Elastica catastrophe machine: theory, design and experiments

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- SUPPLEMENTARY MATERIAL -

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1 Complementary equations for the theoretical framework of the *elastica catastrophe machines*

This Section is devoted to further considerations regarding the theoretical framework of the *elastica catastrophe machines*, with emphasis on the reduction of the dependencies of the control parameters on the physical and kinematical coordinates. Moreover, suggested initial values for such control parameters are proposed to initiate the evolution of the deformed shape within the inextensibility set and in the neighbourhood of an (if existing) effective catastrophe locus. Finally, a detailed description of the algorithm for the numerical evaluation of the catastrophe loci is presented.

1.1 Reducing the dependencies between coordinates

As discussed in the main document, the generic configuration of the rod can be equivalently described (i.) through the three coordinates $\{X_l, Y_l, \Theta_l\}$ expressing the position of the rod's final end, (ii.) through three primary kinematical quantities $\{d, \theta_A, \theta_S\}$ or (iii.) through the two control parameters $\{p_1, p_2\}$. The discrepancy in the number of the variables in each of these three representations suggests that the coordinates p_j in the equation

$$p_j = \widetilde{p}_j(X_l, Y_l, \Theta_l, \mathbf{q}), \qquad j = 1, 2, \tag{SM 1}$$

can be also expressed as function of only two coordinates of the rod's final end, namely

$$p_j = \widetilde{p}_j(Y_l, \Theta_l, \mathbf{q}) \qquad \text{or} \qquad p_j = \widetilde{p}_j(X_l, \Theta_l, \mathbf{q}), \qquad j = 1, 2, \tag{SM 2}$$

so that the substitution into[1] eqn (21) provides one of the two coordinates in terms of the other coordinate and the rotation as

$$X_l = X_l(Y_l, \Theta_l, \mathbf{q}), \qquad \text{or} \qquad Y_l = Y_l(X_l, \Theta_l, \mathbf{q}). \tag{SM 3}$$

[1]Equation numbering "(X)" is referred to the equations of the main document while "(SM X)" to those of the supplementary material as "(SM X)".

Considering the two relations (SM 3) into eqns (20) leads to two implicit relations involving the symmetric and antisymmetric angles together with one of the two physical coordinates,

$$G(Y_l, \theta_A, \theta_S, \mathbf{q}) = X_l\left(Y_l, 2\theta_S, \mathbf{q}\right)\tan(\theta_S - \theta_A) - Y_l = 0$$
$$\text{or} \tag{SM 4}$$
$$H(X_l, \theta_A, \theta_S, \mathbf{q}) = X_l \tan(\theta_S - \theta_A) - Y_l\left(X_l, 2\theta_S, \mathbf{q}\right) = 0.$$

Assuming further that the kinematic rule (21) defined by the machine is such that the implicit function (G or H) has continuous and non-null partial derivative of the implicit function (G or H) with respect to the involved coordinate ($\partial G/\partial Y_l$ or $\partial H/\partial X_l$), such coordinate can be described as a function of the symmetric and antisymmetric angles,

$$Y_l = Y_l(\theta_A, \theta_S, \mathbf{q}) \qquad \text{or} \qquad X_l = X_l(\theta_A, \theta_S, \mathbf{q}). \tag{SM 5}$$

From this property it follows that, considering eqn (20) and (SM 2), the distance d and the control parameters p_j ($j = 1, 2$) can be expressed in turn as functions of the antisymmetric and symmetric angles only

$$d = d(\theta_A, \theta_S, \mathbf{q}), \qquad p_j = \widehat{p}_j(\theta_A, \theta_S, \mathbf{q}), \qquad j = 1, 2. \tag{SM 6}$$

being the latter equation equivalent to

$$p_j = \widehat{p}_j(d, \theta_A, \theta_S, \mathbf{q}), \qquad j = 1, 2, \tag{SM 7}$$

but expressed as a function of a lower number of coordinates. In the cases when the function $G(Y_l, \theta_A, \theta_S, \mathbf{q})$ or $H(X_l, \theta_A, \theta_S, \mathbf{q})$ does not satisfy the aforementioned properties, a different pair of primary kinematical quantities can be taken as variable to express the remaining kinematical quantity and the control parameter vector, namely, as

$$\theta_A = \theta_A(d, \theta_S, \mathbf{q}), \qquad p_j = \widehat{p}_j(d, \theta_S, \mathbf{q}), \qquad j = 1, 2, \tag{SM 8}$$

or equivalently as

$$\theta_S = \theta_S(d, \theta_A, \mathbf{q}), \qquad p_j = \widehat{p}_j(d, \theta_A, \mathbf{q}), \qquad j = 1, 2. \tag{SM 9}$$

It is also worth remarking that, in order to overcome periodicity issues inherent to the trigonometric function arctan, the antisymmetric rotation $\widetilde{\theta}_A$ (as well as the other angular quantities) is evaluated as integration over the time-like parameter t, from an initial and a current state during the controlled ends' history and corresponding to $t = \tau_0$ and $t = \tau$

$$\widetilde{\theta}_A(\tau) = \widetilde{\theta}_A(\tau_0) + \frac{\Theta_l(\tau)}{2} - \int_{\tau_0}^{\tau} \frac{X_l(t)\dot{Y}_l(t) - Y_l(t)\dot{X}_l(t)}{X_l(t)^2 + Y_l(t)^2}\mathrm{d}t, \tag{SM 10}$$

where the superimposed dot stands for the derivative with respect to the time-like parameter t.

1.2 Suggested initial values for the control parameters

In order to initiate the evolution from a configuration in the 'surroundings' of the (if existing) catastrophe locus, the initial values of control parameters $\mathbf{p}(\tau_0)$ are suggested to be selected satisfying the following initial conditions (IC):

(IC1) the end rotation at the final coordinate is null ($\Theta_l(\tau_0) = 0$), implying a null initial value of the symmetric angle

$$\theta_S(\tau_0) = 0; \qquad \text{(SM 11)}$$

(IC2) the initial deformed configuration displays two inflection points, implying the satisfaction of the modulus restriction for the antisymmetric angle ($|\theta_A(\tau_0)| < \pi$) provided by eqn (26).

Being the symmetric angle θ_S only dependent on Θ_l, eqn $(19)_3$, it follows that imposing the initial condition (IC1) through eqn (SM 11) is equivalent to impose

$$p_2(\tau_0) = p_2^0 = -\upsilon. \qquad \text{(SM 12)}$$

1.3 Numerical algorithm for the evaluation of catastrophe sets

The evaluation of the 'catastrophe set' $\mathcal{C}_K$, as the elastica set $\mathcal{E}_K$ intersection with the 'snap-back set' $\mathcal{S}_K$, can be performed only numerically. A specific algorithm was developed in *Mathematica*, whose steps are:

Step 1. to define the discrete set of control parameter vectors $\mathbf{p} \subseteq \mathcal{E}_C$. This is performed as a fine discretization assuming a maximum spacing equal to 10^{-3} for each of the two control parameters;

Step 2. to relate the discrete set of $\mathcal{E}_C$ (introduced at the previous step) to the corresponding projections in the primary kinematical space and the physical plane, as the discretization of the elastica sets $\mathcal{E}_K$ and $\mathcal{E}_P$;

Step 3. to identify the catastrophe set $\mathcal{C}_C$ through the evaluation of the (approximated) critical control parameter vector $\mathbf{p}^{\mathcal{C}}$, belonging to discretization of the elastica set $\mathcal{E}_C$ and numerically found by imposing

$$\mathbf{p}^{\mathcal{C}}: \qquad \left| \theta_S^{sb(\pm)}\left(\overline{d}(\mathbf{p},\mathbf{q}), \overline{\theta}_A(\mathbf{p},\mathbf{q})\right) - \overline{\theta}_S(\mathbf{p},\mathbf{q}) \right| < 6 \cdot 10^{-4}\,\pi; \qquad \text{(SM 13)}$$

Step 4. to evaluate catastrophe sets $\{d^{\mathcal{C}}, \theta_A^{\mathcal{C}}, \theta_S^{\mathcal{C}}\}$ and $\{X_l^{\mathcal{C}}, Y_L^{\mathcal{C}}\}$ as the projection of the critical control parameters $\mathbf{p}^{\mathcal{C}}$ (achieved at the previous step) in the kinematical space and in the physical plane.

2 Complementary equations and considerations about the proposed *elastica catastrophe machines*

The equations presented in Sect. 1 are detailed here for both the families of *elastica catastrophe machines*. In particular, the specific cases of rotation centre coincident ($\kappa_R = \lambda_R = 0$) and infinitely far away ($\sqrt{\kappa_R^2 + \lambda_R^2} \to \infty$) from the the origin are analyzed for the ECM-I respectively in Sects. 2.1.1 and 2.1.2. The limit case of rigid bar with infinite length is developed for ECM-II in Sect. 2.2.1.

2.1 ECM-I

The description of the primary kinematic quantities $\{d, \theta_A, \theta_S\}$ for ECM-I as functions of the control parameter vector $\mathbf{p}$ follows from eqns (19) and (20) in the main text as

$$\begin{aligned}\overline{d}(\mathbf{p}, \mathbf{q}^I) &= \sqrt{(\kappa_R + p_1 \cos p_2)^2 + (\lambda_R + p_1 \sin p_2)^2}\, l,\\ \overline{\theta}_A(\mathbf{p}, \mathbf{q}^I) &= \frac{p_2 + \upsilon}{2} - \arctan \frac{\lambda_R + p_1 \sin p_2}{\kappa_R + p_1 \cos p_2},\\ \overline{\theta}_S(\mathbf{p}, \mathbf{q}^I) &= \frac{p_2 + \upsilon}{2}.\end{aligned} \tag{SM 14}$$

By inverting eqn (38), the control parameters can be expressed as functions of the physical coordinates, in particular the first control parameter is given by

$$\begin{aligned}\widetilde{p}_1(Y_l, \Theta_l, \mathbf{q}^I) &= \frac{Y_l/l - \lambda_R}{\sin(\Theta_l - \upsilon)}, &&\text{for} \quad \Theta_l \neq \upsilon + k\pi,\ k \in \mathbb{Z},\\ \widetilde{p}_1(X_l, \Theta_l, \mathbf{q}^I) &= \frac{X_l/l - \kappa_R}{\cos(\Theta_l - \upsilon)}, &&\text{for} \quad \Theta_l \neq \upsilon + (k + \tfrac{1}{2})\pi,\ k \in \mathbb{Z},\end{aligned} \tag{SM 15}$$

the former equation equivalent to the latter, to be used when the rigid bar is not parallel to the X and Y axis, respectively, while the second control parameter is given by

$$\widetilde{p}_2(\Theta_l, \mathbf{q}^I) = \Theta_l - \upsilon. \tag{SM 16}$$

Exploiting the inverse relations (SM 15) and (SM 16) into eqn (38), one of the two coordinates can be expressed as a function of the other coordinate and the rotation as

$$\begin{aligned}X_l(Y_l, \Theta_l, \mathbf{q}^I) &= \left[\kappa_R + \frac{\frac{Y_l}{l} - \lambda_R}{\tan(\Theta_l - \upsilon)}\right] l, &&\text{for} \quad \Theta_l \neq \upsilon + k\pi,\ k \in \mathbb{Z},\\ Y_l(X_l, \Theta_l, \mathbf{q}^I) &= \left[\lambda_R + \tan(\Theta_l - \upsilon)\left(\frac{X_l}{l} - \kappa_R\right)\right] l, &&\text{for} \quad \Theta_l \neq \upsilon + (k + \tfrac{1}{2})\pi,\ k \in \mathbb{Z}.\end{aligned} \tag{SM 17}$$

Inverting eqn (SM 14) leads to express the control parameters as functions of two of the primary kinematical quantities. In particular, while the second control parameter is only dependent on the antisymmetric angle

$$\widehat{p}_2(\theta_S, \mathbf{q}^I) = 2\theta_S - \upsilon, \tag{SM 18}$$

the first control parameter can be expressed as a function of both the symmetric and antisymmetric angle only when the rotation center of the rigid bar does not lay along the straight line connecting the two rod's ends,

$$\widehat{p}_1(\theta_A, \theta_S, \mathbf{q}^I) = \frac{\kappa_R \tan(\theta_S - \theta_A) - \lambda_R}{\sin(2\theta_S - \upsilon) - \cos(2\theta_S - \upsilon)\tan(\theta_S - \theta_A)}, \quad \text{for} \quad \theta_S + \theta_A \neq \upsilon + k\pi,\ k \in \mathbb{Z}. \tag{SM 19}$$

Differently, in the case when the rotation center of the rigid bar and the two rod's ends are aligned along a straight line, the first control parameter can be expressed as a function of the distance and the symmetric angle as

$$\begin{aligned} \widehat{p}_1(d,\theta_S,\mathbf{q}^I) &= (-1)^k\frac{d}{l} - \frac{\kappa_R}{\cos 2\theta_S - \upsilon}, \\ \widehat{p}_1(d,\theta_S,\mathbf{q}^I) &= (-1)^k\frac{d}{l} - \frac{\lambda_R}{\sin 2\theta_S - \upsilon}, \end{aligned} \qquad \text{for} \qquad \theta_S + \theta_A = \upsilon + k\pi,\ k \in \mathbb{Z}. \tag{SM 20}$$

Finally, whenever the rigid bar is not parallel to the straight line connecting the rod's ends, the distance d for ECM-I can be described as a function of the symmetric and antisymmetric angles as

$$d(\theta_A,\theta_S,\mathbf{q}^I) = \frac{\kappa_R \tan(2\theta_S - \upsilon) - \lambda_R}{\cos(\theta_S - \theta_A)\tan(2\theta_S - \upsilon) - \sin(\theta_S - \theta_A)} l, \quad \text{for} \quad \theta_S + \theta_A \neq \upsilon + k\pi,\ k \in \mathbb{Z}, \tag{SM 21}$$

differently, when the rigid bar becomes parallel, the distance d becomes a free parameter and the angles are constrained to each other

$$\theta_S(\theta_A,\mathbf{q}^I) = \upsilon + k\pi - \theta_A, \qquad \text{or equivalently} \qquad \theta_A(\theta_S,\mathbf{q}^I) = \upsilon + k\pi - \theta_S,\ k \in \mathbb{Z}, \tag{SM 22}$$

where the parameter k for these last parametrizations is given by evolution continuity, similarly to eqn (SM 10).

It is also worth noting that a sufficient condition for the connectivity of the inextensibility set $\mathcal{I}_C$ is given by a position of the rotation center R within such a set,

$${\kappa_R}^2 + {\lambda_R}^2 < 1, \tag{SM 23}$$

a property however not fundamental for the realization of an 'effective' *elastica catastrophe machine*. All the cases reported in Figs. 6, 7 and 8 consider a rotation center R of the rigid bar within the inextensibility set, eqn (SM 23). It is observed that when the design parameters are such that an 'effective catastrophe machine' is realized, the rotation center R is never inside the catastrophe locus. It is also worth remarking that the effectiveness of the catastrophe locus does not imply its simple connection in the control parameters plane. This is indeed the case when the rotation center R lays along the catastrophe set $\mathcal{C}_P$ at the physical coordinates corresponding to the junction point of $\mathcal{C}_P^{(+)}$ with $\mathcal{C}_P^{(-)}$. Its projection in the control parameters plane share only the first coordinate $p_1 = 0$ while has two different values in the second coordinate p_2 (as it can be noted in the left column of Fig. 6).

Finally, the recommendation for the initial value of the control parameters vector $\mathbf{p}(\tau_0) = \{p_1^0, p_2^0 = -\upsilon\}$ to belong to the inextensibility set $\mathcal{I}_C$ constrains the first control parameter to the range $p_1^0 \in \left[p_{1,\min}^0, p_{1,\max}^0\right]$, with

$$\begin{aligned} p_{1,\min}^0 &= \max\left[0, -\kappa_R\cos\upsilon + \lambda_R\sin\upsilon - \sqrt{1 - \kappa_R^2 - \lambda_R^2 + \left(\kappa_R\cos\upsilon - \lambda_R\sin\upsilon\right)^2}\right], \\ p_{1,\max}^0 &= -\kappa_R\cos\upsilon + \lambda_R\sin\upsilon + \sqrt{1 - \kappa_R^2 - \lambda_R^2 + \left(\kappa_R\cos\upsilon - \lambda_R\sin\upsilon\right)^2}. \end{aligned} \tag{SM 24}$$

From imposing the existence of a range inclusive of positive values in the previous equation, the design parameters κ_R and λ_R are constrained to satisfy

$$\begin{cases} \kappa_R^2 + \lambda_R^2 - \left(\kappa_R\cos\upsilon - \lambda_R\sin\upsilon\right)^2 \leq 1, \\ -\kappa_R\cos\upsilon + \lambda_R\sin\upsilon + \sqrt{1 - \kappa_R^2 - \lambda_R^2 + \left(\kappa_R\cos\upsilon - \lambda_R\sin\upsilon\right)^2} > 0. \end{cases}$$

2.1.1 ECM-I with rotation centre R coincident with the origin

Null coordinates for the rotation center ($\kappa_R = \lambda_R = 0$) imply that the distance d reduces to a linear function of the first control parameter p_1 and that the antisymmetric angle θ_A reduces to a linear function of the second control parameter p_2,

$$d = p_1 l, \qquad \theta_A = \frac{\upsilon - p_2}{2} - \left\lfloor \frac{\frac{\upsilon}{\pi} + 1}{2} \right\rfloor 2\pi, \tag{SM 25}$$

where the floor operator $\lfloor\ \rfloor$ provides the greatest integer less than or equal to the relevant argument. Under this condition, the elastica set $\mathcal{E}_K$ reduces to the portion of plane

$$\theta_A + \theta_S = \upsilon - \left\lfloor \frac{\frac{\upsilon}{\pi} + 1}{2} \right\rfloor 2\pi, \tag{SM 26}$$

under the constraint (26). From the analysis of the intersection of such elastica set $\mathcal{E}_K$ and the snap-back surface $\mathcal{S}_K$, it follows that if $\lambda_R = \kappa_R = 0$ then an effective machine is generated only when

$$\left| \upsilon - \left\lfloor \frac{\frac{\upsilon}{\pi} + 1}{2} \right\rfloor 2\pi \right| \lesssim 0.726\pi. \tag{SM 27}$$

2.1.2 ECM-I with rotation centre R infinitely far away from the origin

For large values of $\sqrt{\kappa_R^2 + \lambda_R^2}$, the inextensibility domain approaches an ellipse within the control parameter plane $p_1 - p_2$, described by

$$\mathcal{I}_C = \left\{ \mathbf{p} : \left[p_1 - \sqrt{\kappa_R^2 + \lambda_R^2} \right]^2 + \left(\kappa_R^2 + \lambda_R^2 \right) \left[p_2 - \pi - \arctan \frac{\lambda_R}{\kappa_R} \right]^2 \leq 1 \right\}. \tag{SM 28}$$

In order to satisfy the aforemmentioned inequality, the second control parameter has to assume the approximately constant value $p_2 \simeq \pi + \arctan \lambda_R/\kappa_R$. Therefore, the symmetric angle θ_S takes the approximately constant values

$$\theta_S = \frac{\pi + \upsilon + \arctan \lambda_R/\kappa_R}{2}. \tag{SM 29}$$

It follows that the evolution of the final curvilinear coordinate is just a rigid-body translation within the physical plane, at constant end's rotation $\Theta_l = 2\theta_S$. The equation (SM 29) implies that a centre of rotation very distant from the origin never makes effective ECM-I. In the particular case when $\theta_S = \Theta_l = 0$, attained for $\upsilon + \arctan \lambda_R/\kappa_R = -\pi$, all the points of the catastrophe set of ECM-I become pitchfork bifurcations [1]. Under these hypotheses, the ECM-I could be exploited as a *elastica pitchfork bifurcation machine* to show the infinite possible pitchfork bifurcations of the system, as shown in Fig. SM 1.

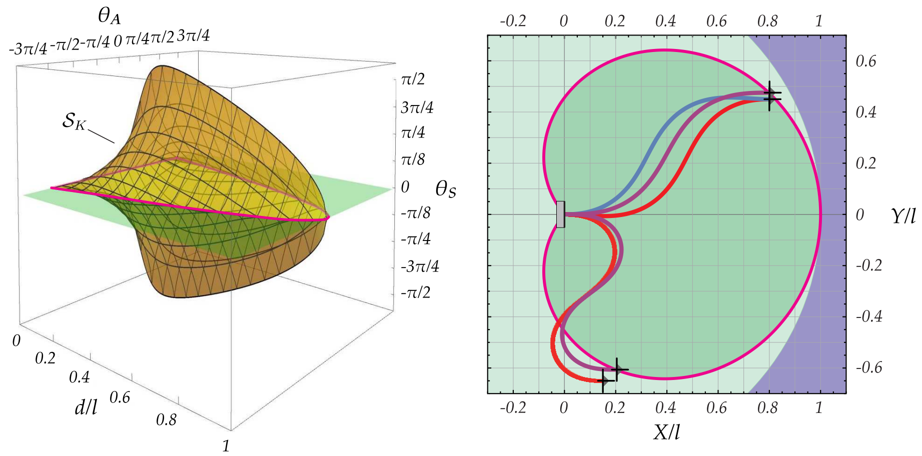

Fig. SM 1: Projection of the bifurcations set for the *elastica pitchfork bifurcation machine* within the primary kinematical space (left) and the physical plane (right). The elastica pitchfork bifurcation machine can be generated from ECM-I with infinitely far rotation center R ($\kappa_R^2 + \lambda_R^2 \to \infty$) and $\upsilon + \arctan \lambda_R/\kappa_R = -\pi$ or from ECM-IIa and ECM-IIb with rigid bar of infinite length ($\rho \to \infty$) and $\upsilon = -\alpha$ and $\upsilon = -\alpha - \pi$, respectively. Deformed configurations are reported within the physical plane for some end position, with those laying along the bifurcation set that display $m = 3$ inflections points (the mid-point and the two ends, elastica reported in purple). Elastica in red (blue) colour has negative (positive) curvature at the initial coordinate, $\Theta'(s = 0)$.

2.2 ECM-II

By considering eqn (42) in eqns (19) and (20), the primary kinematic quantities $\{d, \theta_A, \theta_S\}$ can be expressed for ECM-II as functions of the control parameters vector $\mathbf{p}$ as

$$
\begin{aligned}
\overline{d}(\mathbf{p}, \mathbf{q}^{II}) &= \sqrt{(\kappa_D + p_1 \cos\alpha + \rho \cos p_2)^2 + (\lambda_D + p_1 \sin\alpha + \rho \sin p_2)^2}\; l,\\
\overline{\theta}_A(\mathbf{p}, \mathbf{q}^{II}) &= \frac{p_2 + \upsilon}{2} - \arctan \frac{\lambda_D + p_1 \sin\alpha + \rho \sin p_2}{\kappa_D + p_1 \cos\alpha + \rho \cos p_2},\\
\overline{\theta}_S(\mathbf{p}, \mathbf{q}^{II}) &= \frac{p_2 + \upsilon}{2}.
\end{aligned}
\qquad \text{(SM 30)}
$$

The first control parameter can be expressed as a function of the physical coordinates by inverting eqn (42) as

$$
\begin{aligned}
\widetilde{p_1}(Y_l, \Theta_l, \mathbf{q}^{II}) &= \frac{Y_l/l - \lambda_D - \rho \sin(\Theta_l - \upsilon)}{\sin\alpha}, \qquad \text{for} \quad \alpha \neq k\pi,\\
\widetilde{p_1}(X_l, \Theta_l, \mathbf{q}^{II}) &= \frac{X_l/l - \kappa_D - \rho \cos(\Theta_l - \upsilon)}{\cos\alpha}, \qquad \text{for} \quad \alpha \neq \left(k + \tfrac{1}{2}\right)\pi,
\end{aligned}
\qquad \text{(SM 31)}
$$

where $k \in \mathbb{Z}$. The previous two equations, equivalent to each other, can be used indifferently except for the case of a designed straight line (where the rigid bar rotation center may move) parallel to the X (Y) axis where only the second (first) equation is feasible.
Relations (SM 15) and (SM 31) together with eqn (42) provide one of the position coordinates

as a function of the two remaining physical coordinates as

$$X_l(Y_l, \Theta_l, \mathbf{q}^{II} = \frac{Y_l}{\tan\alpha} + l\left[\kappa_D + \rho\cos(\Theta_l - \upsilon) - \frac{\lambda_D + \rho\sin(\Theta_l - \upsilon)}{\tan\alpha}\right], \qquad \text{for} \quad \alpha \neq k\pi, \tag{SM 32}$$

$$Y_l(X_l, \Theta_l, \mathbf{q}^{II}) = X_l \tan\alpha + \left[\lambda_D + \rho\sin(\Theta_l - \upsilon) - \tan\alpha\left(\kappa_D + \rho\cos(\Theta_l - \upsilon)\right)\right], \qquad \text{for} \quad \alpha \neq (k + \frac{1}{2})\pi.$$

The inversion of eqns (37) and eqn (SM 30) provides the same equation obtained for the ECM-I expressing the second control parameter as a function of the physical rotation and the symmetric angle, respectively

$$\widetilde{p}_2(\Theta_l, \mathbf{q}^{II}) = \Theta_l - \upsilon, \qquad \widehat{p}_2(\theta_S, \mathbf{q}^{II}) = 2\theta_S - \upsilon. \tag{SM 33}$$

The inversion of eqn (SM 30) also provides the expression of the first control parameter as function only of the antisymmetric and symmetric angles

$$\widehat{p}_1(\theta_A, \theta_S, \mathbf{q}^{II}) = \frac{(\kappa_D + \rho\cos(2\theta_S - \upsilon))\tan(\theta_S - \theta_A) - \lambda_D - \rho\sin(2\theta_S - \upsilon))}{\sin\alpha - \cos\alpha\tan(\theta_S - \theta_A)}, \qquad \text{for} \quad \theta_S - \theta_A \neq \alpha + k\pi \tag{SM 34}$$

holding (for $k \in \mathbb{Z}$ and) except when the straight line connecting the two rod's ends is parallel to the designed straight line (where the rigid bar rotation center may move). Differently, under this condition (equivalent to $\alpha + k\pi = \arctan Y_l/X_l$), the first control parameter can be expressed as a function of the distance and symmetric angle as

$$\begin{aligned} \widehat{p}_1(d, \theta_S, \mathbf{q}^{II}) &= (-1)^k\frac{d}{l} - \frac{\kappa_D + \rho\cos(2\theta_S - \upsilon)}{\cos\alpha}, \\ \widehat{p}_1(d, \theta_S, \mathbf{q}^{II}) &= (-1)^k\frac{d}{l} - \frac{\lambda_D + \rho\sin(2\theta_S - \upsilon)}{\sin\alpha}, \end{aligned} \qquad \text{for} \quad \theta_S - \theta_A = \alpha + k\pi, \ k \in \mathbb{Z}. \tag{SM 35}$$

Lastly, one of the three primary kinematical quantities can be expressed as a function of the other two. The distance d for ECM-II can be described as a function of the symmetric and antisymmetric angles as

$$d(\theta_A, \theta_S, \mathbf{q}^{II}) = \frac{\kappa_D + \rho\cos(2\theta_S - \upsilon)\tan\alpha - (\lambda_D + \rho\sin(2\theta_S - \upsilon))}{\cos(\theta_S - \theta_A)\tan\alpha - \sin(\theta_S - \theta_A)} l. \qquad \text{for} \quad \theta_S - \theta_A \neq \alpha + k\pi. \tag{SM 36}$$

As the straight line connecting the two rod's ends is parallel to the designed straight line, the aforementioned equation can not be used and one of the angles θ_A or θ_S can be expressed as a function of the other one (leaving the distance d as free variable)

$$\theta_S(\theta_A, \mathbf{q}^{II}) = \alpha + k\pi + \theta_A, \qquad \text{or equivalently} \qquad \theta_A(\theta_S, \mathbf{q}^{II}) = \theta_S - \alpha - k\pi, \ k \in \mathbb{Z}. \tag{SM 37}$$

Finally, it is worth remarking that the initial value of the control parameter vector is constrained by eqn (SM 12) and the condition of belonging to the inextensibility set $\mathcal{I}_C$. The latter condition constrains the first control parameter within the range $p_1^0 \in \left[p_{1,\min}^0, p_{1,\max}^0\right]$ with

$$p_{1,\min}^0 = -\kappa_D \cos\alpha - \lambda_D \sin\alpha - \rho\cos(\alpha+\upsilon) - \sqrt{(\lambda_D \sin\alpha + \rho\cos(\alpha+\upsilon) + \kappa_D\cos\alpha)^2 - \left({\kappa_D}^2 + {\lambda_D}^2 + \rho^2 - 2\rho(\lambda_D\sin\upsilon - \kappa_D\cos\upsilon) - 1\right)},$$

$$p_{1,\max}^0 = -\kappa_D \cos\alpha - \lambda_D \sin\alpha - \rho\cos(\alpha+\upsilon) + \sqrt{(\lambda_D \sin\alpha + \rho\cos(\alpha+\upsilon) + \kappa_D\cos\alpha)^2 - \left({\kappa_D}^2 + {\lambda_D}^2 + \rho^2 - 2\rho(\lambda_D\sin\upsilon - \kappa_D\cos\upsilon) - 1\right)}, \tag{SM 38}$$

where, in order to have a non-null set for the possible initial vector $\mathbf{p}^0$, the design parameters κ_D, λ_D, α, ρ, and υ have to satisfy the following inequality

$$1 + (\lambda_D \sin\alpha + \rho\cos(\alpha+\upsilon) + \kappa_D\cos\alpha)^2 \geq {\kappa_D}^2 + {\lambda_D}^2 + \rho^2 - 2\rho(\lambda_D\sin\upsilon - \kappa_D\cos\upsilon). \tag{SM 39}$$

2.2.1 ECM-II with the rigid bar of infinite length

In the case of ECM-II with very large values of rigid bar length ($\rho \to \infty$), the inextensibility domain can be approximated by two ellipses within the control parameter plane $p_1 - p_2$. The ellipse for ECM-IIa is defined by

$$\mathcal{I}_C^{IIa} = \left\{\mathbf{p} : \left[p_1 + \left(\rho + \kappa_D\cos\alpha + \lambda_D\sin\alpha - \frac{(\kappa_D\sin\alpha - \lambda_D\cos\alpha)^2}{2\rho}\right)\right]^2 + \rho^2\left[p_2^{(a)} - \alpha - \frac{\kappa_D\sin\alpha - \lambda_D\cos\alpha}{\rho}\right]^2 < 1\right\}. \tag{SM 40}$$

while that for ECM-IIb by

$$\mathcal{I}_C^{IIb} = \left\{\mathbf{p} : \left[p_1 - \left(\rho - \kappa_D\cos\alpha - \lambda_D\sin\alpha - \frac{(\kappa_D\sin\alpha - \lambda_D\cos\alpha)^2}{2\rho}\right)\right]^2 + \rho^2\left[p_2^{(b)} - \alpha - \pi + \frac{\kappa_D\sin\alpha - \lambda_D\cos\alpha}{\rho}\right]^2 < 1\right\}. \tag{SM 41}$$

These two inextensibility domains imply an approximately constant value for the second control parameter, $p_2^{(a)} \simeq \alpha$ and $p_2^{(b)} \simeq \alpha + \pi$, so that the symmetric angle θ_S is also constant

$$\theta_S^{(a)} = \frac{\alpha + \upsilon}{2} \qquad \theta_S^{(b)} = \frac{\alpha + \pi + \upsilon}{2}. \tag{SM 42}$$

Similarly to ECM-I with rotation center infinitely far away from the origin, ECM-II with infinitely long rigid bars is never an effective catastrophe machine and in some special case ($\upsilon = -\alpha$ for ECM-IIa or $\upsilon = -\alpha - \pi$ for ECM-IIb) displays pitchfork bifurcation points as catastrophe set. The projections of this bifurcation set within the primary kinematical space and within the physical plane are coincident with the special cases of ECM-I and reported in Fig. SM 1.

Finally, although the realization of an 'effective' ECM-II is not strictly related to this property, it is noted that the connectivity of its inextensibility domain $\mathcal{I}_C$, eqn (44), is attained when

$$|\kappa_D \sin\alpha - \lambda_D\cos\alpha| + \rho < 1. \tag{SM 43}$$

3 Additional experimental results

Similar to the experimental results shown in Figs. 2 and 17 of the main text, photos taken at specific stages are reported here for ECM-IIb with $\kappa_D = \lambda_D = \alpha = 0$, $\rho = 1$ and $\upsilon = \pi$. In particular, all the stable equilibrium configurations are reported at three stages for in Fig. 2. The clamp position moves from inside the bistable region (left photo) to inside the monostable region (right photo), by crossing the catastrophe locus (central photo) region (from left to right in Figs. 16 and 17). The evolution of the clamp position is ruled by varying the control parameter p_2 at fixed value of p_1. Moreover, the transition of the deformed configuration during

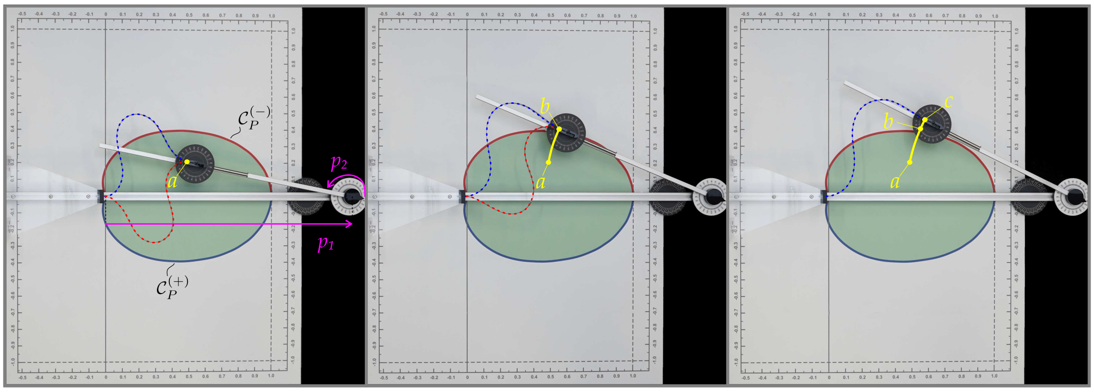

Fig. SM 2: As for Fig. 17 but for ECM-IIb (with $\kappa_D = \lambda_D = \alpha = 0$, $\rho = 1$ and $\upsilon = \pi$), involving a different definition of the control parameters $\{p_1, p_2\}$ (left).

a continuous evolution can be also appreciated in Fig. 3 where the fast motion at snapping is shown in the central photo (each two consecutive snapshots are referred to a time interval of approximately 0.15 sec).

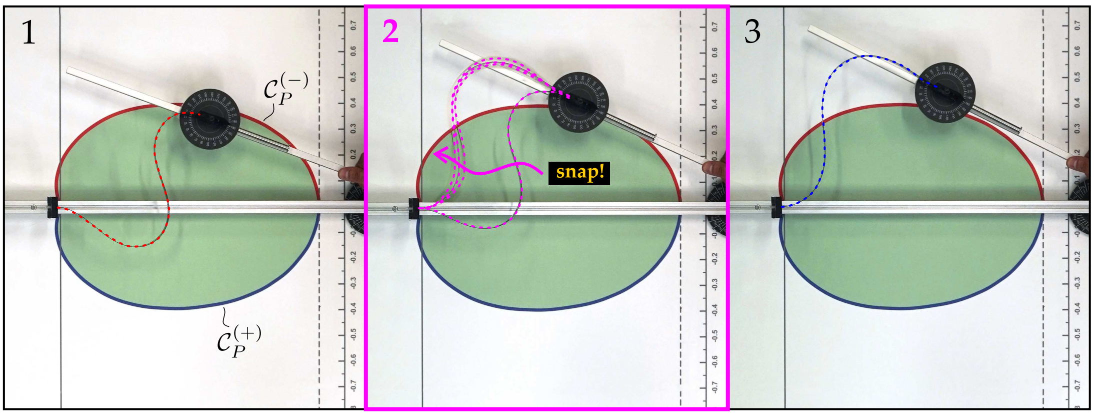

Fig. SM 3: As for Fig. 2, but for ECM-IIb (with $\kappa_D = \lambda_D = \alpha = 0$, $\rho = 1$, $\upsilon = \pi$) with increasing the second control parameter p_2 (rotation angle) and fixed value of p_1. Snapping for a configuration with negative curvature at the two ends (highlighted with a red dashed line) is shown at crossing the part $\mathcal{C}_P^-$ (red line) of catastrophe locus. Due to the symmetry, snapping is not reported for crossing the part $\mathcal{C}_P^+$ (blue line) of catastrophe locus.

References

[1] Cazzolli, A., Dal Corso, F. (2019) Snapping of elastic strips with controlled ends. *Int. J. Sol. Str.* 162, 285–303