# Accretion Mechanics of Nonlinear Elastic Circular Cylindrical Bars Under Finite Torsion*

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**Abstract**

In this paper we formulate the initial-boundary value problem of accreting circular cylindrical bars under finite torsion. It is assumed that the bar grows as a result of printing stress-free cylindrical layers on its boundary while it is under a time-dependent torque (or a time-dependent twist) and is free to deform axially. In a deforming body, accretion induces eigenetrains, and consequently residual stresses. We formulate the anelasticity problem by first constructing the natural Riemannian metric of the growing bar. This metric explicitly depends on the history of deformation during the accretion process. To simplify the kinematics, we consider incompressible solids. For the example of incompressible neo-Hookean solids, we solve the governing equations numerically. We also linearize the governing equations and compare the linearized solutions with the numerical solutions of the neo-Hookean bars.

**Keywords:** Accretion mechanics, surface growth, finite torsion, nonlinear elasticity, residual stress, geometric mechanics.

## 1 Introduction

There are many examples of structures built by accretion in nature (formation of planetary objects, volcanic and sedimentary rock formation, the growth of biological tissues, etc.) and engineering applications (built up of concrete dams in successive layers, solidification of metals, electrolytic deposition, thermal and laser-based 3D printing, etc.). The first theoretical study of accretion mechanics was an analysis of thick-walled cylinders manufactured by wire winding of an initial elastic tube by Southwell [1941]. As examples of notable subsequent contributions one can mention [Brown and Goodman, 1963, Metlov, 1985, Arutyunyan et al., 1990, Manzhirov, 1995, Skalak et al., 1997, Drozdov, 1998a,b]. In recent years there has been a renewed interest in the mechanics of accretion, and specifically the large deformation analysis of accreting bodies. There are several works in the recent literature [Ong and O'Reilly, 2004, Hodge and Papadopoulos, 2010, Lychev, 2011, Lychev and Manzhirov, 2013a, Manzhirov, 2014, Lychev and Manzhirov, 2013a,b, Tomassetti et al., 2016, Sozio and Yavari, 2017, Zurlo and Truskinovsky, 2017, 2018, Abi-Akl et al., 2019, Truskinovsky and Zurlo, 2019, Sozio and Yavari, 2019, Sozio et al., 2020, Abi-Akl and Cohen, 2020, Bergel and Papadopoulos, 2021, Lychev et al., 2021, Yavari et al., 2022]. For detailed reviews of the mechanics of accretion see [Naumov, 1994] and [Sozio and Yavari, 2017].

In classical finite elasticity, a body has a fixed reference configuration and motion is a time-dependent map from the reference configuration to the ambient space. For growing bodies the notion of reference

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configuration needs to be modified. There are two types of growth: bulk and surface growth. For a body undergoing bulk growth material points are fixed but their relaxed (natural) states change due to growth. In the literature this has been modeled using a multiplicative decomposition of deformation gradient into elastic and growth parts: $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{g}{\mathbf{F}}$.[1] Geometrically, in bulk growth the reference configuration is a Riemannian manifold $(\mathcal{B}, \mathbf{G}_t)$, where $\mathcal{B}$ is a fixed 3-manifold that is equipped with a time-dependent Riemannian metric $\mathbf{G}_t$ [Yavari, 2010].[2] For a body undergoing growth on its boundary (or a subset of its boundary) while in motion, the reference configuration is a time-dependent set $\mathcal{B}_t$. Material (stress-free or pre-stressed) can be either added (accretion) or removed (ablation) from the boundary. The natural configuration of the growing body depends on its initial natural configuration (the natural configuration before accretion started) and the state of deformation at the time of attachment of new material points. Accretion induces residual stress, in general.[3] This is due to the non-flatness of the material metric. A geometric analysis of finite deformations of accreting bodies was presented in [Sozio and Yavari, 2017, 2019]. Recently, Yavari et al. [2022] formulated and solved the nonlinear initial-boundary value problem of accreting circular cylindrical bars under finite extension. In this paper we analyze circular cylindrical shafts that undergo finite torsion, are free to deform axially, and are simultaneously growing symmetrically. The classical analogue of this problem (without accretion) has been studied extensively in the literature and is a subset of Family 3 universal deformations [Ericksen, 1954], see Remark 3.2.

This paper is organized as follows. In §2, we tersely review some elements of Riemannian geometry and the nonlinear mechanics of accretion. In §3, the nonlinear accretion problem of a circular cylindrical shaft that is under finite torsion while it is free to deform axially is formulated. The natural configuration (material manifold) of the growing shaft is constructed, and stresses and residual stresses are calculated assuming that during the accretion process either a time-dependent applied torque or a time-dependent twist per unit length is given. Several numerical examples are solved and discussed. The kinematics, stresses, and residual stresses are calculated in the setting of linear accretion mechanics. The linear and nonlinear solutions are compared in a numerical example. Conclusions are given in §4.

## 2 Nonlinear Mechanics of Accretion

In this section, we briefly review some elements of Riemannian geometry, nonlinear elasticity and anelasticity, and accretion mechanics. For more detailed discussions, see [Marsden and Hughes, 1983, Yavari, 2010, Yavari and Goriely, 2012, Sozio and Yavari, 2019].

**Riemannian geometry.** Let us consider a smooth $n$-manifold $\mathcal{B}$ (this is identified with the body in its reference configuration). Its tangent space at a point $X \in \mathcal{B}$ is denoted $T_X\mathcal{B}$. Let $\mathcal{S}$ be another $n$-manifold (this is the Euclidean ambient space) and $\varphi : \mathcal{B} \to \mathcal{S}$ a smooth and invertible mapping (this is the deformation mapping). A smooth vector field $\mathbf{W}$ on $\mathcal{B}$ at every $X \in \mathcal{B}$ assigns a vector $\mathbf{W}_X$ such that $X \mapsto \mathbf{W}_X \in T_X\mathcal{B}$ varies smoothly. For $\mathbf{W}$ a vector field on $\mathcal{B}$, $\varphi_*\mathbf{W} = T\varphi \cdot \mathbf{W} \circ \varphi^{-1}$ is a vector field on $\mathcal{C} = \varphi(\mathcal{B}) \subset \mathcal{S}$—the push-forward of $\mathbf{W}$ by $\varphi$. Similarly, if $\mathbf{w}$ is a vector field on $\mathcal{C} = \varphi(\mathcal{B})$, the pull-back of $\mathbf{w}$ by $\varphi$ is defined as $\varphi^*\mathbf{w} = T(\varphi^{-1}) \cdot \mathbf{w} \circ \varphi$, which is a vector field on $\mathcal{B}$. The derivative map of $\varphi$ is denoted by $\mathbf{F} = T\varphi$, and is a two-point tensor. When $\varphi$ is a deformation map, $\mathbf{F}$ has traditionally been called deformation gradient in the finite elasticity literature. One should note that $\mathbf{F}$ (unlike the gradient operator) is metric independent. It has the following representation

$$\mathbf{F} = F^a{}_A \, \frac{\partial}{\partial x^a} \otimes dX^A, \qquad F^a{}_A = \frac{\partial \varphi^a}{\partial X^A}\,, \tag{2.1}$$

[1]This decomposition is due to Kondaurov and Nikitin [1987], Takamizawa and Hayashi [1987], Takamizawa and Matsuda [1990], and Takamizawa [1991]. One can find similar ideas in [Tranquillo and Murray, 1992, 1993]. This decomposition was popularized in the literature of biomechanics by Rodriguez et al. [1994]. For a historical account of this decomposition in different fields see [Sadik and Yavari, 2017, Yavari and Sozio, 2022].

[2]Growing bodies are non-Euclidean in the sense that their natural configuration is not Euclidean, in general. Non-Euclidean solids—a term that was coined by Henri Poincaré [Poincaré, 1905]—has been used interchangeably for anelastic bodies in the recent literature [Zurlo and Truskinovsky, 2017, 2018, Truskinovsky and Zurlo, 2019].

[3]This was first observed in the setting of linear accretion mechanics in the seminal work of Brown and Goodman [1963] who studied accreting planets under self-gravity.

where $\{X^A\}$ and $\{x^a\}$ are local coordinate charts for $\mathcal{B}$ and $\mathcal{S}$, respectively. Note that $\{\frac{\partial}{\partial x^a}\}$ is a basis for $T_x\mathcal{C}$ ($x = \varphi(X)$) and $\{dX^A\}$ is a basis for $T_X^*\mathcal{B}$, the co-tangent space, i.e., the dual space of $T_X\mathcal{B}$, or the space of 1-forms. The push-forward and pull-back of vectors have the coordinate representations $(\varphi_*\mathbf{W})^a = F^a{}_A\, W^A$, and $(\varphi^*\mathbf{w})^A = (F^{-1})_a{}^A\, w^a$. A $\binom{0}{2}$-tensor at $X \in \mathcal{B}$ is a bilinear map $\mathbf{T} : T_X\mathcal{B} \times T_X\mathcal{B} \to \mathbb{R}$, and in a local coordinate chart $\{X^A\}$ for $\mathcal{B}$ one has $\mathbf{T}(\mathbf{U},\mathbf{W}) = T_{AB}\, U^A\, W^B$, where $\mathbf{U}$ and $\mathbf{W}$ are vectors, i.e., are elements of $T_X\mathcal{B}$. Let $\mathcal{B}$ be a smooth manifold that is equipped with an inner product $\mathbf{G}_X$ on the tangent space $T_X\mathcal{B}$. Assume that $\mathbf{G}_X$ varies smoothly, i.e., if $\mathbf{U}$ and $\mathbf{W}$ are vector fields on $\mathcal{B}$, then $X \mapsto \mathbf{G}_X(\mathbf{U}_X, \mathbf{W}_X) = \langle\!\langle \mathbf{U}_X, \mathbf{W}_X \rangle\!\rangle_{\mathbf{G}_X}$, where $\langle\!\langle ., . \rangle\!\rangle_{\mathbf{G}_X}$ is the inner product induced by the metric $\mathbf{G}_X$, is a smooth function. In this case $(\mathcal{B}, \mathbf{G})$ is called a Riemannian manifold.

For two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{C}, \mathbf{g})$, and for a diffeomorphism (a smooth map with smooth inverse) $\varphi : \mathcal{B} \to \mathcal{C}$, push-forward of the metric $\mathbf{G}$ is denoted by $\varphi_*\mathbf{G}$. It is a metric on $\mathcal{C} = \varphi(\mathcal{B})$, and is defined as

$$\langle\!\langle \mathbf{u}_x, \mathbf{w}_x \rangle\!\rangle_{(\varphi_*\mathbf{G})_x} = \langle\!\langle (\varphi^*\mathbf{u})_X, (\varphi^*\mathbf{w})_X \rangle\!\rangle_{\mathbf{G}_X}\,, \tag{2.2}$$

where $x = \varphi(X)$. In components, $(\varphi_*\mathbf{G})_{ab} = (F^{-1})_a{}^A\, (F^{-1})_b{}^B\, G_{AB}$. The pull-back of the metric $\mathbf{g}$ is a metric in $\varphi^{-1}(\mathcal{C}) = \mathcal{B}$, and is denoted by $\mathbf{C}^\flat = \varphi^*\mathbf{g}$—the right Cauchy-Green strain. It is defined as

$$\langle\!\langle \mathbf{U}_X, \mathbf{W}_X \rangle\!\rangle_{(\varphi^*\mathbf{g})_X} = \langle\!\langle (\varphi_*\mathbf{U})_x, (\varphi_*\mathbf{W})_x \rangle\!\rangle_{\mathbf{g}_x}\,, \qquad (\varphi^*\mathbf{g})_{AB} = F^a{}_A\, F^b{}_B\, g_{ab}\,. \tag{2.3}$$

If $\mathbf{G} = \varphi^*\mathbf{g}$, or equivalently, $\mathbf{g} = \varphi_*\mathbf{G}$, $\varphi$ is called an isometry and the Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{C}, \mathbf{g})$ are isometric.

**Kinematics.** In an accretion process, the material manifold that represents the growing body is time dependent; new material points are attached to part of the boundary of the body that we call the *growth surface*. Let us identify the accreting body with a time-dependent 3-manifold $\mathcal{B}_t$. The *initial body* is denoted by $\mathcal{B} = \mathcal{B}_0$. Accretion occurs in a time interval $[0, t_a]$. We follow [Sozio and Yavari, 2019] and define an accreting body to be a 3-manifold $\mathcal{M}$—the *material ambient space*—that is embedded in the Euclidean ambient space along with a smooth *time of attachment map* $\tau : \mathcal{M} \to [0, t_a]$.[4] Note that for all points in the initial body $\mathcal{B}$, $\tau(X) = 0$. The body at time $t$, $\mathcal{B}_t$, is defined as

$$\mathcal{B}_t = \{X \in \mathcal{M} \mid \tau(X) \leq t\}\,. \tag{2.4}$$

Note that the growth surface at time $t$ is given as $\Omega_t = \tau^{-1}(t)$. For an accreting body, motion is a time-dependent map $\varphi_t : \mathcal{B}_t \to \mathcal{S}$, $t \in [0, t_a]$, where $\mathcal{S}$ is the Euclidean ambient space. Consider the map $\bar{\varphi} : \mathcal{M} \to \mathcal{S}$ defined as $\bar{\varphi}(X) = \varphi(X, \tau(X))$. For points in the initial body $\bar{\varphi}(X) = X$. For a point $X$ in the *secondary body* $\mathcal{B}_t \setminus \mathcal{B}_0$, $\bar{\varphi}(X)$ is the placement of $X$ at its time of attachment. Notice that for each layer $\Omega_t$, $\bar{\varphi}|_{\Omega_t} = \varphi_t|_{\Omega_t}$ because for $\tau(X) = t$, $\bar{\varphi}(X) = \varphi(X, t)$. This implies that $\bar{\varphi}$ records the placement of the deformed configuration $\omega_t = \varphi_t(\Omega_t) = \bar{\varphi}(\Omega_t)$ of the layer $\Omega_t$ at its time of attachment. It should be noted that the map $\bar{\varphi}$ is not one-to-one, in general. In other words, $\bar{\varphi}$ is not a deformation mapping. $T\bar{\varphi}$ need not be injective either.

Deformation gradient is the derivative of $\varphi_t : \mathcal{B}_t \to \mathcal{S}$, see (2.1). The *frozen deformation gradient* is defined as $\overline{\mathbf{F}}(X) = \mathbf{F}(X, \tau(X))$; it is the deformation gradient of point $X$ at its time of attachment $\tau(X)$. It can be shown that $T\bar{\varphi} = \overline{\mathbf{F}} + \mathbf{V} \otimes d\tau$, where $\mathbf{V}(X,t) = \frac{\partial}{\partial t}\varphi(X,t)$ is the material velocity. The frozen deformation gradient $\overline{\mathbf{F}}(X)$ is compatible on each single layer $\Omega_t$. However, it is not the tangent map of any embedding; $\overline{\mathbf{F}}$ is incompatible, in general. In accreting bodies, the incompatibility of the frozen deformation gradient is the source of anelasticity, and hence residual stresses [Sozio and Yavari, 2019].

The growth surface in the deformed configuration $\omega_t = \varphi_t(\Omega_t)$ is that part of the deformed boundary where new material points are added. The growth velocity is a vector field $\mathbf{u}_t$ on $\omega_t$ that describes the rate and direction at which new material points are being added to the boundary. The material growth velocity $\mathbf{U}_t$ describes the time evolution of the layers $\Omega_t$ in the material ambient space. It turns out that $\mathbf{U}_t$, and consequently the material motion, is not unique. In other words, there is some freedom in choosing $\mathbf{U}_t$, and all these equivalent $\mathbf{U}_t$'s lead to isometric material manifolds [Sozio and Yavari, 2019]. Natural distances in

[4]The idea of a time of attachment map is due to Metlov [1985].

the material manifold are measured using a material metric $\mathbf{G}$. This metric is not known a priori in accretion problems; it depends on the state of deformation of the body during the accretion process. It is determined after solving the accretion initial-boundary-value problem. The *accretion tensor* $\mathbf{Q}$ is a time-independent two-point tensor that is defined as

$$\mathbf{Q}(X) = \bar{\mathbf{F}}(X) + \left[\mathbf{u}(\bar{\varphi}(X), \tau(X)) - \bar{\mathbf{F}}(X)\mathbf{U}(X)\right] \otimes \mathrm{d}\tau(X)\,, \quad X \in \mathcal{M}\,. \tag{2.5}$$

Because $\langle d\tau, \mathbf{U}\rangle = 1$, $\mathbf{Q}\mathbf{U} = \mathbf{u}$. Notice that the accretion tensor $\mathbf{Q}$ is not the tangent map of any embedding, although it is compatible on each single layer. Also note that, $\mathbf{Q}|_\Omega = \bar{\mathbf{F}}|_\Omega = T\bar{\varphi}|_\Omega$. The Euclidean metric of the ambient space is denoted by $\mathbf{g}$. The material metric of the accreting body is defined as the pull-back of the Euclidean ambient metric $\mathbf{g}$ using $\mathbf{Q}$, i.e., $\mathbf{G}(X) = \mathbf{Q}^\star(X)\,\mathbf{g}(\bar{\varphi}(X))\,\mathbf{Q}(X)$. In components, $G_{AB}(X) = Q^a{}_A(X)\, g_{ab}(\bar{\varphi}(X))\, Q^b{}_B(X)$. One can show that if the energy function $W$ of the material is rank-one convex, and if the growth surface is traction-free, then $\bar{\mathbf{F}} = \mathbf{Q}$ [Sozio and Yavari, 2019].

Transpose of the deformation gradient $\mathbf{F}^\mathsf{T} : T_x\mathcal{C} \to T_X\mathcal{B}$ is defined as $\langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v}\rangle\!\rangle_{\mathbf{g}} = \langle\!\langle \mathbf{V}, \mathbf{F}^\mathsf{T}\mathbf{v}\rangle\!\rangle_{\mathbf{G}}$, $\forall \mathbf{V} \in T_X\mathcal{B}$, $\mathbf{v} \in T_x\mathcal{C}$. In components, $(F^\mathsf{T}(X))^A{}_a = g_{ab}(x)\, F^b{}_B(X)\, G^{AB}(X)$. The right Cauchy-Green deformation tensor is defined as $\mathbf{C}(X) = \mathbf{F}^\mathsf{T}(X)\mathbf{F}(X) : T_X\mathcal{B} \to T_X\mathcal{B}$, and in components, $C^A{}_B = (F^\mathsf{T})^A{}_a\, F^a{}_B$. Note that $\mathbf{C}^\flat = \varphi^*\mathbf{g}$ ($\flat$ is the flat operator induced by the metric $\mathbf{g}$), and has components $C_{AB} = F^a{}_A\, F^b{}_B\, g_{ab} \circ \varphi$. The left Cauchy-Green deformation tensor is defined as $\mathbf{B}^\sharp = \varphi^*\mathbf{g}^\sharp$ ($\sharp$ is the sharp operator induced by the metric $\mathbf{g}$), and has components $B^{AB} = (F^{-1})^A{}_a\, (F^{-1})^B{}_b\, g^{ab}$. The deformation tensors $\mathbf{c}^\flat$ and $\mathbf{b}^\sharp$ (the Finger deformation tensor) are the spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$, respectively, and are defined as

$$\begin{aligned} \mathbf{c}^\flat &= \varphi_*\mathbf{G}\,, \qquad c_{ab} = \left(F^{-1}\right)^A{}_a \left(F^{-1}\right)^B{}_b\, G_{AB}\,, \\ \mathbf{b}^\sharp &= \varphi_*\mathbf{G}^\sharp\,, \qquad b^{ab} = F^a{}_A\, F^b{}_B\, G^{AB}. \end{aligned} \tag{2.6}$$

It is straightforward to see that $b^{ac}\, c_{cb} = b^a{}_m\, c^m{}_b = \delta^a_b$, i.e., $\mathbf{b} = \mathbf{c}^{-1}$. The strain tensors $\mathbf{C}$ and $\mathbf{b}$ have the principal invariants $I_1$, $I_2$, and $I_3$, which are defined as [Ogden, 1984]: $I_1 = \operatorname{tr}\mathbf{b} = b^a{}_a = b^{ab}\, g_{ab}$, $I_2 = \frac{1}{2}\left(I_1^2 - \operatorname{tr}\mathbf{b}^2\right) \frac{1}{2}\left(I_1^2 - b^a{}_b\, b^b{}_a\right) = \frac{1}{2}\left(I_1^2 - b^{ab}b^{cd}\, g_{ac}\, g_{bd}\right)$, and $I_3 = \det\mathbf{b}$.

**Constitutive equations.** For an isotropic hyperelastic solid, the energy function depends on deformation through the principal invariants: $W = W(X, I_1, I_2, I_3)$. For an incompressible ($I_3 = 1$) isotropic hyperelastic solid energy function only depends on $I_1$ and $I_2$: $W = W(X, I_1, I_2)$. The $X$-dependence of the energy function models material inhomogeneity. In this paper, we restrict our calculations to homogeneous bodies. The Cauchy stress has the following representation [Doyle and Ericksen, 1956, Simo and Marsden, 1984]

$$\boldsymbol{\sigma} = -p\,\mathbf{g}^\sharp + 2W_1\,\mathbf{b}^\sharp - 2W_2\,\mathbf{c}^\sharp\,, \quad \sigma^{ab} = -p\,g^{ab} + 2W_1\,b^{ab} - 2W_2\,c^{ab}\,, \tag{2.7}$$

where $p$ is the Lagrange multiplier associated with the incompressibility constraint $J = \sqrt{I_3} = 1$, and $W_i = \frac{\partial W}{\partial I_i}$, $i = 1, 2$. Notice that $\mathbf{b}^\sharp$ and $\mathbf{c}^\sharp$, and consequently $\boldsymbol{\sigma}$, explicitly depend on the material metric $\mathbf{G}$. It is assumed that the material points of the accreting body are isotropic in their relaxed configuration. However, in its current configuration the accreting body may not be isotropic.

**Equilibrium equations.** Accretion is usually a slow process, and hence one can ignore inertial effects. In the absence of body forces, the balance of linear momentum in local form, and in terms of the Cauchy stress, reads: $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$, where $\operatorname{div} = \operatorname{div}^{\mathbf{g}}$ is divergence with respect to the spatial metric. In components, one writes $(\operatorname{div}\boldsymbol{\sigma})^a = \sigma^{ab}{}_{|b} = \frac{\partial\sigma^{ab}}{\partial x^b} + \sigma^{ac}\gamma^b{}_{cb} + \sigma^{cb}\gamma^a{}_{cb}$, where $\gamma^a{}_{bc}$ is the Christoffel symbol of the Levi-Civita connection $\nabla^{\mathbf{g}}$ in the local coordinate chart $\{x^a\}$, and is defined as $\nabla^{\mathbf{g}}{}_{\partial_b}\partial_c = \gamma^a{}_{bc}\,\partial_a$. More explicitly, $\gamma^a{}_{bc} = \frac{1}{2}g^{ak}\left(g_{kb,c} + g_{kc,b} - g_{bc,k}\right)$.

# 3 Torsion of an accreting circular cylindrical bar

In this section we formulate the initial-boundary value problem of symmetric accretion of a circular cylindrical bar made of an incompressible isotropic hyperelastic solid that is undergoing finite torsion while it is free to

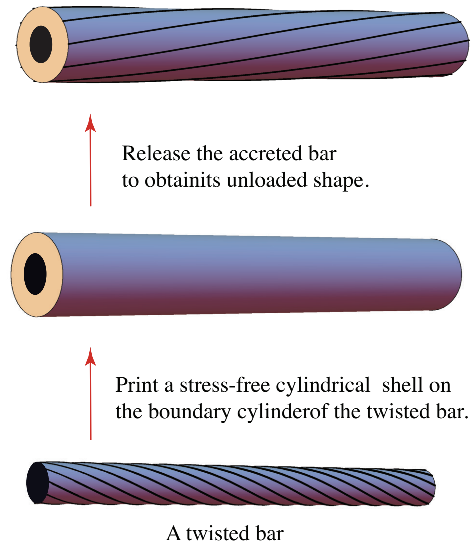


Figure 1: *The twist-fit problem: A cylindrical bar is first twisted. In the deformed configuration, a stress-free cylindrical shell is printed on its cylinder boundary. When the accreted bar is released, the unloaded bar is residually stressed.*

deform axially. In order to motivate the continuous accretion problem, let us first discuss a discrete accretion problem, which is a twist-fit problem [Yavari and Goriely, 2015]. Consider a circular cylindrical bar with radius $R_1$ that is finitely twisted, see Fig.1. While the bar is twisted a cylindrical shell with thickness $R_2 - R_1$ is printed on its boundary cylinder. In other words, we start with a stress-free solid cylinder with radius $R_2$, remove a concentric solid cylinder of radius $R_1$, and replace it with the twisted bar with radius $R_1$, and then glue them. After removal of external loads, the accreted bar is residually stressed. This is because the natural configurations of the core and the shell are incompatible. In the following, we will formulate the continuous analogue of this problem. We will calculate the metric of the natural configuration, the stress distribution during accretion, and the residual stress distribution after removal of the external loads.

**Kinematics and the material metric.** Let us consider a circular cylindrical bar with initial length $L$ and radius $R_0$ that is made of a homogeneous isotropic and incompressible material with energy function $W = W(I_1, I_2)$. We use the cylindrical coordinates $(R, \Theta, Z)$ in the reference configuration, and cylindrical coordinates $(r, \theta, z)$ in the current configuration. The metrics of the reference and current configurations have the following representations $(0 \leq R \leq R_0)$

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.1}$$

Let us consider a time-dependent torsion of the circular cylindrical bar such that it is slow enough for the inertial effects to be negligible. Torsion of circular cylindrical bars is a subset of Family 3 deformations that are universal for incompressible isotropic solids [Ericksen, 1954], and have the following form[5]

$$r = r(R, t), \quad \theta = \Theta + \psi(t) Z, \quad z = \lambda^2(t) Z, \tag{3.2}$$

where $\psi(t)$ is twist per unit length, and $\lambda^2(t)$ is the axial stretch, see Fig.2. Under a twist-control loading $\psi(t)$ is given while $\lambda(t)$ needs to be calculated. Under a torque-control loading the applied torque is given while

[5]Family 3 deformations are universal for certain inhomogeneous and anisotropic bars as well [Yavari, 2021, Yavari and Goriely, 2021, 2022]. In this paper, we restrict our calculations to isotropic and homogeneous bars.

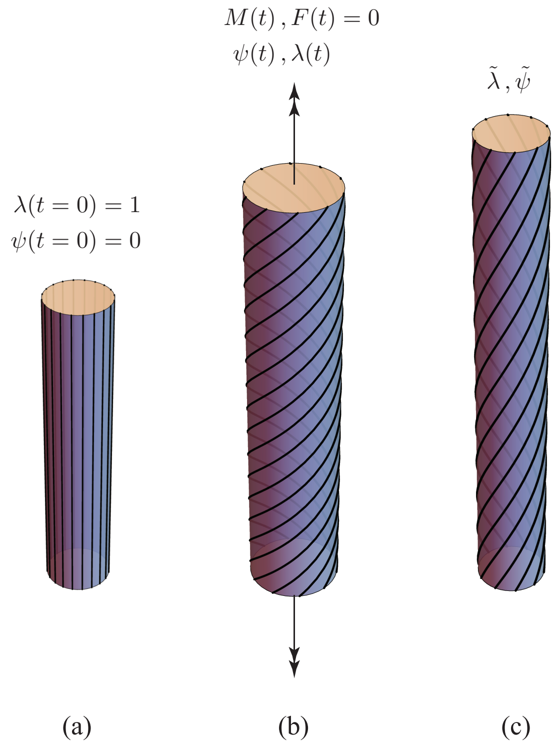


Figure 2: *An accreting circular cylindrical bar undergoing finite torsion while it is free to deform axially. (a) The initial bar, (b) the accreting bar at time t, and (c) the residually-stressed accreted bar after the completion of accretion and removal of the external forces.*

both $\psi(t)$ and $\lambda(t)$ are unknown functions to be determined. In the numerical examples we will consider both cases. The deformation gradient reads

$$\mathbf{F} = \mathbf{F}(R,t) = \begin{bmatrix} r'(R,t) & 0 & 0 \\ 0 & 1 & \psi(t) \\ 0 & 0 & \lambda^2(t) \end{bmatrix} , \tag{3.3}$$

where $r'(R,t) = \frac{\partial r(R,t)}{\partial R}$. The incompressibility condition is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{\lambda^2(t)\, r(R,t)\, r'(R,t)}{R} = 1\,. \tag{3.4}$$

This condition, together with $r(0,t) = 0$, gives us

$$r(R,t) = \frac{R}{\lambda(t)}\,, \quad 0 \leq R \leq R_0\,. \tag{3.5}$$

We assume that while the cylindrical bar is under the time-dependent deformation (3.2) cylindrical layers of stress-free material are printed continuously on its boundary (see Fig.3). The growth velocity is assumed to be normal to the boundary in the current configuration and has magnitude $u_g(t)$. This means that in the time interval $[t, t+dt]$ a stress-free circular cylindrical shell of thickness $u_g(t)dt$ is attached to the deformed body. We also assume that this accretion process is continuous in the time interval $t \in [0, t_a]$. Let us assign a time of accretion $\tau(R)$ to each layer with the radial coordinate $R$ in the reference configuration. For $0 \leq R \leq R_0$, $\tau(R) = 0$. We assume that there is no ablation during the accretion process, and hence $\tau(R)$ is invertible for $R > R_0$. Its inverse is denoted as $s = \tau^{-1}$, and it assigns to the time $t$ the radial coordinate

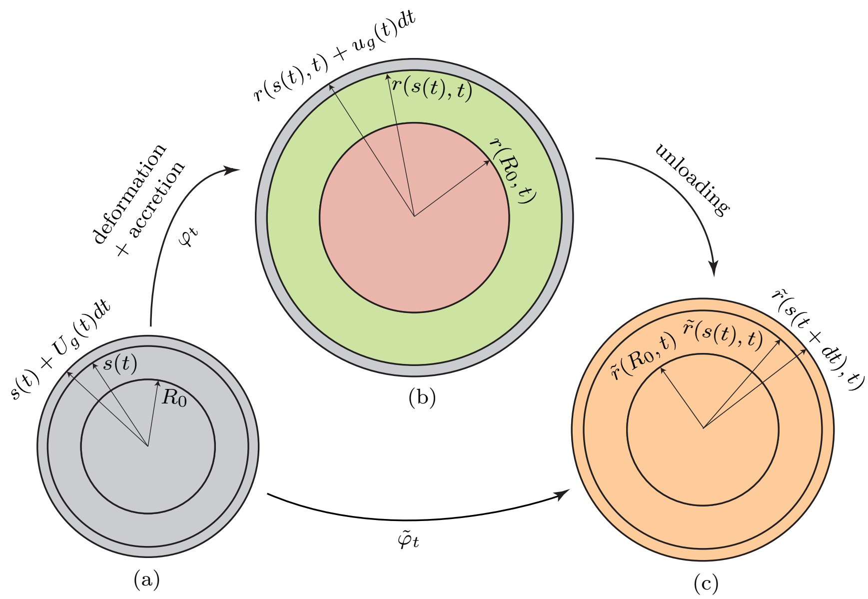


Figure 3: *Cross section of a circular cylindrical bar undergoing symmetric accretion and torsion simultaneously. (a) The material manifold $(\mathcal{B}, \mathbf{G})$. The radial coordinate of the boundary of the accreting bar at time $t$ is $s(t)$. At a later time $t+dt$ the radial coordinate changes to $s(t)+U_g(t)dt$. (b) The deformed bar under torsion with a layer of stress-feee material of thickness $u_g(t)dt$ joining its boundary during the time interval $[t, t+dt]$. (c) The residually-stressed accreted bar after the removal of the external torque.*

of the accreted cylinder in the reference configuration. The growth surfaces in the reference and the current configurations are defined as

$$\begin{aligned} \Omega_t &= \{(s(t), \Theta, Z) : 0 \leq \Theta < 2\pi\,, 0 \leq Z \leq L\}\,, \\ \omega_t &= \left\{(r(s(t),t), \Theta + \psi(t)\,Z, \lambda^2(t)Z) : 0 \leq \Theta < 2\pi\,, 0 \leq Z \leq L\right\}\,. \end{aligned} \tag{3.6}$$

Note that

$$\frac{d}{dt} r(s(t),t) = \frac{\partial r}{\partial R}(s(t),t)\dot{s}(t) + \frac{\partial r}{\partial t}(s(t),t) = r'(s(t),t)\; U_g(t) + V^r(s(t),t), \tag{3.7}$$

where $U_g(t) = \dot{s}(t)$, and $V^r = \frac{\partial r}{\partial t}$ is the radial component of the material velocity on the growth surface. In the absence of accretion, the spatial velocity of the material points lying on the boundary is $V^r(s(t),t)$, and this implies that

$$u_g(t) = r'(s(t),t)\, U_g(t)\,. \tag{3.8}$$

Following [Sozio and Yavari, 2017], we choose $U_g(t) = u_g(t)$. Sozio and Yavari [2017] showed that other choices for $U_g(t)$ will result in isometric material metrics. In other words, this choice will not affect the calculation of stresses, see Remark 3.1.

From (3.8), the choice $U_g(t) = u_g(t)$ imposes the following constraint on $r(R,t)$:

$$r'(s(t),t) = 1\,, \text{ or } \qquad r'(R, \tau(R)) = 1\,. \tag{3.9}$$

Note that $s(t) = R_0 + \int_0^t u_g(\xi)d\xi$. In order to simplify the calculations, let us assume that the spatial growth velocity is constant, i.e., $u_g(t) = u_0 > 0$. Thus

$$s(t) = R_0 + u_0 t\,, \text{ or } \qquad \tau(R) = \frac{R - R_0}{u_0}\,. \tag{3.10}$$

The constraint (3.9) is simplified to read

$$r'(R_0 + u_0 t, t) = 1\,, \text{ or } \qquad r'\left(R, \frac{R-R_0}{u_0}\right) = 1\,. \tag{3.11}$$

For the initial body, i.e., for $0 \leq R \leq R_0$, the material metric has the representation $(3.1)_1$. For $R_0 \leq R \leq s(t)$, we assume that the accreted cylindrical layer at any instant of time $t$ is stress-free (generalizing our analysis to the case of pre-stressed material is straightforward [Sozio and Yavari, 2017]). This implies that the material metric at $R = s(t)$ is the pull-back of the metric of the (Euclidean) ambient space, i.e.,

$$\mathbf{G}(s(t)) = \varphi_t^* \mathbf{g}(r(s(t),t)) \,, \text{ or } \qquad \mathbf{G}(R) = \varphi_{\tau(R)}^* \, \mathbf{g}(r(R,\tau(R))). \tag{3.12}$$

In components, one has $G_{AB}(s(t)) = G_{AB}(R) = F^a{}_A(R,\tau(R))\, F^b{}_B(R,\tau(R))\, g_{ab}(r(R,\tau(R)))$. Therefore

$$\begin{aligned}\mathbf{G}(R) &= \begin{bmatrix} r'^2(R,\tau(R)) & 0 & 0 \\ 0 & r^2(R,\tau(R)) & \psi(\tau(R))\, r^2(R,\tau(R)) \\ 0 & \psi(\tau(R))\, r^2(R,\tau(R)) & \psi^2(\tau(R))\, r^2(R,\tau(R)) + \lambda^4(\tau(R)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2(R,\tau(R)) & \psi(\tau(R))\, r^2(R,\tau(R)) \\ 0 & \psi(\tau(R))\, r^2(R,\tau(R)) & \psi^2(\tau(R))\, r^2(R,\tau(R)) + \lambda^4(\tau(R)) \end{bmatrix},\end{aligned} \tag{3.13}$$

where use was made of (3.9), and $\tau(R)$ is given in $(3.10)_2$.

For this accretion problem, the material manifold is an evolving Riemannian manifold $(\mathcal{B}_t, \mathbf{G})$, where

$$\mathcal{B}_t = \{(R,\Theta,Z) : 0 \leq \Theta < 2\pi \,,\; R_0 \leq R \leq s(t) = R_0 + u_0 t \,,\; 0 \leq Z \leq L\} \,, \tag{3.14}$$

and[6]

$$\begin{aligned} 0 \leq R \leq R_0 :\; \mathbf{G} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ R_0 \leq R \leq R_0 + u_0 t :\; \mathbf{G} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2(R,\tau(R)) & \psi(\tau(R))\, r^2(R,\tau(R)) \\ 0 & \psi(\tau(R))\, r^2(R,\tau(R)) & \psi^2(\tau(R))\, r^2(R,\tau(R)) + \lambda^4(\tau(R)) \end{bmatrix}. \end{aligned} \tag{3.15}$$

The incompressibility constraint for $R \geq R_0$ is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\, \det \mathbf{F} = \frac{r(R,t)}{r(R,\tau(R))\, \lambda^2(\tau(R))}\, r'(R,t)\, \lambda^2(t) = 1. \tag{3.16}$$

Thus

$$r(R,t)\, r'(R,t) = \bar{r}(R)\, \frac{\lambda^2(\tau(R))}{\lambda^2(t)} \,, \tag{3.17}$$

where $\bar{r}(R) := r(R,\tau(R)) = r\left(R, \frac{R-R_0}{u_0}\right)$. Hence

$$r^2(R,t) = \frac{R_0^2}{\lambda^2(t)} + \frac{2}{\lambda^2(t)} \int_{R_0}^{R} \bar{r}(\xi)\, \lambda^2(\tau(\xi))\, d\xi, \quad R_0 \leq R \leq R_0 + u_0 t \,, \tag{3.18}$$

where use was made of (3.5). Thus

$$\lambda^2(t)\, r^2(R,t) = R_0^2 + 2 \int_{R_0}^{R} \bar{r}(\xi)\, \lambda^2(\tau(\xi))\, d\xi \,. \tag{3.19}$$

[6]Note that as soon as a layer is deposited it becomes part of the body and participates in the deformation process. If the load is fixed, one would have a classical twist-fit problem (Fig.1). The time dependence of the load (or twist) makes the natural state of the body (the material metric) inhomogeneous. In other words, after completion of accretion if each cylindrical layer is allowed to relax independently of the rest of the body the collection of relaxed thin cylindrical shells can not be put back together in the Euclidean ambient space without local elastic deformations. This incompatibility of the local rest configurations depends on the state of deformation during accretion and indirectly on the applied load during accretion.

The right-hand side is time independent, and hence, $\lambda^2(t)\, r^2(R,t)$ is independent of time. In particular, $\lambda^2(t)\, r^2(R,t) = \lambda^2(\tau(R))\, r^2(R,\tau(R))$, and hence

$$r(R,t) = \frac{\lambda(\tau(R))}{\lambda(t)}\, \bar{r}(R)\,. \tag{3.20}$$

The constraint (3.9) gives the following ODE for the unknown function $\bar{r}(R)$:

$$\bar{r}'(R) + \frac{\lambda'(\tau(R))\, \tau'(R)}{\lambda(\tau(R))}\, \bar{r}(R) = 1\,. \tag{3.21}$$

This ODE has has the following solution:

$$\bar{r}(R) = \frac{1}{\lambda(\tau(R))} \left[ R_0 + \int_{R_0}^{R} \lambda(\tau(\xi))\, d\xi \right]. \tag{3.22}$$

Therefore[7]

$$r(R,t) = \frac{1}{\lambda(t)} \left[ R_0 + \int_{R_0}^{R} \lambda(\tau(\xi))\, d\xi \right]. \tag{3.23}$$

For $0 \le R \le R_0$:

$$\mathbf{b}^\sharp(R,t) = \begin{bmatrix} \frac{1}{\lambda^2(t)} & 0 & 0 \\ 0 & \frac{1}{R^2} + \psi^2(t) & \lambda^2(t)\psi(t) \\ 0 & \lambda^2(t)\psi(t) & \lambda^4(t) \end{bmatrix}, \qquad \mathbf{c}^\sharp(R,t) = \begin{bmatrix} \lambda^2(t) & 0 & 0 \\ 0 & \frac{\lambda^4(t)}{R^2} & -\psi(t) \\ 0 & -\psi(t) & \frac{R^2\psi^2(t)+1}{\lambda^4(t)} \end{bmatrix}. \tag{3.24}$$

The principal invariants of $\mathbf{b}$ read

$$I_1(R,t) = \frac{2 + R^2\psi^2(t) + \lambda^6(t)}{\lambda^2(t)}\,, \qquad I_2(R,t) = \frac{1 + R^2\psi^2(t) + 2\lambda^6(t)}{\lambda^4(t)}\,. \tag{3.25}$$

The Cauchy stress has the following non-zero components

$$\begin{aligned} \sigma^{rr}(R,t) &= -p(R,t) + \frac{\alpha(R,t)}{\lambda^2(t)} - \beta(R,t)\, \lambda^2(t)\,, \\ \sigma^{\theta\theta}(R,t) &= -p(R,t)\, \frac{\lambda^2(t)}{R^2} + \alpha(R,t) \left[ \frac{1}{R^2} + \psi^2(t) \right] - \frac{\beta(R,t)\, \lambda^4(t)}{R^2}\,, \\ \sigma^{zz}(R,t) &= -p(R,t) + \alpha(R,t)\, \lambda^4(t) - \beta(R,t)\, \frac{1 + R^2\psi^2(t)}{\lambda^4(t)}\,, \\ \sigma^{\theta z}(R,t) &= \psi(t) \left[ \alpha(R,t)\lambda^2(t) + \beta(R,t) \right], \end{aligned} \tag{3.26}$$

where $\alpha = 2\frac{\partial W}{\partial I_1}$ and $\beta = 2\frac{\partial W}{\partial I_2}$. Using the circumferential and axial equilibrium equations one concludes that $p = p(R,t)$. The radial equilibrium equation reads $\frac{\partial \sigma^{rr}}{\partial r} + \frac{1}{r}\sigma^{rr} - r\sigma^{\theta\theta} = 0$. This can be rewritten in terms of the referential coordinates as

$$\frac{\partial \sigma^{rr}}{\partial R} - \frac{\psi^2(t)}{\lambda^2(t)} \alpha R = 0\,. \tag{3.27}$$

Thus

$$\sigma^{rr}(R,t) = \sigma_0(t) - \frac{\psi^2(t)}{\lambda^2(t)} \int_R^{R_0} \xi\, \alpha(\xi,t)\, d\xi\,, \tag{3.28}$$

where $\sigma_0(t) = \sigma^{rr}(R_0,t)$. This implies that for the initial body one has

$$-\, p(R,t) = \sigma_0(t) - \frac{\psi^2(t)}{\lambda^2(t)} \int_R^{R_0} \xi\, \alpha(\xi,t)\, d\xi - \frac{\alpha(R,t)}{\lambda^2(t)} + \beta(R,t)\, \lambda^2(t)\,. \tag{3.29}$$

[7] This is identical to what was obtained in [Yavari et al., 2022] in the case of accreting bars under finite extension.

For the *secondary body*, i.e., for $R_0 \leq R \leq s(t)$:

$$
\begin{aligned}
\mathbf{b}^\sharp(R,t) &= \begin{bmatrix} \frac{\lambda^2(\tau(R))}{\lambda^2(t)} & 0 & 0 \\ 0 & \frac{\lambda^4(\tau(R))+\bar{r}^2(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^4(\tau(R))\bar{r}^2(R)} & \frac{\lambda^2(t)(\psi(t)-\psi(\tau(R)))}{\lambda^4(\tau(R))} \\ 0 & \frac{\lambda^2(t)(\psi(t)-\psi(\tau(R)))}{\lambda^4(\tau(R))} & \frac{\lambda^4(t)}{\lambda^4(\tau(R))} \end{bmatrix}, \\
\mathbf{c}^\sharp(R,t) &= \begin{bmatrix} \frac{\lambda^2(t)}{\lambda^2(\tau(R))} & 0 & 0 \\ 0 & \frac{\lambda^4(t)}{\lambda^4(\tau(R))\bar{r}^2(R)} & \frac{\psi(\tau(R))-\psi(t)}{\lambda^2(\tau(R))} \\ 0 & \frac{\psi(\tau(R))-\psi(t)}{\lambda(\tau(R))^2} & \frac{\lambda^4(\tau(R))+\bar{r}^2(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^4(t)} \end{bmatrix}.
\end{aligned}
\tag{3.30}
$$

The principal invariants of $\mathbf{b}$ read

$$
\begin{aligned}
I_1(R,t) &= \frac{\lambda^4(t)}{\lambda^4(\tau(R))} + \frac{2\lambda^2(\tau(R))}{\lambda^2(t)} + \frac{\bar{r}^2(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^2(\tau(R))\,\lambda^2(t)}, \\
I_2(R,t) &= \frac{\lambda^4(\tau(R))}{\lambda^4(t)} + \frac{2\lambda^2(t)}{\lambda^2(\tau(R))} + \frac{\bar{r}^2(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^4(t)}.
\end{aligned}
\tag{3.31}
$$

The non-zero components of the Cauchy stress are

$$
\begin{aligned}
\sigma^{rr}(R,t) &= -p(R,t) + \alpha(R,t)\,\frac{\lambda^2(\tau(R))}{\lambda^2(t)} - \beta(R,t)\,\frac{\lambda^2(t)}{\lambda^2(\tau(R))}, \\
\sigma^{\theta\theta}(R,t) &= -p(R,t)\,\frac{\lambda^2(t)}{\lambda^2(\tau(R))\,\bar{r}^2(R)} + \frac{\alpha(R,t)}{\bar{r}^2(R)} - \frac{\beta(R,t)\,\lambda^4(t)}{\lambda^4(\tau(R))\,\bar{r}^2(R)} + \frac{\alpha(R,t)\,(\psi(t)-\psi(\tau(R)))^2}{\lambda^4(\tau(R))}, \\
\sigma^{zz}(R,t) &= -p(R,t) + \frac{\alpha(R,t)\,\lambda^4(t)}{\lambda^4(\tau(R))} - \frac{\beta(R,t)\,\lambda^4(\tau(R))}{\lambda^4(t)} - \frac{\beta(R,t)\,\bar{r}^2(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^4(t)}, \\
\sigma^{\theta z}(R,t) &= \frac{\psi(t)-\psi(\tau(R))}{\lambda^4(\tau(R))}\left[\alpha(R,t)\,\lambda^2(t) + \beta(R,t)\,\lambda^2(\tau(R))\right].
\end{aligned}
\tag{3.32}
$$

The equilibrium equation reads

$$
\frac{\partial\sigma^{rr}(R,t)}{\partial R} - \alpha(R,t)\,\frac{\bar{r}(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^2(\tau(R))\,\lambda^2(t)} = 0\,. \tag{3.33}
$$

Thus

$$
\sigma^{rr}(R,t) = \sigma_0(t) + \int_{R_0}^{R} \alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi)))^2}{\lambda^2(\tau)\,\lambda^2(t)} d\xi\,. \tag{3.34}
$$

This implies that for $R_0 \leq R \leq s(t)$:

$$
-p(R,t) = \sigma_0(t) + \int_{R_0}^{R} \alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi)))^2}{\lambda^2(\tau)\,\lambda^2(t)} d\xi - \alpha(R,t)\,\frac{\lambda^2(\tau(R))}{\lambda^2(t)} + \beta(R,t)\,\frac{\lambda^2(t)}{\lambda^2(\tau(R))}\,. \tag{3.35}
$$

Thus on the growth surface, one has

$$
-p(s(t),t) = \sigma_0(t) + \int_{R_0}^{s(t)} \alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))\,\lambda^2(t)} d\xi - \alpha(s(t),t) + \beta(s(t),t)\,. \tag{3.36}
$$

Note that for $R = s(t)$, $\tau(R) = \tau(s(t)) = t$, and hence $\psi(t) = \psi(\tau(R))$. Thus

$$
\boldsymbol{\sigma}(s(t),t) = [-p(s(t),t) + \alpha(s(t),t) - \beta(s(t),t)] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\bar{r}^2(R)} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.37}
$$

We know that $\boldsymbol{\sigma}(s(t),t)=\mathbf{0}$ (note that stress-free material is added on the boundary and this means that the stress tensor vanishes on the boundary), and hence $-p(s(t),t)+\alpha(s(t),t)-\beta(s(t),t)=0$. Therefore

$$\sigma_0(t)=-\int_{R_0}^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))\,\lambda^2(t)}\,d\xi\,. \tag{3.38}$$

Thus, for $R_0\leq R\leq s(t)$ we have

$$-p(R,t)=-\frac{1}{\lambda^2(t)}\int_R^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}d\xi-\alpha(R,t)\,\frac{\lambda^2(\tau(R))}{\lambda^2(t)}+\beta(R,t)\,\frac{\lambda^2(t)}{\lambda^2(\tau(R))}\,. \tag{3.39}$$

From (3.29), for $0\leq R\leq R_0$:

$$\begin{aligned}-p(R,t)=&-\frac{\psi^2(t)}{\lambda^2(t)}\int_R^{R_0}\xi\,\alpha(\xi,t)\,d\xi-\frac{1}{\lambda^2(t)}\int_{R_0}^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\\&-\frac{\alpha(R,t)}{\lambda^2(t)}+\beta(R,t)\,\lambda^2(t)\,.\end{aligned} \tag{3.40}$$

Therefore, the non-zero physical components of the Cauchy stress for the initial body ($0\leq R\leq R_0$) are[8]

$$\begin{aligned}\bar{\sigma}^{rr}(R,t)&=-\frac{\psi^2(t)}{\lambda^2(t)}\int_R^{R_0}\xi\,\alpha(\xi,t)\,d\xi-\frac{1}{\lambda^2(t)}\int_{R_0}^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\,,\\ \bar{\sigma}^{\theta\theta}(R,t)&=-\frac{\psi^2(t)}{\lambda^2(t)}\int_R^{R_0}\xi\,\alpha(\xi,t)\,d\xi-\frac{1}{\lambda^2(t)}\int_{R_0}^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi+\alpha(R,t)\,\frac{R^2\,\psi^2(t)}{\lambda^2(t)}\,,\\ \bar{\sigma}^{zz}(R,t)&=-\frac{\psi^2(t)}{\lambda^2(t)}\int_R^{R_0}\xi\,\alpha(\xi,t)\,d\xi-\frac{1}{\lambda^2(t)}\int_{R_0}^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\\&\qquad+\alpha(R,t)\left[\lambda^4(t)-\frac{1}{\lambda^2(t)}\right]+\beta(R,t)\left[\lambda^2(t)-\frac{1+R^2\psi^2(t)}{\lambda^4(t)}\right],\\ \bar{\sigma}^{\theta z}(R,t)&=\frac{R\,\psi(t)}{\lambda(t)}\left[\alpha(R,t)\lambda^2(t)+\beta(R,t)\right]\,.\end{aligned} \tag{3.41}$$

For the secondary body ($R_0\leq R\leq s(t)$) they read

$$\begin{aligned}\bar{\sigma}^{rr}(R,t)&=-\frac{1}{\lambda^2(t)}\int_R^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\,,\\ \bar{\sigma}^{\theta\theta}(R,t)&=-\frac{1}{\lambda^2(t)}\int_R^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi+\frac{\alpha(R,t)\,\bar{r}^2(R)\,(\psi(t)-\psi(\tau(R)))^2}{\lambda^2(t)\,\lambda^2(\tau(R))}\,,\\ \bar{\sigma}^{zz}(R,t)&=-\frac{1}{\lambda^2(t)}\int_R^{s(t)}\alpha(\xi,t)\,\frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi-\frac{\beta(R,t)\,\bar{r}^2(R)(\psi(t)-\psi(\tau(R)))^2}{\lambda^4(t)}\\&\qquad+\alpha(R,t)\left[\frac{\lambda^4(t)}{\lambda^4(\tau(R))}-\frac{\lambda^2(\tau(R))}{\lambda^2(t)}\right]+\beta(R,t)\left[\frac{\lambda^2(t)}{\lambda^2(\tau(R))}-\frac{\lambda^4(\tau(R))}{\lambda^4(t)}\right],\\ \bar{\sigma}^{\theta z}(R,t)&=\frac{\bar{r}(R)(\psi(t)-\psi(\tau(R)))}{\lambda(t)\,\lambda^3(\tau(R))}\left[\alpha(R,t)\,\lambda^2(t)+\beta(R,t)\,\lambda^2(\tau(R))\right]\,.\end{aligned} \tag{3.42}$$

At the two ends of the bar ($Z=0,L$), the axial force is assumed to be zero and the applied torque is given, i.e.,

$$\begin{aligned}F(t)&=2\pi\int_0^{s(t)}P^{zZ}(R,t)R\,dR=0\,,\\ M(t)&=2\pi\int_0^{s(t)}\bar{P}^{\theta Z}(R,t)R^2\,dR=2\pi\int_0^{s(t)}P^{\theta Z}(R,t)\,r(R,t)R^2\,dR\,,\end{aligned} \tag{3.43}$$

[8]The physical components of the Cauchy stress are defined as $\bar{\sigma}^{ab}=\sigma^{ab}\sqrt{g_{aa}\,g_{bb}}$ (no summation) [Truesdell, 1953].

where $\bar{P}^{zZ} = P^{zZ}$ is the $zZ$-component of the first Piola-Kirchhoff stress and $\bar{P}^{\theta Z} = rP^{\theta Z}$ is the physical $\theta Z$ component of the first Piola-Kirchhoff stress. Note that

$$P^{zZ}(R,t) = \begin{cases} -\frac{\psi^2(t)}{\lambda^4(t)} \int_R^{R_0} \xi\, \alpha(\xi,t)\, d\xi - \frac{1}{\lambda^4(t)} \int_{R_0}^{s(t)} \alpha(\xi,t)\, \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\, d\xi \\ \quad +\alpha(R,t)\left[\lambda^2(t) - \frac{1}{\lambda^4(t)}\right] + \beta(R,t)\left[1 - \frac{1+R^2\psi^2(t)}{\lambda^6(t)}\right], & 0 \le R \le R_0\,, \\ \\ -\frac{1}{\lambda^4(t)} \int_R^{s(t)} \alpha(\xi,t)\, \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\, d\xi - \frac{\beta(R,t)\,\bar{r}^2(R)(\psi(t)-\psi(\tau))^2}{\lambda^6(t)} \\ \quad +\alpha(R,t)\left[\frac{\lambda^2(t)}{\lambda^4(\tau(R))} - \frac{\lambda^2(\tau(R))}{\lambda^4(t)}\right] + \beta(R,t)\left[\frac{1}{\lambda^2(\tau(R))} - \frac{\lambda^4(\tau(R))}{\lambda^6(t)}\right], & R_0 \le R \le s(t)\,, \end{cases} \tag{3.44}$$

and

$$P^{\theta Z}(R,t) = \begin{cases} \left[\alpha(R,t) + \frac{\beta(R,t)}{\lambda^2(t)}\right] \psi(t)\,, & 0 \le R \le R_0\,, \\ \\ \frac{\psi(t)-\psi(\tau(R))}{\lambda^2(t)\,\lambda^4(\tau(R))} \left[\alpha(R,t)\,\lambda^2(t) + \beta(R,t)\,\lambda^2(\tau)\right], & R_0 \le R \le s(t)\,. \end{cases} \tag{3.45}$$

**Remark 3.1.** Instead of the choice $U_g(t) = u_g(t) = u_0$, let us assume that $U_g(t) = U_0 > 0$. In this case, instead of the constraint (3.9), one has

$$r'(s(t),t) = \frac{u_0}{U_0}\,, \text{ or } \qquad r'(\hat{R},\hat{\tau}(\hat{R})) = \frac{u_0}{U_0}\,, \tag{3.46}$$

where $\hat{R}$ is the radial coordinate of the new material manifold (for $0 \le R \le R_0$, $\hat{R} = R$). Note that in the two material manifolds the time of attachment of the same layer should be the same, i.e., $\hat{\tau}(\hat{R}) = \tau(R)$. This implies that

$$\hat{R} = \left(1 - \frac{U_0}{u_0}\right) R_0 + \frac{U_0}{u_0} R\,. \tag{3.47}$$

With this choice, the new time dependent material manifolds is

$$\mathcal{B}_t = \left\{(\hat{R},\Theta,Z) : 0 \le \Theta < 2\pi\,,\ R_0 \le \hat{R} \le s(t) = R_0 + U_0\,t\,,\ 0 \le Z \le L\right\}. \tag{3.48}$$

Let us denote the radial component of the deformation mapping with respect to the new material manifold by $\hat{r}(\hat{R},t)$. The material metric has the following representation

$$\begin{aligned} 0 \le R \le R_0 : \ \mathbf{G} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ R_0 \le \hat{R} \le R_0 + U_0\,t : \ \mathbf{G} &= \begin{bmatrix} \left(\frac{u_0}{U_0}\right)^2 & 0 & 0 \\ 0 & \hat{r}^2(\hat{R},\hat{\tau}(\hat{R})) & \psi(\hat{\tau}(\hat{R}))\,\hat{r}^2(\hat{R},\hat{\tau}(\hat{R})) \\ 0 & \psi(\hat{\tau}(\hat{R}))\,\hat{r}^2(\hat{R},\hat{\tau}(\hat{R})) & \psi^2(\hat{\tau}(\hat{R}))\,\hat{r}^2(\hat{R},\hat{\tau}(\hat{R})) + \lambda^4(\hat{\tau}(\hat{R})) \end{bmatrix}. \end{aligned} \tag{3.49}$$

With respect to the new material manifold

$$\mathbf{F} = \hat{\mathbf{F}}(\hat{R},t) = \begin{bmatrix} \hat{r}'(\hat{R},t) & 0 & 0 \\ 0 & 1 & \psi(t) \\ 0 & 0 & \lambda^2(t) \end{bmatrix}. \tag{3.50}$$

For $0 \le R \le R_0$, we have $\hat{R} = R$, and $\hat{r}(\hat{R},t) = r(R,t) = \frac{R}{\lambda(t)}$. For $R \ge R_0$, incompressibility implies that

$$\hat{J} = \frac{\hat{r}(\hat{R},t)}{\frac{u_0}{U_0}\,\hat{r}(\hat{R},\hat{\tau}(\hat{R}))\,\lambda^2(\hat{\tau}(\hat{R}))}\,\hat{r}'(\hat{R},t)\,\lambda^2(t) = 1\,. \tag{3.51}$$

Therefore

$$\lambda^2(t)\,\hat{r}^2(\hat{R},t) = R_0^2 + 2\frac{u_0}{U_0} \int_{R_0}^{\hat{R}} \bar{r}(\eta)\,\lambda^2(\hat{\tau}(\eta))\,d\eta\,, \tag{3.52}$$

where $\bar{\hat{r}}(\eta) = \hat{r}(\eta, \hat{\tau}(\eta))$. The right-hand side of the above relation is time independent, and hence $\lambda^2(t)\,\hat{r}^2(\hat{R},t) = \lambda^2(\hat{\tau}(\hat{R}))\,\hat{r}^2(\hat{R},\hat{\tau}(\hat{R}))$, or

$$\hat{r}^(\hat{R},t) = \frac{\lambda(\hat{\tau}(\hat{R}))}{\lambda(t)}\,\bar{\hat{r}}(\hat{R})\,. \tag{3.53}$$

The constraint (3.46) gives the following ODE for the unknown function $\bar{\hat{r}}(\hat{R})$:

$$\bar{\hat{r}}\,'(\hat{R}) + \frac{\left[\lambda(\hat{\tau}(\hat{R}))\right]'}{\lambda(\hat{\tau}(\hat{R}))}\,\bar{\hat{r}}(\hat{R}) = \frac{u_0}{U_0}\,. \tag{3.54}$$

This ODE has has the following solution:

$$\bar{\hat{r}}(\hat{R}) = \frac{1}{\lambda(\hat{\tau}(\hat{R}))}\left[R_0 + \frac{u_0}{U_0}\int_{R_0}^{\hat{R}}\lambda(\hat{\tau}(\eta))\,d\eta\right] = \frac{1}{\lambda(\tau(R))}\left[R_0 + \frac{u_0}{U_0}\int_{R_0}^{\hat{R}}\lambda(\hat{\tau}(\eta))\,d\eta\right]. \tag{3.55}$$

Note that $d\hat{R} = \frac{U_0}{u_0}dR$, and hence $\frac{u_0}{U_0}\int_{R_0}^{\hat{R}}\lambda(\hat{\tau}(\eta))\,d\eta = \int_{R_0}^{R}\lambda(\tau(\xi))\,d\xi$. Substituting this relation back into (3.55), and comparing this with (3.23), we observe that $\hat{r}(\hat{R},t) = r(R,t)$. This means that kinematics is not affected by the choice $U_g(t) = U_0 > 0$. Consequently, stresses are not affected either.

**Remark 3.2.** In [Goodbrake et al., 2020] for each of the six known families of universal deformations of incompressible isotropic solids [Ericksen, 1954, Singh and Pipkin, 1965, Klingbeil and Shield, 1966] the corresponding universal eigenstrains (or equivalently material metrics) were found. However, there may be many more pairs of universal deformations and their corresponding universal eigerstrains (material metrics). In [Yavari et al., 2022] one such family of universal deformations and eigestrains was found. In this paper, we have found another family of universal deformations and eigenstrains. More specifically, we have shown that the following pair of deformations and material metrics $(\varphi, \mathbf{G})$

$$(r,\theta,z) = \varphi(R,\Theta,Z):\quad \begin{cases} r = r(R,t) = \begin{cases} \dfrac{R}{\lambda(t)}\,, & 0 \le R \le R_0\,,\\[2mm] \dfrac{1}{\lambda(t)}\left[R_0 + \displaystyle\int_{R_0}^{R}\lambda(\tau(\xi))\,d\xi\right], & R_0 \le R \le s(t)\,, \end{cases}\\ \theta = \Theta + \psi(t)Z\,,\\ z = \lambda^2(t)Z\,, \end{cases} \tag{3.56}$$

and

$$\mathbf{G} = \begin{cases} \begin{bmatrix} 1 & 0 & 0\\ 0 & R^2 & 0\\ 0 & 0 & 1 \end{bmatrix}, & 0 \le R \le R_0\,,\\[4mm] \begin{bmatrix} 1 & 0 & 0\\ 0 & r^2(R,\tau(R)) & \psi(\tau(R))\,r^2(R,\tau(R))\\ 0 & \psi(\tau(R))\,r^2(R,\tau(R)) & \psi^2(\tau(R))\,r^2(R,\tau(R)) + \lambda^4(\tau(R)) \end{bmatrix}, & R_0 \le R \le s(t)\,, \end{cases} \tag{3.57}$$

are universal.

**Example 3.3.** For neo-Hookean solids $\alpha(R) = \mu(R) > 0$ and $\beta(R) = 0$. Let us also assume a uniform shear modulus $\mu(R) = \mu_0$. Therefore, the non-zero physical components of the Cauchy stress for the initial body

$(0 \le R \le R_0)$ are

$$
\begin{aligned}
\bar{\sigma}^{rr}(R,t) &= -\mu_0\,\frac{\psi^2(t)}{\lambda^2(t)}\,\frac{R_0^2-R^2}{2} - \frac{\mu_0}{\lambda^2(t)}\int_{R_0}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\,,\\
\bar{\sigma}^{\theta\theta}(R,t) &= -\mu_0\,\frac{\psi^2(t)}{\lambda^2(t)}\,\frac{R_0^2-R^2}{2} - \frac{\mu_0}{\lambda^2(t)}\int_{R_0}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi + \mu_0\,\frac{R^2\,\psi^2(t)}{\lambda^2(t)}\,,\\
\bar{\sigma}^{zz}(R,t) &= -\mu_0\,\frac{\psi^2(t)}{\lambda^2(t)}\,\frac{R_0^2-R^2}{2} - \frac{\mu_0}{\lambda^2(t)}\int_{R_0}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi + \mu_0\left[\lambda^4(t) - \frac{1}{\lambda^2(t)}\right]\,,\\
\bar{\sigma}^{\theta z}(R,t) &= \mu_0\,R\,\psi(t)\,\lambda(t)\,.
\end{aligned}
\tag{3.58}
$$

For the secondary body $(R_0 \le R \le s(t))$ they read

$$
\begin{aligned}
\bar{\sigma}^{rr}(R,t) &= -\frac{\mu_0}{\lambda^2(t)}\int_{R}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi))}\,d\xi\,,\\
\bar{\sigma}^{\theta\theta}(R,t) &= -\frac{\mu_0}{\lambda^2(t)}\int_{R}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi + \mu_0\,\frac{\bar{r}^2(R)\,(\psi(t)-\psi(\tau(R)))^2}{\lambda^2(t)\,\lambda^2(\tau(R))}\,,\\
\bar{\sigma}^{zz}(R,t) &= -\frac{\mu_0}{\lambda^2(t)}\int_{R}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi + \mu_0\left[\frac{\lambda^4(t)}{\lambda^4(\tau(R))} - \frac{\lambda^2(\tau(R))}{\lambda^2(t)}\right]\,,\\
\bar{\sigma}^{\theta z}(R,t) &= \mu_0\,\frac{\bar{r}(R)(\psi(t)-\psi(\tau(R)))}{\lambda^3(\tau(R))}\,\lambda(t)\,.
\end{aligned}
\tag{3.59}
$$

Thus

$$
P^{zZ}(R,t) = \mu_0 \begin{cases} -\frac{\psi^2(t)}{2\lambda^4(t)}\left(R_0^2-R^2\right) + \lambda^2(t) - \frac{1}{\lambda^4(t)} - \frac{1}{\lambda^4(t)}\int_{R_0}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\,, & 0 \le R \le R_0\,,\\[2ex] \frac{\lambda^2(t)}{\lambda^4(\tau(R))} - \frac{\lambda^2(\tau(R))}{\lambda^4(t)} - \frac{1}{\lambda^4(t)}\int_{R}^{s(t)} \frac{\bar{r}(\xi)(\psi(t)-\psi(\tau(\xi))^2}{\lambda^2(\tau(\xi)}\,d\xi\,, & R_0 \le R \le s(t)\,, \end{cases}
\tag{3.60}
$$

and

$$
P^{\theta Z}(R,t) = \mu_0 \begin{cases} \psi(t)\,, & 0 \le R \le R_0\,,\\[2ex] \frac{\psi(t)-\psi(\tau(R))}{\lambda^4(\tau(R))}\,, & R_0 \le R \le s(t)\,. \end{cases}
\tag{3.61}
$$

The applied torque is calculated as

$$
M(t) = \frac{\pi\mu_0 R_0^4}{2}\,\frac{\psi(t)}{\lambda(t)} + 2\pi\mu_0 R_0\big[\psi(t)\,h_1(t) - h_2(t)\big] + 2\pi\mu_0\big[\psi(t)\,h_3(t) - h_4(t)\big]\,,
\tag{3.62}
$$

where

$$
\begin{aligned}
h_1(t) &= \int_{R_0}^{s(t)} \frac{R^2}{\lambda^5(\tau(R))}\,dR\,, \quad h_2(t) = \int_{R_0}^{s(t)} \frac{R^2\,\psi(\tau(R))}{\lambda^5(\tau(R))}\,dR\,,\\
h_3(t) &= \int_{R_0}^{s(t)} \frac{R^2\,\gamma(R)}{\lambda^5(\tau(R))}\,dR\,, \quad h_4(t) = \int_{R_0}^{s(t)} \frac{R^2\,\psi(\tau(R))\,\gamma(R)}{\lambda^5(\tau(R))}\,dR\,, \quad \gamma(R) = \int_{R_0}^{R} \lambda(\tau(\xi))\,d\xi\,.
\end{aligned}
\tag{3.63}
$$

Thus

$$
\begin{aligned}
h_1'(t) &= \frac{u_0\,s^2(t)}{\lambda^5(t)}\,, \qquad h_2'(t) = \frac{u_0\,s^2(t)\,\psi(t)}{\lambda^5(t)}\,,\\
h_3'(t) &= \frac{u_0\,s^2(t)\,h_5(t)}{\lambda^5(t)}\,, \quad h_4'(t) = \frac{u_0\,s^2(t)\,\psi(t)\,h_5(t)}{\lambda^5(t)}\,, \quad h_5'(t) = u_0\,\lambda(t)\,,
\end{aligned}
\tag{3.64}
$$

where $h_5(t) = \gamma(s(t))$. We assume that $M(0) = 0$, $\lambda(0) = 1$, and $\psi(0) = 0$. Note also that $h_j(0) = 0$, $j = 1, ..., 5$.

The zero applied force condition is written as

$$
\begin{aligned}
&\left[\lambda^2(t) - \frac{1}{\lambda^4(t)}\right]\frac{R_0^2}{2} - \frac{R_0^4\,\psi^2(t)}{8\lambda^4(t)} + \lambda^2(t)\,k_1(t) - \frac{k_2(t)}{\lambda^4(t)} \\
&- \frac{R_0^2}{2\lambda^4(t)}\Big[\psi^2(t)\left(R_0k_3(t)+k_4(t)\right) - 2\psi(t)\left(R_0k_5(t)+k_6(t)\right) + R_0k_7(t)+k_8(t)\Big] \\
&- \frac{1}{\lambda^4(t)}\Big[\psi^2(t)\left(R_0\hat{k}_3(t)+\hat{k}_4(t)\right) - 2\psi(t)\left(R_0\hat{k}_5(t)+\hat{k}_6(t)\right) + R_0\hat{k}_7(t)+\hat{k}_8(t)\Big] = 0\,,
\end{aligned}
\tag{3.65}
$$

where

$$
\begin{aligned}
k_1(t) &= \int_{R_0}^{s(t)} \frac{R}{\lambda^4(\tau(R))}\,dR\,, & k_2(t) &= \int_{R_0}^{s(t)} R\,\lambda^2(\tau(R))\,dR\,, \\
k_3(t) &= \int_{R_0}^{s(t)} \frac{R}{\lambda^3(\tau(R))}\,dR\,, & k_4(t) &= \int_{R_0}^{s(t)} \frac{\gamma(R)}{\lambda^3(\tau(R))}\,dR\,, \\
k_5(t) &= \int_{R_0}^{s(t)} \frac{\psi(\tau(R))}{\lambda^3(\tau(R))}\,dR\,, & k_6(t) &= \int_{R_0}^{s(t)} \frac{\psi(\tau(R))\,\gamma(R)}{\lambda^3(\tau(R))}\,dR\,, \\
k_7(t) &= \int_{R_0}^{s(t)} \frac{\psi^2(\tau(R))}{\lambda^3(\tau(R))}\,dR\,, & k_8(t) &= \int_{R_0}^{s(t)} \frac{\psi^2(\tau(R))\,\gamma(R)}{\lambda^3(\tau(R))}\,dR\,, \\
\hat{k}_3(t) &= \int_{R_0}^{s(t)} R\int_{R}^{s(t)} \frac{1}{\lambda^3(\tau(\xi))}\,d\xi\,dR\,, & \hat{k}_4(t) &= \int_{R_0}^{s(t)} R\int_{R}^{s(t)} \frac{\gamma(\xi)}{\lambda^3(\tau(\xi))}\,d\xi\,dR\,, \\
\hat{k}_5(t) &= \int_{R_0}^{s(t)} R\int_{R}^{s(t)} \frac{\psi(\tau(\xi))}{\lambda^3(\tau(\xi))}\,d\xi\,dR\,, & \hat{k}_6(t) &= \int_{R_0}^{s(t)} R\int_{R}^{s(t)} \frac{\psi(\tau(\xi))\,\gamma(\xi)}{\lambda^3(\tau(\xi))}\,d\xi\,dR\,, \\
\hat{k}_7(t) &= \int_{R_0}^{s(t)} R\int_{R}^{s(t)} \frac{\psi^2(\tau(\xi))}{\lambda^3(\tau(\xi))}\,d\xi\,dR\,, & \hat{k}_8(t) &= \int_{R_0}^{s(t)} R\int_{R}^{s(t)} \frac{\psi^2(\tau(\xi))\,\gamma(\xi)}{\lambda^3(\tau(\xi))}\,d\xi\,dR\,.
\end{aligned}
\tag{3.66}
$$

Thus

$$
\begin{aligned}
&k_1'(t) = \frac{u_0\,s(t)}{\lambda^4(t)}\,, \quad k_2'(t) = u_0\,s(t)\,\lambda^2(t)\,, \quad k_3'(t) = \frac{u_0\,s(t)}{\lambda^3(t)}\,, \quad k_4'(t) = \frac{u_0\,\gamma(s(t))}{\lambda^3(t)}\,, \\
&k_5'(t) = \frac{u_0\,\psi(t)}{\lambda^3(t)}\,, \quad k_6'(t) = \frac{u_0\,\psi(t)\,\gamma(s(t))}{\lambda^3(t)}\,, \quad k_7'(t) = \frac{\psi^2(t)}{\lambda^3(t)}\,, \quad k_8'(t) = \frac{u_0\,\psi^2(t)\,\gamma(s(t))}{\lambda^3(t)}\,.
\end{aligned}
\tag{3.67}
$$

Note that[9]

$$
\hat{k}_3'(t) = \frac{u_0}{2\lambda^3(t)}\left(s^2(t) - R_0^2\right). \tag{3.68}
$$

[9]This is a simple application of the Leibniz integral rule:

$$
\hat{k}_3'(t) = \frac{d}{dt}\int_{R_0}^{s(t)} f(t,R)\,dR = s'(t)\,f(t,s(t)) + \int_{R_0}^{s(t)} \frac{\partial f(t,R)}{\partial t}\,dR\,,
$$

where

$$
f(t,R) = R\int_{R}^{s(t)} \frac{d\xi}{\lambda^3(\tau(\xi))}\,.
$$

Note that

$$
f(t,s(t)) = s(t)\int_{s(t)}^{s(t)} \frac{d\xi}{\lambda^3(\tau(\xi))} = 0\,, \qquad \frac{\partial f(t,R)}{\partial t} = R\,s'(t)\frac{1}{\lambda^3(\tau(s(t)))} = \frac{R\,u_0}{\lambda^3(t)}\,.
$$

Thus

$$
\hat{k}_3'(t) = \int_{R_0}^{s(t)} \frac{R\,u_0}{\lambda^3(t)}\,dR = \frac{u_0}{2\lambda^3(t)}\left(s^2(t) - R_0^2\right).
$$

Similarly,

$$
\begin{aligned}
&\hat{k}_4'(t) = \frac{u_0\,\gamma(s(t))}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,, && \hat{k}_5'(t) = \frac{u_0\,\psi(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,,\\
&\hat{k}_6'(t) = \frac{u_0\,\psi(t)\,\gamma(s(t))}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,, && \hat{k}_7'(t) = \frac{u_0\,\psi^2(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,,\\
&\hat{k}_8'(t) = \frac{u_0\,\psi^2(t)\,\gamma(s(t))}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,.
\end{aligned}
\tag{3.69}
$$

Note that $k_1(0) = \cdots = k_8(0) = 0$, and $\hat{k}_3(0) = \cdots = \hat{k}_8(0) = 0$. Therefore, we have the following system of nonlinear first-order ODEs:

$$
\begin{cases}
\left[\lambda^2(t) - \dfrac{1}{\lambda^4(t)}\right]\dfrac{R_0^2}{2} - \dfrac{R_0^4\,\psi^2(t)}{8\lambda^4(t)} + \lambda^2(t)\,k_1(t) - \dfrac{k_2(t)}{\lambda^4(t)}\\
\quad - \dfrac{R_0^2}{2\lambda^4(t)}\Big[\psi^2(t)\,(R_0k_3(t)+k_4(t)) - 2\psi(t)\,(R_0k_5(t)+k_6(t)) + R_0k_7(t)+k_8(t)\Big]\\
\quad - \dfrac{1}{\lambda^4(t)}\Big[\psi^2(t)\left(R_0\hat{k}_3(t)+\hat{k}_4(t)\right) - 2\psi(t)\left(R_0\hat{k}_5(t)+\hat{k}_6(t)\right) + R_0\hat{k}_7(t)+\hat{k}_8(t)\Big] = 0\,,\\
\dfrac{\pi R_0^4}{2}\,\mu_0\,\dfrac{\psi(t)}{\lambda(t)} + 2\pi\mu_0R_0\big[\psi(t)\,h_1(t) - h_2(t)\big] + 2\pi\mu_0\big[\psi(t)\,h_3(t) - h_4(t)\big] = M(t)\,,\\
h_1'(t) = \dfrac{u_0\,s^2(t)}{\lambda^5(t)}\,, \quad h_2'(t) = \dfrac{u_0\,s^2(t)\,\psi(t)}{\lambda^5(t)}\,, \quad h_3'(t) = \dfrac{u_0\,s^2(t)\,h_5(t)}{\lambda^5(t)}\,,\\
h_4'(t) = \dfrac{u_0\,s^2(t)\,\psi(t)\,h_5(t)}{\lambda^5(t)}\,, \quad h_5'(t) = u_0\,\lambda(t)\,,\\
k_1'(t) = \dfrac{u_0\,s(t)}{\lambda^4(t)}\,, \quad k_2'(t) = u_0\,s(t)\,\lambda^2(t)\,, \quad k_3'(t) = \dfrac{u_0\,s(t)}{\lambda^3(t)}\,, \quad k_4'(t) = \dfrac{u_0\,h_5(t)}{\lambda^3(t)}\,,\\
k_5'(t) = \dfrac{u_0\,\psi(t)}{\lambda^3(t)}\,, \quad k_6'(t) = \dfrac{u_0\,\psi(t)\,h_5(t)}{\lambda^3(t)}\,, \quad k_7'(t) = \dfrac{u_0\,\psi^2(t)}{\lambda^3(t)}\,, \quad k_8'(t) = \dfrac{u_0\,\psi^2(t)\,h_5(t)}{\lambda^3(t)}\,,\\
\hat{k}_3'(t) = \dfrac{u_0}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,, \quad \hat{k}_4'(t) = \dfrac{u_0\,h_5(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,,\\
\hat{k}_5'(t) = \dfrac{u_0\,\psi(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,, \quad \hat{k}_6'(t) = \dfrac{u_0\,\psi(t)\,h_5(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,,\\
\hat{k}_7'(t) = \dfrac{u_0\,\psi^2(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,, \quad \hat{k}_8'(t) = \dfrac{u_0\,\psi^2(t)\,h_5(t)}{2\lambda^3(t)}\big(s^2(t)-R_0^2\big)\,,\\
\lambda(0) = 1,\ \psi(0) = h_1(0) = \cdots = h_5(0) = k_1(0) = \cdots = k_8(0) = \hat{k}_3(0) = \cdots = \hat{k}_8(0) = 0\,.
\end{cases}
\tag{3.70}
$$

Let us assume that $R_0 = 1$, $u_0 = 1$, and $t_a = 1$. We first consider the following twist-control loadings:

$$
\psi_1(t) = \pi\,\sin\left(\frac{2\pi\,t}{t_a}\right), \quad \psi_2(t) = \pi\,\sin^2\left(\frac{2\pi\,t}{t_a}\right), \quad \psi_3(t) = \pi\,\sin\left(\frac{8\pi\,t}{t_a}\right), \quad \psi_4(t) = \pi\,\sin^2\left(\frac{8\pi\,t}{t_a}\right). \tag{3.71}
$$

The corresponding $\lambda^2(t)$ distribution for each loading is shown in Fig.4. Next we consider the following torque-control loadings.

$$
\begin{aligned}
&M_1(t) = \pi\,R_0^3\,\sin\left(\frac{2\pi\,t}{t_a}\right), && M_2(t) = \pi\,R_0^3\,\sin^2\left(\frac{2\pi\,t}{t_a}\right),\\
&M_3(t) = \pi\,R_0^3\,\sin\left(\frac{8\pi\,t}{t_a}\right), && M_4(t) = \pi\,R_0^3\,\sin^2\left(\frac{8\pi\,t}{t_a}\right).
\end{aligned}
\tag{3.72}
$$

The corresponding $\lambda^2(t)$ and $\psi(t)$ distributions are shown in Fig.5.

**Remark 3.4.** Note that in (3.62), $M(t)$ is a linear function of $\psi(t)$. Consequently, in (3.62) and (3.63) the transformation $\psi(t) \to -\psi(t)$ changes the sign of $M(t)$. Note also that (3.65) is unchanged under

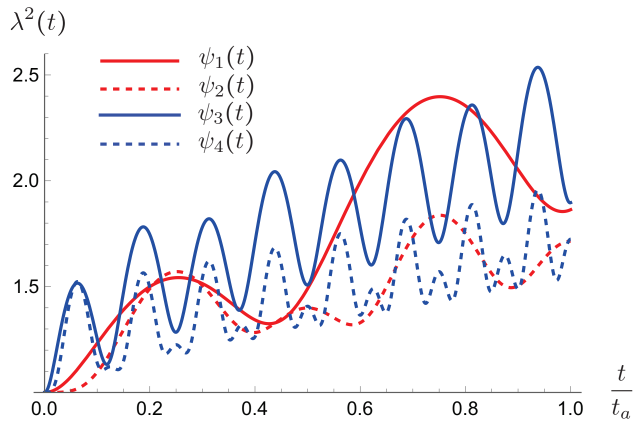


Figure 4: *The axial stretch distribution for a bar under the four different twist-control loadings given in* (3.71) *during accretion.*

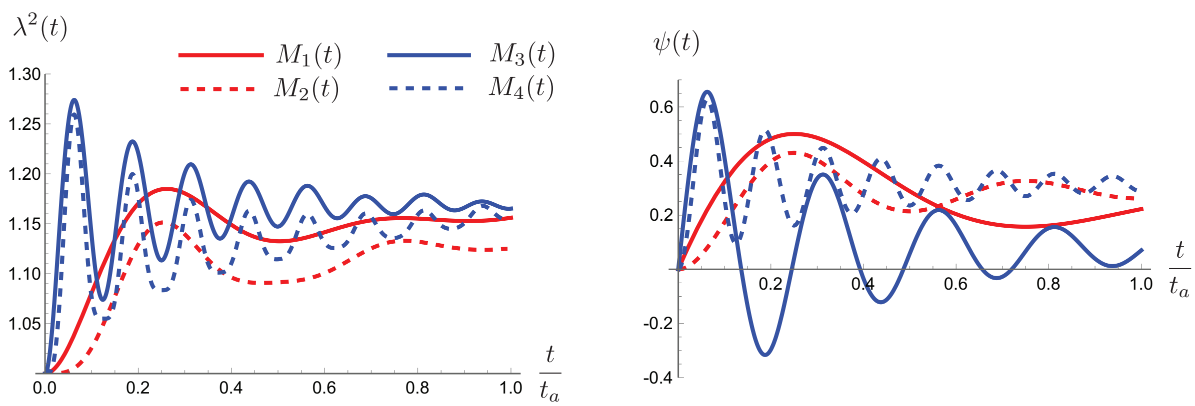


Figure 5: *The time-dependent axial stretch and twist per unit length for a bar under four different applied torques given in* (3.72) *during accretion.*

the transformation $\psi(t) \to -\psi(t)$. This implies that if $(\lambda(t), \psi(t))$, is a solution for $M(t)$, $t \in [0, t_a]$, then $(\lambda(t), -\psi(t))$, is a solution for $-M(t)$, $t \in [0, t_a]$. Consequently, from (3.58) and (3.59), if $\sigma^{rr}(R,t)$, $\sigma^{\theta\theta}(R,t)$, $\sigma^{zz}(R,t)$, and $\sigma^{\theta z}(R,t)$ is the stress distribution for $M(t)$, $t \in [0, t_a]$, then $\sigma^{rr}(R,t)$, $\sigma^{\theta\theta}(R,t)$, $\sigma^{zz}(R,t)$, and $-\sigma^{\theta z}(R,t)$ is the stress distribution for $-M(t)$, $t \in [0, t_a]$.

## 3.1 Residual stresses

Let us assume that after the completion of accretion at time $t_a$ the accreted body is unloaded, i.e., for $t > t_a$, $F(t) = M(t) = 0$. In this section we calculate the residual stretch $\tilde{\lambda}^2$, residual twist $\tilde{\psi}$, and residual stresses. The material metric of the accreted body has the following representation:

$$
\begin{aligned}
0 \leq R \leq R_0: \quad \mathbf{G} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
R_0 \leq R \leq R_a: \quad \mathbf{G} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{r}^2(R) & \psi(\tau(R))\, \bar{r}^2(R) \\ 0 & \psi(\tau(R))\, \bar{r}^2(R) & \psi^2(\tau(R))\, \bar{r}^2(R) + \lambda^4(\tau(R)) \end{bmatrix},
\end{aligned} \tag{3.73}
$$

where $R_a = s(t_a)$. Note that for a given loading during accretion the material manifold $(\mathcal{B}, \mathbf{G})$, where $\mathcal{B} = \mathcal{B}_{t_a}$, has already been constructed. The map from the natural configuration of the accreted body to its residually-stressed configuration with no external loads is denoted by $\tilde{\varphi} : \mathcal{B} \to \tilde{\mathcal{C}} \subset \mathcal{S}$. In cylindrical coordinates it has the representation $\tilde{\varphi}(R, \Theta, Z) = (\tilde{r}, \tilde{\theta}, \tilde{z}) = (\tilde{r}(R), \Theta + \tilde{\psi} Z, \tilde{\lambda}^2 Z)$. Using the incompressibility constraint one obtains

$$\tilde{r}(R) = \begin{cases} \dfrac{R}{\tilde{\lambda}}\,, & 0 \le R \le R_0\,, \\ \dfrac{R_0^2}{\tilde{\lambda}^2} + \dfrac{2}{\tilde{\lambda}^2} \displaystyle\int_{R_0}^{R} \bar{r}(\xi)\, \lambda^2(\tau(\xi))\, d\xi\,, & R_0 \le R \le R_a\,. \end{cases} \tag{3.74}$$

The residual Cauchy stress has the following distribution for the initial body ($0 \le R \le R_0$)

$$\begin{aligned}
\bar{\tilde{\sigma}}^{rr}(R) &= -\frac{\tilde{\psi}^2}{\tilde{\lambda}^2} \int_R^{R_0} \xi\, \alpha(\xi)\, d\xi - \frac{1}{\tilde{\lambda}^2} \int_{R_0}^{R_a} \alpha(\xi)\, \frac{\bar{r}(\xi)(\tilde{\psi} - \psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\, d\xi\,, \\
\bar{\tilde{\sigma}}^{\theta\theta}(R) &= -\frac{\tilde{\psi}^2}{\tilde{\lambda}^2} \int_R^{R_0} \xi\, \alpha(\xi)\, d\xi - \frac{1}{\tilde{\lambda}^2} \int_{R_0}^{R_a} \alpha(\xi)\, \frac{\bar{r}(\xi)(\tilde{\psi} - \psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\, d\xi + \alpha(R)\, \frac{R^2\, \tilde{\psi}^2}{\tilde{\lambda}^2}\,, \\
\bar{\tilde{\sigma}}^{zz}(R) &= -\frac{\tilde{\psi}^2}{\tilde{\lambda}^2} \int_R^{R_0} \xi\, \alpha(\xi)\, d\xi - \frac{1}{\tilde{\lambda}^2} \int_{R_0}^{R_a} \alpha(\xi)\, \frac{\bar{r}(\xi)(\tilde{\psi} - \psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\, d\xi \\
&\qquad + \alpha(R) \left[\tilde{\lambda}^4 - \frac{1}{\tilde{\lambda}^2}\right] + \beta(R) \left[\tilde{\lambda}^2 - \frac{1 + R^2\, \tilde{\psi}^2}{\tilde{\lambda}^4}\right], \\
\bar{\tilde{\sigma}}^{\theta z}(R) &= \frac{R\, \tilde{\psi}}{\tilde{\lambda}} \left[\alpha(R) \tilde{\lambda}^2 + \beta(R)\right],
\end{aligned} \tag{3.75}$$

and for the secondary body ($R_0 \le R \le R_a$)

$$\begin{aligned}
\bar{\tilde{\sigma}}^{rr}(R) &= -\frac{1}{\tilde{\lambda}^2} \int_R^{R_a} \alpha(\xi)\, \frac{\bar{r}(\xi)(\tilde{\psi} - \psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\, d\xi\,, \\
\bar{\tilde{\sigma}}^{\theta\theta}(R) &= -\frac{1}{\tilde{\lambda}^2} \int_R^{R_a} \alpha(\xi)\, \frac{\bar{r}(\xi)(\tilde{\psi} - \psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\, d\xi + \frac{\alpha(R)\, \bar{r}^2(R)\, (\tilde{\psi} - \psi(\tau(\xi)))^2}{\tilde{\lambda}^2\, \lambda^2(\tau(\xi))}\,, \\
\bar{\tilde{\sigma}}^{zz}(R) &= -\frac{1}{\tilde{\lambda}^2} \int_R^{R_a} \alpha(\xi)\, \frac{\bar{r}(\xi)(\tilde{\psi} - \psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\, d\xi - \frac{\beta(R)\, \bar{r}^2(R)(\tilde{\psi} - \psi(\tau(R)))^2}{\tilde{\lambda}^4} \\
&\qquad + \alpha(R) \left[\frac{\tilde{\lambda}^4}{\tilde{\lambda}^4(\tau(R))} - \frac{\lambda^2(\tau(R))}{\tilde{\lambda}^2}\right] + \beta(R) \left[\frac{\tilde{\lambda}^2}{\lambda^2(\tau(R))} - \frac{\lambda^4(\tau(R))}{\tilde{\lambda}^4}\right], \\
\bar{\tilde{\sigma}}^{\theta z}(R) &= \frac{\bar{r}(R)(\tilde{\psi} - \psi(\tau(R)))}{\tilde{\lambda}\, \lambda^3(\tau(R))} \left[\alpha(R)\, \tilde{\lambda}^2 + \beta(R)\, \lambda^2(\tau)\right].
\end{aligned} \tag{3.76}$$

**Example 3.5.** For a homogeneous neo-Hookean solid, the zero applied torque and force conditions are written as the following system of nonlinear algebraic equations

$$\begin{aligned}
& R_0^4\, \frac{\tilde{\psi}}{\tilde{\lambda}} + 4R_0 \big[\tilde{\psi}\, \tilde{h}_1 - \tilde{h}_2\big] + 4\big[\tilde{\psi}\, \tilde{h}_3 - \tilde{h}_4\big] = 0\,, \\
& \left[\tilde{\lambda}^2 - \frac{1}{\tilde{\lambda}^4}\right] \frac{R_0^2}{2} - \frac{R_0^4\, \tilde{\psi}^2}{8 \tilde{\lambda}^4} + \tilde{\lambda}^2\, \tilde{k}_1 - \frac{\tilde{k}_2}{\tilde{\lambda}^4} - \frac{R_0^2}{2 \tilde{\lambda}^4} \Big[\tilde{\psi}^2 \left(R_0\, \tilde{k}_3 + \tilde{k}_4\right) - 2\tilde{\psi} \left(R_0\, \tilde{k}_5 + \tilde{k}_6\right) + R_0\, \tilde{k}_7 + \tilde{k}_8\Big] \\
& \quad - \frac{1}{\tilde{\lambda}^4} \Big[\tilde{\psi}^2 \left(R_0\, \tilde{\hat{k}}_3 + \tilde{\hat{k}}_4\right) - 2\tilde{\psi} \left(R_0\, \tilde{\hat{k}}_5 + \tilde{\hat{k}}_6\right) + R_0\, \tilde{\hat{k}}_7 + \tilde{\hat{k}}_8\Big] = 0\,,
\end{aligned} \tag{3.77}$$

where $\tilde{h}_i = h_i(t_a),\ i = 1, \ldots, 4,\ \tilde{k}_i = k_i(t_a),\ i = 1, \ldots, 8$, and $\tilde{\hat{k}}_i = \hat{k}_i(t_a),\ i = 3, \ldots, 8$.

For $R_0 = 1$, $u_0 = 1$, and $t_a = 1$, the residual twists and stretches for the four applied torques (3.72) are given in Table 1. The residual Cauchy stress components have the following distributions. For $0 \le R \le R_0$:

| | $M_1(t)$ | $M_2(t)$ | $M_3(t)$ | $M_4(t)$ |
|---|---|---|---|---|
| $\tilde{\lambda}^2$ | 1.24866 | 1.20544 | 1.26215 | 1.26174 |
| $\frac{\tilde{\psi}}{\pi}$ | 0.18626 | 0.23904 | 0.022172 | 0.29500 |

Table 1: *Residual stretch and twist for the four different torque-control loadings given in* (3.72).

$$\begin{aligned}
\bar{\bar{\sigma}}^{rr}(R) &= -\mu_0\,\frac{\tilde{\psi}^2}{\tilde{\lambda}^2}\frac{R_0^2-R^2}{2} - \frac{\mu_0}{\tilde{\lambda}^2}\int_{R_0}^{R_a}\frac{\bar{r}(\xi)(\tilde{\psi}-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\,d\xi\,,\\
\bar{\bar{\sigma}}^{\theta\theta}(R) &= -\mu_0\,\frac{\tilde{\psi}^2}{\tilde{\lambda}^2}\frac{R_0^2-R^2}{2} - \frac{\mu_0}{\tilde{\lambda}^2}\int_{R_0}^{R_a}\frac{\bar{r}(\xi)(\tilde{\psi}-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\,d\xi + \mu_0\,\frac{R^2\,\tilde{\psi}^2}{\tilde{\lambda}^2}\,,\\
\bar{\bar{\sigma}}^{zz}(R) &= -\mu_0\,\frac{\tilde{\psi}^2}{\tilde{\lambda}^2}\frac{R_0^2-R^2}{2} - \frac{\mu_0}{\tilde{\lambda}^2}\int_{R_0}^{R_a}\frac{\bar{r}(\xi)(\tilde{\psi}-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\,d\xi + \mu_0\left[\tilde{\lambda}^4-\frac{1}{\tilde{\lambda}^2}\right],\\
\bar{\bar{\sigma}}^{\theta z}(R) &= \mu_0\,\tilde{\psi}\,\tilde{\lambda}\,R\,.
\end{aligned} \tag{3.78}$$

For $R_0 \leq R \leq R_a$:

$$\begin{aligned}
\bar{\bar{\sigma}}^{rr}(R) &= -\frac{\mu_0}{\tilde{\lambda}^2}\int_{R}^{R_a}\frac{\bar{r}(\xi)(\tilde{\psi}-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\,d\xi\,,\\
\bar{\bar{\sigma}}^{\theta\theta}(R) &= -\frac{\mu_0}{\tilde{\lambda}^2}\int_{R}^{R_a}\frac{\bar{r}(\xi)(\tilde{\psi}-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\,d\xi + \frac{\mu_0\,\bar{r}^2(R)\,(\tilde{\psi}-\psi(\tau(R)))^2}{\tilde{\lambda}^2\,\lambda^2(\tau(R))}\,,\\
\bar{\bar{\sigma}}^{zz}(R) &= -\frac{\mu_0}{\tilde{\lambda}^2}\int_{R}^{R_a}\frac{\bar{r}(\xi)(\tilde{\psi}-\psi(\tau(\xi)))^2}{\lambda^2(\tau(\xi))}\,d\xi + \mu_0\left[\frac{\tilde{\lambda}^4}{\lambda^4(\tau(R))}-\frac{\lambda^2(\tau(R))}{\tilde{\lambda}^2}\right],\\
\bar{\bar{\sigma}}^{\theta z}(R) &= \mu_0\,\frac{\tilde{\lambda}\,\bar{r}(R)(\tilde{\psi}-\psi(\tau(R)))}{\lambda^3(\tau(R))}\,.
\end{aligned} \tag{3.79}$$

For $R_0 = 1$, $u_0 = 1$, and $t_a = 1$, in Fig.6 we show the residual stress distributions for the loading $M(t) = 2\pi R_0^3\big(\frac{t}{t_a}\big)^3$. It is observed that the shear stress is an order of magnitude larger than the normal stresses.

## 3.2 Linearized accretion mechanics

In this section we linearize the governing equations of the nonlinear accretion theory and find those of the linearized accretion mechanics. We assume that linearization is with respect to an undeformed stress-free configuration of the bar. More precisely, let us consider a reference motion $\mathring{\varphi}_t$, and a one-parameter family of motions $\varphi_{t,\epsilon}$ such that $\varphi_{t,0} = \mathring{\varphi}_t$ [Marsden and Hughes, 1983, Yavari and Ozakin, 2008, Sozio and Yavari, 2017]. For the combined torsion and extension of a bar we consider the following one-parameter family of motions

$$\varphi_\epsilon(R,\Theta,Z,t) = (r_\epsilon(R,t), \Theta + \psi_\epsilon(t)Z, \lambda_\epsilon^2(t)Z)\,. \tag{3.80}$$

We will linearize the governing equations with respect to the reference motion $\mathring{\varphi}_t(R,\Theta,Z,t) = (R,\Theta,Z)$, which corresponds to the motion of a cylindrical bar that is under no external forces or torques while stress-free cylindrical layers are added to its boundary in the time interval $[0,t_a]$. The variation field is defined as

$$\delta\varphi_t(R,\Theta,Z) = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\varphi_\epsilon(R,\Theta,Z,t) = (\delta r(R,t), \delta\psi(t)\,Z, 2\delta\lambda(t)\,Z)\,. \tag{3.81}$$

From

$$\delta r(R,t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} r_\epsilon(R,t)\,, \tag{3.82}$$

one concludes that $\delta\bar{r}(R) = \delta r\left(R, \frac{R-R_0}{u_0}\right)$. The displacement field is defined as

$$\mathbf{U}(R,\Theta,Z,t) = \delta\varphi_t(R,\Theta,Z) - \delta\varphi_{\tau(R)}(R,\Theta,Z)\,. \tag{3.83}$$

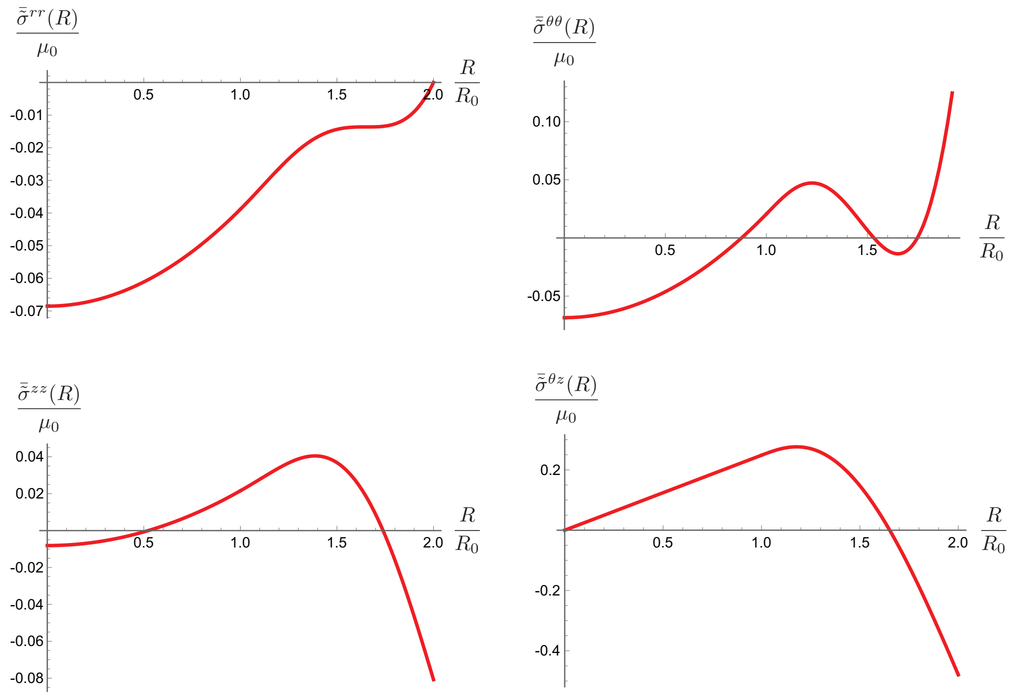


Figure 6: *Residual stresses in a bar under the applied torque* $M(t) = 2\pi R_0^3 \left(\frac{t}{t_a}\right)^3$ *during accretion.*

Assuming that $\psi(0) = 0$ and $\lambda(0) = 1$, for the initial body ($0 \leq R \leq R_0$), $\varphi_\epsilon(R, \Theta, Z, 0) = (r_\epsilon(R, 0), \Theta, Z) = (R, \Theta, Z)$, and hence $\delta\varphi_0(R, \Theta, Z) = (0, 0, 0)$. Thus, for $0 \leq R \leq R_0$, $\mathbf{U}(R, \Theta, Z, t) = \delta\varphi_t(R, \Theta, Z)$. However, for the new material points ($R_0 \leq R \leq s(t) = R_0 + u_0 t$) the displacement field is defined with respect to their positions at the time of attachment.

**Linearized kinematics.** For $0 \leq R \leq R_0$, the incompressibility condition for the perturbed motion is written as $\lambda_\epsilon^2(t)\, r_\epsilon(R, t)\, r'_\epsilon(R, t)/R = 1$, which along with $r_\epsilon(0, t) = 0$, implies that

$$r_\epsilon(R, t) = \frac{R}{\lambda_\epsilon(t)}\,, \quad 0 \leq R \leq R_0\,. \tag{3.84}$$

Taking derivative with respect to $\epsilon$ on both sides, evaluating at $\epsilon = 0$, and noting that $\lambda_{\epsilon=0}(t) = 1$, one obtains

$$\delta r(R, t) = -R\, \delta\lambda(t)\,. \tag{3.85}$$

Knowing that $\lambda_\epsilon(0) = 1$, $\delta\lambda(0) = 0$, and hence $\delta r(R, 0) = 0$.

For $R_0 \leq R \leq s(t)$:

$$r_\epsilon(R, t) = \frac{1}{\lambda_\epsilon(t)} \left[ R_0 + \int_{R_0}^{R} \lambda_\epsilon(\tau(\xi))\, d\xi \right]. \tag{3.86}$$

Thus

$$\delta r(R, t) = -R\, \delta\lambda(t) + \int_{R_0}^{R} \delta\lambda(\tau(\xi))\, d\xi\,. \tag{3.87}$$

**Linearized stresses.** For $0 \leq R \leq R_0$, one has

$$\begin{aligned}
\bar{\sigma}_\epsilon^{rr}(R,t) &= -\mu_0 \frac{\psi_\epsilon^2(t)}{\lambda_\epsilon^2(t)} \frac{R_0^2 - R^2}{2} - \frac{\mu_0}{\lambda_\epsilon^2(t)} \int_{R_0}^{s(t)} \frac{\bar{r}_\epsilon(\xi)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))^2}{\lambda_\epsilon^2(\tau(\xi)} \, d\xi \,, \\
\bar{\sigma}_\epsilon^{\theta\theta}(R,t) &= -\mu_0 \frac{\psi_\epsilon^2(t)}{\lambda_\epsilon^2(t)} \frac{R_0^2 - R^2}{2} - \frac{\mu_0}{\lambda_\epsilon^2(t)} \int_{R_0}^{s(t)} \frac{\bar{r}_\epsilon(\xi)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))^2}{\lambda_\epsilon^2(\tau(\xi)} \, d\xi + \mu_0 \frac{R^2\, \psi_\epsilon^2(t)}{\lambda_\epsilon^2(t)} \,, \\
\bar{\sigma}^{zz}(R,t) &= -\mu_0 \frac{\psi_\epsilon^2(t)}{\lambda_\epsilon^2(t)} \frac{R_0^2 - R^2}{2} - \frac{\mu_0}{\lambda_\epsilon^2(t)} \int_{R_0}^{s(t)} \frac{\bar{r}_\epsilon(\xi)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))^2}{\lambda_\epsilon^2(\tau(\xi)} \, d\xi + \mu_0 \left[\lambda_\epsilon^4(t) - \frac{1}{\lambda_\epsilon^2(t)}\right] \,, \\
\bar{\sigma}^{\theta z}(R,t) &= \mu_0\, R\, \psi_\epsilon(t)\, \lambda_\epsilon(t) \,.
\end{aligned} \tag{3.88}$$

For $R_0 \leq R \leq s(t)$

$$\begin{aligned}
\bar{\sigma}_\epsilon^{rr}(R,t) &= -\frac{\mu_0}{\lambda_\epsilon^2(t)} \int_R^{s(t)} \frac{\bar{r}_\epsilon(\xi)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))^2}{\lambda_\epsilon^2(\tau(\xi)} \, d\xi \,, \\
\bar{\sigma}_\epsilon^{\theta\theta}(R,t) &= -\frac{\mu_0}{\lambda_\epsilon^2(t)} \int_R^{s(t)} \frac{\bar{r}_\epsilon(\xi)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))^2}{\lambda_\epsilon^2(\tau(\xi)} \, d\xi + \mu_0 \frac{\bar{r}_\epsilon^2(R)\, (\psi_\epsilon(t) - \psi_\epsilon(\tau(R)))^2}{\lambda_\epsilon^2(t)\, \lambda_\epsilon^2(\tau(R))} \,, \\
\bar{\sigma}_\epsilon^{zz}(R,t) &= -\frac{\mu_0}{\lambda_\epsilon^2(t)} \int_R^{s(t)} \frac{\bar{r}_\epsilon(\xi)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))^2}{\lambda_\epsilon^2(\tau(\xi)} \, d\xi + \mu_0 \left[\frac{\lambda_\epsilon^4(t)}{\lambda_\epsilon^4(\tau(R))} - \frac{\lambda_\epsilon^2(\tau(R))}{\lambda_\epsilon^2(t)}\right] \\
\bar{\sigma}_\epsilon^{\theta z}(R,t) &= \mu_0 \frac{\bar{r}_\epsilon(R)(\psi_\epsilon(t) - \psi_\epsilon(\tau(\xi))}{\lambda_\epsilon^3(\tau(\xi)} \lambda_\epsilon(t) \,.
\end{aligned} \tag{3.89}$$

Thus, for $0 \leq R \leq R_0$

$$\begin{aligned}
\delta\bar{\sigma}^{rr}(R,t) &= \delta\bar{\sigma}^{\theta\theta}(R,t) = 0 \,, \\
\delta\bar{\sigma}^{zz}(R,t) &= 6\mu_o\, \delta\lambda(t) \,, \\
\delta\bar{\sigma}^{\theta z}(R,t) &= \mu_0\, R\, \delta\psi(t) \,,
\end{aligned} \tag{3.90}$$

and for $R_0 \leq R \leq s(t)$

$$\begin{aligned}
\delta\bar{\sigma}^{rr}(R,t) &= \delta\bar{\sigma}^{\theta\theta}(R,t) = 0 \,, \\
\delta\bar{\sigma}^{zz}(R,t) &= 6\mu_0\big[\delta\lambda(t) - \delta\lambda(\tau(R))\big] \,, \\
\delta\bar{\sigma}^{\theta z}(R,t) &= \mu_0\, R\big[\delta\psi(t) - \delta\psi(\tau(R))\big] \,.
\end{aligned} \tag{3.91}$$

For the perturbed motion (3.62) reads

$$\begin{aligned}
M_\epsilon(t) = \frac{\pi R_0^4}{2}\, \mu_0 \frac{\psi_\epsilon(t)}{\lambda_\epsilon(t)} &+ 2\pi\mu_0 R_0 \left[\psi_\epsilon(t) \int_{R_0}^{s(t)} \frac{R^2}{\lambda_\epsilon^5(\tau(R))} \, dR - \int_{R_0}^{s(t)} \frac{R^2\, \psi_\epsilon(\tau(R))}{\lambda_\epsilon^5(\tau(R))} \, dR\right] \\
&+ 2\pi\mu_0 \left[\psi_\epsilon(t) \int_{R_0}^{s(t)} \frac{R^2\, \gamma_\epsilon(R)}{\lambda_\epsilon^5(\tau(R))} \, dR - \int_{R_0}^{s(t)} \frac{R^2\, \psi_\epsilon(\tau(R))\, \gamma_\epsilon(R)}{\lambda_\epsilon^5(\tau(R))} \, dR\right] \,,
\end{aligned} \tag{3.92}$$

where $\gamma_\epsilon(R) = \int_{R_0}^R \lambda_\epsilon(\tau(\xi))\, d\xi$. Notice that $\gamma_{\epsilon=0}(R) = \int_{R_0}^R d\xi = R - R_0$. Thus

$$\begin{aligned}
\frac{\delta M(t)}{2\pi\, \mu_0} &= \frac{R_0^4}{4}\, \delta\psi(t) + \delta\psi(t)\, R_0 \int_{R_0}^{s(t)} R^2\, dR - R_0 \int_{R_0}^{s(t)} R^2\, \delta\psi(\tau(R))\, dR \\
&\quad + \delta\psi(t) \int_{R_0}^{s(t)} R^2(R - R_0)\, dR - \int_{R_0}^{s(t)} R^2(R - R_0)\, \delta\psi(\tau(R))\, dR \\
&= \frac{R_0^4}{4}\, \delta\psi(t) + \delta\psi(t)\, R_0 \frac{s^3(t) - R_0^3}{3} - R_0 \int_{R_0}^{s(t)} R^2\, \delta\psi(\tau(R))\, dR \\
&\quad + \frac{\delta\psi(t)}{12}\big(3s^4(t) - 4R_0\, s^3(t) + R_0^4\big) - \int_{R_0}^{s(t)} R^2(R - R_0)\, \delta\psi(\tau(R))\, dR \\
&= \frac{s^4(t)}{4}\, \delta\psi(t) - R_0 \int_{R_0}^{s(t)} R^2\, \delta\psi(\tau(R))\, dR - \int_{R_0}^{s(t)} R^2(R - R_0)\, \delta\psi(\tau(R))\, dR \,.
\end{aligned} \tag{3.93}$$

Taking time derivative of both sides one finds

$$\frac{\dot{\overline{\delta M(t)}}}{2\pi\,\mu_0} = \frac{s^4(t)}{4}\,\dot{\overline{\delta\psi(t)}}\,. \tag{3.94}$$

Knowing that $\delta\psi(0) = 0$, one obtains

$$\delta\psi(t) = \frac{2}{\pi\,\mu_0}\int_0^t \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\,dx\,. \tag{3.95}$$

Similarly, for the perturbed motion (3.65) reads

$$\begin{aligned}
&\left[\lambda_\epsilon^2(t) - \frac{1}{\lambda_\epsilon^4(t)}\right]\frac{R_0^2}{2} - \frac{R_0^4\,\psi_\epsilon^2(t)}{8\lambda_\epsilon^4(t)} + \lambda_\epsilon^2(t)\,k_{1\epsilon}(t) - \frac{k_{2\epsilon}(t)}{\lambda_\epsilon^4(t)}\\
&- \frac{R_0^2}{2\lambda_\epsilon^4(t)}\Big[\psi_\epsilon^2(t)\,(R_0k_{3\epsilon}(t)+k_{4\epsilon}(t)) - 2\psi_\epsilon(t)\,(R_0k_{5\epsilon}(t)+k_{6\epsilon}(t)) + R_0k_{7\epsilon}(t)+k_{8\epsilon}(t)\Big]\\
&- \frac{1}{\lambda_\epsilon^4(t)}\Big[\psi_\epsilon^2(t)\left(R_0\hat{k}_{3\epsilon}(t)+\hat{k}_{4\epsilon}(t)\right) - 2\psi_\epsilon(t)\left(R_0\hat{k}_{5\epsilon}(t)+\hat{k}_{6\epsilon}(t)\right) + R_0\hat{k}_{7\epsilon}(t)+\hat{k}_{8\epsilon}(t)\Big] = 0\,,
\end{aligned} \tag{3.96}$$

where

$$\begin{aligned}
k_{1\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{R}{\lambda_\epsilon^4(\tau(R))}\,dR\,, & k_{2\epsilon}(t) &= \int_{R_0}^{s(t)} R\,\lambda_\epsilon^2(\tau(R))\,dR\,,\\
k_{3\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{R}{\lambda_\epsilon^3(\tau(R))}\,dR\,, & k_{4\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{\gamma_\epsilon(R)}{\lambda_\epsilon^3(\tau(R))}\,dR\,,\\
k_{5\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{\psi_\epsilon(\tau(R))}{\lambda_\epsilon^3(\tau(R))}\,dR\,, & k_{6\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{\psi_\epsilon(\tau(R))\,\gamma_\epsilon(R)}{\lambda_\epsilon^3(\tau(R))}\,dR\,,\\
k_{7\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{\psi_\epsilon^2(\tau(R))}{\lambda_\epsilon^3(\tau(R))}\,dR\,, & k_{8\epsilon}(t) &= \int_{R_0}^{s(t)} \frac{\psi_\epsilon^2(\tau(R))\,\gamma_\epsilon(R)}{\lambda_\epsilon^3(\tau(R))}\,dR\,,\\
\hat{k}_{3\epsilon}(t) &= \int_{R_0}^{s(t)}\int_R^{s(t)} \frac{\xi}{\lambda_\epsilon^3(\tau(\xi))}\,d\xi\,dR\,, & \hat{k}_{4\epsilon}(t) &= \int_{R_0}^{s(t)}\int_R^{s(t)} \frac{\gamma_\epsilon(\xi)}{\lambda_\epsilon^3(\tau(\xi))}\,d\xi\,dR\,,\\
\hat{k}_{5\epsilon}(t) &= \int_{R_0}^{s(t)}\int_R^{s(t)} \frac{\psi_\epsilon(\tau(\xi))}{\lambda_\epsilon^3(\tau(\xi))}\,d\xi\,dR\,, & \hat{k}_{6\epsilon}(t) &= \int_{R_0}^{s(t)}\int_R^{s(t)} \frac{\psi_\epsilon(\tau(\xi))\,\gamma_\epsilon(\xi)}{\lambda_\epsilon^3(\tau(\xi))}\,d\xi\,dR\,,\\
\hat{k}_{7\epsilon}(t) &= \int_{R_0}^{s(t)}\int_R^{s(t)} \frac{\psi_\epsilon^2(\tau(\xi))}{\lambda_\epsilon^3(\tau(\xi))}\,d\xi\,dR\,, & \hat{k}_{8\epsilon}(t) &= \int_{R_0}^{s(t)}\int_R^{s(t)} \frac{\psi_\epsilon^2(\tau(\xi))\,\gamma_\epsilon(\xi)}{\lambda_\epsilon^3(\tau(\xi))}\,d\xi\,dR\,.
\end{aligned} \tag{3.97}$$

Thus, linearizing (3.96), one obtains

$$s^2(t)\,\delta\lambda(t) = 2\int_{R_0}^{s(t)} R\,\delta\lambda(\tau(R))\,dR\,. \tag{3.98}$$

Taking time derivative of both sides one obtains $s^2(t)\,\dot{\overline{\delta\lambda(t)}} = 0$, and hence $\dot{\overline{\delta\lambda(t)}} = 0$. Knowing that $\delta\lambda(0) = 0$, one concludes that $\delta\lambda(t) = 0$. Therefore, the only nonzero linearized stress has the following distribution:

$$\delta\bar{\sigma}^{\theta z}(R,t) = \begin{cases}\mu_0\,R\,\delta\psi(t)\,, & 0 \le R \le R_0\,,\\ \mu_0\,R\big[\delta\psi(t) - \delta\psi(\tau(R))\big]\,, & R_0 \le R \le s(t)\,.\end{cases} \tag{3.99}$$

Or

$$\delta\bar{\sigma}^{\theta z}(R,t) = \begin{cases}\dfrac{2R}{\pi}\displaystyle\int_0^t \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\,dx\,, & 0 \le R \le R_0\,,\\[2ex] \dfrac{2R}{\pi}\left[\displaystyle\int_0^t \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\,dx - \int_0^{\tau(R)} \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\,dx\right]\,, & R_0 \le R \le s(t)\,.\end{cases} \tag{3.100}$$

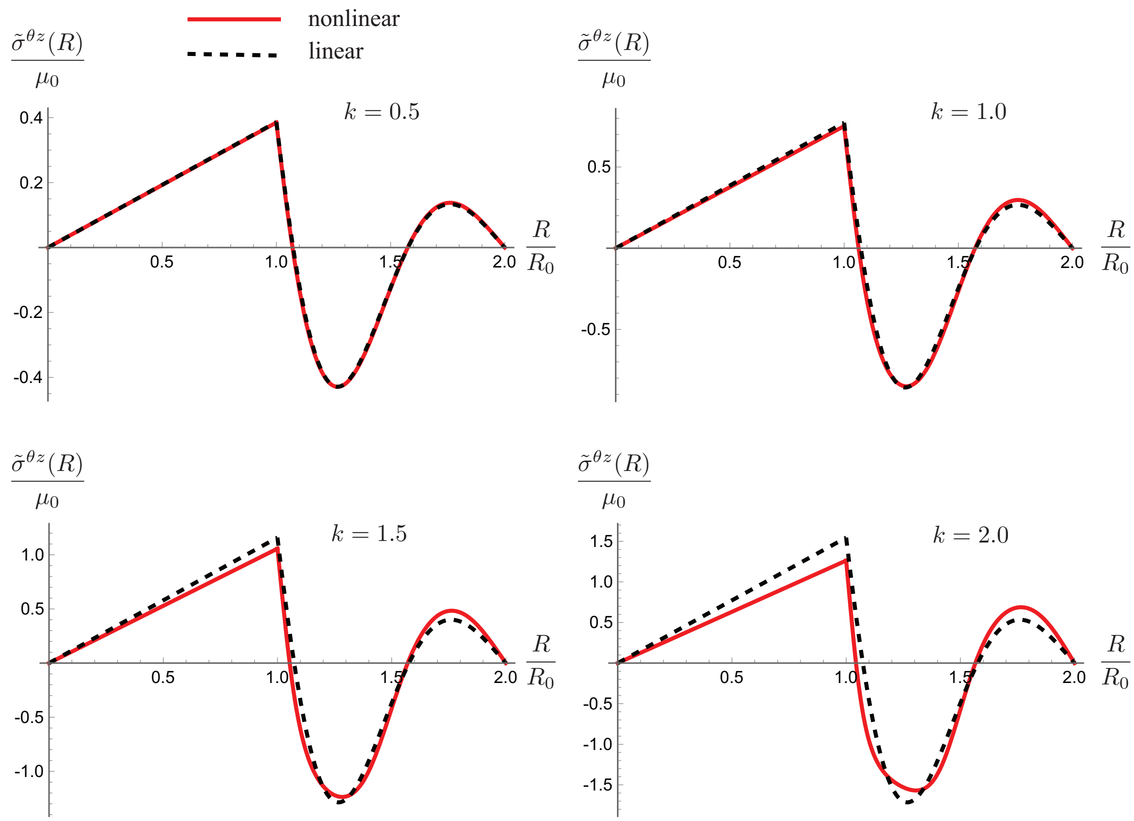


Figure 7: *Residual shear stress and linearized residual shear stress for the loading* $M(t) = k\pi R_0^3 \sin\left(\frac{2\pi t}{t_a}\right)$ *for four different values of* $k$.

This can equivalently be written as

$$\delta\bar{\sigma}^{\theta z}(R,t) = \begin{cases} \dfrac{2R}{\pi} \displaystyle\int_0^t \dfrac{\dot{\overline{\delta M(x)}}}{s^4(x)}\, dx\,, & 0 \leq R \leq R_0\,, \\ \dfrac{2R}{\pi} \displaystyle\int_{\tau(R)}^t \dfrac{\dot{\overline{\delta M(x)}}}{s^4(x)}\, dx\,, & R_0 \leq R \leq s(t)\,. \end{cases} \tag{3.101}$$

**Linearized residual stresses.** Linearizing the zero-force condition $(3.77)_2$, one finds $\delta\tilde{\lambda} = 0$. Similarly, linearizing the zero-torque condition $(3.77)_1$, one obtains

$$\delta\tilde{\psi} = \frac{8}{\pi\mu_0 R_a^4} \int_{R_0}^{R_a} \xi^3 \int_0^{\tau(\xi)} \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\, dx\, d\xi\,. \tag{3.102}$$

The only nonzero linearized residual stress has the following distribution:

$$\delta\bar{\bar{\sigma}}^{\theta z}(R) = \begin{cases} \mu_0\, R\, \delta\tilde{\psi}\,, & 0 \leq R \leq R_0\,, \\ \mu_0\, R\big[\delta\tilde{\psi} - \delta\psi(\tau(R))\big]\,, & R_0 \leq R \leq R_a\,, \end{cases} \tag{3.103}$$

or

$$\delta\bar{\bar{\sigma}}^{\theta z}(R) = \begin{cases} \dfrac{8R}{\pi R_a^4} \displaystyle\int_{R_0}^{R_a} \xi^3 \int_0^{\tau(\xi)} \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\, dx\, d\xi\,, & 0 \leq R \leq R_0\,, \\ \dfrac{8R}{\pi R_a^4} \displaystyle\int_{R_0}^{R_a} \xi^3 \int_0^{\tau(\xi)} \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\, dx\, d\xi - \frac{2R}{\pi} \int_0^{\tau(R)} \frac{\dot{\overline{\delta M(x)}}}{s^4(x)}\, dx\,, & R_0 \leq R \leq R_a\,. \end{cases} \tag{3.104}$$

For $R_0 = 1$, $u_0 = 1$, and $t_a = 1$, in Fig.7 the residual shear stress and the linearized residual shear stress distributions for the loading $M(t) = k\pi R_0^3 \sin\left(\frac{2\pi t}{t_a}\right)$ and four different values of $k$ are shown. As expected, as $k$ increases the difference between the nonlinear and linear solutions increases.

# 4 Conclusions

In this paper, we formulated the initial-boundary-value problem of finite torsion and extension of an accreting circular cylindrical bar. The bar is assumed to be homogeneous and is made of an arbitrary incompressible isotropic solid. It is also assumed that accretion is symmetric, i.e., the accreting bar is a solid circular cylinder at all times. Assuming a generalized Family 3 kinematics (3.2), we showed that radial deformation is a functional of the time-dependent axial stretch $\lambda^2(t)$, see $(3.56)_1$. Assuming that stress-free material is added to the boundary of the deforming bar (generalizing our analysis to the case of pre-stressed material is straightforward), we calculated the material metric of the accreting bar. We noted that this metric is unique up to isometry. The kinematics is completely specified as soon as the time-dependent axial stretch $\lambda^2(t)$ and the twist per unit length $\psi(t)$ are known. The applied toque $M(t)$ and the axial force $F(t)$ explicitly depend on these two functions. We assumed that there is no applied axial force, i.e., $F(t) = 0$; the bar is free to deform axially. We considered both twist-control ($\psi(t)$ is given) and torque-control ($M(t)$ is given) loadings. We calculated the corresponding stresses. It was observed that the kinematics (3.2) together with its corresponding material metric are universal for incompressible isotropic solids (see Remark 3.2) in the sense that equilibrium equations are satisfied in the absence of body forces and for any energy function $W(I_1, I_2)$. We also calculated the residual stresses that are induced by accretion. Finally, we calculated the deformations and stresses in the setting of linear accretion mechanics by linearizing the nonlinear fields. The nonlinear and linear solutions were numerically compared for a few examples. As expected, as the applied torque increases the difference between the linear and nonlinear solutions becomes more appreciable.

The analysis presented in this paper can be extended to inhomogeneous and anisotropic bars. In the case of incompressible transversely isotropic, orthotropic, and monoclinic solids, we expect the kinematics ansatz given in (3.2) to be universal for circular cylindrical bars with the universal material preferred directions found in [Yavari and Goriely, 2021]. We also suspect that for either isotropic or the three anisotropy classes (transversely isotropic, orthotropic, and monoclinic solids), the cylindrical bar can have radial inhomogeneity, i.e., its energy function can explicitly depend on the radial coordinate: $W = W(R, I_1, I_2)$ [Yavari, 2021, Yavari and Goriely, 2022].

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