Impact of Cellular Materials

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Abstract Cellular materials are used as impact energy absorbers due to their large densification strain at the plateau stress during the plastic compression. For a cellular rod struck by a rigid object, the critical impact velocity is determined. If the impact velocity is higher than the critical impact velocity, the elastic wave will be followed by plastic shock waves. Plastic shock waves and shock arrest are investigated analytically for longitudinal impacts. Shock behaviors are characterized for material design purpose and will be used for impact protection.

1 Introduction

Cellular materials, such as metal foams, are used as impact energy absorbers in crash and blast protection due to their unique constitutive behavior, e.g., Ashby et al. (2000). Three stages can be identified in the stress-strain curve of the uniaxial compression of cellular materials, as shown schematically by the solid line in Fig. 1.

- Stage I: For a closed-cell cellular material, deformation is in the form of bending of the cell walls and edges and is, in general, reversible. At the end of this stage some cells suffer collapse. This may be due to elastic buckling, plastic deformation or fracture.
- Stage II: The almost constant compressive stress appears in a wide range of strain. Buckling and plastic collapse occurs successively until all cells are collapsed. The deformation in this stage is unrecoverable.
- Stage III: Cell walls and edges contact each other and are crushed; giving rise to a steeply rising stress.

For impact analysis, Reid and Peng (1997) firstly treated cellular materials subject to uniaxial compression using a simplified rigid, perfectly-plastic, locking (RPPL) model as shown by the dashed lines in Fig. 1. The constitutive behavior in stage - I is simplified as rigid. Stage - II is treated as perfect plastic with the yielding plateau stress σ_{pl}. The second stage ends with the

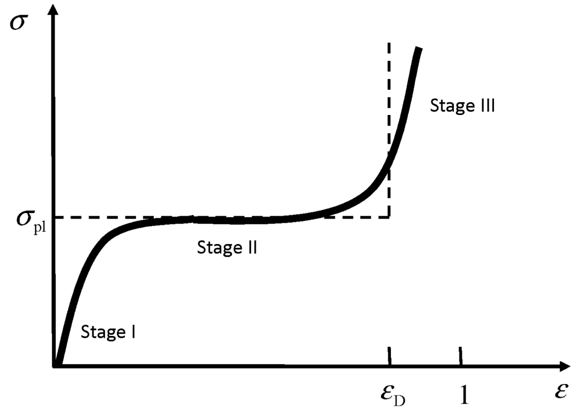

Figure 1. Constitutive behavior of a typical cellular material. The solid line shows schematically the relation between stress σ and strain ε. The dashed line shows the idealization of the stress-strain curve in the rigid, perfect-plastic, locking (RPPL) model. In stage - I, the material is treated as rigid. In stage - II, the material is in perfect plasticity with yield stress σ_{pl}. In stage - III, the material is again treated as rigid, with the densification strain ε_{D}

locking (densification) strain ε_{D}. Stage - III is again idealized as rigid. Tan et al. (2002, 2005a,b) used a RPPL idealization of foam materials to study the inertia effects and shock enhancement in uniaxial dynamic compression. Radford et al. (2005) used RPPL model to study the shock behavior in a metal foam projectiles. Harrigan et al. (2009) compared RPPL with other analytical approaches on shock wave based models.

Alporas aluminium foam developed by the Shinko Wire Company has a plateau stress σ_{pl} in the range of $1.3 \sim 1.7$ MPa, the Young's modulus E in the rage of $0.4 \sim 1.0$ GPa, and densification strain ε_{D} in the range of 0.7 ~ 0.82 (Ashby et al., 2000). The strain at the end of first stage is of the order σ_{pl}/E, which is around 0.2%; compared with the densification strain ε_{D} it is reasonable to ignore this part of deformation and treat the material in stage I as rigid, as shown in Fig. 1.

To keep the mathematics and models simple, let us only consider uniaxial compressive stress and strain. In the following, impact of a one-dimensional cellular material as shown in Fig. 2 will be studied. For simplicity, the RPPL model with be adopted.

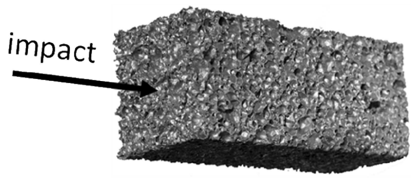

Figure 2. Impact on a cellular rod. Planar shock wave will be generated from the impact end, and propagates along the impact direction

2 Wave Propagation in a Cellular Rod

To design the crush and blast protection using cellular materials, we need to understand the wave behaviors in those materials under different impact velocities. Lopatnikov et al. (2004) considered in the general case of four significantly different possibilities of high-velocity impact on a cellular material that separated by three characteristic velocities: sound velocity of the constituent material, sound velocity of the cellular material and sound velocity during the plateau and up to the complete densification of the foam.

After impacted by a striking object, as shown in Fig. 2, there are two kinds of waves generated in the cellular rod, first an elastic wave, then followed by a plastic shock wave. However for the generation of the plastic shock wave, there is a minimum requirement for the impact velocity v_{m} to be higher than a critical value $v_{\mathrm{m}}^{\mathrm{c}}$. This section finds the critical impact velocity for the generation of plastic shock waves.

The case of a rigid mass impacting a semi-infinite rod, as shown in Fig. 3, is studied. A one-dimensional coordinate system is set along the rod, with

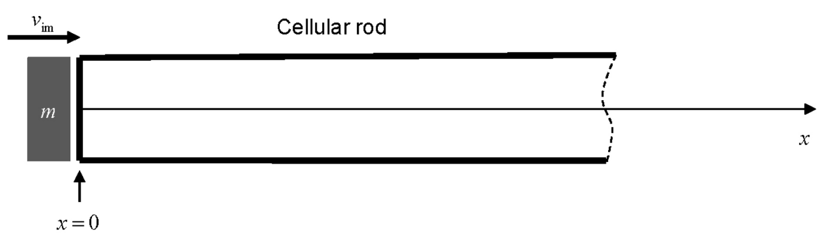

Figure 3. One-dimensional coordinate system on a cellular rod, with impact from a striking object of mass m and velocity v_m

the origin at the impact end. The general one-dimensional wave equation is

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where σ and u are the stress and displacement along the x-axis, ρ is the density, and t is time with $t = 0$ corresponds to the moment of the impact commencement.

When the impact level is low, the density of the cellular material is ρ_0, and the relation between the uniaxial stress σ and strain ε is elastic that takes the form

$$\sigma = E\varepsilon \tag{2}$$

where E is the Young's modulus. The strain ε relates to the axial displacement u through

$$\varepsilon = \frac{\partial u}{\partial x} \tag{3}$$

Substituting the above relations between the stress, strain and displacement into Eq. (1), the displacement wave equation can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_{\mathrm{el}}^2} \frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where

$$c_{\mathrm{el}}^2 = E/\rho_0 \tag{5}$$

The solution of the above wave equation can be expressed in a general form as

$$u(x,t) = f_{\mathrm{forward}}(x - c_{\mathrm{el}}t) + f_{\mathrm{backward}}(x + c_{\mathrm{el}}t) \tag{6}$$

which gives the meaning of c_{el} as the elastic wave speed, $f_{\mathrm{forward}}(x - c_{\mathrm{el}}t)$ and $f_{\mathrm{backward}}(x + c_{\mathrm{el}}t)$ are two arbitrary functions representing waves that propagate in the forward and backward directions along the x-axis, respectively. Detailed expressions of $f_{\mathrm{forward}}(x - c_{\mathrm{el}}t)$ and $f_{\mathrm{backward}}(x + c_{\mathrm{el}}t)$ need to be determined by detailed physical conditions.

The displacement waves generated from the impact end $x = 0$ propagate in the positive x direction and are initially elastic, which entails the general wave expression as

$$u(x,t) = H(c_{\mathrm{el}}t - x) f(c_{\mathrm{el}}t - x), \tag{7}$$

where $H(\xi)$ is the Heaviside function. For Alporas metal foam, the Young's modulus E and the density ρ_0 are in the rage of $0.4 \sim 1.0$ GPa and $0.2 \sim 0.25\ 10^3$ kg/m^3, respectively (Ashby et al., 2000); the elastic wave speed is, calculated from Eq. (5), in the range $1.4 \sim 2.2$ km/s. The impact velocity we consider, such as during the car crash accident, is normally on the order of several tens of meters per second, much slower compared with the elastic wave speed in typical cellular protection materials.

Detailed expression of the function f, for the case of a rod struck by a rigid object of impact velocity v_{m} as shown in Fig. 3, will be explored in the following. The areal mass m of the striking body relates to the total mass of the striking object M_{impact} through the relation

$$m = \frac{M_{\mathrm{impact}}}{A} \tag{8}$$

where A the cross section area of the rod. The impulse-momentum relation applied to the striking mass gives

$$m \mathrm{d} v_{\mathrm{m}} = \sigma_{\mathrm{m}} \mathrm{d} t, \tag{9}$$

where σ_{m} is the stress in the striking object. Assuming that the striking object is attached to the rod after impact, i.e., the striking object and the impact end of the rod share the same velocity and stress so that

$$v_{\mathrm{m}} = \frac{\partial u(0,t)}{\partial t}, \qquad \sigma_{\mathrm{m}} = E\frac{\partial u(0,t)}{\partial x} \tag{10}$$

With the general expression of $u(x,t)$ in Eq. (7), the velocity v_{m} and stress σ_{m} can be expressed using the function f as

$$v_{\mathrm{m}} = c_{\mathrm{el}} f'(c_{\mathrm{el}} t), \qquad \sigma_{\mathrm{m}} = -E f'(c_{\mathrm{el}} t) \tag{11}$$

Substitution of the expressions for v_{m} and σ_{m} into Eq. (9) generates the differential equation

$$m c_{\mathrm{el}} \mathrm{d} f'(c_{\mathrm{el}} t) + E f'(c_{\mathrm{el}} t) \mathrm{d} t = 0 \tag{12}$$

The initial conditions

$$v_{\mathrm{m}}\big|_{t=0} = v_{\mathrm{m}}, \qquad u(0,0) = 0 \tag{13}$$

give the initial conditions of function $f(\xi)$ as

$$f(0) = 0, \qquad f'(0) = \frac{v_{\mathrm{m}}}{c_{\mathrm{el}}} \tag{14}$$

The solution of the differential equation on $f(\xi)$ with the above initial conditions gives

$$f(\xi) = \frac{m c_{\mathrm{el}} v_{\mathrm{m}}}{E}\left[1 - \exp\left(-\frac{E}{m c_{\mathrm{el}}^2}\xi\right)\right], \tag{15}$$

Using Eq. (7), the general displacement field in the rod is given by

$$u(x,t) = \frac{mv_{\mathrm{m}}}{c_{\mathrm{el}}\rho_0}\left[1 - \exp\left(-\frac{E}{mc_{\mathrm{el}}^2}(c_{\mathrm{el}}t - x)\right)\right] H(c_{\mathrm{el}}t - x) \tag{16}$$

Using

$$v(x,t) = \frac{\partial u(x,t)}{\partial t}, \tag{17}$$

the evolution of velocity field is

$$v(x,t) = v_{\mathrm{m}} \exp\left[-\frac{\rho_0}{m}(c_{\mathrm{el}}t - x)\right] H(c_{\mathrm{el}}t - x), \tag{18}$$

Using

$$\sigma(x,t) = E\frac{\partial u(x,t)}{\partial t}, \tag{19}$$

the evolution of velocity field is

$$\sigma(x,t) = -c_{\mathrm{el}}\rho_0 v_{\mathrm{m}} \exp\left[-\frac{\rho_0}{m}(c_{\mathrm{el}}t - x)\right] H(c_{\mathrm{el}}t - x) \tag{20}$$

The relation between the velocity and stress field is

$$\frac{v(x,t)}{\sigma(x,t)} = \frac{1}{c_{\mathrm{el}}\rho_0} \tag{21}$$

Figure 4 shows the time evolution of the stress fields. The vertical axis is the compressive stress nondimensionalized as $-\sigma/c_{\mathrm{el}}\rho_0 v_m$. The horizontal axis is the nondimensionalized position $x\rho_0/m$. Stress waves are drawn for three moments of $m/\rho_0 c_{\mathrm{el}}$, $2m/\rho_0 c_{\mathrm{el}}$ and $3m/\rho_0 c_{\mathrm{el}}$. The compressive stresses at the wave front keep the same magnitude of $c_{\mathrm{el}}\rho_0 v_m$, which is independent of the mass of the striking object.

At the impact end $x = 0$, the time evolution of the displacement, velocity and stress fields are

$$\begin{aligned} u(0,t) &= \frac{mv_{\mathrm{m}}}{c_{\mathrm{el}}\rho_0}\left[1 - \exp\left(-\frac{c_{\mathrm{el}}\rho_0}{m}t\right)\right], \\ v(0,t) &= v_{\mathrm{m}} \exp\left(-\frac{c_{\mathrm{el}}\rho_0}{m}t\right), \\ \sigma(0,t) &= -c_{\mathrm{el}_{\rho_0}} v_{\mathrm{m}} \exp\left(-\frac{c_{\mathrm{el}}\rho_0}{m}t\right), \end{aligned} \tag{22}$$

respectively. At time $t = 0$, the compressive stress at the impact end of the cellular rod is $\rho_0 c_{\mathrm{el}} v_{\mathrm{m}}$. When

$$c_{\mathrm{el}}\rho_0 v_{\mathrm{m}} > \sigma_{\mathrm{pl}}, \tag{23}$$

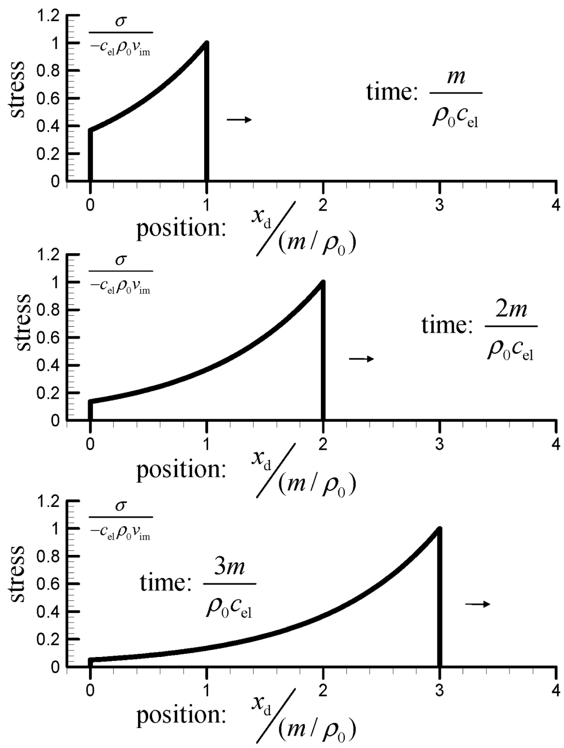

Figure 4. Impact induced compressive stress wave in an elastic rod. The wave propagates along x (rod axis) direction. In the figure, the position is nondimensionalized as $x\rho_0/m$, and the stress is nondimensionalized as $-\sigma/c_{\mathrm{el}}\rho_0 v_m$. The wave is shown at 3 different times: $m/\rho_0 c_{\mathrm{el}}, 2m/\rho_0 c_{\mathrm{el}}$ and $3m/\rho_0 c_{\mathrm{el}}$

the plastic shock wave will follow the elastic wave.

The response of materials and structures to suddenly applied loads, such as shown in Fig. 3, can be quite different from their response when subjected to loads which increases slowly. The critical impact velocity is

$$v_{\mathrm{im}}^{\mathrm{c}} = \frac{\sigma_{\mathrm{pl}}}{\rho_0 c_{\mathrm{el}}} \tag{24}$$

If the impact velocity is slower than $v_{\mathrm{im}}^{\mathrm{c}}$, only an elastic wave will be generated from the impact end. If the impact velocity is higher than $v_{\mathrm{im}}^{\mathrm{c}}$, the elastic wave will be followed by plastic shock waves. A plastic shock wave is made by a rapid, continuous push, and is characterized by an abrupt, nearly discontinuous change in the mechanical properties of the cellular rod: stress, mass density, particle velocity.

For Alporas metal foam the plateau stress σ_{pl}, density ρ_0 and elastic wave speed c_{el} are in the ranges of $1.3 \sim 1.7$ MPa, $0.2 \sim 0.25\ 10^3$ kg/m^3 and $1.4 \sim 2.2$ km/s, respectively. Therefore, the critical impact velocity $v_{\mathrm{im}}^{\mathrm{c}}$ is in the range of $2.6 \sim 6.0$ m/s . According to Eq. (21), when the wave stress reaches $-\sigma_{\mathrm{pl}}$ the particle velocity is $\sigma_{\mathrm{pl}}/(c_{\mathrm{el}}\rho_0)$. This particle velocity of the elastic wave is relatively small compared with the impact velocity v_{im} which can be around 30 m/s during cases such as a highway car crashing.

Shock-wave analysis is used in the following sections for high speed impact. We study two cases on the impact of cellular materials: a rigid object striking on a cellular rod with fixed end, and a rigid object striking on a free cellular rod. The kinetic energy of the striking object is transmitted to the rod and absorbed by deformation and damage in the cellular material.

3 Rigid Object Strikes on a Cellular Rod of Fixed End

In crash protection the absorber must absorb the kinetic energy of the moving object without reaching complete densification, and the stress it transmits never exceeds the plateau stress.

Consider the impact of one end of a stationary rod by a rigid striker at a high velocity v_{m}, as denoted in Fig. 5(a). The rod is homogeneous and made from a RPPL material. After impact a shock wave moves from the impacted end to the opposite fixed end of the rod, as shown in Fig 5(b). The stress ahead of the shock wave is compressive with magnitude σ_{pl} brought up by the fast travelling elastic wave. According to the argument in the previous section particle velocity of the elastic wave is relatively small compared with the impact velocity v_{m}, therefore, the material ahead of the shock front can be treated as static with zero particle velocity. The material behind the shock front has attained a strain ε_{D}, its particle velocity is v_{d}, its density has been raised from the initial value ρ_0 to the densification value ρ_{D}, and the compressive stress has been raised to σ_{d}.

3.1 Basic Assumptions

Assumptions are made for analytically investigating the shock behavior in a one-dimensional cellular rod:

1. There is a sharp shock front separating the compressed and undeformed regions of the foam. The micrograph in Fig. 6 clearly shows a

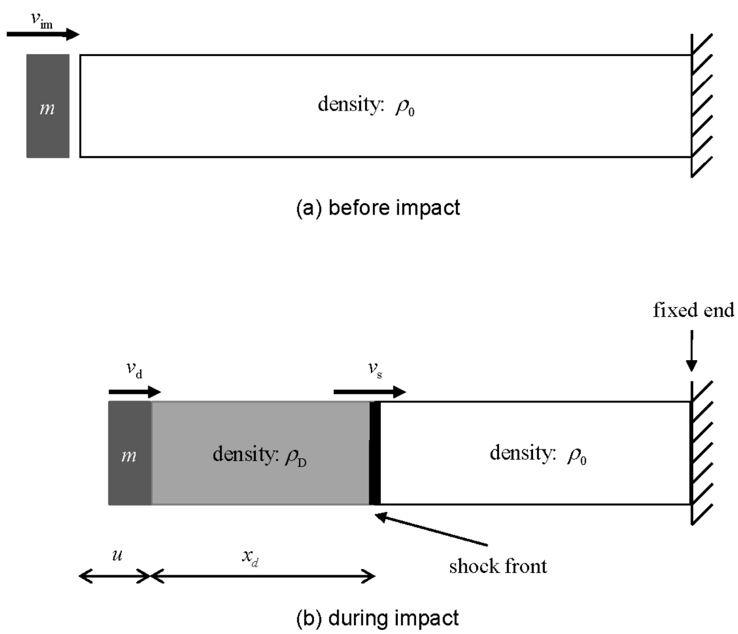

Figure 5. A fixed cellular rod struck by a rigid object. The object has an areal mass m and impact velocity v_m. The cellular rod is separated by the shock front into two portions, the uncondensed portion is in white colour, and the condensed portion is in gray colour. For the uncondensed portion the particle velocity is 0 and the material density is ρ_0. For the densified portion the particle velocity is v_d and the density is ρ_{D}. The velocity of shock front is v_{s}

sharp shock front separating the compressed and undeformed regions of the foam.

2. After the densification, the particle velocities of the densified materials are equal to each other and equal to the velocity of the striking body.
3. The densified layer has a density which is spatially constant.
4. The rate effects that affect the deformation and failure modes of cellular materials (e.g., Calladine and English (1984)) are ignored.

3.2 Shock Wave Analysis

Here we consider the case where the impact speed is lower than the elastic wave speed Therefore, in the undeformed region ahead of the shock wave,

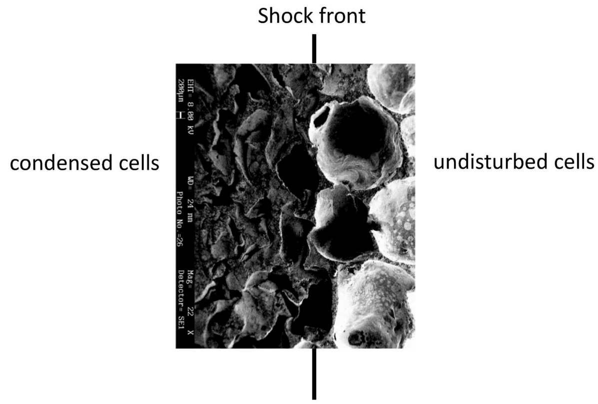

Figure 6. A scanning electron micrograph of the metal foam specimen sectioned along its impact axis. The impact is acting from left to the right (Radford et al., 2005)

the stress is equal to the plateau stress σ_{pl}, while the strain and particle velocity can be ignored.

According to definition, the densification strain ε_{D}

$$\varepsilon_{\mathrm{D}} = \frac{u}{x_{\mathrm{d}} + u}, \tag{25}$$

where, as shown Fig. 5, u is the displacement of the rigid mass at time t, and x_{d} is the deformed length of the crushed cellular material.

Conservation of mass across the shock front gives

$$\frac{\rho_0}{\rho_{\mathrm{D}}} = 1 - \varepsilon_{\mathrm{D}} \tag{26}$$

As shown in Fig. 1, the densification strain ε_{D} in the RPPL model cannot be exactly defined from the measured stress-strain cure. Equation (26) provides a method to determine ε_{D} experimentally, that is to measure the material density before and after densification, the densification strain ε_{D} can thus be determined through $\varepsilon_{\mathrm{D}} = 1 - \rho_0/\rho_{\mathrm{D}}$.

The cellular material behind the shock front is idealized as rigid. The instantaneous velocity (reducing) of the rigid mass with respect to a stationary frame is the same as the particle velocity of the condensed part, and is denoted as

$$v_{\mathrm{d}} = \frac{\mathrm{d}u}{\mathrm{d}t} \tag{27}$$

The shock velocity v_s is faster than the particles velocity v_d, and this gives the increase of x_d with respect of time,

$$\frac{\mathrm{d}x_\mathrm{d}}{\mathrm{d}t} = v_\mathrm{s} - v_\mathrm{d} \tag{28}$$

This provides the relation between the shock velocity v_s and the condensed part velocity v_d as

$$v_\mathrm{s} = \frac{v_\mathrm{d}}{\varepsilon_\mathrm{D}}, \tag{29}$$

and

$$\frac{\mathrm{d}x_\mathrm{d}}{\mathrm{d}t} = \frac{1-\varepsilon_\mathrm{D}}{\varepsilon_\mathrm{D}} v_\mathrm{d} \tag{30}$$

Considering the momentum change of the body consists of the striking object and the cellular material behind the shock front, one has

$$\mathrm{d}[(m + \rho_\mathrm{D} x_\mathrm{d}) v_\mathrm{d}] = -\sigma_\mathrm{pl}\mathrm{d}t, \tag{31}$$

where m is the areal mass of the impact object. The above equation gives

$$v_\mathrm{d}\mathrm{d}[(m + \rho_\mathrm{D} x_\mathrm{d}) v_\mathrm{d}] + \sigma_\mathrm{pl}\frac{1-\varepsilon_\mathrm{D}}{\varepsilon_\mathrm{D}}\mathrm{d}x_\mathrm{d} = 0,$$

which can be rewritten as

$$\frac{1-\varepsilon_\mathrm{D}}{\sigma_\mathrm{pl}\varepsilon_\mathrm{D} + \rho_0 v_\mathrm{d}^2}\mathrm{d}v_\mathrm{d}^2 + \frac{2}{m + \rho_0 x_\mathrm{d}} = 0$$

The above differential equation, together with the initial condition

$$v_\mathrm{d} = v_\mathrm{m} \qquad \text{when} \qquad x_\mathrm{d} = 0 \tag{32}$$

provides

$$(\rho_0 v_\mathrm{d}^2 + \sigma_\mathrm{pl}\varepsilon_\mathrm{D})(m + \rho_0 x_\mathrm{d})^2 = (\rho_0 v_\mathrm{d}^2 + \sigma_\mathrm{pl}\varepsilon_\mathrm{D}) m^2 \tag{33}$$

This further gives the particle velocity behind the shock as

$$v_\mathrm{d} = v_\mathrm{C}\sqrt{\left[1 + \left(\frac{v_\mathrm{m}}{v_\mathrm{C}}\right)^2\right]\left(\frac{m}{m + \rho_\mathrm{D} x_\mathrm{d}}\right)^2 - 1}, \tag{34}$$

where the characteristic velocity v_C is defined as

$$v_\mathrm{C} = \sqrt{\frac{\varepsilon_\mathrm{D}\sigma_\mathrm{pl}}{\rho_0}} \tag{35}$$

The evolution of x_d can be solved from the following differential equation

$$\frac{\mathrm{d}x_\mathrm{d}}{\mathrm{d}t} = \frac{1-\varepsilon_\mathrm{D}}{\varepsilon_\mathrm{D}} v_\mathrm{C} \sqrt{\frac{1+\tilde{v}^2}{\left(1+\frac{\rho_\mathrm{D} x_\mathrm{d}}{m}\right)^2} - 1},$$

where

$$\tilde{v} = \frac{v_\mathrm{m}}{v_\mathrm{C}} \tag{36}$$

The solution of the above differential equation, together with the initial condition

$$x = 0 \quad \text{at} \quad t = 0, \tag{37}$$

gives

$$\frac{1-\varepsilon_\mathrm{D}}{\varepsilon_\mathrm{D}} v_\mathrm{C} t + \sqrt{\frac{m^2 \tilde{v}^2}{\rho_\mathrm{D}^2} - \frac{2m}{\rho_\mathrm{D}} x_\mathrm{d} - x_\mathrm{d}^2} = \frac{m\tilde{v}}{\rho_\mathrm{D}}$$

Solve the equation for x_d, one has the time evolution of the length of the crushed foam as

$$x_\mathrm{d} = -\frac{m}{\rho_\mathrm{D}} + \sqrt{\left(\frac{m}{\rho_\mathrm{D}}\right)^2 - \left(\frac{1-\varepsilon_\mathrm{D}}{\varepsilon_\mathrm{D}} v_\mathrm{C} t\right)^2 + 2\frac{m}{\rho_\mathrm{D}} \frac{1-\varepsilon_\mathrm{D}}{\varepsilon_\mathrm{D}} v_\mathrm{im} t}, \tag{38}$$

which can be written in a nondimensionalized form as

$$\frac{x_\mathrm{d}}{m/\rho_\mathrm{D}} = \sqrt{1 - \tilde{t}^2 + 2\tilde{v}\tilde{t}} - 1, \tag{39}$$

where the nondimensionalized time $\tilde{t}$ is defined as

$$\tilde{t} = \frac{t}{m\varepsilon_\mathrm{D}/\rho_0 v_\mathrm{C}} \tag{40}$$

Figure 7 shows the time evolution of the crushed length x_d affected by the impact velocity.

Shock arrest, which means complete stopping of the shock wave, requires

$$\frac{\mathrm{d}x_\mathrm{d}}{\mathrm{d}t} = 0, \tag{41}$$

which gives the arrest distance $x_\mathrm{d}^\mathrm{arrest}$, or the maximum length of the crushed cellular material, as

$$\frac{x_\mathrm{d}^\mathrm{arrest}}{m/\rho_\mathrm{D}} = \sqrt{1 - \tilde{v}^2} - 1 \tag{42}$$

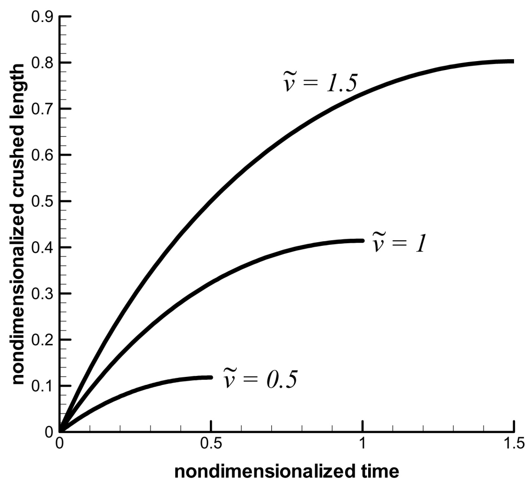

Figure 7. Nondimensionalized crushed-length, $x_{\mathrm{d}}/(m/\rho_{\mathrm{D}})$, evolves as a function of nondimensionalized time, $t/(\varepsilon_{\mathrm{D}} m/\rho_0 v_{\mathrm{C}}$, for different impact velocities v_m at $0.5\sqrt{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}/\rho_0}$, $\sqrt{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}/\rho_0}$ and $1.5\sqrt{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}/\rho_0}$, respectively

Figure 8 shows the relation between the arrest distance and the impact velocity.

According to Eq. (39), the time to reach shock arrest is

$$\tilde{t} = \tilde{v}, \tag{43}$$

which means

$$t = \frac{m v_{\mathrm{im}}}{\sigma_{\mathrm{pl}}} \tag{44}$$

The above relation can also be derived from the momentum-impulse relations during the time period from impact commencement to the shock arrest.

According to Eq. (42), to crush a impact velocity of v_{im}, the mass of the cellular material for energy absorbing, M_{cellular}, should be at least

$$\frac{M_{\mathrm{cellular}}}{M_{\mathrm{impact}}} = \sqrt{1 + \left(\frac{v_{\mathrm{m}}}{v_{\mathrm{C}}}\right)^2} - 1, \tag{45}$$

where M_{impact} is the mass of the impact object. For the case of $v_{\mathrm{im}} = v_{\mathrm{C}}$, $M_{\mathrm{cellular}}/M_{\mathrm{impact}} = \sqrt{2} - 1$, i.e, for the complete protection (shock arrest)

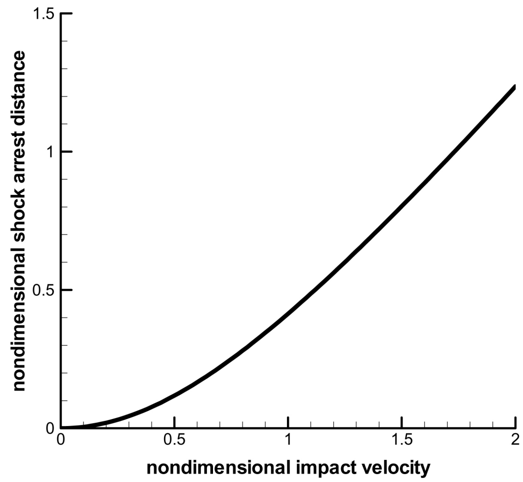

Figure 8. Nondimensionalized arrest distance, $x_{\mathrm{d}}^{\mathrm{arrest}}/(m/\rho_{\mathrm{D}})$, as a function of the nondimensionalized impact velocity $\tilde{v} = v_{\mathrm{m}}/v_{\mathrm{C}}$

against a impact velocity of v_{C}, the ratio of the mass of the cellular material for energy absorbing and that of the impact object should be at least 1.4. From Eq. (45), a fixed cellular rod can fully arrest a striking object of mass M_{impact} with impact velocity up to

$$v_{\mathrm{C}}\sqrt{\left(1+\frac{M_{\mathrm{cellular}}}{M_{\mathrm{impact}}}\right)^2-1}$$

For car crash, the impact velocity is about $v_{\mathrm{m}} = 30$ m/s. Table 1 gives the material properties (density, densification strain and plateau stress) for several typical cellular materials. The characteristic velocity is given through Eq. (35).

Stress behind the shock front. For a shock wave passing a cellular material, the shock enhancement effect under high speed impact (>100 m/s) was originally proposed by Reid and Peng (1997). Afterwards, a number of authors also reported this effect for various cellular materials at high impact speeds (Lopatnikov et al., 2003, 2004; Tan et al., 2002, 2005a,b; Radford et al., 2005). For relatively low impact speeds, there is the so-called critical velocity under which shock enhancement is not significant (∼50m/s).

Table 1. Material properties for some typical cellular materials

Foam	Density ρ_0 (kg/m^3)	Plateau stress σ_{pl} (MPa)	Densification strain ε_{D}	Characteristic velocity v_{C} (m/s)
Polyurethane	34	0.25	0.55	64
Cork	164	1	0.55	58
Alporas metal	200 ~ 250	1.3 ~ 1.7	0.7 ~ 0.82	60.3 ~ 83.4

Stress behind the shock front is derived in the following. Conservation of momentum across the shock front gives the compressive stress behind the shock σ_{d} as

$$\sigma_{\mathrm{d}} = -\left(\sigma_{\mathrm{pl}} + \frac{\rho_0}{\varepsilon_{\mathrm{D}} v_{\mathrm{d}}^2}\right) \tag{46}$$

Using Eq. (33), the stress behind the shock can be written as a function of the length of the crushed cellular rod x_{d}

$$\sigma_{\mathrm{d}} = -\sigma_{\mathrm{pl}}\left[1 + \left(\frac{v_{\mathrm{m}}}{v_{\mathrm{C}}}\right)^2\right]\left(\frac{m}{m + \rho_{\mathrm{D}} x_{\mathrm{d}}}\right)^2 \tag{47}$$

Using Eq. (38) for x as a function of time t, and the nondimensionalized notation for impact velocity $\tilde{v} = v_{\mathrm{m}}/v_{\mathrm{C}}$ and time $\tilde{t} = t/(m\varepsilon_{\mathrm{D}}/\rho_0 v_{\mathrm{C}})$, Eq. (47) provides

$$\frac{\sigma_{\mathrm{d}}}{\sigma_{\mathrm{pl}}} = -\frac{1 + \tilde{v}^2}{1 + 2\tilde{v}\tilde{t} - \tilde{t}^2} \tag{48}$$

Figure 9 shows the time evolution of the stress behind the shock front affected by impact velocity. In a dimensional form

$$\frac{\sigma_{\mathrm{d}}}{\sigma_{\mathrm{pl}}} = -\frac{\sigma_{\mathrm{pl}}\varepsilon_{\mathrm{D}} + \rho_0 v_{\mathrm{im}}^2}{\varepsilon_{\mathrm{D}} m^2 \sigma_{\mathrm{pl}} + 2m v_{\mathrm{im}} \rho_0 \sigma_{\mathrm{pl}} t - \rho_0 \sigma_{\mathrm{pl}} \sigma_{\mathrm{pl}} t^2} m^2 \tag{49}$$

At time $t = 0$,

$$\sigma_{\mathrm{d}} = -\left(\sigma_{\mathrm{pl}} + \frac{\rho_0 v_{\mathrm{im}}^2}{\varepsilon_{\mathrm{D}}}\right), \tag{50}$$

which shows that higher densification strain and lower material density of the cellular rod can reduce the impact stress.

Stress on the striking object. Substitute the expression for x_{d} as a function of time into Eq. (25), one has

$$u = \frac{m\varepsilon_{\mathrm{D}}}{\rho_0}\left(-1 + \sqrt{1 + 2\tilde{v}\tilde{t} - \tilde{t}^2}\right) \tag{51}$$

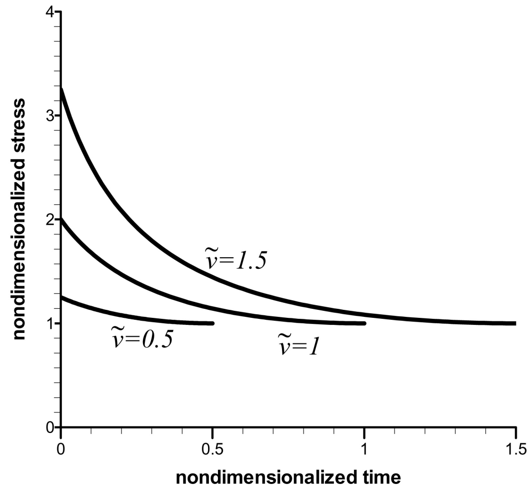

Figure 9. Nondimensionalized stress behind the shock front, $\sigma_\mathrm{d}/\sigma_\mathrm{pl}$, evolves as a function of nondimensionalized, $t/(\varepsilon_\mathrm{D} m/\rho_0 v_\mathrm{C})$, for different impact velocities v_m at $0.5\sqrt{\varepsilon_\mathrm{D}\sigma_\mathrm{pl}/\rho_0}$, $\sqrt{\varepsilon_\mathrm{D}\sigma_\mathrm{pl}/\rho_0}$ and $1.5\sqrt{\varepsilon_\mathrm{D}\sigma_\mathrm{pl}/\rho_0}$, respectively

The stress on the mass can be calculated from

$$\sigma_\mathrm{m} = m\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}, \tag{52}$$

which gives

$$\frac{\sigma_\mathrm{m}}{\sigma_\mathrm{pl}} = -\frac{1+\tilde{v}_\mathrm{m}^2}{(1+2\tilde{v}_\mathrm{m}\tilde{t}-\tilde{t}^2)^{3/2}} \tag{53}$$

In a dimensional form,

$$\frac{\sigma_\mathrm{m}}{\sigma_\mathrm{pl}} = -\frac{1+\dfrac{\rho_0 v_\mathrm{im}^2}{\varepsilon_\mathrm{D}\sigma_\mathrm{pl}}}{\left(1+2\dfrac{\rho_0}{\varepsilon_\mathrm{D} m}v_\mathrm{im}t-\dfrac{\rho_0\sigma_\mathrm{pl}}{\varepsilon_\mathrm{D} m^2}t^2\right)^{3/2}} \tag{54}$$

Figure 10 shows the time evolution of the stress acted on the mass affected by the impact velocity. At time $t=0$,

$$\sigma_\mathrm{m} = -\left(\sigma_\mathrm{pl}+\frac{\rho_0 v_\mathrm{im}^2}{\varepsilon_\mathrm{D}}\right), \tag{55}$$

which shows that higher densification strain and lower material density of the cellular rod can reduce the impact stress.

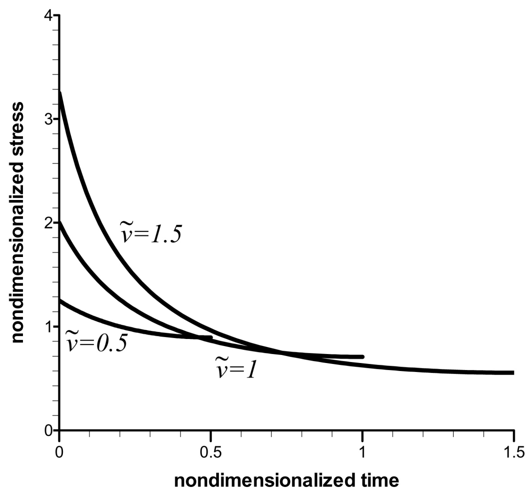

Figure 10. Nondimensionalized stress on the striking object, $-\sigma_{\mathrm{m}}/\sigma_{\mathrm{pl}}$, evolves as a function of nondimensionalized, $t/(\varepsilon_{\mathrm{D}} m/\rho_0 v_{\mathrm{C}}$, for different impact velocities v_m at $0.5\sqrt{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}/\rho_0}$, $\sqrt{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}/\rho_0}$ and $1.5\sqrt{\varepsilon_{\mathrm{D}}\sigma_{\mathrm{pl}}/\rho_0}$, respectively

4 Rigid Object Strikes on a Free Cellular Rod

This section investigates the behavior of a cellular rod struck by a rigid object as shown in Fig. 11(a). The rod is homogeneous and made from a RPPL cellular material. The original length of the rod is L. The striking object has areal mass m and is at a high impact velocity v_{im}. After impact a shock wave is generated from the impacted end and moves to the opposite free end of the rod, as shown in Fig. 11(b). The shock front moves at velocity v_{s}. The material ahead of the shock wave (uncondensed) is treated as rigid that moves at velocity v_0. The material behind the shock front has attained a strain ε_{D} and a particle velocity v_{d}; its density has been raised from the initial value ρ_0 to the densification value ρ_{D}, and the compressive stress has been raised to σ_{d}. The length of the condensed cellular rod is denoted as x_{d}.

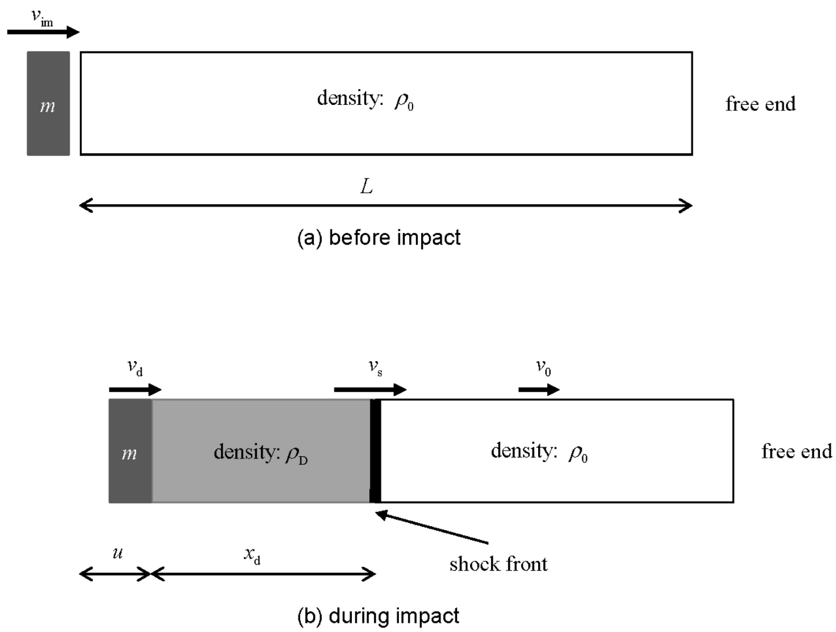

Figure 11. A free stationary cellular-rod struck by a rigid object of areal mass m and impact velocity v_m. Shock wave is generated from the impact end and propagates at a velocity v_s. The shock front separate the rod into two parts, denoted as state "0" (with initial density ρ_0, zero stress and strain, and particle velocity v_0) and state "d" (with condensed strain ε_D, density ρ_D, and particle velocity v_d). During the impact, the displacement of the striker is u, and the length of the crushed rod is x_d

The usage of the subscripts in this section is in consistent with that in the last section. The capital letter "D" (such as that in ρ_D and ε_D) denotes material constants during the densification, and the small letter "d" (such as that in $x + \mathrm{d}$, $v + \mathrm{d}$ and σ_d) denotes variables that change during the densification. The subscript "0" includes both material constants (such as ρ_0) and variables (such asv_0) before densification.

The displacement of the striking object is u. The relation between the shock velocity v_s, condensed particle velocity v_d, and the increasing rate of the condensed length $\mathrm{d}x_d/\mathrm{d}t$ can be written as

$$v_s = \frac{\mathrm{d}(x_d + u)}{\mathrm{d}t} = \frac{\mathrm{d}x_d}{\mathrm{d}t} + v_d \tag{56}$$

Conservation of mass across the shock front gives the relation between the material densities before and after densification as $\rho_0 = \rho_\mathrm{D}(1 - \varepsilon_\mathrm{D})$.

A shock coordinate system that sits on the shock front can be established as in Fig. 12. In this reference system particles enter into the shock front at

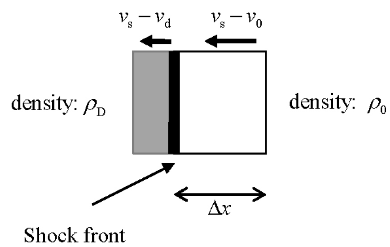

Figure 12. Local coordinate system at the shock front that travels with the shock wave. Viewing in this reference coordinate system, the particles move across the shock front at the speed of $v_\mathrm{s} - v_0$ to the left, and comes out of the shock wave front at the speed of $v_\mathrm{s} - v_\mathrm{d}$

velocity $v_\mathrm{s} - v_0$, and come out at velocity $v_\mathrm{s} - v_\mathrm{d}$. During a short time period Δt, an element of length $\Delta X = (v_\mathrm{s} - v_0)\Delta t$ (shown as a white block in Fig. 12) and density ρ_D enters into the shock front. Meanwhile, an element of length $(v_\mathrm{s} - v_\mathrm{d})\Delta t$ (shown as a gray block in Fig. 12) and density ρ_D moves out of the shock front. The conservation of mass across the shock front provides

$$\rho_\mathrm{D}(v_\mathrm{s} - v_\mathrm{d}) = \rho_0(v_\mathrm{s} - v_0),$$

which gives

$$v_\mathrm{s} = \frac{v_\mathrm{d} - v_0}{\varepsilon_\mathrm{D}} + v_\mathrm{d} \tag{57}$$

The mass of the foam behind the shock is $\rho_\mathrm{D} x$, and the mass of the foam ahead the shock is $\rho_0 L - \rho_\mathrm{D} x$. Conservation of momentum of the whole system that includes both impact object and the cellular rod gives

$$(m + \rho_\mathrm{D} x_\mathrm{d})v_\mathrm{d} + (\rho_0 L - \rho_\mathrm{D} x_\mathrm{d})v_0 = m v_\mathrm{im} \tag{58}$$

For the region ahead of the shock wave front, we have

$$\sigma_\mathrm{pl} = (\rho_0 L - \rho_\mathrm{D} x_\mathrm{d})\frac{\mathrm{d}v_0}{\mathrm{d}t} \tag{59}$$

From Eq. (56) to (59), a non-dimensional equation system that contains 3 equations for 3 time-varying variables, $\tilde{x}_{\mathrm{d}}$, $\tilde{v}_{\mathrm{d}}$ and $\tilde{v}_0$, can be derived as

$$(\tilde{m} + \tilde{x}_{\mathrm{d}})\tilde{v}_{\mathrm{d}} + (1 - \tilde{x}_{\mathrm{d})})\tilde{v}_0 = \tilde{m},$$
$$\frac{\mathrm{d}\tilde{x}_{\mathrm{d}}}{\mathrm{d}\tilde{t}} = \tilde{v}_{\mathrm{d}} - \tilde{v}_0, \qquad \frac{\mathrm{d}\tilde{v}_d}{\mathrm{d}\tilde{t}} = \frac{1}{\eta(1 - \tilde{x}_{\mathrm{d}}}, \tag{60}$$

where the mass coefficient

$$\tilde{m} = \frac{1}{\rho_0 L} m \tag{61}$$

is the ratio of the impact mass versus the foam mass; and the impact coefficient

$$\eta = \frac{\rho_0 v_{\mathrm{im}}^2}{\sigma_{\mathrm{pl}}\varepsilon_{\mathrm{D}}} = \left(\frac{v_{\mathrm{im}}}{v_{\mathrm{C}}}\right)^2 \tag{62}$$

characterizes the impact strength, where the characteristic velocity of the cellular material v_{C} is defined in Eq. (35); the time is expressed in a non-dimensional form that combines the impact velocity v_{im}, the length of the foam L L and densification strain ε_{D} as

$$\tilde{t} = \frac{v_{\mathrm{im}}}{L\varepsilon_{\mathrm{D}}} t; \tag{63}$$

and the 3 nondimensionalized time-varying variables ($\tilde{x}_{\mathrm{d}}$, $\tilde{v}_{\mathrm{d}}$ and $\tilde{v}_0$) are defined as

$$\tilde{x}_{\mathrm{d}} = \frac{x_{\mathrm{d}}}{(1 - \varepsilon_{\mathrm{D}})L}, \qquad \tilde{v}_{\mathrm{d}=} \frac{v_{\mathrm{d}}}{v_{\mathrm{im}}}, \qquad \tilde{v}_0 = \frac{\tilde{v}_0}{v_{\mathrm{im}}} \tag{64}$$

The range for $\tilde{x}_{\mathrm{d}}$ is [0,1] where $\tilde{x}_{\mathrm{d}} = 0$ corresponds to the commencement for impact, and $\tilde{x}_{\mathrm{d}} = 1$ corresponds to complete densification of the whole foam rod.

As an example, we use Alporas metal foams for the protection from car crashing. For this type of foams, the characteristic v_{C} is in the range 60.3 $\sim$ 83.4 m/s. The impact velocity during the highway car crash is around 30 m/s. For this impact velocity, the impact coefficient η is in the range 0.13 $\sim$ 0.25.

The relation between $\tilde{v}_0$ and $\tilde{x}_d$ is derived in the following. Equations (60) generate

$$\eta[\tilde{m} - \tilde{v}_0(1 + \tilde{m})]\mathrm{d}\tilde{v}_0 = \left(\frac{1 + \tilde{m}}{1 - \tilde{x}_{\mathrm{d}}} - 1\right)\mathrm{d}\tilde{x}_{\mathrm{d}}$$

The integration gives

$$\tilde{m}\tilde{v}_0 = \frac{1}{2}\tilde{v}_0^2(1 + \tilde{m}) + \frac{1}{\eta}[\tilde{x}_{\mathrm{d}} + (1 + \tilde{m})\ln(1 - \tilde{x}_d)] = \text{ const}$$

With the initial condition

$$\tilde{v}_0 = 0 \quad \text{when} \quad \tilde{x}_\mathrm{d} = 0,$$

the above equation generates

$$\frac{1}{2}(1+\tilde{m})\tilde{v}_0^2 - \tilde{m}\tilde{v}_0 - \frac{1}{\eta}[\tilde{x}_\mathrm{d} + (1+\tilde{m})\ln(1-\tilde{x}_d)] = 0$$

Solving the above equation for $\tilde{v}_0$ one has

$$\tilde{v}_0 = \frac{1}{1+\tilde{m}}\left[\tilde{m} - \sqrt{\tilde{m}^2 + \frac{2}{\eta}(1+\tilde{m})[\tilde{x}_\mathrm{d} + (1+\tilde{m})\ln(1-\tilde{x}_\mathrm{d})]}\right] \tag{65}$$

The relation between $\mathrm{d}\tilde{x}_\mathrm{d}/\mathrm{d}t$ and $\tilde{x}_\mathrm{d}$ is derived in the following. Eq. (60) gives the relation

$$\frac{\mathrm{d}\tilde{x}_\mathrm{d}}{\mathrm{d}\tilde{t}} = \frac{\tilde{m} - (1-\tilde{m})\tilde{v}_0}{\tilde{m} + \tilde{x}_\mathrm{d}} \tag{66}$$

Substitution of the expression for $\tilde{v}_0$ into the above equation generates

$$\frac{\mathrm{d}\tilde{x}_\mathrm{d}}{\mathrm{d}\tilde{t}} = \frac{1}{\tilde{m} + \tilde{x}_\mathrm{d}}\sqrt{\tilde{m}^2 + \frac{2}{\eta}(1+\tilde{m})[\tilde{x}_\mathrm{d} + (1+\tilde{m})\ln(1-\tilde{x}_\mathrm{d})]} \tag{67}$$

The above equation gives the time evolution of the length of the crushed cellular rod.

Figure 13 shows the time evolution of $\tilde{x}_\mathrm{d}$ for several impact coefficients at $\eta = 0.5$, 1, 2 and 4. When both condensed part and uncondensed part of the cellular rod reach the same particle velocity, i.e.,

$$\tilde{v}_\mathrm{d} = \tilde{v}_0, \tag{68}$$

shock wave disappears. This is termed as shock arrest. The arrest distance (nondimensionalized) to the shock wave, which is the non-dimensional length of crushed foam at this moment, is denoted as $\tilde{x}_\mathrm{d}^\mathrm{arrest}$. Using Eq. (60), the nondimensionalized arrest distance $\tilde{x}_\mathrm{d}^\mathrm{arrest}$ can be determined from

$$\frac{\mathrm{d}\tilde{x}_\mathrm{d}}{\mathrm{d}\tilde{t}} = 0 \tag{69}$$

Further using Eq. (67), $\tilde{x}_\mathrm{d}^\mathrm{arrest}$ can be determined as the root of the following equation

$$F(\tilde{x}_\mathrm{d}) = \tilde{x}_\mathrm{d} + (1+\tilde{m})\ln(1-\tilde{x}_\mathrm{d}) + \frac{\eta\tilde{m}^2}{2(1+\tilde{m})} = 0 \tag{70}$$

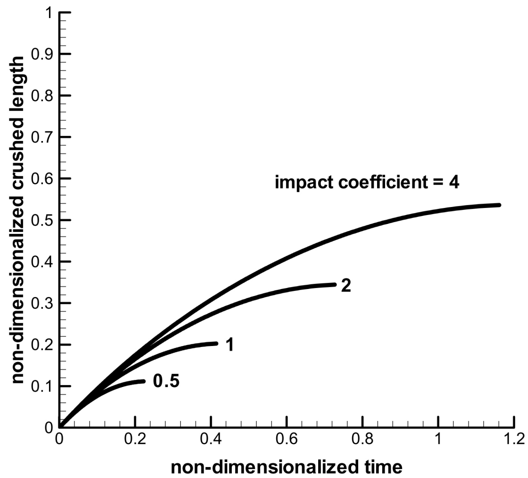

Figure 13. Nondimensionalized length of the crushed cellular rod, $\tilde{x}_{\mathrm{d}} = x/(1-\varepsilon_{\mathrm{D}})L$, as a function of nondimensionalized time, $\tilde{t} = v_m \mathrm{t}/L\varepsilon_{\mathrm{D}}$, for several impact coefficients η at 0.5,1, 2 and 4, respectively. The mass coefficients is set at $\tilde{m} = 1$

For the function $F(\tilde{x}_{\mathrm{d}})$, the derivative is $\mathrm{d}F/\mathrm{d}\tilde{x}_{\mathrm{d}} = -(\tilde{x}_{\mathrm{d}} + \tilde{m})/(1-\tilde{x}_{\mathrm{d}})$. In the range (0,1), $F(0) > 0$, $F(1) < 1$ and $\mathrm{d}F/\mathrm{d}\tilde{x}_{\mathrm{d}} < 0$. So there exists a root $\tilde{x}_{\mathrm{d}}^{\mathrm{arrest}}$ for the equation $F(\tilde{x}_{\mathrm{d}}) = 0$, which corresponds to the $\mathrm{d}\tilde{x}_{\mathrm{d}}/\mathrm{d}\tilde{t} = 0$, i.e., the end for the shock wave. Figure 14 shows the relation between the nondimensionalized arrest distance $\tilde{x}_{\mathrm{d}}^{\mathrm{arrest}}$ and the impact coefficient η for several mass coefficients at $\tilde{m} = 0.5$, 1 and 2.

Stress on the striking object. The stress on the striking object σ_{m} can be calculated from

$$\sigma_{\mathrm{m}} = m\frac{\mathrm{d}v_{\mathrm{d}}}{\mathrm{d}t} \tag{71}$$

The non-dimensional form can be written as

$$\frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{pl}}} = \eta\tilde{m}\frac{\mathrm{d}\tilde{v}_{\mathrm{d}}}{\mathrm{d}\tilde{t}} \tag{72}$$

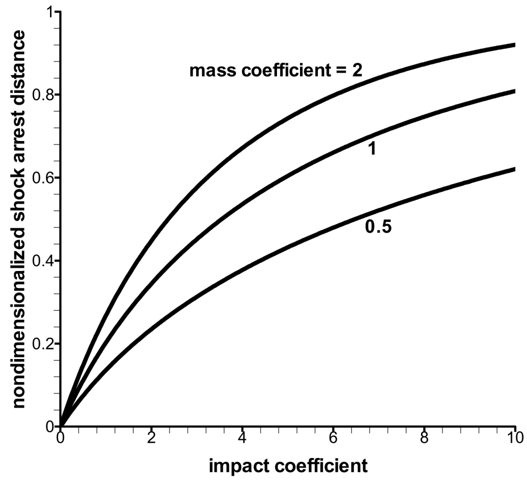

Figure 14. Relation between the nondimensionalized arrest distance $\tilde{x}_{\mathrm{d}}^{\mathrm{arrest}}$ and the impact coefficient η for several mass coefficients $\tilde{m}$: 0.5, 1 and 2, respectively

Since $\tilde{v}_{\mathrm{d}} = (\mathrm{d}\tilde{x}_{\mathrm{d}}/\mathrm{d}\tilde{t}) + \tilde{v}_0$, substitution of the expression for $\mathrm{d}\tilde{x}_{\mathrm{d}}/\mathrm{d}\tilde{t}$ and $\tilde{v}_0$ gives

$$\begin{aligned}\tilde{v}_{\mathrm{d}} &= \left(\frac{1}{\tilde{m}+\tilde{x}_{\mathrm{d}}} + \frac{1}{1+\tilde{m}}\right)\sqrt{\tilde{m}^2 + \frac{2}{\eta}(1+\tilde{m})[\tilde{x}_{\mathrm{d}} + (1+\tilde{m})\ln(1-\tilde{x}_{\mathrm{d}})]} \\ &+ \frac{\tilde{m}}{1+\tilde{m}}\end{aligned} \tag{73}$$

The derivative of $\tilde{v}_{\mathrm{d}}$ over time t gives

$$\frac{\mathrm{d}\tilde{v}_{\mathrm{d}}}{\mathrm{d}\tilde{t}} = -\frac{1}{(\tilde{m}+\tilde{x}_{\mathrm{d}})^2}\sqrt{\tilde{m}^2 + \frac{2}{\eta}(1+\tilde{m})[\tilde{x}_{\mathrm{d}} + (1+\tilde{m})\ln(1-\tilde{x}_{\mathrm{d}})]} - \frac{1}{\eta}\frac{1}{\tilde{m}+\tilde{x}_{\mathrm{d}}} \tag{74}$$

Therefore,

$$-\frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{pl}}} = \frac{\tilde{m}}{\tilde{m}+\tilde{x}_{\mathrm{d}}} + \frac{\tilde{m}}{(\tilde{m}+\tilde{x}_{\mathrm{d}})^2}\sqrt{\eta^2\tilde{m}^2 + 2\eta(1+\tilde{m})[\tilde{x}_{\mathrm{d}} + (1+\tilde{m})\ln(1-\tilde{x}_{\mathrm{d}})]} \tag{75}$$

At the commencement of the impact, $\tilde{x}_{\rm d} = 0$, the stress on the striking object is $\sigma_{\rm m} = -(1-\eta)\sigma_{\rm pl}$. Substitute the expression of the impact coefficient η, the stress on the striking object is

$$\sigma_{\rm m} = -\left(\sigma_{\rm pl} + \frac{\rho_0 v_{\rm im}^2}{\varepsilon_{\rm D}}\right)$$

Higher densification strain and lower material density of the cellular rod reduce impact stress.

Stress behind the shock front. Shock wave brings an abrupt raise of the stress at the shock front. During the time period Δt the impulse acting on the mass element ΔX in Fig. 12 is $(-\sigma_{\rm d} - \sigma_{\rm pl})$, which changes the velocity of the mass element $\rho_0 \Delta x$ from v_0 to $v_{\rm d}$. The impulse-momentum equation gives

$$(-\sigma_{\rm d} - \sigma_{\rm pl})\Delta t = \rho_0 \Delta x (v_{\rm d} - v_0)$$

Remove Δt from the two sides of the equation, one has

$$-\sigma_{\rm d} - \sigma_{\rm pl} = \rho_0 (v_{\rm s} - v_0)(v_{\rm d} - v_0),$$

which can be further written as

$$-\sigma_{\rm pl} = \sigma_{\rm d} + \rho_0 \frac{(v_{\rm d} - v_0)^2}{\varepsilon_{\rm D}} \tag{76}$$

The stress behind the shock front is compressive, hence the negative sign of $\sigma_{\rm d}$. The nondimensionalization of the above equation gives

$$-\frac{\sigma_{\rm d}}{\sigma_{\rm pl}} = 1 + \eta(\tilde{v}_{\rm d} - \tilde{v}_0)^2, \tag{77}$$

which relates to the length change rate of crushed cellular rod as

$$-\frac{\sigma_{\rm d}}{\sigma_{\rm pl}} = 1 + \eta \left(\frac{{\rm d}\tilde{x}_{\rm d}}{{\rm d}\tilde{t}}\right)^2 \tag{78}$$

Substituting the expression for ${\rm d}\tilde{x}_{\rm d}/{\rm d}\tilde{t}$ gives

$$-\frac{\sigma_{\rm d}}{\sigma_{\rm pl}} = 1 + \frac{1}{(\tilde{m} + \tilde{x}_{\rm d})^2}\{\eta \tilde{m}^2 + 2(1+\tilde{m})[\tilde{x}_{\rm d} + (1+\tilde{m})\ln(1-\tilde{x}_{\rm d})]\} \tag{79}$$

At the commencement of the impact, $\tilde{x}_{\rm d} = 0$, the stress behind the shock front is $\sigma_{\rm d} = -(1+\eta)\sigma_{\rm pl}$. This is the same as the stress on the striking object. After impact the two stresses becomes different.

5 Concluding Remarks

Cellular materials are used as impact energy absorbers due to their large densification strain ε_{D} at the plateau stress σ_{pl} during the plastic compression.

For a cellular rod struck by a object of areal mass m, the critical impact velocity is determined as

$$v_{\mathrm{im}}^{\mathrm{c}} = \frac{\sigma_{\mathrm{pl}}}{\rho_0 c_{\mathrm{el}}},$$

where ρ_0 is the density of the cellular material before. If the impact velocity is slower than the critical impact velocity $v_{\mathrm{im}}^{\mathrm{c}}$, only elastic wave will be generated from the impact end. If the impact velocity is higher than $v_{\mathrm{im}}^{\mathrm{c}}$, the elastic wave will be followed by plastic shock waves.

To fully absorb the kinetic energy of a striking object with mass M_{impact} and impact velocity v_{im}, the mass of the protecting rod should be higher than

$$M_{\mathrm{impact}} \left[\sqrt{1 + \left(\frac{v_{\mathrm{im}}}{v_{\mathrm{c}}} \right)^2} - 1 \right],$$

where $v_{\mathrm{c}} = \sqrt{\varepsilon_{\mathrm{D}} \sigma_{\mathrm{pl}} / \rho_0}$ is the characteristic velocity of the cellular material. A fix-ended cellular rod can fully arrest a striking object of mass M_{impact} with impact velocity up to

$$v_{\mathrm{c}} \sqrt{1 + \left(\frac{M_{\mathrm{cellular}}}{M_{\mathrm{impact}}} \right)^2 - 1},$$

At the commencement of the impact the stress on the striking object is

$$-\left(\sigma_{\mathrm{pl}} + \frac{\rho_0 v_{\mathrm{im}}^2}{\varepsilon_{\mathrm{D}}} \right)$$

Higher densification strain and lower material density of the cellular rod can reduce the impact stress.

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