(\mathbf{C}\mathbf{G}^\sharp\mathbf{C}) = \mathbf{G}^\sharp : \mathbf{C}\mathbf{G}^\sharp\mathbf{C}\,,\\ \mathrm{tr}\,\mathring{\mathbf{g}}^2 &= \mathrm{tr}_{\mathbf{g}^\sharp}(\mathring{\mathbf{g}}\mathbf{g}^\sharp\mathring{\mathbf{g}}) = \mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star : \mathring{\mathbf{g}}\mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star\mathring{\mathbf{g}} = \mathbf{G}^\sharp : \mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F} = \mathbf{G}^\sharp : \mathbf{C}\mathbf{G}^\sharp\mathbf{C}\,. \end{aligned} \tag{3.34}$$

Hence, $\widetilde{I}_2 = I_2$. Finally

$$\begin{aligned} I_3 &= \det\mathbf{C} = \det(\mathbf{C}^\flat\mathbf{G}^\sharp) = \det(\mathbf{F}^\star\mathring{\mathbf{g}}\mathbf{F})\det\mathbf{G}^\sharp = (\det\mathbf{F})^2\det\mathring{\mathbf{g}}\,(\det\mathbf{G})^{-1}\,,\\ \widetilde{I}_3 &= \det(\mathring{\mathbf{g}}\mathbf{g}^\sharp) = \det\mathring{\mathbf{g}}\det(\mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star) = (\det\mathbf{F})^2\det\mathring{\mathbf{g}}\,(\det\mathbf{G})^{-1}\,, \end{aligned} \tag{3.35}$$

and hence, $\widetilde{I}_3 = I_3$. For a compressible elastically isotropic anelastic solid, $W = W(X, I_1, I_2, I_3)$, where I_1, I_2, and I_3 are the principal invariants of the right Cauchy-Green deformation tensor calculated using $\mathbf{G}$ as in (3.33), (3.34), and (3.35). With respect to the global material intermediate configuration and the deformation map $\varphi : (\mathcal{B}, \mathbf{G}) \to (\mathcal{C}, \mathring{\mathbf{g}})$, and using (2.12) one writes $\mathbf{S} = 2W_1\mathbf{G}^\sharp + 2W_2(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}) + 2W_3I_3\mathbf{C}^{-1}$, where $W_i = \frac{\partial W}{\partial I_i}$, $i = 1, 2, 3$. Similarly, the Cauchy stress has the following representation [Doyle and Ericksen, 1956, Truesdell and Noll, 2004]

$$\boldsymbol{\sigma} = \frac{2}{\sqrt{I_3}}\left[W_1\,\mathbf{b}^\sharp + (I_2W_2 + I_3W_3)\mathring{\mathbf{g}}^\sharp - I_3W_2\,\mathbf{c}^\sharp\right]\,. \tag{3.36}$$

With respect to the global spatial intermediate configuration and the map $\mathrm{id}_{\mathcal{C}} : (\mathcal{C}, \mathbf{g}) \to (\mathcal{C}, \mathring{\mathbf{g}})$, the Cauchy stress has the following representation

$$\widetilde{\boldsymbol{\sigma}} = \frac{2}{\sqrt{I_3}}\left[W_1\,\widetilde{\mathbf{b}}^\sharp + (I_2W_2 + I_3W_3)\mathring{\mathbf{g}}^\sharp - I_3W_2\,\widetilde{\mathbf{c}}^\sharp\right]\,, \tag{3.37}$$

where $\widetilde{\mathbf{c}}^\flat = (\mathrm{id}_{\mathcal{C}})_* \mathbf{g} = \mathbf{g} = \varphi_* \mathbf{G} = \mathbf{c}^\flat$, and $\widetilde{\mathbf{b}}^\sharp = (\mathrm{id}_{\mathcal{C}})_* \mathbf{g}^\sharp = \mathbf{g}^\sharp$. Note that $\mathbf{g} = \mathbf{F}^{-\star}\,\mathbf{G}\,\mathbf{F}^{-1}$, and hence $\mathbf{g}^\sharp = \mathbf{F}\,\mathbf{G}^\sharp\,\mathbf{F}^\star$. Thus, $\widetilde{\mathbf{b}}^\sharp = \mathbf{b}^\sharp$. This means that, as expected, $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$.

Example 3.7 (Elastically transversely isotropic anelastic solids)**.** As the simplest example of an elastically anisotropic anelastic solid let us consider transverse isotropy. A transversely isotropic solid at every point $X \in \mathcal{B}$ has a plane of isotropy. When there are no anelastic strains, the plane of isotropy has the unit normal vector $\mathring{\mathbf{N}}$ such that $\langle\!\langle \mathring{\mathbf{N}}, \mathring{\mathbf{N}} \rangle\!\rangle_{\mathring{\mathbf{G}}} = 1$. The energy function has the form $W = \mathsf{W}(X, \overset{e}{\mathbf{C}}^\flat, \mathring{\mathbf{G}}, \mathring{\boldsymbol{\zeta}})$, where $\mathring{\boldsymbol{\zeta}} = \mathring{\mathbf{N}} \otimes \mathring{\mathbf{N}}$ is a structural tensor [Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000]. Thus, knowing that the energy function is materially covariant, one has

$$W = \mathsf{W}(X, \overset{e}{\mathbf{C}}^\flat, \mathring{\mathbf{G}}, \mathring{\boldsymbol{\zeta}}) = \mathsf{W}(X, \overset{a}{\mathbf{F}}_* \mathbf{C}^\flat, \overset{a}{\mathbf{F}}_* \mathbf{G}, \mathring{\boldsymbol{\zeta}}) = \mathsf{W}(X, \mathbf{C}^\flat, \mathbf{G}, \overset{a}{\mathbf{F}}{}^* \mathring{\boldsymbol{\zeta}}) = \mathsf{W}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta})\,, \tag{3.38}$$

where $\boldsymbol{\zeta} = \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{N}} \otimes \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{N}} = \mathbf{N} \otimes \mathbf{N}$. Note that $\mathbf{N} = \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{N}} = \overset{a}{\mathbf{F}}{}^{-1} \mathring{\mathbf{N}}$. This implies that when $\overset{a}{\mathbf{F}}$ evolves, i.e., when $\overset{a}{\mathbf{F}} = \overset{a}{\mathbf{F}}(X,t)$, although the anelastic body remains elastically transversely isotropic, the plane of symmetry at every point evolves as well. Note that in the presence of anelastic strains, $\mathbf{N}(X)$ is a unit vector with respect to the material metric $\mathbf{G}$ because $\langle\!\langle \mathbf{N}, \mathbf{N} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{N}}, \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{N}} \rangle\!\rangle_{\overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{G}}} = \langle\!\langle \mathring{\mathbf{N}}, \mathring{\mathbf{N}} \rangle\!\rangle_{\mathring{\mathbf{G}}} = 1$. The integrity basis consists of five members $\{I_1, I_2, I_3, I_4, I_5\}$, where the first three are the principal invariants of $\mathbf{C}$. With respect to the global material intermediate configuration $(\mathcal{B}, \mathbf{G})$ the extra invariants are defined as

$$I_4 = \mathbf{C}^\flat(\mathbf{N}, \mathbf{N}) = N^A N^B\, C_{AB}\,, \quad I_5 = (\mathbf{C}^\flat \mathbf{G}^\sharp \mathbf{C}^\flat)(\mathbf{N}, \mathbf{N}) = N^A N^B\, C_{BM}\, C^M{}_A\,. \tag{3.39}$$

We know that $\widetilde{\boldsymbol{\zeta}} = \varphi_* \boldsymbol{\zeta} = \varphi_* \mathbf{N} \otimes \varphi_* \mathbf{N} = \mathbf{n} \otimes \mathbf{n}$, where $\mathbf{n} = \mathbf{F}\mathbf{N}$. Thus, with respect to the global spatial intermediate configuration $(\mathcal{C}, \mathbf{g})$, the energy function has the functional form $\mathsf{w} = \widehat{\mathsf{w}}(x, \mathring{\mathbf{g}}, \mathbf{g}, \mathbf{n} \otimes \mathbf{n})$. Note that $\widetilde{I}_4 = \mathring{\mathbf{g}}(\mathbf{n}, \mathbf{n}) = \mathring{\mathbf{g}}(\varphi_* \mathbf{N}, \varphi_* \mathbf{N}) = \varphi^* \mathring{\mathbf{g}}(\mathbf{N}, \mathbf{N}) = \mathbf{C}^\flat(\mathbf{N}, \mathbf{N}) = I_4$. Also, $\widetilde{I}_5 = (\mathring{\mathbf{g}}\, \mathbf{g}^\sharp\, \mathring{\mathbf{g}})(\mathbf{n}, \mathbf{n}) = \varphi^*(\mathring{\mathbf{g}}\, \mathbf{g}^\sharp\, \mathring{\mathbf{g}})(\mathbf{N}, \mathbf{N}) = (\varphi^* \mathring{\mathbf{g}}\, \varphi^* \mathbf{g}^\sharp\, \varphi^* \mathring{\mathbf{g}})(\mathbf{N}, \mathbf{N}) = (\mathbf{C}^\flat \mathbf{G}^\sharp \mathbf{C}^\flat)(\mathbf{N}, \mathbf{N}) = I_5$. Therefore, as expected, the Cauchy stresses calculated using the two intermediate configurations are identical.

4 Concluding Remarks

In this paper we studied the geometry of the reverse multiplicative decomposition $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$. Over the years, intermediate configuration and its interpretation has been a controversial topic in anelasticity. Following the early works of Eckart [1948] and Kondo [1949] for elastically isotropic anelastic solids one can bypass this configuration and directly start with a material manifold—a Riemannian manifold whose metric encodes the anelastic strains. However, for elastically anisotropic solids, in addition to the material metric $\mathbf{G}$, $\overset{a}{\mathbf{F}}$ (or $\overset{a}{\mathbb{F}}$) is needed in describing the constitutive equations. Constructing global intermediate configurations and understanding their connections with the material manifold is a fundamental problem in nonlinear anelasticity. Goodbrake et al. [2021] provided an interpretation of the material manifold $(\mathcal{B}, \mathbf{G})$ as a global intermediate configuration. We call $(\mathcal{B}, \mathbf{G})$ the global material intermediate configuration. It is exactly what one would call material manifold and can be identified with the natural configuration of a residually-stressed anelastic body. By construction the intermediate configuration $(\mathcal{B}, \mathbf{G})$ is unique up to isometry.

In the literature there have been discussions on other possible decompositions, and particularly, the reverse decomposition of the deformation gradient $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}}$. It has been shown that under certain assumptions the reverse decomposition is equivalent to the Bilby-Kröner-Lee decomposition for isotropic solids. In this paper we showed the equivalence of the two decompositions for anisotropic solids. First, we constructed a global spatial intermediate configuration $(\mathcal{C}, \mathbf{g})$ without assuming any relationship between $\overset{e}{\mathbb{F}}$ and $\overset{e}{\mathbf{F}}$. In the spatial intermediate configuration $\mathcal{C} = \varphi(\mathcal{B})$, and $\mathbf{g} = \overset{e}{\mathbb{F}}_* \mathring{\mathbf{G}}$, where $\mathring{\mathbf{G}}$ is the flat metric of the Euclidean reference configuration $(\mathcal{B}, \mathring{\mathbf{G}})$. Next, we noted that the two decompositions $\mathbf{F} = \overset{a}{\mathbb{F}}\overset{e}{\mathbb{F}} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$ locally represent the same anelastic deformation if and only if they induce the same elastic strain (and consequently anelastic strain). We proved that this is equivalent to $\overset{e}{\mathbb{F}}$ and $\overset{e}{\mathbf{F}}$ being equal up to isometry, i.e., $\overset{e}{\mathbb{F}}(X) = \overset{e}{\mathbf{F}}(X)\boldsymbol{\mathcal{Q}}(X)$, where $\boldsymbol{\mathcal{Q}}$ is an isometry of $(T_X\mathcal{B}, \mathring{\mathbf{G}})$. We also showed that this is equivalent to the two manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{C}, \mathbf{g})$ being isometric.

The constitutive equations of elastically isotropic anelastic solids can be written with respect to $(\mathcal{B}, \mathbf{G})$; in the classical constitutive equations the Euclidean metric $\mathring{\mathbf{G}}$ is replaced by the Riemannian metric $\mathbf{G}$ that encodes the anelastic strains. In the case of elastically anisotropic anelastic solids, in addition to $\mathbf{G}$, $\overset{a}{\mathbf{F}}$ explicitly enters the constitutive equations through the structural tensors. It was shown that the constitutive equations with respect to $(\mathcal{C}, \mathbf{g})$ are push forward of those with respect to $(\mathcal{B}, \mathbf{G})$ by the deformation mapping, which is an isometry between the two intermediate configurations. This, in particular, implies that the Cauchy stresses calculated with respect to the two intermediate configurations are identical. In this sense, the two decompositions of deformation gradient are equivalent even for arbitrary anisotropic solids.

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