3.3 A TRUSS ELEMENT BASED ON GREEN'S STRAIN

In devising the governing equations for the various truss elements, we will not necessarily adopt the most computationally efficient formulation. Instead, we intend to introduce the concepts in forms that can be readily extended to continua, beams and shells. Hence, we will adopt standard finite element procedures using shape functions etc., although such procedures are not strictly necessary for these simple elements. Detail will be given for two-dimensional 'planar truss elements', but it will be shown in Section 3.7 that the procedures and formulae are easily extendible to three-dimensional 'space truss elements'.

3.3.1 Geometry and the strain–displacement relationships

Figure 3.5 shows a truss element P_0Q_0 in its original configuration with a non-dimensional coordinate, ξ, being used to define the position of a point A_0 lying

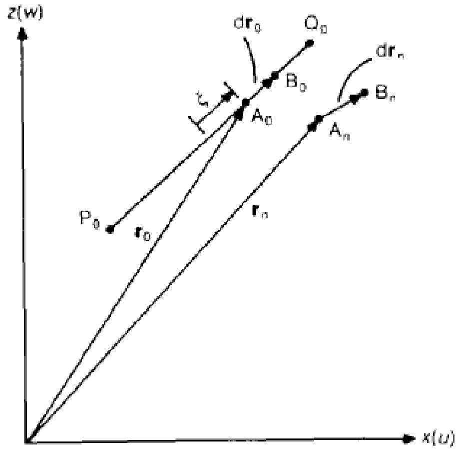

Figure 3.5 Deformation of general truss element.

between P_0 and Q_0. As the truss experiences deformation, points A_0 and the adjacent B_0 move to A_n and B_n respectively. During this process, the position vector, $\mathbf{r}_0$, of point A_0 moves to the position vector, $\mathbf{r}_n$, of A_n, where:

$$\mathbf{r}_n = \mathbf{r}_0 + \mathbf{u} \tag{3.45}$$

and, in two dimensions,

$$\mathbf{r}^T = \{x, z\}, \qquad \mathbf{u}^T = \{u, w\}. \tag{3.46}$$

Equivalent nodal coordinates will be written as

$$\mathbf{x}_n = \mathbf{x}' = \mathbf{x}_o + \mathbf{p} = \mathbf{x} + \mathbf{p} \tag{3.47}$$

where the initial coordinates $\mathbf{x}$ (or $\mathbf{x}_o$, but the subscript o will often be omitted) are

$$\mathbf{x}^T = (x_1, x_2, z_1, z_2) \tag{3.48}$$

and the nodal displacements are (see Figure 3.6)

$$\mathbf{p}^T = (u_1, u_2, w_1, w_2). \tag{3.49}$$

(Note the non-standard ordering of the components of $\mathbf{p}$ and see the footnote on page 25.)

In Figures 3.5 and 3.6, we have introduced the non-dimensional coordinate, ξ, for use with standard finite element shape functions. However, we will initially avoid the use of such shape functions which are not strictly necessary for these simple elements.

By Pythagoras' theorem, the initial length of the element is given by

$$l_o^2 = 4\alpha_o^2 = (x_{21}^2 + z_{21}^2) = \mathbf{x}_{21}^T \mathbf{x}_{21} \tag{3.50}$$

where

$$x_{21} = x_2 - x_1, \qquad z_{21} = z_2 - z_1 \tag{3.51}$$

and

$$\mathbf{x}_{21}^T = (x_{21}, z_{21}). \tag{3.52}$$

matrix

In (3.50), we have, for compatibility with later developments using shape functions (Section 3.3.4), introduced the original 'length parameter', α_o, which is half the original

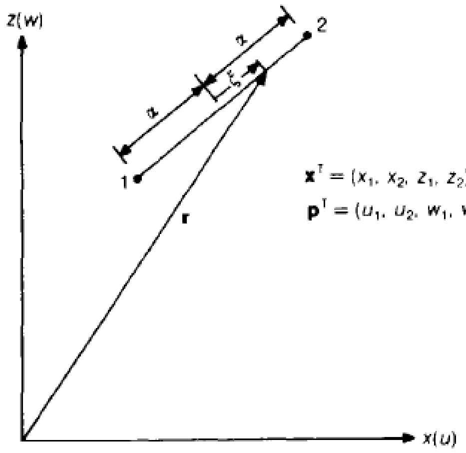

Figure 3.6 Geometry and modes for general truss element.

length, l_o. For the current length, l_n, the equivalent of (3.50) is

$$l_n^2 = 4\alpha_n^2 = (x_{21} + u_{21})^2 + (z_{21} + w_{21})^2 = (\mathbf{x}_{21} + \mathbf{p}_{21})^T(\mathbf{x}_{21} + \mathbf{p}_{21}) \tag{3.53}$$

where in a similar manner to (3.52), $\mathbf{p}_{21}^T = (u_{21}, w_{21})$. Using (3.9), (3.50) and (3.53), Green's strain is given by

$$\varepsilon = \frac{l_n^2 - l_o^2}{2l_o^2} = \frac{(\mathbf{x}_{21} + \mathbf{p}_{21})^T(\mathbf{x}_{21} + \mathbf{p}_{21}) - \mathbf{x}_{21}^T\mathbf{x}_{21}}{2\mathbf{x}_{21}^T\mathbf{x}_{21}}$$

$$= \frac{1}{4\alpha_o^2}(\mathbf{x}_{21}^T\mathbf{p}_{21} + \tfrac{1}{2}\mathbf{p}_{21}^T\mathbf{p}_{21}). \tag{3.54}$$

Equation (3.54) can be re-expressed as

how this is got?

$$\varepsilon = \mathbf{b}_1^T\mathbf{p} + \frac{1}{2\alpha_o^2}\mathbf{p}^T\mathbf{A}\mathbf{p} \tag{3.55}$$

where

$$\mathbf{b}_1^T = \frac{1}{4\alpha_o^2}(-x_{21}, x_{21}, -z_{21}, z_{21}) = \frac{1}{4\alpha_o^2}\mathbf{c}(\mathbf{x})^T \tag{3.56}$$

and

$$A = \frac{1}{4}\begin{bmatrix} 1 & \text{symmetric} & & \\ -1 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{3.57}$$

From (3.54)–(3.57), the incremental Green strain (caused by $\Delta\mathbf{p}$) is given by

$$\Delta\varepsilon = \frac{1}{4\alpha_o^2}((\mathbf{x}_{21} + \mathbf{p}_{21})^T\Delta\mathbf{p}_{21} + \tfrac{1}{2}\Delta\mathbf{p}_{21}^T\Delta\mathbf{p}_{21})$$

$$= (\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p}))^T\Delta\mathbf{p} + \frac{1}{2\alpha_o^2}\Delta\mathbf{p}^T\mathbf{A}\,\Delta\mathbf{p} = \mathbf{b}(\mathbf{p})^T\Delta\mathbf{p} + \frac{1}{2\alpha_o^2}\Delta\mathbf{p}^T\mathbf{A}\,\Delta\mathbf{p} \tag{3.58}$$

where (compare (3.56))

$$\mathbf{b}_2(\mathbf{p})^T = \frac{1}{4\alpha_o^2}(-u_{21}, u_{21}, -w_{21}, w_{21}) = \frac{1}{4\alpha_o^2}\mathbf{c}(\mathbf{p})^T = \frac{1}{\alpha_o^2}(\mathbf{A}\mathbf{p})^T. \tag{3.59}$$

Comparing (3.58) with a Taylor series expansion for $\Delta\varepsilon$,

$$\Delta\varepsilon = \frac{\partial\varepsilon}{\partial\mathbf{p}}\Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^T\frac{\partial\varepsilon}{\partial\mathbf{p}\Delta\mathbf{p}}\Delta\mathbf{p} = \mathbf{b}(\mathbf{p})^T\Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^T\frac{\partial\mathbf{b}}{\partial\mathbf{p}}\Delta\mathbf{p} \tag{3.60}$$

we can see that $(1/\alpha_o^2)\mathbf{A}$ is the second partial derivative of ε with respect to the displacements, $\mathbf{p}$ or the first partial derivative of $\mathbf{b}$ with respect to $\mathbf{p}$.

For a small virtual displacement, with $\delta\mathbf{p}_v$ instead of $\Delta\mathbf{p}$, the last term in (3.58) becomes negligible and

$$\delta\varepsilon_v = \frac{\partial\varepsilon}{\partial\mathbf{p}}\delta\mathbf{p}_v = (\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p}))^T\delta\mathbf{p}_v = \mathbf{b}(\mathbf{p})^T\delta\mathbf{p}_v. \tag{3.61}$$

3.3.2 Equilibrium and the internal force vector

The principle of virtual work (Sections 1.3.2, 2.1 and 3.1) can now be used to provide internal nodal forces, $\mathbf{q}_i$, that are in a weighted average sense [C2.2], in equilibrium with a set of stresses, σ, that relate to total displacements, $\mathbf{p}$. To this end, using (3.61),

$$\sum_e \delta\mathbf{p}_v^T\mathbf{q}_i = \sum_e \int \sigma_G \delta\varepsilon_v \,\mathrm{d}V_o = \sum_e \delta\mathbf{p}_v^T \int \sigma_G \mathbf{b}\,\mathrm{d}V_o \tag{3.62}$$

where $\sum_e$ involves a 'summation' over the elements. For the following developments, we will drop this summation sign and hence will only directly deal with force vectors or stiffness matrices at the element level. The 'merging process' to the structural level is identical to that adopted for linear analysis [C2.2].

The strain–displacement vector $\mathbf{b}$ in (3.62) is given by (3.61) (with (3.56) and (3.59)) while the subscript G on σ follows the work of Section 3.1.5, where it was shown that we must take note of the *type* of stress. The stress σ_G is the stress that is work conjugate to the Green strain (later—Chapter 4—to be called the second Piola–Kirchhoff stress).

Equation (3.62) must stand for arbitrary $\delta\mathbf{p}_v$ and hence using (3.61), (3.56) and (3.59),

$$\mathbf{q}_i = \int \sigma_G \mathbf{b}\,\mathrm{d}V_o = 2\alpha_o A_o \sigma_G \mathbf{b} = 2\alpha_o A_o \sigma_G(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})) = \frac{\sigma_G A_o}{2\alpha_o}(\mathbf{c}(\mathbf{x}) + \mathbf{c}(\mathbf{p})) = \mathbf{q}_{i1} + \mathbf{q}_{i2}. \tag{3.63}$$

Using (3.62), the procedure for computing the internal forces, $\mathbf{q}_i$, from a set of nodal displacements, $\mathbf{p}$, is as follows:

Using (3.62), the procedure for computing the internal forces, $\mathbf{q}_i$, from a set of nodal displacements, $\mathbf{p}$, is as follows:

(1) compute the strain from (3.54) or (3.55);
(2) compute the stress, σ_G (here, constant over the element), assuming a linear material response from $\sigma_G = E\varepsilon$;

(3) compute the internal forces, $\mathbf{q}_i$, from (3.63) with $\mathbf{b}_1$ and $\mathbf{b}_2$ being defined in (3.56) and (3.59).

3.3.3 The tangent stiffness matrix

From (2.20) and (3.63)

how?

$$\mathbf{K}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{q}_i}{\partial \mathbf{p}} = 2\alpha_o A_o \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{p}} + 2\alpha_o A_o \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \sigma_G. \tag{3.64}$$

Using (3.64) and the non-virtual form of (3.61),

$$\frac{\partial \sigma_G}{\partial \mathbf{p}} = E \frac{\partial \varepsilon}{\partial \mathbf{p}} = E\{\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\}^T = E\mathbf{b}(\mathbf{p})^T. \tag{3.65}$$

From (3.64) and (3.65), the first term of (3.64) can be written as

$$2\alpha_o A_o \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{p}} = 2E\alpha_o A_o \mathbf{b}\mathbf{b}^T = \mathbf{K}_{t1} + \mathbf{K}_{t2} \tag{3.66}$$

where

$$K_{t1} = 2EA_o\alpha_o \mathbf{b}_1\mathbf{b}_1^T = \frac{EA}{8\alpha_o^3}\mathbf{c}(\mathbf{x})\mathbf{c}(\mathbf{x})^T \tag{3.67}$$

$$\mathbf{K}_{t2} = 2EA\alpha_o[\mathbf{b}_1\mathbf{b}_2^T + \mathbf{b}_2\mathbf{b}_1^T + \mathbf{b}_2\mathbf{b}_2^T] = \mathbf{K}_{t2a} + \mathbf{K}_{t2a}^T + \mathbf{K}_{t2b}. \tag{3.68}$$

Equation (3.67) provides the standard linear stiffness matrix while (3.68) gives the 'initial displacement (or slope) matrix' (compare (1.10)). The 'geometric' or 'initial-stress matrix' (Section 1.2) comes from the second term in (3.64). Noting that, of the constituents of $\mathbf{b}$ (see (3.58)), only $\mathbf{b}_2$ is a function of $\mathbf{p}$, from (3.64) and (3.59),

$$\mathbf{K}_{t\sigma} = 2\alpha_o A_o \frac{\partial \mathbf{b}}{\partial \mathbf{p}}\sigma_G = 2\alpha_o A_o \frac{\partial \mathbf{b}_2}{\partial \mathbf{p}}\sigma_G = \frac{2A_o\sigma_G}{\alpha_o}\mathbf{A} = \frac{A_o\sigma_G}{2\alpha_o}\begin{bmatrix} 1 & & \text{symmetric} & \\ -1 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{3.69}$$

Equations (3.67) and (3.68) can be expanded to give

$$\mathbf{K}_{t1} = \frac{EA}{8\alpha_o^3}\begin{bmatrix} x_{21}^2 & & \text{symmetric} & \\ -x_{21}^2 & x_{21}^2 & & \\ x_{21}z_{21} & -x_{21}z_{21} & z_{21}^2 & \\ -x_{21}z_{21} & x_{21}z_{21} & -z_{21}^2 & z_{21}^2 \end{bmatrix} = \frac{EA_o}{8\alpha_o^3}\mathbf{c}(\mathbf{x})\mathbf{c}(\mathbf{x})^T \tag{3.70}$$

$$\mathbf{K}_{t2a} = 2EA_o\alpha_o\mathbf{b}_1\mathbf{b}_2^T = \frac{EA_o}{8\alpha_o^3}\begin{bmatrix} x_{21}u_{21} & -x_{21}u_{21} & x_{21}w_{21} & -x_{21}w_{21} \\ -x_{21}u_{21} & x_{21}u_{21} & -x_{21}w_{21} & x_{21}w_{21} \\ z_{21}u_{21} & -z_{21}u_{21} & z_{21}w_{21} & -z_{21}w_{21} \\ -z_{21}u_{21} & z_{21}u_{21} & -z_{21}w_{21} & z_{21}w_{21} \end{bmatrix} \tag{3.71}$$

$$\mathbf{K}_{t2b} = \frac{EA}{8\alpha_o^3}\begin{bmatrix} u_{21}^2 & & & \text{symmetric} \\ -u_{21}^2 & u_{21}^2 & & \\ u_{21}w_{21} & -u_{21}w_{21} & w_{21}^2 & \\ -u_{21}w_{21} & u_{21}w_{21} & -w_{21}^2 & w_{21}^2 \end{bmatrix} \tag{3.72}$$

with the final tangent stiffness matrix being given by

$$\mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t2} + \mathbf{K}_{t\sigma} = \mathbf{K}_{t1} + \mathbf{K}_{t2a} + \mathbf{K}_{t2a}^{T} + \mathbf{K}_{t2b} + \mathbf{K}_{t\sigma}. \tag{3.73}$$

The internal force vector, $\mathbf{q}_i$, tangent stiffness matrix, $\mathbf{K}_t$, and strain/displacement relationships that have just been derived can be incorporated into a computer program using a very similar procedure to that adopted for a shallow-truss theory in Chapter 2. (This is discussed further in Section 3.9.) The technique is known as 'total Lagrangian' because all measures are related back to the initial configuration. While the detail has been given in relation to a two-dimensional analysis, the concepts are equally valid in three dimensions—see Section 3.7.