2013

THE LEHANNEUR-REBUFFEL MODEL FOR CABLE BENDING ANALYSIS

Presentation and translation of the 1949 Lehanneur paper in which he develops a stick-slip model first proposed by Rebuffel. While specifically oriented towards wire rope problems, model could be applied to other helical strand systems, and in particular to Overhead Electrical Conductors. The paper may be considered a precursor to more recent models.

Two, more cable-oriented, sections of the paper have been omitted.

Alain Cardou, Ph.D.
GREMCA, Université Laval, Québec
01/01/2013

The Lehanneur-Rebuffel model for cable bending analysis

By Alain Cardou, Ph.D.

INTRODUCTION

Depending on the application at hand, various models have been developed for cable bending situations, the latest being Finite Element Analysis (FEA). One class of models, used for bending stiffness prediction, as well as corresponding wire stresses, is based on Coulomb's friction laws and can be called "stick-slip" models, although this term has also been used for more complex models where contact elastic properties are included.

While stick-slip models can be traced back to German engineer H. Ernst, in 1933-34, a major contribution was published in 1949 by French engineers A. Rebuffel and M. Lehanneur, in the "*Annales des Ponts et Chaussées*" a technical journal sponsored by the prestigious engineering school "*École Nationale des Ponts et Chaussées*," near Paris. The exact references are:

Lehanneur, M. *La flexion des câbles métalliques* (Bending of metal cables). Annales des Ponts et Chaussées. May-June 1949, pp. 321-386 and 439-454.

Rebuffel, A. *La flexion d'un câble eu égard aux frottements* (Cable bending, taking friction into account). Annales des Ponts et Chaussées. July-August 1949, pp. 455-466.

While Rebuffel's paper was published after Lehanneur's, the latter is, in fact, a follow up to the former. Lehanneur clearly states his indebtedness to Rebuffel. His approach being based on Rebuffel's with some modifications, this is the reason why their model is called in this work the "Lehanneur-Rebuffel" model (or, for conciseness, the LR model).

The LR model has been completely ignored in later works on cable bending. This may be explained by the fact that it was published in French and in a journal which is probably not widely circulated. Also, the extremely well polished language used in Lehanneur's paper is rather unusual in current technical journals. The notations (such as r for a stress and m for a radius) are far from standard for contemporary readers. Besides, meaning of a given symbol may vary from one section to the other (e.g. symbol θ, which represents an angle and a tensile force). In general, the context is clear enough to avoid any ambiguity.

Thus, the object of this work is to render Lehanneur's paper accessible to a wider readership, in particular to those students who are embarking in the development of new models for this technical problem, which apparently is still of current interest. Indeed, although addressing cable problems, the paper can also apply to other helical strand systems, such as overhead electrical conductors, the translator's domain of interest.

The translation follows the original text as closely as possible. Also, the notations have been kept (a list of symbols is given below), as well as equation and figure numbering. The interested reader should have no difficulty to turn to the original in order to compare with the proposed translation.

Discussing some of Lehanneur's results and comparing them with more recent ones is out of the scope of this work. This has been done in the following separate report:

Cardou, A., *Stick-Slip mechanical models for overhead electrical conductors in bending (with Matlab applications),*ISBN 978-2-9812337-2-1, Québec, Canada, 2013. 97 pages.

CONTENTS OF LEHANNEUR'S PAPER

Section I
The general conditions of wire rope bending. Qualitative presentation. Each wire is under tensile and shear forces as well as bending and torsional moments.

Section II
Wire bending and torsional moments

Recalls Baticle's 1912 paper. A critical examination of his hypotheses. Shows that the geodesic curve hypothesis is not valid. Instead, proposes the toroidal helix (a hypothesis which is adopted by most contemporary authors).

Section III
Calculation of moments

Determines total bending moment $\mathfrak{M}$ which is the contribution of each wire, independent of its position within cable cross-section. Shows contribution of wire bending stiffness and, also, torsion stiffness.

Section IV
Multiple strand cables

Extension of single strand results to multiple strand case. Each strand is taken as a single wire, whose bending and torsional properties are obtained through single strand formulas.

Section V
A study of wire axial force and displacement

Study of force distribution in wires when cable is straight and under axial load only. Results are similar to Hruska's (1951).

Section VI
Rebuffel's Theory

Stick-slip theory of cable bending. Lehanneur details Rebuffel's approach with appropriate comments. Here, the term "loop" is used to designate one complete turn of a helical wire around the cable axis. Starting in its straight (curvature $1/R = 0$), axially loaded state, cable is given a circular shape. At first, cable behaves as a solid beam. Bending moment has two components $\mathfrak{M}$ and $\mathfrak{M}'$. Component $\mathfrak{M}$ arises from wire bending and torsional properties, and has been determined in Section III. Component $\mathfrak{M}'$ arises from each wire tensile force moment with respect to cable cross-section neutral axis in bending. As long as wires stick together, wire tensile force is obtained using the standard Bernoulli-Euler hypothesis (plane section remains plane).

When imposed curvature reaches $1/R_0$, slip starts in the outer layer (being the layer where bending stresses are maximum). Rebuffel studies slip propagation assuming continuous contact between the first (outer) layer and the second layer, as in parallel lay cables. Contact between adjacent wires in the same layer is neglected. Slip regime propagates in two directions along the contact line. In the slip domain, wire tensile force cannot be obtained with the plane-section hypothesis and a differential equation has to be solved, based on Coulomb's law. When curvature $1/R$ reaches value $1/R_1$ slip is complete between first and second layer.

For inner layers, it is assumed that pressure coming from the outer ones does not influence the slip conditions, that is, slip conditions in a layer depend only on the pressure due to the tensile forces in that particular layer. This hypothesis is valid only if one considers wire diameters as very small compared to the lay cylinder diameter, and, also, this strictly applies to parallel lay only. In cross-lay cables, friction forces are oblique with respect to contact lines. It is assumed that slip starts sequentially, that is, complete slip is reached in one layer before starting in the adjacent one.

A new equation is obtained for $\mathfrak{M}'$. If $1/R$ is increased beyond $1/R_1$, moment from the first layer remains constant. The same applies to inner layers so that, when slip is complete on all layers, $\mathfrak{M}'$ remains constant as $1/R$ is increased beyond that point.

Section VII
A correction in the calculation of R_1 and subsequent Rebuffel formulas

Lehanneur shows that one of the hypotheses is faulty and compares with the corrected one. Results show little numerical difference leading him to keep Rebuffel simpler formulas.

Section VIII
An attempt to generalize Rebuffel's formulas

Rebuffel's formulas are based on several hypotheses. Among them: cable is of the spiral type, curvature is uniform (cable is given a circular shape) and deformation process, starting in the straight state, is monotonous. Lehanneur tries to extend these results to :

A. Non-monotonous process
B. Non-uniform curvature
C. Multi-strand cables

Results in the (A) and (B) cases are more or less qualitative. Recent works, most notably by Papailiou (1995), have tried to tackle these more difficult problems. As for case (C), it is assumed that results obtained in the single strand case can be applied to the multi-strand case, by considering that, in that case, strands play the role of wires in the single strand cable.

Section IX
Friction coefficient determination. Klein's Method

This very extensive section of the paper (more than 14 pages) describes in detail a method proposed in his thesis by E.H. Klein (1934) to deduce cable friction coefficient μ from experimental results. Cable technology has to be considered however, which is outside the area of interest of the translator. This section has been omitted.

Section X
Application to cables of Isaachsen's problem

Isaachsen (1905) solved the problem of a single wire, under axial load and bent over a fixed support point. Knowing the wire bending stiffness, the applied tensile force, and the angle of the wire axis on both sides of the fixed support, he found the equation of the curve taken by the wire middle line, in particular its curvature at that point.

Application to a cable is immediate when curvature is smaller than $1/R_0$, before any slip takes place. Lehanneur goes a step further by simplifying the stick-slip process and reducing it to a bilinear law in which the cable is either in the stick or in the slip state. Transition is supposed to occur at a "virtual" critical curvature. This approach has been generalized independently by Papailiou (1995).

Section XI
Some numerical applications

In this section, the LR formulas are applied to a specific problem. The selected cable is a six-strand cable, each strand being made of 7 identical steel wires. Because of its technological content, <u>this section is not included in the enclosed translation</u>.

NOTATION

Latin symbols

a, a'	constants
b	a constant
d	a distance (Fig. 6)
dl	cable element
dL	cable elongation
ds	wire element (Sec. III)
E	wire material Young's modulus
f	a function (Sec. II)
f	component from wire bending and torsion moments normal to cable axis (Sec. III)
F	a function
F_α	partial derivative of F with respect to α , i.e. $\partial F/\partial\alpha$
$g(t)$	a monotonous function of time, increasing from 0 to 1
G	wire material shear modulus
$G'j'$	multi-strand cable equivalent stiffness in torsion
h	a distance (Fig. 16)
i	wire index in the layer
I	wire inertia in bending (with respect to wire section neutral axis)
j	wire inertia in torsion (wire section polar moment of inertia)
k	a constant
K, K'	constants
l	curvilinear abscissa of a section on curved cable axis (Sec. VIII)
l	length of wire over one half lay length (Sec. VI) (half-loop)
L	cable length
m	radius of a layer lay cylinder (on which wire centerlines are laid helically)
m_j	radius of a layer lay cylinder j
m'	moment of wire tensile forces for a given layer when slip is complete
M	torsional moment on cable
M	total bending moment $M = \mathfrak{M} + \mathfrak{M}'$ (Sec. VIII)
$\mathfrak{M}$	total bending moment on a layer resulting from independent wires
$\mathfrak{M}'$	total bending moment on a layer resulting from wire tensile forces
n	number of wires in a given layer
p	helical wire lay length
P	radius of cable (Sec. VI)
r	radius of a torus parallel line (Sec. II)
r_1	normal stress on outer layer wire cross-section
r_n	normal stress on wire n cross-section
R	radius of curvature of bent cable
R'	derivative dR/dl (Sec. VIII)
R_0	radius of curvature at impending slip
R_1	radius of curvature when slip is complete

s area of wire cross-section
s_n area of wire n cross-section (also s_i)
S_j total wire cross-section area in layer j
t wire tensile force arising from cable axial load T (Sec. VI)
t component from a wire bending and torsion moments parallel to cable axis
t time (Sec. VIII)
T tensile force on cable
T_j tensile force on layer j
$\mathfrak{T}$ total torsional moment on a layer resulting from independent wires (Lehanneur uses a script capital T)
x abscissa along helical wire over on one lay length (Sec. VI)
x, y, z Cartesian coordinates and functions for parametric curve definition (Sec. II)
x', y', z' functions x, y, z first derivatives
x'', y'', z'' functions x, y, z second derivatives
y wire strain (Sec. VI)
Y wire strain function (Sec. VI)

Greek symbols

α polar angle of wire section center in cable cross-section (Sec. II)
α', α'' derivatives of α with respect to polar angle θ
α angle of wire bent over fixed point (Sec. X)
α_0 polar angle of wire section center at which slip starts in outer layer
β angle (Fig. 6)
γ angle (Sec. VI)
γ a constant
ε ratio m/R
θ polar angle of a particular cable cross-section after bending (Sec. II)
θ wire tensile force in single-strand cable (Sec. VI)
φ lay angle
λ a constant and a ratio
μ coefficient of friction
ψ_n lay angle of strand n in multi-strand cable
ρ a distance (Fig. 5)
ρ radius of curvature at point of support (Sec. X)
ρ_i wire section radius (also, in Sec. V, ρ)

BENDING OF WIRE ROPES

By M. Lehanneur, *Ingénieur en Chef des Ponts et Chaussées*

Safety standards regarding ropeways, annexed to the 15 October 1947 Ministry document, nowhere specify the admissible stress levels in a rope individual wire. They just indicate the maximum axial load on a given cable, based on its role and on its own mechanical properties. They also define which conditions it should satisfy when going over sheaves, pulleys, or rollers.

It is to be noted that, by so doing, they have deviated from the usual mistaken rules generally adopted by French authorities regarding safety as far as strength of materials is concerned. Usually, these rules try to determine as exactly as possible maximum stresses at critical points of a structure or element thereof. System safety is then defined by taking the ratio of these stresses to some critical stress values.

Needless to say, this new approach is based on definite justifications. In fact, from a practical point of view, it has real advantages, leading to simpler and easier to control calculations. Besides, this approach has apparently been adopted in several foreign standards.

However, we do not think that one should neglect a problem which has apparently been discarded, that is, the cable bending problem. A complete solution of this problem, even if it has been up to now the object of numerous studies, seems to be far removed. New contributions are thus to be encouraged. We shall see there are situations in which this bending problem cannot be avoided, even within the bounds of the 15 October 1947 regulations.

Our main objective here is to give the state of knowledge on this problem while at the same time giving the reader access to little known French and foreign studies. Occasionally, we intend to present some modifications or extensions to their results. Then, we plan to use these results to derive the stress conditions which correspond to the 15 October 1947 rules. We also intend to give a method which lifts the indeterminacy arising in cases such as those mentioned earlier.

I. The general conditions of wire rope bending

A wire rope is a system exclusively (or mainly) composed of metal wires. There are two main differences with ordinary strength of materials systems:

1. Most of the time, stranding operations have brought the wire material beyond its yield limit, thus inducing residual strains (which are easily evidenced in an unlaying operation); in spite of them, there are certainly very high residual stresses in the wound wires. Unfortunately, the actual stress levels are not known precisely;
2. Wires are not bound rigidly. However, they are under two kinds of bounds imposed by neighbouring wires: geometric and friction conditions.

A consequence is that a cable is a system in which standard strength of materials rules are not *a priori* applicable. For example, when a cable passes over a pulley, it cannot be assumed *a priori* that the axis of the bending moment be parallel to the pulley axis. It is not obvious either that, given a cable under axial load, when its middle axis follows a certain curve, the bending moment on a section is a function of the radius of curvature only. In fact, we shall see that it does depend on the axial load.

Consider first the case of a cable wound over a pulley, radius R. Assume the winding angle is large enough (this assumption will be given a more precise definition later on). The radius R is the distance between cable and pulley axes.

In order to define the internal forces in the cable, a cross-section is considered. It may be a plane cross-section, normal to the cable axis or, rather, a set of plane cross-sections, each one being normal to a wire axis at a point where it intersects the plane normal to the cable axis. Textile core, if any, is neglected.

On each wire cross-section, the usual internal loads can be defined:

A tensile force (here, compressions are not to be considered);

A shear force;

A bending moment;

A torsional moment.

Internal forces on the cable cross-section are the resultants of these wire forces.

Thus, each one of them will be studied separately and, in order to follow the chronological order, moments will be first considered.

II. Wire bending and torsional moments

Apparently, these internal forces have been the first to undergo a systematic study, published in the January-February 1912 issue of the *Annales des Ponts et Chaussées*, a paper contributed by M. Baticle who, at the time, was a mere Ponts et Chaussées engineer.

The simple formulas obtained by M. Baticle were easily applicable, necessitating only the wire traction and torsion elasticity coefficients.

Some of his hypotheses having been criticized, it is justified to re-examine this question.

Baticle's hypotheses, explicit or implicit, were as follows:

1. The *complementary* wire stresses or strains arising from cable bending stay within the elastic domain, at least within certain limits of the imposed bending. The stress-strain relations are thus linear. (M. Baticle even assumes explicitly that total wire stresses and strains stay within the elastic domain. It is easily verified that his most important results, e.g. the cable bending moment determination, still apply when his hypothesis is restricted to *complementary* stresses or strains);
2. In a single strand cable, when the cable axis is a straight line, a given layer wire axes are wound on a circular cylinder, radius *m*. When the cable passes over a pulley, radius R, the wire axes are on a torus with a meridian circle whose radius is *m*;
3. On this torus, wire axes follow the geodesic lines;
4. Instead of a circle, radius R, assume now that the cable is bent in such a way that its axis follows an arbitrary curve C. On a cable cross-section at a point M of its axis, where curve C has a radius of curvature R, stress and strain are calculated as in the case of the circle, radius R;
5. For a multi-strand cable, the preceding hypothesis applies firstly for each strand, secondly for each wire in a strand. In the first stage, each strand is first assumed to be equivalent to a wire concentrated on its own axis;
6. Bending moment axis on cable cross-section is parallel to geometric bending axis (e.g., it is parallel to the pulley rotation axis in the case of a cable passing over a pulley).

FIRST HYPOTHESIS

If complementary stresses and strains are small enough, it corresponds to a well-known mathematical hypothesis. It is found in practice that these conditions are generally respected.

We shall keep this hypothesis, thus following most of previous works on this subject.

SECOND HYPOTHESIS

It looks obvious if one neglects Poisson's effect on wires (an hypothesis generally accepted in strength of materials).

Indeed, a cable cross-section is compact, and position of a given wire is constrained by its contact with its neighbours. Thus, if all wires are supposed to keep the same transverse size, their respective position should not vary.

We shall keep this hypothesis. However, it should be noted that the section compactness may be more or less complete, depending on the type of cable and, for a given cable, depending on the wires.

For example, it is obvious that compactness will be higher when the outer layer is made of "profiled wires", rather than circular section wires, and even more so for strands.

This probably explains why locked coil ropes are harder to bend (i.e. stiffer) than ordinary ones.

THIRD HYPOTHESIS

It is the most objectionable. To our knowledge, published critics have not been rigorous. However, as we intend to show, we believe they are well grounded.

Obviously, a thin, zero bending stiffness, taut wire, wound on a frictionless torus, will take the shape of a geodesic line.

These conditions, however, do not apply to a wire within a cable, because of the other wires. As already mentioned, it is constrained by geometry conditions. Combined with the second hypothesis, these constraints are such that, in a single strand cable, the *n* wires in a given layer will be located so that their axes intersect any normal cross-section plane at points which are the vertices of a regular n-sided regular polygon.

We shall prove that this fact alone suffices to make it impossible that all wires take the shape of a geodesic line.

The torus is defined by radii $\mathrm{OC} = \mathrm{R}$ and $\mathrm{CA} = m$. The position of a point M is given by two polar angles θ and α, the origin of the first one being measured arbitrary (Fig. 1).

Any curve on the surface may be given by a relationship $f(\alpha,\theta) = 0$.

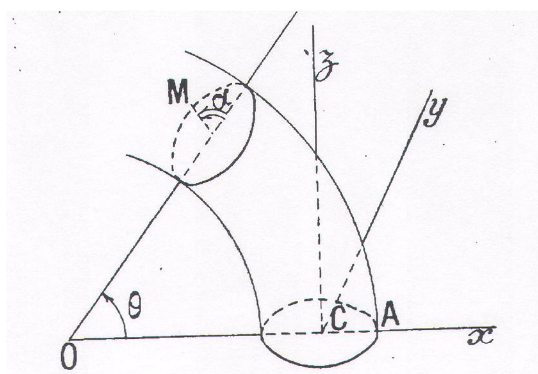

Figure 1

In the cartesian system C*xyz*, point M coordinates are given by :

$$x = (\mathrm{R} + m\cos\alpha)\cos\theta - \mathrm{R}$$
$$y = m\sin\alpha$$
$$z = (\mathrm{R} + m\cos\alpha)\sin\theta$$

In these equations, assume that θ is the independent variable, α being given by equation $f(\alpha,\theta) = 0$. In order for a curve parametrically defined by these equations to be a torus geodesic curve, it is necessary and sufficient that at each point, its osculating plane contains a vector normal to the torus surface. The direction cosines of the normal unit vector are $\cos\alpha\cos\theta$, $\sin\alpha$ and $\cos\alpha\sin\theta$. The condition is thus expressed a:

$$\begin{vmatrix} \cos\alpha\cos\theta & \sin\alpha & \cos\alpha\sin\theta \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

where the last two lines are the first and second derivatives of *x, y* and *z* with respect to θ.

If this equation is developed, it becomes a second order differential equation:

$$\mathrm{F}(\theta, \alpha, \alpha', \alpha'') = 0$$

where α' and α'' are the first and second derivatives of α with respect to θ.

Now, assuming that the *n* wires of a layer follow a geodesic line and, at the same time, intersect a torus meridian section to give the vertices of a regular polygon, one must have the simultaneous equations:

$$\begin{cases} \mathrm{F}(\theta, \alpha, \alpha', \alpha'') = 0 \\ \mathrm{F}(\theta, \alpha + \frac{2\pi}{n}, \alpha', \alpha'') = 0 \\ \mathrm{F}(\theta, \alpha + \frac{4\pi}{n}, \alpha', \alpha'') = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{F}(\theta, \alpha + \frac{2n-2}{n}\pi, \alpha', \alpha'') = 0 \end{cases}$$

In these equations, θ, α' and α'' take the same value, since θ is the same for all the polygon vertices, and adding up a constant to a function does not affect its derivatives.

However, if the hypothesis is correct, it must be so for any integer *n*, which may be arbitrarily large. That is to say, $\alpha, \alpha + \frac{2\pi}{n}, \cdots, \alpha + \frac{2n-2}{n}\pi$ may me arbitrarily close to any value in the interval $[\alpha, \alpha + 2\pi]$. Thus, any system of values $\theta, \alpha, \alpha', \alpha''$ which satisfies $\mathrm{F} = 0$ must also satisfy $\frac{\partial \mathrm{F}}{\partial \alpha} = 0$. It is easy to see that this is not the case.

After some tedious calculations, it can indeed be shown that equation $\mathrm{F} = 0$ reduces to:

$$2\varepsilon(1+\varepsilon\cos\alpha)\alpha'' + 2\varepsilon^2\alpha'^2\sin\alpha + (1+\varepsilon\cos\alpha)^2\sin\alpha = 0$$

in which $\varepsilon = m/\mathrm{R}$.

It is noticed that θ does not appear explicitly in this equation. A result which could be expected, a torus showing a symmetry of revolution: if $f(\alpha,\theta)=0$ is a geodesic curve, it is obvious that $f(\alpha,\theta+\lambda)=0$, where λ is a constant, also represents a geodesic. Thus, $\mathrm{F}=0$ should not depend on θ either.

Therefore, in order for the hypothesis to be valid, the following two-equations system has to be satisfied:

$$\mathrm{F} = 2\varepsilon(1+\varepsilon\cos\alpha)\alpha'' + 2\varepsilon^2\alpha'^2\sin\alpha + (1+\varepsilon\cos\alpha)^2\sin\alpha = 0$$

$$\frac{\partial \mathrm{F}}{\partial \alpha} = -2\varepsilon^2\alpha''\sin\alpha + 2\varepsilon^2\alpha'^2\cos\alpha$$

$$+\cos\alpha(1+\varepsilon\cos\alpha)^2 - 2\varepsilon\sin^2(1+\varepsilon\cos\alpha) = 0$$

That is, both equations must be satisfied for the same values of α'' and α'^2 , since it is possible to give arbitrary values to these quantities. This is possible only if their coefficients are proportional, for any α .

It is obvious that this is not the case.

Thus, the hypothesis has to be discarded.

Apparently, some of the critics have argued that, in the bent cable, wires kept the same lay angle φ with the torus parallel lines.

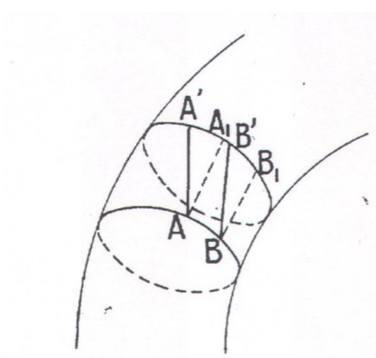

Figure 2

It is immediately noticed that this hypothesis also contradicts the definition of a "geometric constraint" given above.

Indeed, consider a given wire between two neighbouring cable cross-sections. Its axis intersects the corresponding planes at points A and A' (Fig. 2). Segment AA_1 is an element of the parallel between the two planes. According to the hypothesis, $\mathrm{A_1AA'} = \varphi$:

$$\mathrm{A_1A'} = \mathrm{AA_1}\tan\varphi$$

In the same fashion, for the neighbouring wire:

$$\mathrm{B_1B'} = \mathrm{BB_1}\tan\varphi$$

However, in general, AA_1 and BB_1 are not equal. This applies to A_1A' and to B_1B'. This shows that, in general, points A and B, and points A' and B' cannot be simultaneously two consecutive vertices of an n –sided polygon.

There is, however, another hypothesis, which should seem rather natural, and which does not have this problem. It consists in assuming that, in the equation $f(\alpha,\theta)=0$, α and θ are linearly related.

It means that, in cable bending, for wires belonging to a given layer their axes follow the same deformation pattern as in usual strength of materials beam bending theory. With this hypothesis agrees, cable plane cross-sections remain plane.

It also implies that, in the cable region in contact with a pulley, wires belonging to a given layer take the same shape: they can be superposed by rotation with respect to the torus axis. This property seems rather obvious for reasons of symmetry. It will be shown in the sequel that this hypothesis is the only one having such property.

First, it is obvious that it applies. Indeed, equation $f(\alpha,\theta)=0$ may be written $\alpha = \mathrm{K}(\theta-\theta_0)$ where K is a constant, the same for all wires in the same layer. θ_0 is also a constant, but it is different for each wire. Consequently, a rotation of an appropriate angle about the torus axis will make any two curves of this family coincide.

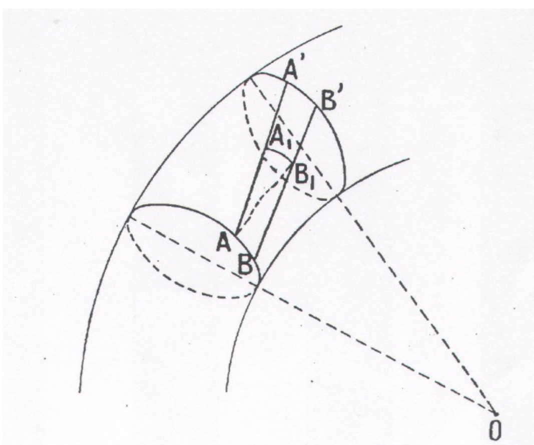

Figure 3

This hypothesis is the only one having this property. Indeed, in order for two neighbouring curves AA' and BB' (Fig. 3) on the torus to coincide after such a rotation, while at the same time meeting the condition $AB = A'B'$ between neighbouring cross-sections, the arc length of the parallel curve AB_1 has to be proportional to this parallel curve radius *r*. Now, letting $\varphi = \widehat{B_1AA_1}$, one has $AB_1 = A_1B_1/\tan\varphi$.

Since $A_1B_1 = 2m\pi/n$, it is a constant and independent of selected point A. Thus, for any point A, $A_1B_1/\tan\varphi = r\gamma$, where γ is a constant, and $r\tan\varphi$ is a constant for all points on the curve.

From the differential equations general theory, condition $r\tan\varphi = c^{st}$ is sufficient to determine a curve on the torus surface, given a point and the tangent at that point. And the solution is unique. As shown above, curve $\alpha = \mathrm{K}(\theta-\theta_0)$ is a solution. It is thus the only solution to the problem.

Thus, in the bent cable it seems logical to adopt the linear relationship hypothesis. Instead of $\varphi = c^{st}$, it implies the condition $r\tan\varphi = c^{st}$.

It will be used in the sequel. However, one point has to be noted, which seems to point to a flaw in the result.

Indeed, when entering onto a pulley at point A (Fig. 4), the cable curvature cannot undergo a jump from 0 to $1/\mathrm{R}$. Such a discontinuity would mean a kink in the wires.

E.g., for a wire located on the convex side, the angle in the rectilinear domain being φ would become $\arctan(\varphi \times R/R+m)$ in the curved part.

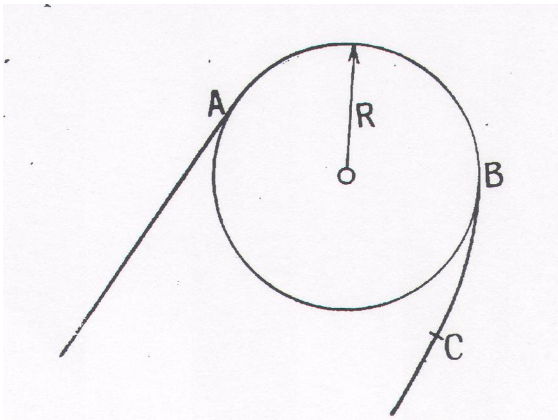

Figure 4

We are thus led to assume that the cable is bent when nearing the pulley, as shown schematically in Fig. 4, between B and C, where curvature is brought progressively from $1/R$ to 0.

To our knowledge, there are no test data available which could contradict our hypothesis. Besides, it is well known that in strength of materials theory, a constant section beam cannot afford a discontinuous curvature.

FOURTH HYPOTHESIS

This hypothesis would be exact if the cable were behaving as a solid prismatic beam, according to standard strength of materials theory.

While this is not the case, it may be noted that with the third hypothesis, under the proposed linear relationship, the wire shape in the bent cable, is the same as with a solid cable. That is, it is the same in a given cable cross-section, whatever the shape of the bent cable, as long as its axis principal center of curvature stays the same.

Thus, retaining the third hypothesis, with the linear relationship modification, implies the validity of the fourth one.

FIFTH HYPOTHESIS

Arguments given in favor of the third hypothesis, under its modified form, apply evidently to a strand belonging to a multistrand cable.

Accordingly, the fifth hypothesis should be retained.

SIXTH HYPOTHESIS

This hypothesis leads to much simpler calculations. However, it is not essential, and it will not be used here.

III. Calculation of moments

Based on the preceding hypotheses, the objective is to determine the bending and torsional moments in each cable wire, at a section where the cable axis is imposed a radius of curvature R.

In this calculation, it is assumed (4th hypothesis) that the cable is wound on a pulley, radius R . The single strand case is first examined, where the problem of one given layer, radius m , with n circular cross-section wires .

The 3rd hypothesis, as reformulated, allows a rather simple treatment. According to the 1st hypothesis, it deals only on the complementary deformations arising from the winding on the pulley.

Consider an axis $y'y$ and wire element MM' . At first, the cable axis is rectilinear, and the wire element is limited by two neighbouring parallel planes, normal to the cable axis. Assume that the plane at M contains axis $y'y$ (Fig. 5). Now, element MM' is wound on a pulley whose axis is $y'y$. Consider wire sections (S) and (S') defined by the preceding parallel planes, plane $My'y$ and corresponding plane through M'.

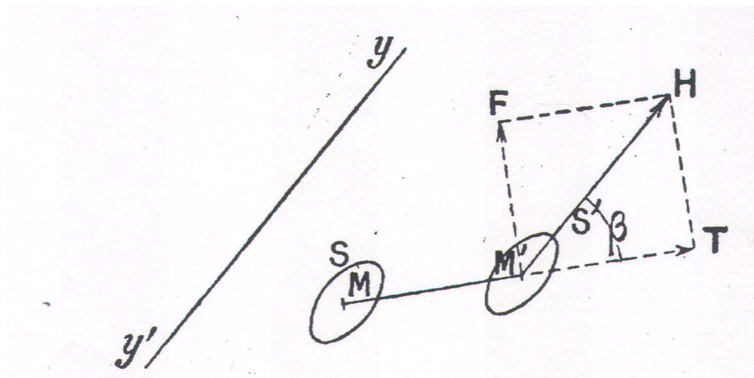

Figure 5

What happens to this element when the cable is wound on a pulley with axis $y'y$? Again, plane $My'y$ contains section (S) while (S') is in a plane parallel to $My'y$ and passing through point M'.

Here we neglect the variation in length of centerline fiber element MM', assuming that it belongs to a cylinder which bends according to the usual strength of materials hypotheses. Thus, section (S') will rotate about an axis parallel to $y'y$ and passing through M'. The rotation is such that the plane containing (S') will contain $y'y$. The corresponding angle of rotation is d/ρ , d being the distance between M' and plane $My'y$ and ρ being the distance between M' and axis $y'y$.

Distance MM' is a differential element, so that the rotation is also infinitesimal. It can be represented as a vector M'H, length d/ρ, parallel to axis $y'y$. As easily seen in Fig. 6, a projection on a plane perpendicular to axis $y'y$, this vector has the same orientation for all wires in a given layer. By convention, it will be taken as positive.

Back to Fig. 5. Rotation M'H may be considered as the resultant of two components, M'T and M'F. M'T is parallel to MM' while M'F is normal to MM', in plane MM'H.

First component is a torsion. Calling β the angle between directions $y'y$ and MM' (a positive quantity), its magnitude is $(d/\rho)\cos\beta$. Component M'F corresponds to bending, magnitude $(d/\rho)\sin\beta$.

Letting $\mathrm{MM'} = ds$, distance d can be expressed as $d = ds\cos\varphi$ (Fig. 6) where, again, φ is the lay angle of the given layer.

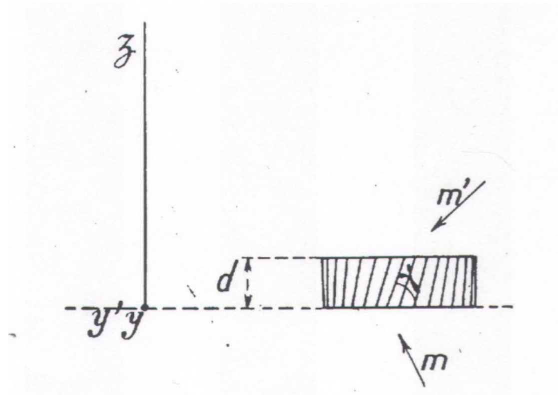

Figure 6

Torsion M'T generates a corresponding torsional moment parallel to MM' and given by $\mathrm{G}j\left(\cos\varphi\cos\beta/\rho\right)$ (G is the material shear modulus, while j is the wire cross-section polar moment of inertia).

The absolute value of infinitesimal angle of rotation M'F due to bending is given by $ds\cos\varphi\sin\beta/\rho$. The corresponding bending moment is $\mathrm{EI}\left(\cos\varphi\sin\beta/\rho\right)$, E being the material Young's modulus and I the wire cross-section moment of inertia with respect to the axis defined by vector M'F.

This moment axis, is parallel to the direction defined by M'F, in the opposite direction. Its component on the M'H axis is given by $\mathrm{EI}\left(\cos\varphi\sin^2\beta/\rho\right)$ (absolute value). This component is oriented opposite to M'H (see Fig. 5).

Thus, its algebraic expression is $-\mathrm{EI}\left(\cos\varphi\sin^2\beta/\rho\right)$.

Also, component of torsional moment $-\mathrm{G}j\left(\cos\varphi\cos\beta/\rho\right)$ on the M'H axis is $-\mathrm{G}j\left(\cos\varphi\cos^2\beta/\rho\right)$.

Thus, the wire "resisting moment" along the M'H axis is given by

$$-\frac{\left(\mathrm{EI}\sin^2\beta+\mathrm{G}j\cos^2\beta\right)\cos\varphi}{\rho}$$

For the given layer, the "resisting moment" $\mathfrak{M}$ acting parallel to the pulley axis is the sum of these components for the n wires of the layer.

In order to perform the summation, wires are indexed as shown in Fig. 7. Since angle $\alpha = 2i\pi/n$, for wire number i one gets:

$$\rho = \mathrm{R} + m\cos\frac{2i\pi}{n}$$

and:

$$\cos\beta = \sin\varphi\cos\frac{2i\pi}{n}$$

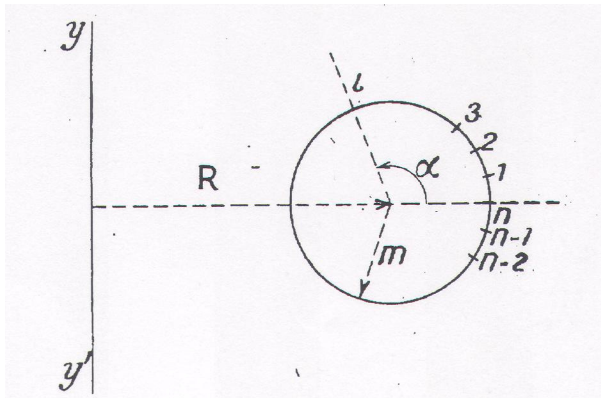

Figure 7

Thus, moment $\mathfrak{M}$ is the sum for all wires in given layer:

$$\mathfrak{M} = -\sum_{i=1}^{i=n}\cos\varphi\frac{\mathrm{EI}\left(1-\sin^2\varphi\cos^2\frac{2i\pi}{n}\right)+\mathrm{G}j\sin^2\varphi\cos^2\frac{2i\pi}{n}}{\mathrm{R}+m\cos\frac{2i\pi}{n}}$$

If the number of wires n is large enough, summation may be replaced by the following integral:

$$\mathfrak{M} = -\cos\varphi\int_0^n\frac{\mathrm{EI}\left(1-\sin^2\varphi\cos^2\frac{2i\pi}{n}\right)+\mathrm{G}j\sin^2\varphi\cos^2\frac{2i\pi}{n}}{\mathrm{R}+m\cos\frac{2i\pi}{n}}di$$

While an exact evaluation of this integral can easily be performed, it is not practically useful. Indeed, some approximations can be made. Layer radius m is always much smaller than pulley's radius R. Thus, both terms of the numerator being essentially positive, the denominator can be replaced by its average value R. The value for $\mathfrak{M}$ becomes (neglecting the minus sign, which is henceforth of no avail):

$$\mathfrak{M} = \frac{\cos\varphi}{\mathrm{R}}\int_0^n\left[\mathrm{EI}+(\mathrm{G}j-\mathrm{EI})\sin^2\varphi\cos^2\frac{2i\pi}{n}\right]di$$

Evaluation of this integral is straightforward:

$$\mathfrak{M} = \frac{n\cos\varphi}{\mathrm{R}}\left[\mathrm{EI}\left(1-\frac{\sin^2\varphi}{2}\right)+\mathrm{G}j\frac{\sin^2\varphi}{2}\right] \tag{1}$$

It is interesting to compare this result with Baticle's. Using the present notations, it yields:

$$\mathfrak{M} = \frac{n\cos\varphi}{\mathrm{R}}\left[\mathrm{EI}\frac{\cos^2\varphi}{2}+\mathrm{G}j\frac{\sin^2\varphi}{2}\right] \tag{2}$$

Eq. (1) differs from Eq. (2) by the single term $\frac{n\mathrm{EI}\cos\varphi}{2\mathrm{R}}$. Eq. (1) yields a larger value of $\mathfrak{M}$, which is not surprising. Indeed, from Baticle's third hypothesis (wires following the geodesics), those wires which correspond to $\alpha = \pi/2$ and $\alpha = 3\pi/2$ have their principal normal parallel to the pulley axis. Thus, they do not participate in the moment whose axis is parallel to this axis. On the contrary, the modified hypothesis includes these wire bending in the torus tangential plane.

The relative difference between both values is rather large. If wire section radius is ρ_i , (yielding $\mathrm{I} = \pi\rho_i^4/4$ and $j = \pi\rho_i^4/2$), and assuming with Baticle that $\mathrm{G} = 0.4\mathrm{E}$ their ratio is:

$$\frac{1+\cos^2\varphi+0.8\sin^2\varphi}{\cos^2\varphi+0.8\sin^2\varphi}$$

It is always larger than 2. For usual cable lays, with lay angles between 15° and 20°, it is not far from this limit value.

Turning back to Fig. 5, consider the bending and torsional moment components which are perpendicular to $y'y$ and in MM'H plane. An orientation is selected on this perpendicular in the direction mm' , where *m* and *m'* are the orthogonal projections of M and M' onto a plane perpendicular to $y'y$ (Fig. 6).

Their algebraic sum is:

$$\frac{\mathrm{EI}-\mathrm{G}j}{\rho}\cos\varphi\sin\beta\cos\beta$$

This vector has two components in the plane shown in Fig. 6. One is in the direction *ym* and the other on the axis *yz*. They are, respectively:

$$\frac{\mathrm{EI}-\mathrm{G}j}{\rho}\cos\varphi\sin\beta\cos\beta\cos\gamma = f$$

and:

$$\frac{\mathrm{EI}-\mathrm{G}j}{\rho}\cos\varphi\sin\beta\cos\beta\sin\gamma = t$$

in which angle γ is the angle between directions *yz* and *mm'*. It is defined by:

$$\cos\gamma = \sin\varphi\sin\frac{2i\pi}{n}$$

Recall, however, that $\cos\beta = \sin\varphi\cos\frac{2i\pi}{n}$. Hence ($\gamma$ being in the $[0\ \pi]$ interval, and, as seen above, $\sin\beta \geq 0$):

$$f = \frac{\mathrm{EI}-\mathrm{G}j}{\rho}\sqrt{1-\sin^2\varphi\cos^2\frac{2i\pi}{n}}\sin^2\varphi\cos\varphi\sin\frac{2i\pi}{n}\cos\frac{2i\pi}{n}$$

$$t = \frac{\mathrm{EI}-\mathrm{G}j}{\rho}\sqrt{1-\sin^2\varphi\cos^2\frac{2i\pi}{n}}\sqrt{1-\sin^2\varphi\sin^2\frac{2i\pi}{n}}\sin\varphi\cos\varphi\cos\frac{2i\pi}{n}$$

Since $\rho = \mathrm{R} + m\cos\frac{2i\pi}{n}$, sign of *f* changes when *i* is changed into $n-i$. Thus, summation over the *n* wires $\sum_{i=1}^{i=n} f_i = 0$. This means that the only bending moment component from a given layer is parallel to the pulley axis. Its magnitude is thus given by Eq. (1).

This is not the case for t . One might be tempted to replace ρ by radius R, as was done in the evaluation of $\mathfrak{M}$. With this simplification, when *i* is changed into $\left(\frac{n}{2}-i\right)$, sign of *t* also changes and summation over the n wires also yields $\sum_{i=1}^{i=n} t_i = 0$. However, here this approach is not valid because the sign of *t* is not uniform.

In order to evaluate expression $\sum_{i=1}^{i=n} t_i$, another simplification is made; *t* may be expressed as:

$$t=\frac{\mathrm{EI}-\mathrm{G}j}{R+m\cos\frac{2i\pi}{n}}\sqrt{1-\sin^2\varphi-\sin^4\varphi\sin^2\frac{2i\pi}{n}\cos^2\frac{2i\pi}{n}}\times\cos\frac{2i\pi}{n}\sin\varphi\cos\varphi$$

In the summation $\sum_{i=1}^{i=n} t_i$, the terms corresponding to i and $\left(\frac{n}{2}-i\right)$ may be combined. They add up to:

$$\left(\mathrm{EI}-\mathrm{G}j\right)\sqrt{1-\sin^2\varphi-\sin^4\varphi\sin^2\frac{2i\pi}{n}\cos^2\frac{2i\pi}{n}}\sin\varphi\cos\varphi$$
$$\times\left(\frac{\cos\frac{2i\pi}{n}}{\mathrm{R}+m\cos\frac{2i\pi}{n}}-\frac{\cos\frac{2i\pi}{n}}{\mathrm{R}-m\cos\frac{2i\pi}{n}}\right)$$

This expression always has the same sign. Indeed, layer radius m being smaller than R, the parenthesis is always negative. Besides, considering that:

$$\mathrm{EI}-\mathrm{G}j=\mathrm{E}\frac{\pi\rho_i^4}{4}-0.4\mathrm{E}\frac{\pi\rho_i^4}{2}=0.2\mathrm{E}\frac{\pi\rho_i^4}{4}>0$$

the expression is always negative.

Thus, $\sum_{i=1}^{i=n} t_i<0$. Besides, in order to evaluate this summation, one may assume that the square root reduces to $\sqrt{1-\sin^2\varphi}=\cos\varphi$. Indeed, lay angle φ is at most of the order of 20°, and the term $\sin^4\varphi\sin^2\frac{2i\pi}{n}\cos^2\frac{2i\pi}{n}$ is negligible compared to $\cos^2\varphi$.

The resultant torsional moment for the given layer may be expressed as:

$$\mathfrak{T}=\sum_{i=1}^{i=n}\left(\mathrm{EI}-\mathrm{G}j\right)\sin\varphi\cos^2\varphi\frac{\cos\frac{2i\pi}{n}}{\mathrm{R}+m\cos\frac{2i\pi}{n}}$$

Again, it is replaced by the integral:

$$\mathfrak{T}=\left(\mathrm{EI}-\mathrm{G}j\right)\sin\varphi\cos^2\varphi\int_0^n\frac{\cos\frac{2i\pi}{n}}{\mathrm{R}+m\cos\frac{2i\pi}{n}}di$$

After integration:

$$\mathfrak{T} = -\frac{nm}{R^2}(EI - Gj)\sin\varphi\cos^2\varphi \qquad (3)$$

Thus, the bent layer generates a torsional moment $\mathfrak{T}$. The minus sign indicates that cross-section *m'* (Fig. 6) exerts a couple on the part above. Its axis is in the *zy* direction. It tightens the helical lay.

This result is of practical interest. Indeed, when a cable comes into contact with a pulley, it is a well-known fact that it has a tendency to rotate in the direction shown in Fig. 8.

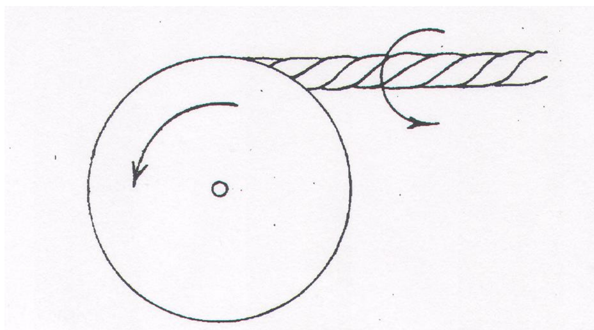

Figure 8

This can explain the severe wear experienced by some cables such as in ski-lifts. In order to mitigate this effect, the previous calculation provides indications on the cable lay. It must be such that the sum of the $nm(EI - Gj)\sin\varphi\cos^2\varphi$ for all layers vanish. In each term, lay angle φ should be taken as positive or negative, depending on the direction of the lay.

However, one should notice that the absolute value of the ratio $\mathfrak{T}/\mathfrak{M}$ is quite small. It is given by:

$$\frac{\mathfrak{T}}{\mathfrak{M}} = -\frac{m\sin\varphi\cos\varphi}{R} \times \frac{EI - Gj}{EI\left(1 - \frac{1}{2}\sin^2\varphi\right) + Gj\frac{1}{2}\sin^2\varphi}$$

This ratio is always small, a result of the smallness of ratio (m/R) .

Thus, in the sequel, torsional moment $\mathfrak{T}$ will in general be neglected and the following rule is adopted:

When the cable is imposed a radius R, a particular layer may be considered as a solid beam with equivalent bending stiffness $E'I'$ given by:

$$\mathrm{E'I'} = n\cos\varphi\left[\mathrm{EI}\left(1-\frac{1}{2}\sin^2\varphi\right)+\mathrm{G}j\frac{1}{2}\sin^2\varphi\right]$$

This is based on the assumption that each wire in the layer contributes only through bending and torsional moments. Each wire contribution is the expression enclosed in the square brackets. It applies to circular cross-section wires. In the non-circular case, one should use equivalent circular sections.

This rule may be extended to all layers in a single strand cable, with the following remark.

Consider a system of *n* parallel independent, frictionless, straight wires. The system bending stiffness is $n\mathrm{EI}$. Letting $\mathrm{G} = 0.4\mathrm{E}$ and $j = 2\mathrm{I}$ (which applies to circular cross-section wires) in $\mathrm{E'I'}$, one gets $\mathrm{E'I'} = n\mathrm{EI}\cos\varphi\left(1-0.1\sin^2\varphi\right)$. While $\mathrm{E'I'}/n\mathrm{EI} < 1$, this ratio is quite close to unity. Take, for example $\sin^2\varphi = 0.1$, a typical practical value, $\mathrm{E'I'}/n\mathrm{EI} \simeq 0.94$.

Thus, a layer bending stiffness may be closely approximated by *n* parallel independent, frictionless, identical straight wires.

This valid simple approach has been used by various authors, particularly German authors, whose contributions are mentioned in the following.

IV. Multi-strand cables

The rule found in the previous section may be applied to a multi-strand cable, as each strand may be replaced with an equivalent wire. However, this applies to the bending stiffness $\mathrm{E'I'}$, not to the torsional stiffness $\mathrm{G'}j'$, which enters in the cable bending stiffness. It is derived below.

Obviously, the cable torsional stiffness comes from both the bending and twisting of its wires, and from the variation in wire tensile forces, arising from their lay angle.

Here, for the sake of consistency, since wire forces have not been considered, the latter effect is neglected.

Consider a cable element, length *dl*, and a corresponding wire element, length $dl/\cos\varphi$, making an angle φ with cable axis.

Elementary rotation of cable is $\alpha\, dl$, which has two components: one rotation around wire axis, $\alpha\, dl\cos\varphi$, the other rotation perpendicular to wire axis $\alpha\, dl\sin\varphi$. The former corresponds to wire torsion, with the corresponding moment:

$$\mathrm{G}j\times\frac{\alpha\, dl\cos\varphi}{dl/\cos\varphi} = \alpha\mathrm{G}j\cos^2\varphi$$

whose axis parallel to cable axis is $\alpha \mathrm{G} j \cos^3 \varphi$. The other component corresponds to wire bending, with a moment:

$$\mathrm{EI} \times \frac{\alpha\, dl \sin\varphi}{dl/\cos\varphi} = \alpha \mathrm{EI} \sin\varphi \cos\varphi$$

whose projection on cable axis is $\alpha \mathrm{EI} \sin^2 \varphi \cos\varphi$.

Wire torsional stiffness within the cable is thus:

$$\left(\mathrm{EI} \sin^2 \varphi + \mathrm{G} j \cos^2 \varphi\right) \cos\varphi$$

The cable torsional stiffness is the sum:

$$\mathrm{G}' j' = \sum \left(\mathrm{EI} \sin^2 \varphi + \mathrm{G} j \cos^2 \varphi\right) \cos\varphi$$

Finally, for a given cable going over a pulley with radius R, it should not be too difficult to determine the bending moment $\mathfrak{M}$ and, possibly, the torsional moment $\mathfrak{T}$. The complementary stresses on any of the individual wires can also be obtained.

Writing down the corresponding expressions would be too tedious. Instead, numerical applications will be given at the end of this work.

V. A study of wire axial force and displacement

Obviously, the cable is not a solid, and when a wire axis undergoes some deformation there may occur some slip between neighbouring wires. Such slip depends on a wire position within the cable cross-section leading to relative axial motion between wires. Assuming frictionless contact, such motion would simply be such as to equalize the tensile force in any given wire over its length. Cable resistance to bending would be restricted to moments $\mathfrak{M}$ and $\mathfrak{T}$. With inter-wire friction, displacements are impeded yielding a variation of tensile force along a given wire. A study of this variation is presented in the sequel.

In fact, there are already several contributions on this subject. The following paper by D[r] Ing[r] Hellmut Ernst is particularly noteworthy: *Beitrag zur Beurteilung der behördlichen Vorschriften für die Seilen von Personenschwebebahnen.* That is: "A contribution to the study of rules and regulations as applied to people-carrying ropeways". The paper was published in the 1933-34 volume of *Fördertechnik und Frachtverkher*, a technical journal, editor A. Ziemsen, Wittenberg, Germany.

This paper contributes a number of interesting results, which shall be referred to in the following. There are however several hypotheses and simplifications, either implicit or explicit, which are not fully justified, leading to some questionable results.

Recently, a new approach has been suggested by Mr A. Rebuffel in an unpublished note. He was kind enough to let this Author report it.

The following is based on Rebuffel's approach, with a difference on one important point. While his results are exact for all practical purpose, there seems to be a flaw, at least theoretical, in his development.

However, before embarking into this analysis, one preliminary question must be settled: the problem of axial load sharing between wires in a straight loaded cable. Indeed, in order to determine friction forces in cable bending, these wire forces are supposed to be known. Hence, they should be determined beforehand.

INDIVIDUAL WIRE LOAD SHARING IN A TAUT CABLE

As in strength of materials theory, cross-sections in a cable under axial load are assumed to remain plane and normal to cable axis. This hypothesis holds except for sections located in the immediate neighbourhood of the cable ends.

Using this hypothesis, it is easy to deduce the tensile force distribution between wires, provided, of course, that all cable material stays in the elastic domain.

As before, in the bending case, it is assumed that the cable cross-section does not vary.

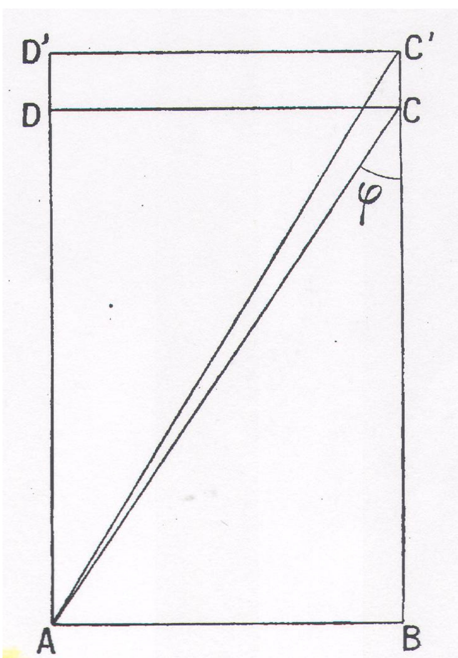

Figure 9

First, in a straight unloaded single-strand cable, consider a given layer, radius m , lay angle φ.

Consider a cylinder, length p, the wire pitch length on this layer, cut it open along one generator and lay it flat, yielding a rectangle ABCD (Fig. 9), base $2\pi m$ and height p. Diagonal AC corresponds to one particular wire in that layer.

Under an axial load on the cable, rectangle undergoes a relative stretch CC'/BC which is the cable relative stretch, or strain dL/L .

From Fig. 9, one sees that the corresponding wire strain is:

$$\frac{CC'\cos\varphi}{AC} = \frac{CC'\cos^2\varphi}{AC\cos\varphi} = \frac{CC'\cos^2\varphi}{BC}$$

Hence:

$$\frac{CC'\cos\varphi}{AC} = \frac{dL}{L}\cos^2\varphi$$

Calling s the area of a wire cross-section, and E its material Young's modulus, the wire tensile force is $\mathrm{E}s\frac{\mathrm{dL}}{\mathrm{L}}\cos^2\varphi$.

The wire tensile force may be considered as the resultant of two perpendicular components, one in the cable axis direction, the other, in a plane normal to the cable axis. Thus, the axial component is $\mathrm{E}s\frac{\mathrm{dL}}{\mathrm{L}}\cos^3\varphi$, while the cross-section component is $\mathrm{E}s\frac{\mathrm{dL}}{\mathrm{L}}\cos^2\varphi\sin\varphi$. This latter component, because of the symmetry of the same layer wires, yields a torsional moment on the cable of $m\mathrm{E}s\frac{\mathrm{dL}}{\mathrm{L}}\cos^2\varphi\sin\varphi$.

The resultant of all axial components is the axial load on the cable, that is:

$$\mathrm{T}=\frac{\mathrm{dL}}{\mathrm{L}}\sum \mathrm{E}s\cos^3\varphi \tag{4}$$

And the resultant torsional moment on the cable is:

$$\mathrm{M}=\frac{\mathrm{dL}}{\mathrm{L}}\sum m\mathrm{E}s\cos^2\varphi\sin\varphi \tag{5}$$

If $\sum m\mathrm{E}s\cos^2\varphi\sin\varphi=0$ the cable is said to be anti-rotating.

It is interesting to note that this condition does not coincide exactly with the condition seen above that the cable does not rotate while passing over a pulley. With the current notations, this condition is:

$$\sum m(\mathrm{EI}-\mathrm{G}j)\cos^2\varphi\sin\varphi=0$$

In fact, these conditions coincide with cables having identical round wires, and same material in a given layer. If material properties are such that $\mathrm{G}=0.4\mathrm{E}$ one gets:

$$\mathrm{EI}-\mathrm{G}j=\frac{\mathrm{E}\pi\rho^4}{4}-0.4\mathrm{E}\frac{\pi\rho^4}{2}=\frac{\mathrm{E}\pi\rho^2}{2}\left(\frac{\rho^2}{2}-0.8\frac{\rho^2}{2}\right)=\frac{\rho^2}{10}\times\mathrm{E}s$$

In the second sum, $\frac{\rho^2}{10}$ can be factored out, and both conditions are identical.

Return to Eq. (4). It yields:

$$\frac{\mathrm{dL}}{\mathrm{L}} = \frac{\mathrm{T}}{\sum \mathrm{E}_i s_i \cos^3 \varphi_i}$$

(Here, index i is shown on parameters E, s and φ, and characterizes any given wire in the cable).

Thus, the tensile stress in any wire whose index is n is:

$$r_n = \frac{\mathrm{TE}_n \cos^2 \varphi_n}{\sum \mathrm{E}_i s_i \cos^3 \varphi_i} \tag{5}$$

and the corresponding tensile force is:

$$\mathrm{T}\frac{\mathrm{E}_n s_n \cos^2 \varphi_n}{\sum \mathrm{E}_i s_i \cos^3 \varphi_i}$$

In the case of a multistrand cable, tensile stress in any wire can be determined from Eq. 4. Indeed, each single strand may be considered as being equivalent to a single wire whose product Es is given by $\mathrm{E}s = \sum \mathrm{E}_i s_i \cos^3 \varphi_i$.

Call s_n the wire cross-section area, E_n its Young's modulus, φ_n its lay angle within the corresponding strand and ψ_n the strand lay angle within the cable ($\psi_n = 0$ for the core strand).

The cable being subjected to a tensile force T, a strand strain is $\frac{\mathrm{dL}}{\mathrm{L}} \cos^2 \psi_n$. A wire strain is $\frac{\mathrm{dL}}{\mathrm{L}} \cos^2 \psi_n \cos^2 \varphi_n$.

The wire tensile force is $\mathrm{E}_n s_n \frac{\mathrm{dL}}{\mathrm{L}} \cos^2 \psi_n \cos^2 \varphi_n$. It is its axial component $\mathrm{E}_n s_n \frac{\mathrm{dL}}{\mathrm{L}} \cos^2 \psi_n \cos^3 \varphi_n$ which gives rise to the strand tensile force θ , and the cable tensile force is itself the resultant of axial components $\theta \cos \psi_n$. Thus:

$$\mathrm{T} = \frac{\mathrm{dL}}{\mathrm{L}} \sum \mathrm{E}_i s_i \cos^3 \psi_i \cos^3 \varphi_i$$

Finally, the resulting wire stress is:

$$r_n = \frac{\mathrm{TE}_n \cos^2 \psi_n \cos^2 \varphi_n}{\sum \mathrm{E}_i s_i \cos^3 \psi_i \cos^3 \varphi_i} \tag{6}$$

The wire axial force is given by $r_n s_n$ which is the sought for result in this study. In fact, combined with some experimental data, it was the basis to the formula given in the 15 October 1947 regulations, which gives an *a priori* value of a cable strength.

VI. Rebuffel's Theory

Rebuffel considers a straight single-strand cable, under axial load. Initially, there are no friction forces between wires. Then, at a given section, a monotonously increasing curvature is imposed, starting at 0 and up to $1/\mathrm{R}$. One seeks to determine the cable resistance to bending in the circular region, without consideration of cable end conditions.

While these hypotheses restrict the results validity, they are necessary. It will be shown how these results are affected when some of them are not satisfied.

First, recall that cable bending resistance includes a component noted $\mathfrak{M}$ which comes from the wire elastic stiffness in bending and torsion. This term expression has been given above.

A second component noted $\mathfrak{M}'$ arises from each wire tensile force. It will be derived in the sequel.

1°) Firstly, assume that $1/\mathrm{R}$ is small enough so that, for a given axial force on the cable, it behaves as a solid beam; the axial force level controls the pressure between wires, and it is related to the limit tangential forces at which slip may occur.

Under the solid beam assumption, wire forces are easily obtained.

In Fig. 1, distance of point M from cable section neutral axis is $m \cos\alpha$. Thus, bending strain on the torus parallel line through M is $m \cos\alpha / \mathrm{R}$.

Using the same argument as in the case of the straight cable under axial load, the corresponding tensile stress on a wire cross-section is[1] $\mathrm{E} m \cos\alpha \cos^2\varphi / \mathrm{R}$. This stress results from a tensile force whose component parallel to the cable axis is $\mathrm{E} ms \cos\alpha \cos^2\varphi / \mathrm{R}$. It is the only component contributing to moment $\mathfrak{M}'$, which is then given by:

$$\mathfrak{M}' = \frac{1}{\mathrm{R}} \sum \mathrm{E}_i s_i m_i^2 \cos^2\alpha_i \cos^3\varphi_i$$

[1] Under the assumption that wire lay angle variation due to bending is negligible. This is reasonable for small curvature.

where the summation extends to all wires in the cable. In fact, it can easily be expressed layer by layer. Indeed, in a given layer, in general, parameters $E_i\ s_i\ m_i$ and φ_i are constants. Besides, the number of wires n_j in layer *j* is such that $\sum \cos^2 \alpha_i = n_j/2$.

Thus, finally:

$$\mathfrak{M}' = \frac{1}{2R} \sum E_j S_j m_j^2 \cos^3 \varphi_j \qquad (7)$$

In this equation, summation extends to all layers *j*, and S_j is the sum of wire cross-section areas in layer *j*. Also, the core wire, if any, corresponds to a null radius layer, and does not enter in the summation.

2°) Now, curvature $1/R$ is supposed to increase. The question is to determine for which value of $1/R$ impending slip between wires will occur.

At point M (Fig. 1), consider wire element *dl*. It may be expressed as $dl = m d\alpha / \sin \varphi$

In the stick phase, before any slip occurs at M, this element must undergo a tangential force equal to the difference between tensile forces at both ends of the element:

$$\frac{d}{d\alpha} \frac{Ems \cos\alpha \cos^2 \varphi}{R} d\alpha = -\frac{Ems \cos^2 \varphi}{R} \sin\alpha \, d\alpha$$

Impending slip occurs at that point when its absolute value reaches the friction limit.

In order to determine this limit value, first consider the outer layer case, numbered layer 1.

Friction forces on the element may come from contact:

a) With the adjacent layer (layer 2)
b) With neighbouring wires in layer 1

Rebuffel, as well as Ernst, assume that friction from contact (b) is negligible. Ernst gives the following justification. Except for the 6 wire innermost layer, in contact with the core wire, identical to those 6 wires, other layers are laid in such a way that their wires are just touching, with little contact pressure. When a tensile force is applied on the cable (Fig. 9), their lay angle decreases. In the cable cross-section, individual wire cross-sections, which are practically ellipses, see a decrease of their semi-major axis (which tends to their own circular cross-section). Thus, the axial load results in loss of contact between same layer wires.

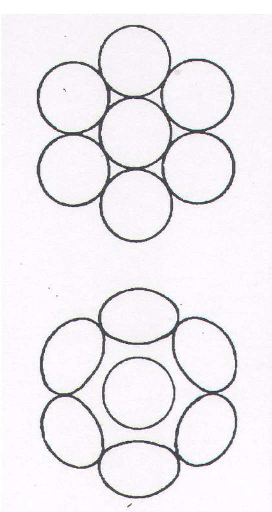

Figure 10

The case of the 6 wire inner layer must be dealt with separately (Fig. 10), assuming that all 7 wires (layer and core) have the same diameter. Indeed, assuming a zero lay angle (parallel wires), they would just touch each other, as shown in the upper drawing of Fig. 10. If the 6 wire layer is laid with a non-zero lay angle, one gets the lower drawing situation, with a loss of contact between core and layer wires, and with non-negligible contact forces within the layer. Applying an axial load on the cable will bring a decrease in these contact forces (which will still persist).

While this explanation is plausible, it may be discussed as practical observations do not seem to corroborate these findings. At any rate, it does seem that friction forces (b) are much smaller than forces (a). Here, only the latter will be considered.

Further observations have to be made concerning forces (a). It is not guaranteed *a priori* that the adjacent layer is itself static while slip is starting in layer 1. However, calculations tend to confirm this hypothesis.

Thus, the wire element is considered to be slipping on the static adjacent layer, with friction coefficient μ .

The limit friction force which can be applied on the element up to impending slip is found in the following way.

Tensile force on the element is θ. Normal forces on the two ends make an angle $\sin\varphi\, d\alpha$. The corresponding resultant on the adjacent layer is $\theta \sin\varphi\, d\alpha$.

The limit friction force is $\mu\theta \sin\varphi\, d\alpha$. The stick condition is:

$$\frac{\mathrm{E}ms \sin\alpha \cos^2\varphi}{\mathrm{R}} \leq \mu\theta \sin\varphi$$

where $\sin\alpha$ is taken as positive, that is, the domain for α is $[0\ \pi]$ corresponding to one half-loop of the wire around the cable axis.

However, θ is itself the sum of two components. One is t arising from the axial load on cable. The other component is due to cable bending. Component t has already been obtained: $t = \mathrm{T}_j / n\cos\varphi$, in which T_j is the total axial load on layer j, and n is the corresponding number of wires. The second component has also been calculated, and is expressed as $\mathrm{E}ms \cos\alpha \cos^2\varphi / \mathrm{R}$. Thus, the above stick condition becomes:

$$\frac{\mathrm{E}ms \sin\alpha \cos^2\varphi}{\mathrm{R}} \leq \mu\left(\frac{\mathrm{T}_j}{n\cos\varphi} + \frac{\mathrm{E}ms \cos\alpha \cos^2\varphi}{\mathrm{R}}\right)\sin\varphi$$

yielding:

$$\mathrm{E}ms \cos^2\varphi\left(\sin\alpha - \mu\cos\alpha\sin\varphi\right) \leq \frac{\mathrm{R}\mu\mathrm{T}_j \sin\varphi}{n\cos\varphi} \tag{8}$$

In this inequality, α is now considered as the unknown. Call R_0 the value of R corresponding to impending slip. Slip occurs for values of α where the inequality becomes an equality. Obviously, these values maximize expression $(\sin\alpha - \mu\cos\alpha\sin\varphi)$

These maxima correspond to a vanishing derivative $(\cos\alpha + \mu\sin\alpha\sin\varphi)$, that is for $\tan\alpha_0 = -1/\mu\sin\varphi$. At such value, it passes from a positive value to a negative one.

Slip starts at a wire section defined by angle α_0 . For all wires of that layer, all impending slip points are located on the same torus parallel line (Fig. 1), whose radius d is smaller than R (as angle α_0 is greater than $\pi/2$). By symmetry, the same result is obtained on the other half-loop, for α in the $[\pi\ 2\pi]$ interval, where impending slip occurs on a parallel circle having the same radius d.

Slip starts on two parallel lines of the torus corresponding to the bent cable outer layer. These parallel lines are circles whose radius is smaller than the cable radius of curvature. That is, they are in the concave side of the curved cable axis.

Making $\alpha = \alpha_0$ (again in the interval $[0\ \pi]$) one gets:

$$\sin\alpha_0 = \frac{1/\mu\sin\varphi}{\sqrt{1+\dfrac{1}{\mu^2\sin^2\varphi}}} = \frac{1}{\sqrt{1+\mu^2\sin^2\varphi}}$$

and:

$$\cos\alpha_0 = -\frac{\mu\sin\varphi}{\sqrt{1+\mu^2\sin^2\varphi}}$$

yielding:

$$\sin\alpha_0 - \mu\cos\alpha_0\sin\varphi = \frac{1+\mu^2\sin^2\varphi}{\sqrt{1+\mu^2\sin^2\varphi}} = \sqrt{1+\mu^2\sin^2\varphi}$$

Radius of curvature R_0 for impending slip is given by:

$$R_0 = \frac{E mns\cos^3\varphi\sqrt{1+\mu^2\sin^2\varphi}}{\mu T_j\sin\varphi}$$

Noting that $T_j/ns\cos\varphi$ is the tensile stress r_1 on outer layer wires (Eq. 5), another expression for R_0 is:

$$R_0 = \frac{E m_1\cos^2\varphi_1\sqrt{1+\mu^2\sin^2\varphi_1}}{\mu r_1\sin\varphi_1} \tag{9}$$

3°) Assume now that curvature $1/R$ increases beyond $1/R_0$, and assume again that layer 2 is still in the stick phase.

Obviously, in the $[0 < \alpha < \pi]$ interval (that is, in the half-loop being considered) the slip region, which started at point M_0, polar angle α_0 , will propagate on both sides of this point.

Now, at this point of his development, Rebuffel assumes that the slip zone boundaries will reach simultaneously, for a certain radius R, the upper and lower points A and B of the half-loop, which correspond to $\alpha = 0$ and $\alpha = \pi$.

This is a mere hypothesis, and we intend to prove that this is not the case.

Consider a wire half-loop AB represented by a straight line AB (Fig. 11). Again, it is subjected to a tensile force *t* , arising from the axial load T on the straight cable. Call Ω the middle point and *2l* its length. The origin is taken at point Ω. The positive direction is from Ω to B.

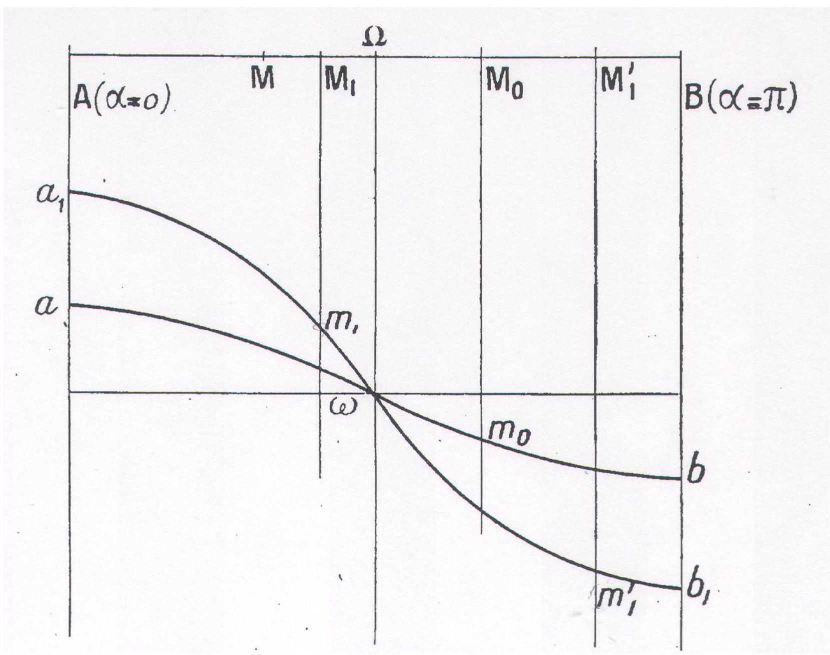

Figure 11

Assume the cable is bent into a circular shape, its radius R being large enough so that no slip occurs. Consider a point M on a wire half-loop, its abscissa being $x = \frac{2l}{\pi}\left(\alpha - \frac{\pi}{2}\right)$. It has been shown that the wire relative elongation (strain) is given by:

$$y = \frac{m \cos\alpha \cos^2\varphi}{R} = -\frac{m\cos^2\varphi}{R}\sin\frac{\pi x}{2l}$$

In Fig. 11, this relative elongation is shown as a sine curve $a\omega b$. It corresponds to a differential normal force on a wire element dx given by:

$$\mathrm{E}s\frac{dy}{dx}dx = -\frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}}\times\frac{\pi}{2l}\cos\frac{\pi x}{2l}$$

which must be balanced by an equal and opposite friction force.

Assume now that radius R reaches a value R_1 slightly smaller than R_0, the impending slip radius. This means that slip has begun at point M_0, which has already been determined, and the slip region has reached points M_1 and M_1' . In Fig. 11, these two points have abscissas x_1 and x_1' which are functions of R_1. They are derived as follows.

First, notice that sine curve $a_1\omega b_1$ which represents function $\mathrm{Y}_1 = -\frac{m\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x}{2l}$ still represents wire relative elongation between points a and m_1 as well as between points m_1' and b , as these regions are still in the stick regime. Thus, wire tensile force at point M_1 is given by:

$$\theta_1 = t - \frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x_1}{2l} \qquad (10)$$

And at point M_1' :

$$\theta_1' = t - \frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x_1'}{2l} \qquad (11)$$

Now, what is the expression for the tensile force between points M_1 and M_1' ? That region is in the slip regime. A wire element dx is subjected to a tensile force θ. Hence the normal force on the adjacent layer arising from that element is $\theta\sin\varphi\, d\alpha$ or else $\theta\frac{\pi}{2l}\sin\varphi\, dx$. Because of slip, there is a corresponding friction tangential force $\frac{\mu\theta\pi}{2l}\sin\varphi\, dx$. Thus, there is a variation in the wire tensile force given by:

$$d\theta = -\frac{\mu\theta\pi}{2l}\sin\varphi\, dx$$

where a minus sign takes into account the fact that tensile force θ decreases from A to B. This differential equation is readily integrated yielding, k being a constant:

$$\theta = ke^{-\frac{\mu\pi}{2l}x\sin\varphi}$$

Combining this equation with Eqs (10) and (11), one gets a first relationship between x_1 x_1' and R_1 :

$$e^{\frac{\mu\pi}{2l}(x_1 - x_1')\sin\varphi} = \frac{t - \dfrac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}\sin\dfrac{\pi x_1'}{2l}}{t - \dfrac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}\sin\dfrac{\pi x_1}{2l}} \tag{12}$$

Another equation is needed. It is obtained from the following geometrical condition. From symmetry considerations, when cable is given a circular shape, the ends A and B of the half-loop must not move with respect to the adjacent layer. In other words, the length AB remains constant in the bending process. Thus, calling Y(x) the relative elongation function, this condition is written:

$$\int_{-l}^{+l} \mathrm{Y}dx = 0$$

However, in the stick domains $-l \le x \le x_1$ and $x_1' \le x \le +l$, function Y is given by:

$$y_1 = -\frac{m\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x}{2l}$$

In the slip domain $x_1 \le x \le x_1'$, function Y is :

$$\frac{\theta - t}{\mathrm{E}s} = \frac{ke^{-\frac{\mu\pi}{2l}x\sin\varphi} - t}{\mathrm{E}s}$$

Besides, constant *k* must satisfy the condition that wire tensile force must be continuous at point M_1, coming from the stick side and from the slip side:

$$t - \frac{Ems\cos^2\varphi}{R_1}\sin\frac{\pi x_1}{2l} = ke^{-\frac{\mu\pi}{2l}x_1\sin\varphi}$$

which yields:

$$k = e^{\frac{\mu\pi}{2l}x_1 \sin\varphi}\left(t - \frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x_1}{2l}\right)$$

Thus, the constant length condition becomes:

$$0 = \int_{-l}^{x_1} -\frac{m\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x}{2l}dx + \int_{x_1'}^{l} -\frac{m\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x}{2l}dx$$
$$+\frac{1}{\mathrm{E}s}\int_{x_1}^{x_1'}\left[e^{\frac{\mu\pi}{2l}(x_1 - x)\sin\varphi}\left(t - \frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}\sin\frac{\pi x_1}{2l}\right) - t\right]dx$$

Using Eq. (12), and after integration, it yields:

$$\begin{aligned} &\frac{2ml\cos^2\varphi}{\pi\mathrm{R}_1}\left(\cos\frac{\pi x_1}{2l} - \cos\frac{\pi x_1'}{2l}\right) + \frac{t}{\mathrm{E}s}\left(x_1 - x_1'\right) \\ &-\frac{2ml\cos^2\varphi}{\mu\pi\mathrm{R}_1\sin\varphi}\left(\sin\frac{\pi x_1}{2l} - \sin\frac{\pi x_1'}{2l}\right) = 0 \end{aligned} \tag{13}$$

It is easy to verify that in Eqs (12) and (13) one cannot have simultaneously $x_1' = l$ and $x_1 = -l$. Indeed, assuming this were the case, Eq. (13) would yield:

$$\frac{2ml\cos^2\varphi}{\mu\pi\mathrm{R}_1\sin\varphi}\times 2 = \frac{2lt}{\mathrm{E}s}$$

yielding the condition:

$$\frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1} = \frac{\mu\pi t\sin\varphi}{2}$$

With this condition, Eq. (12) becomes:

$$e^{-\mu\pi\sin\varphi} = \frac{2 - \mu\pi\sin\varphi}{2 + \mu\pi\sin\varphi}$$

Obviously, one solution is $\mu\pi \sin\varphi = 0$. It is easily checked that there is no corresponding positive solution. Thus, the slip boundaries cannot reach A and B simultaneously.

This result may also be obtained graphically using Fig. 12 , which is similar to Fig. 11. In Fig. 12, curve *ab* represents wire tensile force θ as slip reaches A and B simultaneously. Ordinates are taken with respect to axis O*x*. Horizontal line $t't''$ represents wire uniform tension *t* before cable bending.

From Eqs (10) and (11), one must have $t'a = -t''b$ while Eq. (12) shows that curve *ab* is an exponential with upwards concavity.

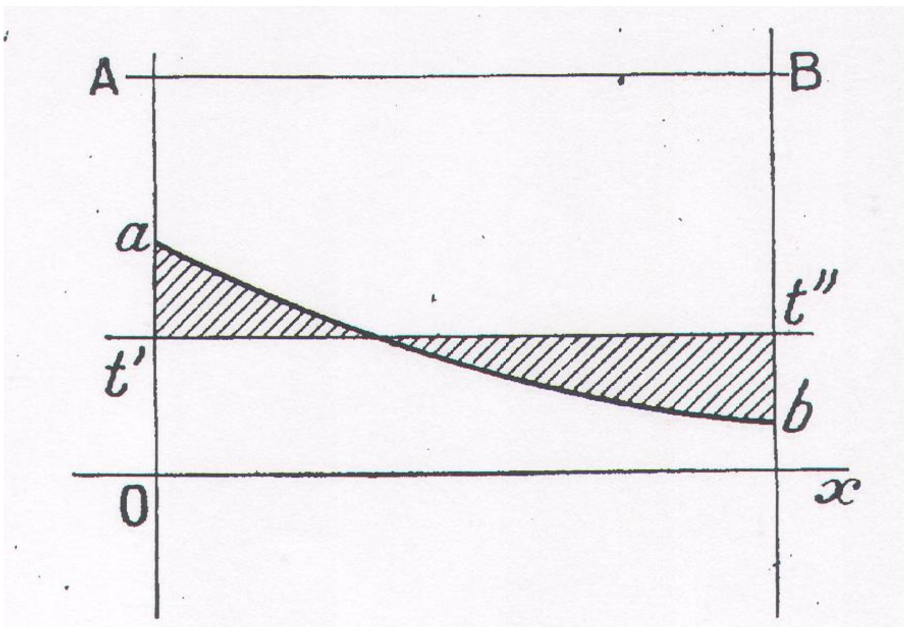

Figure 12

Wire relative elongation *y* generated by cable bending are proportional to $\theta - t$. The constant length condition of Eq. (13) requires that the shaded area in Fig. 12 be equal to zero or, equivalently, the absolute values of the upper and lower areas must be equal. It is obviously not so, because of the exponential curve shape.

It is also found that slip will first reach point B (Fig. 13). Indeed, in order to have equal areas above and below line $t't''$, part of curve *acb* which represents $\theta - t$ should have a downward concavity . This downward concavity region has to occur near point A, not at point B. It must be the sine curve corresponding to the stick domain.

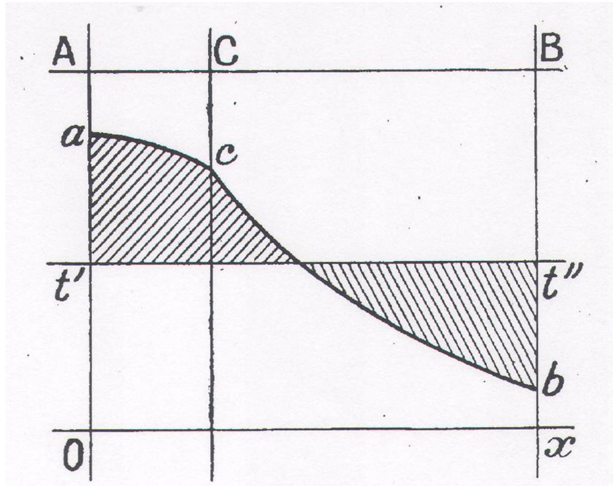

Figure 13

This result was to be expected. Indeed, it has been found that slip starts at a point which is closer to B than to A. Besides, wire tensile force being smaller on the left, pressure is smaller, as well as corresponding friction forces. Hence, slip domain boundary must propagate faster to the right than to the left[2]

Rebuffel's hypothesis is thus to be discarded.

However, before proceeding with the implications of this result, the rest of his development should be given.

By assuming that slip boundaries M_1 and M_1' simultaneously reach points A and B, there is just one equation to determine the radius of curvature R_1 at which the half-loop is completely in the slip regime.

Using Eq. (12), and making $x_1' = -x_1 = l$, he gets:

---

[2] One must be careful not to make the following mistake when determining points M_1 and M_1' bounding the slip region when curvature $1/R_1$ is greater than impending slip curvature $1/R_0$ It is indeed tempting to use the following argument. When slip is impending at points M_1 and M_1', it is also impending at points M_2 and M_2' which are arbitrarily close to M_1 and M_1'. Hence, elements M_1M_2 and $M_1'M_2'$ are themselves at the slip limit. The corresponding condition is:

$$Ems\cos^2\varphi\left(\sin\alpha - \mu\cos\alpha\sin\varphi\right) = \frac{R_1\mu T_j\sin\varphi}{n\cos\varphi}$$

This equation could be solved to get α_1 and α_1' corresponding to points M_1 and M_1'. In fact , the error is to assume that elements M_1M_2 and $M_1'M_2'$ are at the point of impending slip. They are under that limit. Forces on these elements are proportional to the slope of the sine curve which represents relative elongation in the stick region. Instead, friction forces are proportional to the slope at that point of exponential curve, which represents the relative elongation in the slip region. And this slope is larger (in absolute value), than the previous one. These slopes coincide only when slip starts, the exponential curve being a point.

$$R_1 = \frac{Ems\cos^2\varphi}{t} \times \frac{e^{\mu\pi\sin\varphi}+1}{e^{\mu\pi\sin\varphi}-1} = \frac{Em\cos^2\varphi}{r} \times \frac{e^{\mu\pi\sin\varphi}+1}{e^{\mu\pi\sin\varphi}-1}$$

in which *r* is the wire tensile stress, under cable axial load T.

Now, recall equation $\theta = ke^{-\frac{\mu\pi}{2l}x\sin\varphi}$ which gives wire tensile force within the slip region. It will be used to determine bending moment m' arising from friction as slip is complete over the half-loop. At point A:

$$\theta_1 = \left(r + \frac{Em\cos^2\varphi}{R_1}\right) \times s = ke^{\frac{\mu\pi}{2}\sin\varphi}$$

and at any point between A and B:

$$\theta = ke^{-\frac{\mu\pi}{2l}x\sin\varphi}$$

Hence:

$$\theta = s\left(r + \frac{Em\cos^2\varphi}{R_1}\right)\frac{e^{-\frac{\mu\pi}{2l}x\sin\varphi}}{e^{\frac{\mu\pi}{2l}\sin\varphi}} = sr\frac{2e^{\frac{\mu\pi}{2}\sin\varphi}}{1+e^{\mu\pi\sin\varphi}}e^{-\frac{\mu\pi}{2l}x\sin\varphi}$$

And, recalling that $x = \frac{2l}{\pi}\left(\alpha - \frac{\pi}{2}\right)$:

$$\theta = sr\frac{2e^{\frac{\mu\pi}{2}\sin\varphi}}{1+e^{\mu\pi\sin\varphi}}e^{-\mu\left(\alpha-\frac{\pi}{2}\right)\sin\varphi}$$

Yielding:

$$\theta = \frac{2rse^{\mu\pi\sin\varphi}}{1+e^{\mu\pi\sin\varphi}} \times e^{-\mu\alpha\sin\varphi}$$

This wire tensile force component parallel to cable axis is $\theta\cos\varphi$, with a moment with respect to the layer neutral axis of $m\cos\alpha \times \theta\cos\varphi$.

It should be remembered that this expression is valid only for the half-loop, between A and B, i.e. for $0 \le \alpha \le \pi$. In order to get moment m' , one must take the sum of the $m\theta\cos\alpha\cos\varphi$ over one half of the circle, radius *m*, with *α* varying on the corresponding interval, then double the result, as both halves of the layer must yield the same moment. Thus:

$$m' = \frac{4mrs\cos\varphi\, e^{\mu\pi\sin\varphi}}{1+e^{\mu\pi\sin\varphi}} \sum e^{-\mu\alpha\sin\varphi}\cos\alpha$$

When the number n of wires in the layer is large enough, summation $\sum$ may be replaced by an integral:

$$\frac{n}{2\pi}\int_0^\pi e^{-\mu\alpha\sin\varphi}\cos\alpha\, d\alpha$$

After two integrations by part, one gets:

$$\frac{n\mu\sin\varphi}{2\pi} \times \frac{1+e^{-\mu\pi\sin\varphi}}{1+\mu^2\sin^2\varphi}$$

And, finally:

$$m' = \frac{\mu mnsr\sin 2\varphi}{\pi\left(1+\mu^2\sin^2\varphi\right)} \tag{15}$$

4°) Rebuffel assumes that when curvature $1/\mathrm{R}$ is increased beyond the full-slip value $1/\mathrm{R}_1$ (which corresponds to A and B being the boundaries of the slip region), moment m' from that layer will remain constant, and equal to the value given by Eq. (15).

This hypothesis has already been made by several authors, and it looks logical. Indeed, when $1/\mathrm{R}$ goes beyond $1/\mathrm{R}_1$, the half-loop sections will continue to move longitudinally. However, because of symmetry, points A and B cannot move, and it seems that tensile force at A will remain constant. Indeed, it cannot increase because there is no more rigid connection with the adjacent layer. Thus, an increase in cable curvature does not generate any increase in wire relative elongation. And, if there were any tendency to decrease that strain, it would stick and tension value would be re-established. If the tensile force remains constant at A, at any other point on the half-loop, the exponential function which gives the local tensile force will also remain unchanged.

5°) After studying the outer layer, Rebuffel considers inner layer behavior. He assumes that slip phases are sequential, that is, a layer complete slip occurs before slip starts in the next adjacent layer.

He makes the assumption that the only friction forces acting on a layer are those arising from its own tensile forces and the corresponding normal forces on the adjacent layer. *A priori*, this hypothesis seems to be logical. It may be justified as follows.

Obviously, total pressure between layers *i* and *i*+1 has two components. The first one corresponds to pressure coming from layer $i-1$ acting on layer i, and transferred to layer $i+1$. The second component comes from layer *i* tensile forces. However, by hypothesis, layer $i-1$ is in the slip stage. The first component induces on layer *i* two friction force systems which are equal and opposite, since layer $i-1$ tends to induce slippage of layer *i* while layer $i+1$ impedes that slippage. Thus, the only component to be considered is the second one.

Rebuffel thus concludes that the following expression:

$$m' = \frac{\mu mnsr \sin 2\varphi}{\pi\left(1+\mu^2 \sin^2 \varphi\right)}$$

which was found for the outer layer, also applies to inner layers. Consequently, the friction moment $\mathfrak{M}'$, which is the cable resistance to bending after slip is complete on all layers, is obtained simply by adding similar terms for all layers. In particular, if μ, φ and r are identical in all layers, moment $\mathfrak{M}'$ is given by:

$$\mathfrak{M}' = \frac{\mu r \sin 2\varphi}{\pi\left(1+\mu^2 \sin^2 \varphi\right)} \sum mns$$

in which, summation must be made over the whole cable. Further simplification can be made if one assumes that product *ns* ,which is the layer metal area, is practically $3/4$ of the cylinder cross-section, in which the layer is inscribed. Thus, $\sum mns$ is $3/4$ of the cable cross-section first moment taken with respect to its center. Considering the cable as a solid cylinder, radius P, this first, or static, moment is known to be $\pi \mathrm{P}^3/2$, yielding:

$$\mathfrak{M}' = \frac{\mu r \sin 2\varphi}{2\left(1+\mu^2 \sin^2 \varphi\right)} \times \mathrm{P}^3$$

VII. A correction in the calculation of R_1 and subsequent Rebuffel formulas

It has been found that R_1 corresponds to that radius of curvature at which slip domain reaches half-loop end point A. At that instant, end point B has been for a while in the slip state.

When end point B is reached, wire tensile force at B is not known *a priori*. However, at that instant, tensile force at A is still given by $\theta_1 = t + \frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}_1}$.

When slip is complete, at any point on the half-loop, tensile force is given by:

$$\theta = ke^{-\frac{\mu\pi}{2l}x\sin\varphi}$$

Using the condition at A, where $x = -l$:

$$\theta = \frac{\theta_1 e^{-\frac{\mu\pi}{2l}x\sin\varphi}}{e^{\frac{\mu\pi}{2}\sin\varphi}}$$

The half-loop invariant length condition yields:

$$\int_{-l}^{+l} \frac{\theta - t}{\mathrm{E}s} dx = 0 \quad or \quad \int_{-l}^{+l} (\theta - t) dx = 0$$

Using this new equation, radius R_1 is easily obtained:

$$\mathrm{R}_1 = \frac{\mathrm{E}ms\cos^2\varphi}{t} \times \frac{e^{\pi\mu\sin\varphi} - 1}{1 - (1 - \mu\pi\sin\varphi)e^{\pi\mu\sin\varphi}}$$

and, finally:

$$\mathrm{R}_1 = \frac{\mathrm{E}m\cos^2\varphi}{r} \times \frac{e^{\pi\mu\sin\varphi} - 1}{1 - (1 - \mu\pi\sin\varphi)e^{\pi\mu\sin\varphi}} \qquad (14)$$

A layer moment of friction m' is easily obtained from this expression for R_1 . It is not even necessary to proceed with a direct calculation. In the above calculation of m' , where R_1 was Rebuffel's value, it can be noticed that R_1 enters only through the term $r + \frac{Em\cos^2\varphi}{R_1}$.

Depending on the selected expression for R_1, this term takes one of the following forms:

$r \times \frac{2e^{\mu\pi\sin\varphi}}{e^{\mu\pi\sin\varphi}+1}$ using Rebuffel's

$r \times \frac{\mu\pi\sin\varphi e^{\mu\pi\sin\varphi}}{e^{\mu\pi\sin\varphi}-1}$ using Eq. (14)

Now, using Eq. (15)[3], and the second expression, yields:

$$m' = \frac{\mu mnsr\sin 2\varphi}{\left(1+\mu^2\sin^2\varphi\right)} \times \frac{\mu\sin\varphi}{2} \times \frac{e^{\mu\pi\sin\varphi}+1}{e^{\mu\pi\sin\varphi}-1} \qquad (15')$$

Rebuffel's equations for the complete cable remain valid provided they are multiplied by the following factor f:

$$f = \frac{\pi\mu\sin\varphi}{2} \times \frac{e^{\mu\pi\sin\varphi}+1}{e^{\mu\pi\sin\varphi}-1}$$

Factor f is a relatively simple function of $\mu\pi\sin\varphi$. It is easily found that $f = 1$ when $\pi\mu\sin\varphi = 0$. For this value, its derivative $df/d\left(\pi\mu\sin\varphi\right)$ vanishes, and f is a monotonous increasing function for $\pi\mu\sin\varphi > 0$.

For $\pi\mu\sin\varphi = 1$, $f = \frac{e+1}{2(e-1)} \cong 1.08$. Taking typical values $\mu = 0.35$ and $\varphi = 18.5°$, $f \cong 1.02$.

Thus, if one takes $f = 1$, the error on friction moments is negligible, considering in particular the rather imprecise value to be given to friction coefficient μ.

This is equivalent to say that Rebuffel's formulas may be considered as acceptable in practice. Since they are simpler, not containing any exponential, than Eq. (15'), they will be used in the following.

[3] Translator's note: in the paper, Eq. (15) actually precedes Eq. (14)

VIII. An attempt to generalize Rebuffel's formulas

In this section, an attempt is made to revisit Rebuffel's hypotheses, which are recalled:

1°) Cable is bent in such a fashion that, passing from its initial rectilinear state to its final constant curvature $1/\mathrm{R}$ state, all intermediate states correspond to circular shapes of monotonously decreasing radius; that is, at any point on its axis, radius of curvature is given by $\mathrm{R}/g(t)$, in which g is a monotonous function of time t , increasing from 0 to 1 in the bending process, and it takes the same value at any point of cable axis.

2°) It is a single-strand cable (also called a "spiral" cable in the German literature)

In fact, hypothesis (1) may be split in two:

a. Bending deformation is monotonous
b. Radius of curvature is uniform, at a given instant, along cable axis

A. Case when bending deformation is not monotonous

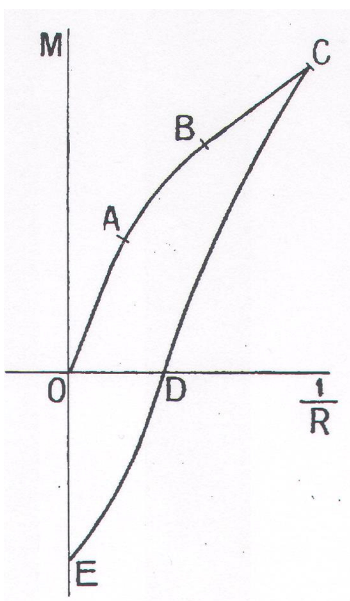

Figure 14

Assume that cable is rectilinear, under tensile force T. Friction is negligible and tensile force in individual wires is uniform over their whole length.

Cable is bent maintaining a circular shape, with decreasing radius R . The corresponding bending moment $\mathrm{M} = \mathfrak{M} + \mathfrak{M}'$ (necessary to overcome the cable resistance to bending) may be shown as a function of curvature $1/\mathrm{R}$ (Fig. 14).

It is assumed that imposed curvature is small enough so that wire material does not reach its yield limit.

From preceding considerations, it is found that the curve in Fig. 14 will comprise two linear segments OA and BC, joined by curve AB, where A and B correspond to the slip initiation and completion within cable.

From B to C, friction moment $\mathfrak{M}'$ remains constant.

At points most remote from flexure axis ("neutral axis"), all wires are subjected to a known tensile force. That force decreases on each side of these points according to a known law, and this force variation is balanced by friction forces which act in the direction opposite to wire element displacements which occur from the lower to the higher tension side.

At point C, a reverse moment is applied to bring a decrease in curvature. This will induce a tendency of wire elements to slip from the higher to the lower tension region.

However, friction forces will impede the relative displacements, and thus impede cable straightening. In fact, when applied moment M vanishes, cable curvature is not zero. (point D in Fig. 14). In order to straighten the cable, a reverse moment has to be applied (point E).

It is obvious then, that internal friction forces induce a classical hysteresis case, and cable bending resistance when passing over a pulley will depend on the cycles previously undergone. Curve OABC, which corresponds to Rebuffel's $\mathfrak{M}'$ (plus the known elastic term $\mathfrak{M}$) is only an approximation of M, which may be higher or lower. The only way to determine its degree of accuracy will be though experiment.

It is most important not to forget these *caveats* when using the theoretical results.

B. Non-uniform curvature bending

In Section VI, the following equation was used to derive the stick condition:

$$\frac{d}{d\alpha}\frac{\mathrm{E}ms\cos^2\varphi\cos\alpha}{\mathrm{R}}d\alpha = -\frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}}\sin\alpha\, d\alpha$$

This equation does not hold if R is a function of α. In fact, the derivative becomes:

$$\frac{d}{d\alpha}\frac{\mathrm{E}ms\cos^2\varphi\cos\alpha}{\mathrm{R}}d\alpha = -\frac{\mathrm{E}ms\cos^2\varphi}{\mathrm{R}}\sin\alpha\, d\alpha - \frac{\mathrm{E}ms\cos^2\varphi\cos\alpha}{\mathrm{R}^2}\frac{d\mathrm{R}}{d\alpha}d\alpha$$

and the no-slip condition [former Eq. (8)] becomes:

$$\mathrm{E}ms\cos^2\varphi\left(\sin\alpha + \frac{\cos\alpha}{\mathrm{R}}\frac{d\mathrm{R}}{d\alpha} - \mu\cos\alpha\sin\varphi\right) \le \frac{\mathrm{R}\mu\mathrm{T}_j\sin\varphi}{n\cos\varphi} \qquad (8')$$

Here, $d\alpha$ is the differential of the polar angle of a point on a wire axis. It may be related to to the differential element dl , l being the curvilinear abscissa on the cable axis of the point at which that axis intersects the torus meridian plane containing point M. Differentials $d\alpha$ and dl are related through equation $md\alpha = dl\tan\varphi$. Hence, Eq. (8') becomes:

$$\mathrm{E}ms\cos^2\varphi\left(\sin\alpha + \frac{m\cos\alpha}{\mathrm{R}\tan\varphi}\frac{d\mathrm{R}}{dl} - \mu\cos\alpha\sin\varphi\right) \le \frac{\mathrm{R}\mu\mathrm{T}_j\sin\varphi}{n\cos\varphi} \qquad (8'')$$

Impending slip in a given cable cross-section will occur at angle α such that the expression within the parenthesis is a maximum. As already mentioned in Section VI, α

is taken between 0 and π. If an impending slip point is reached in that domain on a particular wire, there is a corresponding point in the other half-loop.

The parenthesis depends on R through ratio $d\mathrm{R}/\mathrm{R}dl$. At a given instant in the bending process, it is a constant. Indeed, Rebuffel's process assumes that at time $t=0$ cable axis is a straight line. Then, $1/\mathrm{R}_0$ being the final curvature at a point of cable axis, each intermediate state at that point is $(1/\mathrm{R}_0)\,g(t)$, where $g(t)$ is a monotonous function increasing from 0 to 1. Indeed:

$$\mathrm{R}=\frac{\mathrm{R}_0}{g(t)} \text{ hence } \frac{d\mathrm{R}}{dl}=\frac{d\mathrm{R}_0}{dl}\times\frac{1}{g(t)}$$

Thus, ratio $d\mathrm{R}/\mathrm{R}dl$ does not depend on time and $d\mathrm{R}/\mathrm{R}dl=d\mathrm{R}_0/\mathrm{R}_0dl$. Calling $\mathrm{R}'=d\mathrm{R}/dl$ the derivative, that constant ratio will be noted R'/R in condition (8").

In the left-hand side of (8"), the parenthesis can be written as

$$\sin\alpha-\cos\alpha\left(\mu\sin\varphi-\frac{m}{\tan\varphi}\frac{\mathrm{R}'}{\mathrm{R}}\right)$$

Its maximum is reached at angle α given by:

$$\tan\alpha=-\frac{1}{\mu\sin\varphi-\dfrac{m}{\tan\varphi}\dfrac{\mathrm{R}'}{\mathrm{R}}}$$

The corresponding maximum value is:

$$\sqrt{1+\left(\mu\sin\varphi-\frac{m}{\tan\varphi}\frac{\mathrm{R}'}{\mathrm{R}}\right)^2}$$

Thus, the impending slip condition is given by:

$$\mathrm{R}\mu\mathrm{T}_j\sin\varphi=\mathrm{E}mns\cos^3\varphi\sqrt{1+\left(\mu\sin\varphi-\frac{m}{\tan\varphi}\frac{\mathrm{R}'}{\mathrm{R}}\right)^2} \qquad (9')$$

In this equation, all parameters but R' have a definite sign (with the current sign conventions, they are all positive). A word of caution must be made with respect to the sign of R', which depends on which is the curvilinear abscissa positive direction being selected on the cable axis.

When writing $m d\alpha = dl \tan\varphi$, it is implied that l and α vary in the same direction. Assuming that a wire half-loop goes from the convex to the concave side of the bent cable, the positive direction for l on the cable axis will go from the half-loop origin A to its end point B.

Consider the cable normal projection onto the bent axis osculating plane at point C (Fig. 15). Assume that radius of curvature R increases from A to C to B. In order to determine the point of impending slip in the cross-section CD for the radius m layer, one must consider each one of the half cross-sections, that is, with respect to the middle plane, the one in front and the one on the back of the figure.

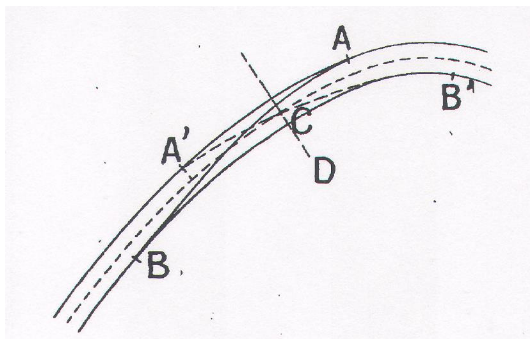

Figure 15

The first one intersects half-loops such as AB, for which $R' > 0$; the other one intersects half-loops such as A'B', for which $R' < 0$.

Eq. (9') shows that impending slip will occur first in the second half cross-section. This means that slip will start at one point in the cross-section, instead of two when $R' = 0$. And impending slip occurs when

$$R \mu T_j \sin\varphi = E m n s \cos^3\varphi \sqrt{1 + \left(\mu \sin\varphi + \frac{m}{\tan\varphi} \left| \frac{R'}{R} \right| \right)^2} \qquad (9'')$$

It occurs earlier, for a given radius of curvature R, than in the circular bending case.

Now, what happens after slip starts at a point? For a given wire, it is still possible to study the slip domain propagation for R values slightly inferior to the impending slip value (with ratio R'/R considered to be constant). One has to find two equations yielding abscissas x_1 and x_1' of the ends M_1 and M_1' of the slip domain. First equation is obtained from the static condition between tensile forces at both ends and friction forces. The second arises from geometry, stipulating that length of wire segment M_1M_1' be kept constant during the slip process.

Obviously, these equations are much more involved than in the circular bending case. The same applies to the calculation of the full slip state. In the circular case, symmetry was used to decide a priori that the half-loop end points A and B were fixed. This is not possible in the variable R case. A wire full-slip stage will be found when point M_1' coincides with point M_2 of a neighboring slip domain in the adjacent half-loop. And, finally, in general, full-slip will not be achieved simultaneously for all wires in the same layer.

Thus, this calculation does not seem to be practical.

Besides, it does not seem to be of such a fundamental importance. Indeed, as will be shown further, the interesting points, for a given cable, is to be able to determine the impending slip curvature – such as given by equation (9”) – and the bending moment which corresponds to the full slip state.

In the circular bending case, according to Rebuffel’s approach, once the full slip state is reached, component $\mathfrak{M}'$ of that moment remains constant. For a given layer, it is given by Eq. (15), in which the only parameters are the layer characteristics, and the applied tensile force, independently of radius of curvature R.

When full slip has been completed on a layer, the action of the layer below will not modify the tensile forces in that layer wires. Thus, this should apply, even with a variable radius of curvature, and the bending moment $\mathfrak{M}'$ should keep the same value.

Consequently, Rebuffel’s formulas, in particular Eq. (15), should apply to a cable in the full-slip state, whatever the shape taken by its axis.

C. Multi-strand cables

Slip may occur in two ways: individual strand slippage within the cable, and wire slip in a particular strand. Both cases may be easily studied using the previous results.

First, strand slippage should be studied, *mutatis mutandis*, following the same approach as the one used with wire slip within a spiral cable. Indeed, in this last case, slip conditions did not consider wire bending or twisting effects. Only the wire tensile force had to be taken into account. It has been shown that under an axial force, a spiral cable – and consequently a cable strand – behaves as a single wire with appropriate characteristics. And this applies with or without slip occurring between wires. The difference between both situations, strand slip or wire slip, will only be quantitative, in particular with respect to friction coefficients, which could be quite different.

Wire slip within a strand will be studied as in the spiral cable case. It has been shown in Section III that, knowing the multi-strand cable axis radius of curvature, individual strand radius of curvature is found, and the slip analysis for spiral cable may be applied. Slip within a strand will modify that strand stiffness, which enters in the multi-strand cable stiffness in the same way as wire stiffness enters in a spiral cable.

When a multi-strand cable is bent and the radius of curvature is small enough that slip is complete both between strands within the cable and between wires within strands, it is thus possible to determine the corresponding moment, as well as the stress in any of the wires.

A numerical example will be worked out in a further section.

IX. Friction coefficient determination. Klein's Method.

Translator's note:

In this section, Lehanneur summarizes a method devised by E.H. Klein and published in the June 1934 issue of "Fördertechnik und Frachtwerkehr", a German technical journal. The title of the paper was : "Die innere Reibung von Drahtseilen", or, in translation, "The friction within cables".

Here, this lengthy section (14 pages) has been omitted .

X. Application to cables of Isaachsen's problem

At the beginning of this paper, it has been mentioned that, in some cases, enforcing the 15 October 1947 recommendations might require cable bending to be taken into account. This is the case when one has to decide if, on a pulley or on a roller, contact has to be considered to be a line or a point contact. Indeed, recommendations differ radically in either case. It is thus crucial to decide which is applicable. Most of the time, the answer is obvious. But it is not always the case.

In 1905, Isaachsen, from Austria, has published a theoretical solution to the single wire problem (Fig. 16).

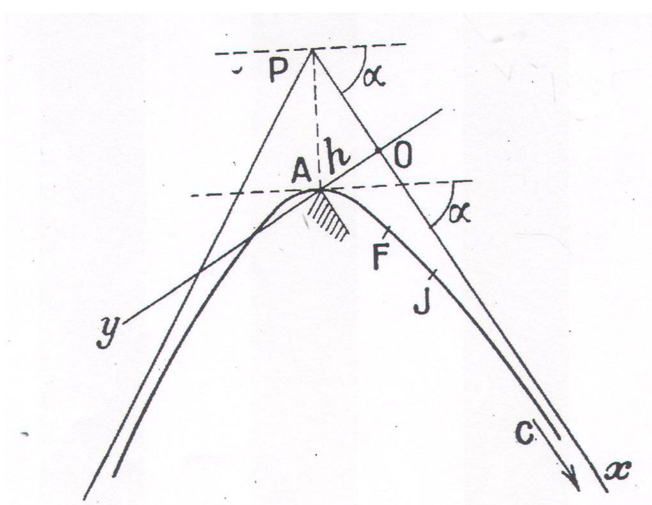

Figure 16

He determines the radius of curvature of such wire passing over a point support at A. That wire has a given bending stiffness and, far enough from A, it is subjected to a tensile force T on both sides of A. These forces make an angle 2α.

Assume now that there is a pulley at A, radius R. Calling ρ Isaachsen's radius of curvature at A. is, it is obvious that there will be point contact if $R \le \rho$, and line contact if $R > \rho$.

Isaachsen's solution is now a classical one. It was in fact rederived by Baticle – who, at that time, was probably unaware of Isaachsen's work – and published in his 1912 paper in the *Annales des Ponts et Chaussées*, which has already been referred to.

Being short and useful for the sequel, this solution is repeated below.

Wire section inertia in bending is noted I, and its material Young's modulus is E. Cartesian axes are defined as shown in Fig. 16. Axis *Ox* is taken along the line of action of one of the applied end forces. Point J is any point on the corresponding half. At that point, bending moment is T*y*, and, from beam theory:

$$\mathrm{T}y = \frac{\mathrm{EI}y''}{\left(1+y'^2\right)^{3/2}}$$

Multiplying both sides by y' one gets a first integral for this differential equation. Hence:

$$\frac{\mathrm{T}y'^2}{2} = -\frac{\mathrm{EI}}{\sqrt{1+y'^2}} + \mathrm{K} \qquad (26)$$

K being a constant such that at the wire end, boundary conditions $y = y' = 0$. (One sees easily that C cannot be at a finite distance). Thus, $\mathrm{K=EI}$.

Now, consider conditions at point A. There, $x = 0 \quad y = \mathrm{OA} = h$. Because of symmetry, bisector of angle between end forces must go through A. Thus, $y' = -\tan\alpha$. Let these values into Eq. (26) and,recalling that $\mathrm{K=EI}$, it yields:

$$\frac{\mathrm{T}h^2}{2} = \mathrm{EI}\left(1-\cos\alpha\right)$$

Whence:

$$h = \sqrt{\frac{2\mathrm{EI}\left(1-\cos\alpha\right)}{\mathrm{T}}}$$

Bending moment at point A is:

$$\mathrm{T}h = \sqrt{2\mathrm{EIT}\left(1-\cos\alpha\right)}$$

from which wire radius of curvature is derived:

$$\rho = \frac{\mathrm{EI}}{\sqrt{2\mathrm{EIT}\left(1-\cos\alpha\right)}}$$

And, finally:

$$\rho = \sqrt{\frac{\mathrm{EI}}{2\mathrm{T}\left(1-\cos\alpha\right)}} \qquad (27)$$

Now, consider the case of a cable. It was found that starting with a straight cable, and increasing curvature $1/\mathrm{R}$ from 0, cable resistance to bending, moment M will take the following values:

1°) As long as $1/\mathrm{R} < 1/\mathrm{R}_0$, where $1/\mathrm{R}_0$ is the outer layer impending slip curvature, moment M is given by $\mathrm{M} = a/\mathrm{R}$, a being a constant.

2°) Calling $1/\mathrm{R}_1$, the curvature at which slip is complete, when $1/\mathrm{R}_0 < 1/\mathrm{R} < 1/\mathrm{R}_1$ the variation of M with respect to $1/\mathrm{R}$ is a non-linear function which depends on the given cable.

3°) As $1/\mathrm{R}_1 \leq 1/\mathrm{R}$, function M has the form $\mathrm{M} = \dfrac{a'}{\mathrm{R}} + b$, where a' and b are constant parameters, with $a' < a$.

The M vs $1/\mathrm{R}$ curve is shown in Fig. 17. It starts with a straight line OB, followed by curve BD, which may present points of slope discontinuity at curvature values corresponding to beginning or end of slip in a layer. Curve BD is followed by straight line D*z*.

This curve may be replaced by a simpler one *o*F*z*. Point F, where *o*B and D*z* intersect will be called a “virtual” complete slip point. With this bilinear function, it is possible to obtain a formal solution to Isaachsen’s problem.

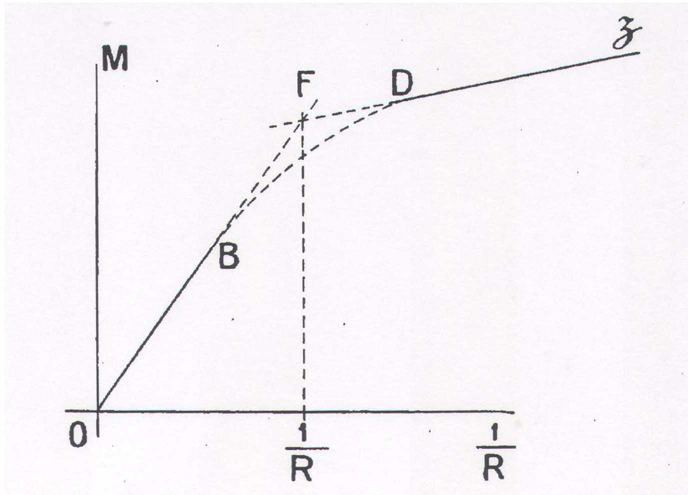

Figure 17

In Fig. 16, point F corresponds to the cable section at which complete slip is supposed to occur.

Thus, in the AF region (Fig. 16):

$$\mathrm{T}y = \frac{a'}{\mathrm{R}} + b$$

While in region FC: $\mathrm{T}y = \dfrac{a}{\mathrm{R}}$

Integration of these equations yields:

$$\frac{\mathrm{T}y^2}{2} = by - \frac{a'}{\sqrt{1+y'^2}} + \mathrm{K}' \tag{28}$$

and:

$$\frac{\mathrm{T}y^2}{2} = -\frac{a}{\sqrt{1+y'^2}} + \mathrm{K}$$

Conditions at C yield $\mathrm{K} = a$

However, there is continuity of y and y' at F. Eliminating $\dfrac{1}{\sqrt{1+y'^2}}$, one gets:

$$\frac{Ty^2}{2}\left(\frac{1}{a'} - \frac{1}{a}\right) = \frac{b}{a'}y + \frac{K'}{a'} - 1 \tag{29}$$

At point F, deflection y is known. Indeed:

$$\mathrm{T}y = \frac{a}{\mathrm{R}} = \frac{a'}{\mathrm{R}} + b$$

yielding:

$$\mathrm{R} = \frac{a - a'}{b} \quad \text{and} \quad y = \frac{ab}{(a - a')}$$

Letting this value into Eq. (29) yields:

$$\mathrm{K}' = a' \times \frac{ab^2 - 2b^2 a + 2\mathrm{T}a'(a - a')}{2\mathrm{T}a'(a - a')}$$

Now, apply Eq. (28) at point A (Fig. 16):

$$\frac{\mathrm{T}h^2}{2} = bh - a' \cos\alpha + \mathrm{K}'$$

Radius of curvature ρ at A is given by $\mathrm{T}h = \frac{a'}{\rho} + b$. Replace h in the above equation:

$$\frac{1}{2\mathrm{T}} \left(\frac{a'}{\rho} + b \right)^2 = \frac{b}{\mathrm{T}} \left(\frac{a'}{\rho} + b \right) - a' \cos\alpha + a' \frac{ab^2 - 2b^2 a + 2\mathrm{T}a'(a - a')}{2\mathrm{T}a'(a - a')}$$

This equation can be solved for ρ :

$$\rho = \sqrt{\frac{a'(a - a')}{2\mathrm{T}(a - a')(1 - \cos\alpha) - b^2}} \qquad (30)$$

This solution is valid only if the denominator is greater than 0. This condition is not satisfied if

$$b^2 > 2\mathrm{T}(a - a')(1 - \cos\alpha)$$

In this case, the following inequality holds:

$$b^2 a > 2\mathrm{T}(a - a')^2 (1 - \cos\alpha)$$

That is:

$$\sqrt{\frac{a}{2\mathrm{T}(1-\cos\alpha)}} > \frac{a-a'}{b}$$

which means that radius of curvature ρ given by Eq. (27) , for a non-slipping cable, whose bending stiffness is $\mathrm{EI}=a$, is bigger than the virtual slip radius (Fig. 17). Thus, curvature at A is too small to induce any slip; in this case, solution is given by Eq. (27).

From Eq. (30), it can be observed that , given two similar cables, with their respective geometrical parameters having a ratio λ, under axial loads T having a ratio λ^2, making the same deviation angle 2α over a point support, their respective minimum radius of curvature ρ will have the same ratio λ. Thus, the wire bending stresses at corresponding points will be equal, as are the initial traction stresses.

This comes from the fact that ρ, a and a' vary as λ^4 , b varies as λ^5 (resulting from the previous equations), and T varying as λ^2.

Thus, taking into account inner friction forces does not modify the mechanical similitude arising, in the single wire case, from Isaachsen's Eq. (27).

Consequently, in this work Section XI , where a 6 strand cable, made of 3 mm diameter wires, the results will be directly applicable to cables made of wires of other diameters.

There still remains a question about the validity of using the bilinear line oFz instead of curve OBDz (Fig. 17).

This same question was raised by Ernst in his memoir (see Sec. V). In his work, ρ was determined using polygonal line OBDz to represent moment M. He found that his results were very close to those obtained with the bilinear case. His answer was thus in favor of this approximation.

However, his calculations are rather tedious. Besides, they apply only to the particular case (even if it is often encountered in practice) where α is small and, consequently, where $y'^2 \ll 1$ .The above second order differential equations, instead of providing only a first integral, can then be fully integrated.

Here, they are just mentioned and won't be repeated.

XI. Some numerical applications

In this section, the LR formulas are applied to a specific problem. The selected cable is a six-strand cable, each strand being made 7 identical steel wires. Being rather technological in scope, the section has not been translated.